Chapter III

Associativity Theorems

In this chapter we obtain some associativity results about alternative algebras. The most important is Artin's Theorem which says that any two elements generate an associative subalgebra. Thus when we are working with products involving only two elements of an alternative algebra we can act just as if we were in an associative algebra.

§1. The nucleus, center, and centroid

The nucleus of an arbitrary linear algebra A is the set N(A) of elements $n \in A$ which associate with all elements of the algebra:

$$[n,x,y] = [x,n,y] = [x,y,n] = 0 .$$

For alternative algebras it is enough if $[n,x,y] = 0$. The nucleus is clearly a linear subspace of A since it is defined by linear conditions, and it is a subalgebra since if n,m are in N(A) then

$$\begin{aligned}[nm,x,y] &= \{(nm)x\} y - (nm)(xy) \\ &= \{n(mx)\} y - n\{m(xy)\} \\ &= n\{(mx)y\} - n\{(mx)y\} = 0 .\end{aligned}$$

and similarly when nm is in the middle or right of an associator. Thus N(A) is an associative subalgebra of A. In a product involving a nuclear element n we can drop parentheses, as in $nxy = (nx)y = n(xy)$.

The center of A is the set C(A) of elements c which both commute and associate with all elements:

$$[c,x] = [c,x,y] = [x,c,y] = [x,y,c] = 0 .$$

This again is a linear subspace of the nucleus, and it too is a subalgebra since if c,d are in C(A) then the product cd belongs to the nucleus and satisfies

$$[cd,x] = cdx - xcd = cxd - cxd = 0 .$$

Thus C(A) is a commutative, associative subalgebra of A.

More generally, the nucleizer of any set $S \subset A$ is

$$N(S) = \{n \mid [n,S,S] = 0\}$$

and the centralizer is

$$C(S) = \{c \mid [c,S] = [c,S,S] = 0\} .$$

In general these will not be subalgebras.

As in associative algebras, the centroid is is often more useful than the center. The centroid of any linear algebra is the set $\Gamma(A)$ of linear transformations T on A which commute with all left and right multiplications L_x, R_x; equivalently, $\Gamma(A)$ is the centralizer of the multiplication algebra M(A). In terms of elements,

$$\Gamma(A) = \{T \mid T(xy) = (Tx)y = x(Ty) \text{ for all } x,y\} .$$

Clearly $\Gamma(A)$ is an algebra of operators containing ΦI. A is naturally a $\Gamma(A)$ - module, and the defining formula shows multiplication is $\Gamma(A)$ - bilinear. Thus (even if $\Gamma(A)$ is not commutative) we can think of elements of the centroid as scalar multiplications ; in fact, the centroid consists precisely of the transformations which behave on the algebra like a scalar multiplication should. For instance, $[Tx,y,z] = [x,Ty,z] = [x,y,Tz] = T[x,y,z]$ and $[Tx,y] = [x,Ty] = T[x,y]$ for $T \in \Gamma(A)$.

In particular, if A is unital we see $[T1,y,z] = [x,T1,z] = [x,y,T1] = 0$ and $[T1,y] = 0$ since we can move the T outside, and any associator or commutator involving a factor 1 vanishes. This shows T(1) lies in the center. Although the centroid consists of operators and the center of elements, the two can be identified in the unital case:

1.1 (Centroid-Center Lemma) The centroid of a unital linear algebra is isomorphic to the center, $\Gamma(A) \cong C(A)$, under the map $T \to T(1)$. In this case the elements of the centroid are just the multiplications $T = L_t = R_t$ by elements $t = T(1)$ of the center.

Proof. We just saw all T(1) are central. The map $T \to T(1)$ is clearly linear, and it is multiplicative since $ST \to S(T(1)) = S(1 \cdot T(1)) = S(1)T(1)$. From $T(x) = T(1 \cdot x) = T(1)x$ we see $T = L_t$ for $t = T(1)$, similarly $T = R_t$. This shows the map is injective, since $t = T(1) = 0$ implies $T = 0$ as operator. It is surjective, for if c is central the linear transformation $T = L_c$ lies in the centroid ($T(xy) = T(x)y = x\,T(y)$ because $c(xy) = (cx)y = x(cy)$) and has $T(1) = L_c\,1 = c$. ◻

The reason the centroid is more useful than the center is that it always exists; for example, a simple algebra may have no center at all, but it always has a nice centroid.

