§8 Bimodules with involution

A bimodule with involution or $*$-bimodule for a $*$-algebra A is a bimodule M together with an endomorphism $m \to \bar{m}$ of period 2 such that

$$a \oplus m \to \bar{a} \oplus \bar{m}$$

induces an involution on the split null extension $E = A \oplus M$, turning it into a $*$-algebra. Because $\overline{xy} = \bar{y}\bar{x}$ already holds for $x, y \in A$ (if A is a $*$-algebra) or for $x, y \in M$ (as $M^2 = 0$), the condition amounts to

$$\overline{am} = \bar{m}\bar{a} \qquad \overline{ma} = \bar{a}\bar{m} . \tag{8.1}$$

This makes it clear that the negative of an involution on M is again an involution. Also, the regular bimodule $M = A$ with the natural involution becomes a $*$-bimodule.

We have obvious notions of $*$-sub-bimodule, $*$-irreducible, $*$-completely reducible, $*$-homomorphism etc. One convenient fact about $*$-homomorphisms: if $M \xrightarrow{F} N$ satisfies $F(am) = aF(m)$ and $F(\bar{m}) = \overline{F(m)}$, it automatically satisfies $F(ma) = F(m)a$: $\overline{F(ma)} = F(\overline{ma}) = F(\bar{a}\bar{m}) = \bar{a}F(\bar{m}) = \bar{a}\overline{F(m)} = \overline{F(m)a}$.

For a composition algebra A we thus obtain the regular $*$-bimodules $reg_+(A)$ and $reg_-(A)$ obtained from the regular bimodule $M = A$ by taking the usual involution or its negative. The Cayley-Dickson $*$-bimodules $cay_+(A)$ and $cay_-(A)$ are obtained from the Cayley-Dickson bimodule $M = A\ell$ by taking the usual involution (namely $-I$) induced from the involution on $C(A) = A \oplus A\ell$, or its negative (namely $+I$).

As an example, we consider the possible involutions on the regular and Cayley bimodules for the composition algebras.

8.2 Proposition. If $\mathbb{C}$ is an ordinary composition algebra of dimension 1, 4, 8 over a field Φ then the only involutions on reg $\mathbb{C}$ are $\pm$ the standard involution; if $\mathbb{C}$ has dimension 2 the involutions are of the form $f(a) = c_f\bar{a}$ where $n(c_f) = 1$, and in this case $\mathbb{C}_f$ is $*$-isomorphic to $\text{reg}_+\mathbb{C}$. If $\mathbb{C} = \Phi e_1 \boxplus \Phi e_2$ is split of dimension 2 there are no involutions on the module Φe_i.

If $\mathbb{C}$ is a division algebra of dimension 1, 2, 4 then the only involutions on cay $\mathbb{C}$ are $+$ the standard involution (i.e. ∓ 1); if $\mathbb{C} = e_1\mathbb{C} + e_2\mathbb{C}$ is split of dimension 2 or 4, the only involution on cay $(e_i\mathbb{C})$ are $\pm$ the restrictions of the standard involutions (i.e. ∓ 1).

Proof. According to the Commuting Criterion 3.19, in the regular bimodule reg $\mathbb{C}$ of dimension 1, 4, 8 the only commuters are the elements of $\Phi 1$, so an involution must have $f(1) = \alpha 1$; since $f^2(1) = 1$ we see $\alpha^2 = 1$, so $\alpha = \pm 1$ and by (8.1) $f(a) = f(a\cdot 1) = f(1)\bar{a} = \pm\bar{a}$. Thus f is $\pm$ the standard involution. In dimension 2, if $f(1) = c$ then $f(a) = f(a\cdot 1) = f(1)\bar{a} = c\bar{a}$ and $1 = f(f(1)) = f(c) = c\bar{c}$. The map $a \mapsto \bar{c}a$ is a $*$-isomorphism $\mathbb{C}_f \overset{F}{\to} \mathbb{C}_+$: it is a linear bijection with $F(a\cdot b) = \bar{c}ab = a\bar{c}b = aF(b)$ and $F(b\cdot a) = \bar{c}ba = F(b)a$ and $F(f(a)) = F(c\bar{a}) = \bar{c}c\bar{a} = \bar{a}$. Since $f(e_i) = f(e_ie_i) = f(e_i)\bar{e}_i = f(e_i)e_j \in \Phi e_ie_j$, there are no involutions on Φe_i.