1.2 (Schurs' Centroid Lemma) The centroid of a simple linear algebra is a field.

Proof. A simple algebra has no proper ideals, ie no subspaces invariant under all multiplications L_x, R_y. Also $A^2 \neq 0$ implies $M(A)A \neq 0$. Therefore A is irreducible as M(A) - module, and by Schur's Usual Lemma the Centralizer $\Gamma(A)$ of M(A) (in $\text{End}_\Phi(A)$) is a division ring. Commutativity follows from a general

1.3 Lemma. If $A^2 = A$ then $\Gamma(A)$ is commutative.

Proof. If S, T lie in the centroid then $(ST - TS)A^2 = 0$ since by the HIDING TRICK $ST(xy) = S\{x\,T(y)\} = S(x)\,T(y) = T\{S(x)y\} = TS(xy)$. When $A^2 = A$ the operator $ST - TS$ is zero, and $ST = TS$. ⊡⊡

Thus a simple linear algebra (over an arbitrary ring of scalars Φ) can always be reinterpreted as a simple algebra over a field $\Gamma(A)$. (Notice that A is even simpler over $\Gamma(A)$ than it was over Φ, since $\Gamma(A) \supset \Phi$ means it is harder to be a $\Gamma(A)$-ideal than a Φ-ideal). Since vector spaces are nicer than Φ-modules, this is very helpful. In some sense $\Gamma(A)$ is the "natural" ring of scalars for A.

An algebra over Φ is centroidal if its centroid is just $\Gamma(A) = \Phi I$, and central if it is unital with center $C(A) = \Phi 1$. Especially important are central simple and centroid simple algebras. A simple algebra A is always centroidal when regarded as an algebra over $\Gamma(A)$.

The centroid behaves correctly under scalar extension.

1.4 Lemma. If $\Omega \supset \Phi$ is free as a Φ-module (for example, if Ω and Φ are fields) then $\Gamma(A_\Omega) = \Gamma(A)_\Omega$.

Proof. If $T \in \Gamma(A)$ commutes with all multiplications by A clearly any $\omega \otimes T$ commutes with all multiplications by $\Omega \otimes A$, so $\Gamma(A)_\Omega = \Omega \otimes \Gamma(A) \subset \Gamma(A_\Omega)$.
For the converse, by freedom we can choose some basis $\{\omega_i\}$ for Ω over Φ, so $\Omega = \oplus_i \Phi\omega_i$ and $A_\Omega = \oplus_i\{\Phi\omega_i \otimes_\Phi A\} = \oplus_i \omega_i \otimes A$ where each $\omega_i \otimes A$ is a Φ-subspace invariant under multiplications by elements of A ($= 1 \otimes A$). If T belongs to the centroid $\Gamma(A_\Omega)$ define endomorphisms T_i of A by the formula $T(1 \otimes a) = \Sigma\, \omega_i \otimes T_i(a)$. Any multiplication operator M on A extends to a multiplication operator $I \otimes M$ on A_Ω (observe $I \otimes L_a = L_{1\otimes a}$, $I \otimes R_a = R_{1\otimes a}$); since $T \in \Gamma(A_\Omega)$ we know $T(1 \otimes M) = (1 \otimes M)T$. But $T(1 \otimes M)\{a\} = T(1 \otimes Ma) = \Sigma\, \omega_i \otimes T_i(Ma)$ and $(1 \otimes M)T\{a\} = (1 \otimes M)\Sigma\, \omega_i \otimes T_i(a) = \Sigma\, \omega_i \otimes M(T_i(a))$, so $T_i(Ma) = M(T_i a)$. This says the endormorphism T_i of A commutes with all $M \in M(A)$, so $T_i \in \Gamma(A)$. Thus T coincides with $\Sigma\, \omega_i \otimes T_i$ on A and both are Ω-linear, they coincide on $\Omega A = A_\Omega$. Thus $T = \Sigma\, \omega_i \otimes T_i$ lies in $\Omega \otimes \Gamma(A)$ and $\Gamma(A_\Omega) \subset \Gamma(A)_\Omega$. ∎

To see why centroid-simple algebras are nice, say that an algebra A over a field Φ is <u>strictly simple</u> if it is simple and stays simple under scalar extension: $A_\Omega = \Omega \otimes_\Phi A$ is simple for all fields $\Omega \supset \Phi$.

1.5 (Strict Simplicity Theorem) A linear algebra over a field Φ is strictly simple iff it is centroid simple. When A is centroid simple over Φ, all extensions A_Ω are centroid simple over Ω.

Proof. Assume A is strictly simple over Φ with centroid $\Gamma = \Gamma(A)$. By strictness, $A_\Gamma = \Gamma \otimes_\Phi A$ is simple. On the other hand, the linear map $\Gamma \otimes A \xrightarrow{F} A$ determined by $T \otimes a \to Ta$ is a homomorphism of Γ algebras since $F(S \cdot T \otimes a) = F(ST \otimes a) = ST(a) = S \cdot F(T \otimes a)$ and $F((S \otimes a)(T \otimes b)) = F(ST \otimes ab) = ST(ab) = S(a\, T(b)) = S(a)\, T(b) = F(S \otimes a)\, F(T \otimes b)$ because of the way the elements S, T in the centroid behave relative to products, and because of the way operations in the Γ-algebra $\Gamma \otimes A$ are defined. The kernel of the homomorphism F is an ideal in A_Γ different from A_Γ since $F(I \otimes a) = a$, so it must be zero. But if $T \in \Gamma$ did not belong to Φ then $T \otimes a - I \otimes Ta$ would be nonzero in A_Γ when $a \neq 0$ (since we tensor over Φ), yet $F(T \otimes a - I \otimes Ta) = T(a) - I(Ta) = Ta - Ta = 0$, contradicting $\operatorname{Ker} F = 0$. Thus all T belong to Φ and $\Gamma(A) = \Phi I$.

From the Lemma 1.4 it is immediate that if A is centroidal over Φ, $\Gamma(A) = \Phi$, then A_Ω is centroidal over Ω, $\Gamma(A_\Omega) = \Gamma(A)_\Omega = \Omega \otimes_\Phi \Phi = \Omega$.

If A is simple in addition to being centroidal over Φ, we show A_Ω is simple too. We will use a minimal length argument.

Suppose B is a nonzero ideal in A_Ω. As before, any element can be expressed uniquely as $b = \Sigma\,\omega_i \otimes b_i$ for some $b_i \in A$; choose $b \neq 0$ in B with minimal possible number of nonzero b_i 's. (What follows should remind you of the Density Theorem). Say $b_o \neq 0$. If $M \in \hat{M}(A)$ is any multiplication we have $(I \otimes M)b = \Sigma\,\omega_i \otimes Mb_i$; if $Mb_o = 0$ then there are fewer nonzero Mb_i than there are nonzero b_i, and $(I \otimes M)b$ is still in B if B is an ideal, so by minimality this can only happen when $(I \otimes M)b = 0$: $Mb_o = 0 \Rightarrow Mb_i = 0$ for all i. But this makes $T_i(Mb_o) = Mb_i$ a well-difined linear map $\hat{M}(A)b_o \xrightarrow{T_i} A$. Since $\hat{M}(A)b_o$ is an ideal in A containing $b_o \neq 0$, by simplicity it is all of A. Thus T_i is an endomorphism of A, which by construction $T_i(M\,M'b_o) = MM'b_i = MT(M'b_o)$ commutes with all $M \in M(A)$, so T_i belongs to the centroid $\Gamma(A) = \Phi I$. Then $T_i = \alpha_i I$, $b_i = T_i(b_o) = \alpha_i b_o$, and $b = \Sigma\,\omega_i \otimes \alpha_i b_o = \Sigma\,\omega_i\alpha_i \otimes b_o = \omega b_o$. Since $b \neq 0$ we have $\omega \neq 0$, so $\omega^{-1}b = b_o$ lies in B and in A, and $B \cap A \neq 0$; but $B \cap A$ is an ideal in A, so by simplicity $B \cap A = A$, $B \supset A$, and (since B is an Ω-ideal) $B \supset \Omega A = A_\Omega$. Thus the only ideals are $B=0$ and $B=A_\Omega$, and $A\,^2 \neq 0$ if $A^2 \neq 0$, so A_Ω is simple.