Cay $\mathbb{C} = \mathbb{C}\ell$ is not an alternative bimodule if $\mathbb{C}$ has dimension 8. Let $\mathbb{C}$ be a division algebra of dimension 1, 2, or 4. If $f(\ell) = c\ell$ then $f(a\ell) = f(\ell)\bar{a} = (c\ell)\bar{a} = (ca)\ell$, where $\ell = f(f\ell) = f(c\ell) = c^2\ell$. But $c^2 = 1$ in a division algebra implies $c = \pm 1$ (as $(c-1)(c+1) = 0$), so $f(a\ell) = \pm a\ell$. In this case the only involutions are $\pm I$. If $\mathbb{C}$ is split of dimension 2 or 4, $\mathbb{C} = e_1\mathbb{C} \oplus e_2\mathbb{C}$, then cay $\mathbb{C}$ is not irreducible and has lots of involutions $f(a\ell) = (ca)\ell$ for $c^2 = 1$. However, cay $(e_i\mathbb{C})$ is irreducible. If f is an involution on $(e_i\mathbb{C})\ell$ we have $f(e_i\ell) = c\ell$ $(c = e_ic)$ and $f((e_ia)\ell) = f(a(e_i\ell)) = f(e_i\ell)\bar{a} = (c\ell)\bar{a} = (ca)\ell$. In particular $c\ell = f(e_i\ell) = f((e_ie_i)\ell) = (ce_i)\ell$, so $c = ce_i$ and $c = e_ice_i$ belongs to the Peirce space $e_i\mathbb{C}e_i = \Phi e_i$: $c = \gamma e_i$. From $f(f(e_i\ell)) = e_i\ell$ we see $c^2 = e_i$, $\gamma^2 = 1$, $\gamma = \pm 1$, $c = \pm e_i$. Thus $f((e_ia)\ell) = \pm(e_ia)\ell$, and f is $\pm I$. ∎

Strange things happen with involutions in characteristic 2 (since + and - are the same), so we first consider the characteristic $\neq 2$ case.

8.3 *-Bimodule Theorem. (First Version) Every *-bimodule for an ordinary composition algebra $\mathbb{C}$ over a field Φ of characteristic $\neq 2$ is completely *-reducible, with *-irreducible sub-bimodules isomorphic to the *-sub-bimodules of the regular and Cayley-Dickson *-bimodules. We thus obtain the following list of *-irreducibles:

I. $\mathbb{C} = \Phi 1$: $\mathrm{reg}_+(\mathbb{C})$, $\mathrm{reg}_-(\mathbb{C})$

IIa. $\mathbb{C} = C(\Phi,\mu_1)$ division algebra: $\mathrm{reg}_+(\mathbb{C})$, $\mathrm{cay}_+(\mathbb{C})$, $\mathrm{cay}_-(\mathbb{C})$

IIb. $\mathbb{C} = C(\Phi,1)$ split: $\mathrm{reg}_+(\mathbb{C})$, $\mathrm{cay}_+(\Phi e_1)$, $\mathrm{cay}_-(\Phi e_1)$, $\mathrm{cay}_+(\Phi e_2)$, $\mathrm{cay}_-(\Phi e_2)$

IIIa. $\mathbb{C} = C(\Phi,\mu_1,\mu_2)$ division algebra: $\mathrm{reg}_+(\mathbb{C})$, $\mathrm{reg}_-(\mathbb{C})$, $\mathrm{cay}_+(\mathbb{C})$, $\mathrm{cay}_-(\mathbb{C})$