Consequently, if A/Φ is centroid simple so is A_Ω/Ω. ☐

In any <u>alternative</u> algebra, an element n of the nucleus can slip in and out of associators:

$$(1.6)\qquad n[x,y,z] = [nx,y,z] = [xn,y,z] = [x,ny,z] = [x,yn,z] = [x,y,nz] = [x,y,zn] = [x,y,z]n.$$

To see this note that $n[x,y,z] = n\{(xy)z\} - n\{x(yz)\} = \{n(xy)\}z - \{nx\}(yz) = \{(nx)y\}z - (nx)\{yz\} = [nx,y,z]$ in any algebra, and similarly $[xn,y,z] = [x,ny,z]$ and $[x,yn,z] = [x,y,nz]$ and $[x,y,zn] = [x,y,z]n$ because n slips in and out of parentheses (being careful to keep the same order). What is unique to alternative algebras is the ability of n to climb over a variable, $[nx,y,z] = [xn,y,z]$ (and similarly it can climb over y or z): $[nx,y,z] = n[x,y,z] = -n[y,x,z] = -[ny,x,z] = [x,ny,z] = [xn,y,z]$ by slipping in and out of parentheses, as well as interchanging variables.

An immediate consequence of this is the fact that in the alternative case the nucleus commutes with associators

(1.7) $[N(A),[A,A,A]] = 0$

and that

(1.8) $[N(A),A] \subset N(A)$

since $[[n,x],y,z] = [nx,y,z] - [xn,y,z] = 0$. Not only does $[n,x]$ lie in the nucleus, it kills associators involving x:

(1.9) $[x,n][x,y,z] = 0$. (Nucleus-Center Identity)

Indeed, $(xn)[xyz] = x\{n[xyz]\} = x\{[xyz]n\} = \{x[xyz]\}n = [x,y,zx]n = n[x,y,zx] = n\{x[x,y,z]\} = (nx)[xyz]$.

This is another example of the Hiding Trick - we hid the variable x inside an associator until we had commuted n past the associator, then brought it out of hiding.

This mutual antagonism between commutators and associators has an important consequence.

1.10 (Nucleus = Center Theorem.) If A is an alternative algebra without zero divisors, then either A is associative, $N(A) = A$, or else its nucleus and center coincide, $N(A) = C(A)$.

Proof. Suppose A is <u>not</u> associative, so there exist elements x not in the nucleus. For such an x some associator $[x,y,z] \neq 0$, so $[x,n][x,y,z] = 0$ and the absence of zero divisors forces $[x,n] = 0$ for any $n \in N(A)$. Thus the nucleus commutes with any $x \notin N(A)$; on the other hand, if $w \in N(A)$ then $x + w \notin N(A)$ implies $[x + w,n] = 0$ by the above, and since $[x,n] = 0$ we have $[w,n] = 0$ for $w \in N(A)$. Thus $[A,N(A)] = 0$ and everything in the nucleus is already in the center. (This is a standard sort of argument: if something linear happens whenever x isn't in some subspace, then it also happens when x is in that subspace).

Another way to argue for arbitrary w is to use the linearization.

$$(1.9)' \qquad [x,n][w,y,z] + [w,n][x,y,z] = 0 ;$$

here $[x,n] = 0$, so $[w,n][x,y,z] = 0$ but $[x,y,z] \neq 0$, forcing $[w,n] = 0$ for all w and all n.

Still another proof uses the Two Kernels Lemma explicitly (see Part I). If $[x,y,z] \neq 0$ then Ker $([\cdot, y,z])$ is not all A, yet $[w,n][w,y,z] = 0$ implies either $[w,n] = 0$ or $[w,y,z] = 0$ for any w, any w lies in one of the two kernels, and $A = \text{Ker } [\cdot,n] \cup \text{Ker } [\cdot,y,z]$. This is impossible if both kernels are proper, and $\text{Ker } [\cdot,y,z]$ is proper, so $\text{Ker } [\cdot,n]$ must not be, and $[A,n] = 0$ for any $n \in N(A)$. ∎

We have seen something like this before - in II. 3.19 we saw that the nucleus and center of a Cayley algebra coincide, $N(A) = C(A) = \Phi 1$. This is the real reason the Nucleus = Center Theorem holds: the only algebras without zero divisors (or merely prime; see Appendix II) which are not associative are Cayley algebras.

One final, unrelated result:

1.11 (Uniqueness Proposition for Isotopes.) Two isotopes $A^{(u,v)}$ and $A^{(u',v')}$ Coincide as algebras iff $u' = un$, $v' = n^{-1}v$ for some invertible element n of the nucleus,

$$A^{(un,n^{-1}v)} = A^{(u,v)}.$$

Proof. Immediately from the definition $x \cdot_{un,v} y = \{x(un)\}\{vy\} = \{(xu)\} n\ (vy) = (xu)\{n(vy)\} = x \cdot_{u,nv} y$ we see

$$A^{(un,v)} = A^{(u,nv)}$$

for nuclear n, proving the "if" part of the proposition.