IIIb. $\mathbb{C} = C(\Phi,1,1)$ split: $\mathrm{reg}_+(\mathbb{C})$, $\mathrm{reg}_-(\mathbb{C})$, $\mathrm{cay}_+((e_1\mathbb{C})\ell)$, $\mathrm{cay}_-((e_1\mathbb{C})\ell)$

IV. $\mathbb{C} = C(\Phi,\mu_1,\mu_2,\mu_3)$: $\mathrm{reg}_+(\mathbb{C})$, $\mathrm{reg}_-(\mathbb{C})$.

Proof. Once more the essential part of the proof is that every *-bimodule is a sum of images of regular and Cayley-Dickson *-bimodules. The difference of the present list from that in the Bimodule Theorem is due to the fact that for $\mathbb{C} = C(\Phi,\mu)$ we don't need both involutions since $reg_+(\mathbb{C}) \cong reg_-(\mathbb{C})$ under the map $a \to ia$ in characteristic $\neq 2$: $reg\ \mathbb{C} = \Phi 1 + \Phi i$, $i^2 = \mu 1$ so $F(a) = ia$ is a homomorphism of bimodules which is also a *-homomorphism since $\overline{ia} = \bar{i}\bar{a} = -i\bar{a}$; moreover this is irreducible as *-bimodule even in the split case as Φe_1, Φe_2 alone don't form *-bimodules.

To fill M up with regular or Cayley-Dickson *-bimodules we need only fill it up with symmetric and skew commuters and *-commuters. Indeed, a symmetric (resp. skew) commuter m generates a *-bimodule $\{m\}$ which is an image of reg_+A (resp. reg_-A) since the module homomorphism $a \to am$ of 7.2 is automatically a *-homomorphism: $\bar{a} \to \bar{a}m = +\bar{a}\bar{m} = \overline{ma} = \overline{am}$ (resp. $\bar{a} \to -\overline{am}$). Similarly a skew (resp. symmetric) *-commuter generates a *-bimodule $\{m\}$ which is an image of cay_+A (resp. cay_-A) since $a\ell \to am$ as in 7.4 is a *-homomorphism: $\overline{a\ell} = -a\ell \to -am = -m\bar{a} = \bar{m}\bar{a} = \overline{am}$ (resp. $\overline{a\ell} \to -\overline{am}$).

In characteristic $\neq 2$ it is easy to fill M with such elements: since $m \to \bar{m}$ is an anti-automorphism on $E = A \oplus M$, if m is a commuter or *-commuter so is its image $\bar{m}$ and hence also its symmetric and skew parts $m_+ = \frac{1}{2}(m + \bar{m})$, $m_- = \frac{1}{2}(m - \bar{m})$. Thus every commuter (resp. *-commuter) is the sum of a symmetric and a skew commuter (resp. *-commuter) $m = m_+ + m_-$, and such elements fill up (i.e. generate) M. ∎

We now develop an alternate approach which works in all characteristics.

Instead of filling a bimodule M up with homomorphic images of the 4 basic bimodules $reg_{\mp}A$ and $cay_{\pm}A$ we fill it up with homomorphic images of two bimodules reg(A) and cay(A) with exchange involution.

Suppose M is any bimodule (not necessarily with involution) for a *-algebra A. Then we can imbed M in the <u>exchange *-bimodule</u>

$$ex(M) = M \oplus M^*$$

with A-module structure

$$a(m,n) = (am, n\bar{a})$$

$$(m,n)a = (ma, \bar{a}n)$$

and <u>exchange involution</u>

$$\overline{(m,n)} = (n,m) \ .$$

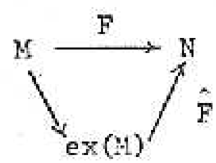
This is indeed an involution of the module structure since $\overline{a(m,n)} = \overline{(am,n\bar{a})} = (n\bar{a}, am) = (n,m)\bar{a} = \overline{(m,n)}\bar{a}$. To see ex(M) is an alternative A-bimodule, notice that $M \subset ex(M)$ carries its given bimodule structure while the representation on M* is given by $\ell^*_a = r_{\bar{a}}$, $r^*_a = \ell_{\bar{a}}$ in terms of ℓ, r on M. Now we know we can give M the structure of an A^{op}-bimodule M^{op} by $\ell^{op}_a = r_a$, $r^{op}_a = \ell_a$ (the split null extension is then just $A^{op} \oplus M^{op} = (A \oplus M)^{op}$, which is alternative if $A \oplus M$ is). Composing this with the isomorphism $A \to A^{op}$ by $a \to \bar{a}$, we get a birepresentation $\ell^*: a \to \bar{a} \to \ell^o_{\bar{a}} = r_{\bar{a}}$ and $r^*: a \to \bar{a} \to r^o_{\bar{a}} = \ell_{\bar{a}}$. Thus as bimodule ex(M) is just the direct sum of the two bimodules M and M*.

Note that this construction is additive,

$$ex(\oplus_i M_i) \cong \oplus_i ex(M_i) .$$

The usefulness of the exchange bimodule resides in its universal property.

8.4 (Universal Property of Exchange Bimodule) Any bimodule homomorphism $M \xrightarrow{F} N$ of a bimodule M into a *-bimodule N extends uniquely to a *-homomorphism $ex(M) \xrightarrow{\hat{F}} N$,



Proof. If we define $\hat{F}(m,n) = F(m) + \overline{F(n)}$ we have a *-homomorphism because $\hat{F}(a(m,n)) = \hat{F}(am,n\bar{a}) = F(am) + \overline{F(n\bar{a})} = aF(m) + a\overline{F(n)} = a\hat{F}(m,n)$ and $\hat{F}(\overline{m,n}) = \hat{F}(n,m) = F(n) + \overline{F(m)} = \overline{\overline{F(n)} + F(m)} = \overline{\hat{F}(m,n)}$. This is unique since it is uniquely determined on the *-generating set M of ex(M). ▨

8.5 Example. If $\mathbb{C} = \Phi e_1 \boxplus \Phi e_2$ is split of dimension 2 then

$$ex(\Phi e_1) \cong ex(\Phi e_2) \cong reg_+(\mathbb{C})$$

since the imbedding $\Phi e_1 \to reg_+ \mathbb{C}$ extends to a *-imbedding $ex(\Phi e_1) \to reg_+ \mathbb{C}$. ▨

Thus we have a universal way of building *-bimodules out of ordinary bimodules. What happens if M already carries an involution, i.e. an endomorphism f of period 2 satisfying $f(am) = f(m)\bar{a}$ and $f(ma) = \bar{a}f(m)$? In this case the set of <u>f-traces</u>

$$t_f(M) = \{(m, f(m))\} \cong M_f$$

is a *-sub-bimodule of ex(M) which is *-isomorphic to M_f: $m \overset{F}{\to} (m, f(m))$ is a linear bijection $M_f \to t_f(M)$ with $F(f(m)) = (f(m), f(f(m)) = (f(m), m) = \overline{(m, f(m))} = \overline{F(m)}$ and $f(am) = (am, f(am)) = (am, f(m)\bar{a}) = a(m, f(m)) = aF(m)$.

Thus if M is a *-bimodule it is *-imbedded in ex(M). What does the the remaining part of ex(M) look like? We claim it looks like M, but relative to the involution $-f$:

$$ex(M)/t_f(M) \cong M_{-f}\ .$$

Indeed, by the Universal Property 8.4 the isomorphism $M \overset{I}{\to} M_{-f}$ induces an epimorphism $ex(M) \overset{\hat{F}}{\to} M_{-f}$ by $\hat{F}(m,n) = F(m) - f(F(n)) = m - f(n)$, with kernel $\{(m,n) | m = f(n)\} = \{(m, f(m))\} = t_f(M)$. Thus F induces *-isomorphism $ex(M)/t_f(M) \to M_{-f}$.