For the "only if" part, suppose

$$\{xu\}\{vy\} = \{xu'\}\{v'y\} \tag{1.12}$$

for all x,y (where u,v,u',v' are invertible). Then $u' = un$ for some n (namely $n = u^{-1}u'$); setting $x = u'^{-1}$ and $y = 1$ in (1.12) leads to $n^{-1}v = v'$. We must show n belongs to the nucleus. Setting $y = v'^{-1}$ in (1.12) yields $(xu)n = xu'$ for all x, and similarly $x = u'^{-1}$ yields $n^{-1}(vy) = v'y$, so that (1.12) becomes $\{xu\}\{vy\} = \{(xu)n\}\{n^{-1}(vy)\}$. Given x', y' we set $x = x'u^{-1}$, $y = v^{-1}(ny')$ (so $xu = x', vy = ny'$) and obtain $x'\{ny'\} = \{x'n\}y'$. Thus $n \in N(A)$. ∎

Exercises

1.1 The <u>left nucleus</u> in a linear algebra A is $N_\ell(A) = \{n \mid [n,A,A] = 0\}$, the <u>middle nucleus</u> is $N_m(A) = \{n \mid [A,n,A] = 0\}$, and the <u>right nucleus</u> is $N_r(A) = \{n \mid [A,A,n] = 0\}$; thus $N(A) = N_\ell(A) \cap N_m(A) \cap N_r(A)$. Show all three nuclei are associative subalgebras. If z belongs to one of the nuclei, and commutes with everything, does it belong to the other nuclei as well?

1.2. If c is central in a linear algebra (unital or not) show $R_c = L_c$ belongs to the centroid.

1.3 If c centralizes a subalgebra $B \subset A$ (A alternative) and $d \in B$ also centralizes B, show cd and dc also centralize B. If $d \notin B$ this need no longer be true: the centralizer in A of a subalgebra B is not usually a subalgebra. Construct an example with $B = \Phi[b]$ generated by a single element b, so everything in A automotically nucleizes B: $[A,B,B] = 0$, but $[x,b] = [y,b] = 0$ without $[xy,b] = 0$.

1.4 In an alternative algebra A show $[x,n]\,x \in N(A)$ for all $n \in N(A)$. Conclude $[x,n][x,y,z] = 0$ for all x,y,z and nuclear n.

1.5 Show that the left, right, two-sideal ideals generated by a nuclear element n are $\hat{A}n$, $n\hat{A}$, $\hat{A}n\hat{A}$ as in associative algebras. If n is trivial, $U_n \hat{A} = 0$, show the ideal AnA is also trivial.

1.6 Show $M(A_\Omega) = M(A)_\Omega$ for any scalar extension $\Omega \supset \Phi$, and any linear algebra A. If we regard A as algebra over $\Gamma(A)$, show the multiplications $M(A/\Gamma)$ of A as Γ-algebra coincide with those $M(A/\Phi)$ of A as Φ-algebra.

1.7 If A is finite -dimensional simple linear algebra, show $M(A) = \mathrm{End}_{\Gamma(A)}(A)$. In particular, $1 \in M(A)$ and $M(A) = \hat{M}(A)$.

1.8 Re-prove the Strict Simplicity Theorem by means of the Density Theorem of Part I.

1.9 In the case of a unital algebra, reprove the Strict Simplicity Theorem in terms of central simplicity. Avoid Density (prove an element is central by showing it commutes and associates with everything).

1.10 Prove (as in 1.4, but not using 1.4 explicitly) that $C(A_\Omega) = C(A)_\Omega$ if $\Omega \supset \Phi$ is a free extension.

1.11 If A is algebraic over a field Φ but $\omega \notin \Phi$, show $\Phi(\omega) \otimes_\Phi \Phi(\omega)$ has zero divisors. Conclude that if $\Gamma(A)$ contains an element algebraic over (but not in) Φ then A_Γ has zero divisors in its centroid and is not simple. If A is finite-dimensional over a field Φ, show $\Gamma(A)$ is algebraic.

1.12 If Ω is a proper field extension of Φ, show $\Omega \otimes_\Phi \Omega$ is never a field. Conclude that if $\Gamma(A) > \Phi$ then $\Gamma(A) \otimes_\Phi A$ is not simple.

#10. Problem Set on Nuclear Isotopes

1. If n lies in the nucleus of an alternative algebra A and $1^{(u,v)} = v^{-1}u^{-1}$ is the unit of the isotope $A^{(u,v)}$, show $n1^{(u,v)} \cdot_{u,v} x = nx$ and $x \cdot_{u,v} 1^{(u,v)}n = xn$.

2. Show $[n1^{(u,v)}, x, y]^{(u,v)} = 0$ (associator in $A^{(u,v)}$) so that $N \cdot 1^{(u,v)}$ is contained in the nucleus $N^{(u,v)}$ of $A^{(u,v)}$. Show $N^{(u,v)} = N \cdot 1^{(u,v)}$.

3. Show $C^{(u,v)} = C \cdot 1^{(u,v)}$

4. Conclude that an isotope $A^{(u,v)}$ is associative iff A is associative, and commutative associative iff A is.

5. Prove $T_w(x) = w\,x\,w^{-1}$ defines an automorphism of A iff $w^3 \in N(A)$.

6. Even though T_w need not be an automorphism, prove $T_w(N(A)) = N(A)$. Conclude $wN = Nw$.

7. Establish the relations $N^{(u,v)} = N \cdot 1^{(u,v)} = v^{-1} \cdot N \cdot u^{-1} = 1^{(u,v)} \cdot N$, $C^{(u,v)} = C \cdot 1^{(u,v)} = v^{-1} \cdot C \cdot u^{-1} = 1^{(u,v)} \cdot C$.

Slater: Nuclear [illegible]

#11. Problem Set on Nuclear Radicals

We would like to break an alternative algebra into an associative part and a purely alternative part.

1. The <u>associator ideal</u> A(A) of any linear algebra A is the ideal generated by all associators $[x,y,z]$. Use (2.4) to show $A(A) = \hat{A}[A,A,A] = [A,A,A]\hat{A}$. Show A(A) is the smallest ideal B such that A/B is associative. Conclude A(A) = 0 iff A is associative, and A(A) = A iff A is purely exceptional (has no associative homomorphic images).

2. Show the following are equivalent for an element $z \in N(A)$ in an alternative algebra A: (i) all za are nuclear, (ii) all az are nuclear, (iii) $z[A,A,A] = 0$, (iv) $[A,A,A]z = 0$, (v) $zA(A) = 0$, (vi) $A(A)z = 0$. A nuclear element is <u>properly nuclear</u> if it is nuclear and all multiples za stay nuclear. Show the set of properly nuclear elements forms a nuclear ideal, the <u>nuclear radical</u> , which for reasons of euphony is called Nurd (A). Show Nurd (A) contains all one-sided nuclear ideals of A.

Conclude if [illegible] then Nurd(A) = [illegible]

3. Show $\mathrm{Nurd}(A)\,A(A) = A(A)\,\mathrm{Nurd}(A) = 0$, so that if A is a prime algebra ($BC = 0 \Rightarrow B = 0$ or $C = 0$ when $B, C \triangleleft A$) either A is associative or it is <u>purely alternative</u> in the sense that it has no nuclear ideals: Nurd(A) = 0. Show $\mathrm{Nurd}(A) \cap A(A)$ is a trivial ideal, so if A is merely semiprime at least A is <u>unmixed</u> in the sense that $A(A) \cap \mathrm{Nurd}(A) = 0$; if a one-sided ideal B is nuclear show $B \cap A(A) = 0$. Conversely, in general

if $B \cap A(A) = 0$ then B is nuclear. Conclude that if A is purely alternative then A(A) hits all nonzero ideals B.

4. Show that if z is properly nuclear in A, any homomorphic image F(z) is properly nuclear in F(A). Conclude F(Nurd(A)) $\subset$ Nurd(F(A)). If A is unmixed, show Nurd (A/Nurd(A)) = 0 .

5. Show [N(A), N(A)] and any [x,N(A)][x,N(A)] are contained in Nurd(A). Conclude that if A is purely purely alternative, its nucleus N(A) is commutative. (In the next problem set we show N(A) not only commutes with itself, but with all of A).

6. Just as A/A(A) is the maximal associative image of A, show if A is semiprime that ^{A}/Nurd(A) is the maximal purely alternative image of A.

In some sense, A/A(A) is the associative "part" of A and ^{A}/Nurd(A) the purely alternative "part."

Slater

#12. Problem Set on Nucleus and Center

How far the nucleus is from being central is measured by the ideal $CN(A)$ generated by all commutators $[a,n]$ for $a \in A$ and $n \in N(A)$: $CN(A) = 0$ iff all $[a,n] = 0$ iff $N(A) = C(A)$.

1. Show $CN(A) = \hat{A}[A,N(A)] = [A,N(A)]\hat{A}$.
2. Show $CN(A) \subset Nurd(A) \Longleftrightarrow CN(A)\ A(A) = 0 \Longleftrightarrow [A,N(A)][A,A,A] = 0 \Longleftrightarrow [CN(A),A,A,] = 0$. Usually $CN(A)$ will be contained in $Nurd(A)$. Show that although $[CN(A),A,A]$ may not always be zero, at least it is always contained in $N(A)$.

3. Show any $m = [a[x,n],b,c]$ (for $a,b,c,x \in A, n \in N(A)$) is a trivial element of the nucleus. Conclude that either $CN(A) \subset Nurd(A)$ or else there is a trivial ideal $I(m)$ where m is contained in $CN(A) \cap A(A) \cap N(A)$.

4. Deduce <u>Slater's General Nuclear Theorem</u>: If $CN(A) \cap A(A) \cap Nurd(A)$ contains no trivial elements then $CN(A) \subset Nurd(A)$, so that if A is also purely alternative then $N(A) = C(A)$.

5. Deduce <u>Slater's Nuclear Theorem</u>: If A is semiprime then $CN(A) \subset Nurd(A)$. If A is semiprime and purely alternative then its nucleus and center coincide, $N(A) = C(A)$.

6. Deduce $N(A) = C(A)$ also if A has no associative ideals, or is semiprime with no associative images (=purely exceptional).

7. If A has no trivial nuclear elements show: (i) A is unmixed, (ii) $CN(A) \subset Nurd(A)$, (iii) $N(A) = C(A)$ if A is purely alternative, (iv) $A(A) \cap N(A) = A(A) \cap C(A)$, (v) $[N(A),A(A)] = 0$.

#13. Problem Set on Slater's Nuclear Conjectures

Michael Slater has made the following conjectures about the distance of the nucleus from the center in the purely alternative case:

(SN 1) $CN(A) \cap A(A)$ is zero or contains trivial nuclear ideals

(SN 2) $CN(A) \subset N(A)$ or else $CN(A) \cap A(A)$ contains trivial nuclear ideals

(SN 3) A unmixed $\Rightarrow CN(A) \subset N(A)$

(SN 4) A purely alternative $\Rightarrow N(A) = C(A)$

(SN 5) If $Nurd(A)$ is commutative without nilpotent elements, then $N(A) = C(A)$.

1. Show in general $1 \Rightarrow 2 \Rightarrow 3 \Longleftrightarrow 4 \Longleftarrow 5$ for alternative algebras.

2. Let $g(x_i, y_i, z_i; w_j; n_j) = \Pi_{i=1}^{r} [x_{ij} y_{ij} z_i] \, \Pi_{j=1}^{s} [w_j, n_j]$ $(r, s \geq 1)$ for $x_i \, y_i \, z_i, w_j \in A$ and $n_j \in N(A)$. Show g is independent of the order and association of the factors $[x_i, y_i, z_i]$ and $[w_j, n_j]$. Show its values lie in $CN(A) \cap N(A)$.

3. Show g is an alternating function of its arguments x_i, y_i, z_i, w_j, which vanishes if one of these variables lies in $N(A)$. If $w_s = w_s' w_s''$ show $g(x,y,z,w) = g(x,y,z,w',n) w_s'' + g(x,y,z,w'',n) w_s' + g(x,y,z,w'',n'')$ where $n_s'' = [w_s'', n_s]$. Conclude that if g vanishes on a set of generators x, y, z, w for A modulo $N(A)$, it vanishes everywhere.

4. Conclude that if A is finitely generated mod N(A) then $g \equiv 0$ for large enough r and s. Show 2-5 hold if A is finitely generated mod N(A).

5. If A is generated mod N(A) by 3 elements, show $CN(A) \subseteq N(A)$.