In characteristic $\neq 2$ the exchange bimodule decomposes into the direct sum

$$(8.6) \qquad ex(M) = t_f(M) \oplus t_{-f}(M) \cong M_f \oplus M_{-f}$$

of one copy of M under its given involution, and one copy with the negative of this involution. One way to see this splits is to observe that the bimodule isomorphism $M \overset{f}{\to} M \overset{in}{\to} ex(M) \overset{*}{\to} ex(M)$ $(F(m) = (0, f(m))$ extends to a *-automorphism $ex(M) \overset{\hat{F}}{\to} ex(M)$ of period 2 $(\hat{F}(m,n) = (f(n), f(m))$ by universality, so the ± 1 eigenspaces of $\hat{F}$ are the *-submodules $t_f(M) = \{(m,n) | n = f(m)\} = \{(m,n) | (fn, fm) = (m,n)\}$ and $t_{-f}(M) = \{(m,n) | n = -f(m)\} = \{(m,n) | (fn, fm) = -(m,n)\}$, having ex(M) as their direct sum.

In characteristic 2, +1 coincides with −1 so $t_f(M) = t_{-f}(M)$ and ex(M) does not break up into their direct sum. All we can say is

$$t_f(M) \cong M_f \cong ex(M)/t_f(M) .$$

Next we investigate to what extent ex(M) preserves irreducibility.

8.7 Proposition. Let M be an irreducible A-bimodule. Then the only proper *-submodules of ex(M) are the submodules $t_f(M)$ and the only nonzero *-homomorphic images of ex(M) are ex(M) and all possible $t_f(M)$ for all involutions f on M (if such exist).

Proof. We begin by recalling the basic fact (Vol. I) that the only proper submodules of $M_1 \oplus M_2$ when the M_i are irreducible are

$$M_1, M_2, t_f(M) = \{(m, f(m)\}$$

for all possible isomorphisms $M_1 \overset{f}{\to} M_2$. In our case an isomorphism $M \overset{f}{\to} M^*$ satisfies $f(am) = f(m)\bar{a}$, $f(ma) = \bar{a}f(m)$. If we demand proper *-sub-bimodules, M and M* are ruled out, and $t_f(M)$ is a *-bimodule only for those f of period 2: $\overline{(m, f(m))} = (f(m), m) \in t_f(M)$ implies $f(f(m)) = m$. Thus the only proper *-submodules are the $t_f(M)$ for the involutions f on M.

Since a *-homomorphic image of ex(M) is isomorphic to ex(M)/K for some *-submodule K, for K = 0, $t_f(M)$, ex M we get ex(M), $t_{-f}(M)$, 0 respectively. ∎

For all characteristics, we can at least fill up a given *-bimodule M for a composition algebra $\mathbb{C}$ with images of

$$ex(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C}^*, \ ex(\mathbb{C}\ell) = \mathbb{C}\ell \oplus \mathbb{C}\ell^* .$$

Just as we can represent $\mathrm{cay}(\mathbb{C}) = \mathbb{C}\ell$ as $\mathbb{C}$ with operations $\ell_a = R_a$, $r_a = R_{\bar{a}}$, the formulas $c(a\ell, b\ell) = ((ac)\ell, (bc)\ell)$, $(a\ell, b\ell)c = ((a\bar{c})\ell, (b\bar{c})\ell)$ show we can represent $\mathrm{ex}(\mathbb{C}\ell)$ as $\mathbb{C} \oplus \mathbb{C}$ with action $c(a,b) = (ac, bc)$, $(a,b)c = (a\bar{c}, b\bar{c})$.

8.8 *-Bimodule Theorem (2nd Version) Every *-bimodule for an ordinary composition algebra $\mathbb{C}$ over a field Φ is a sum of *-homomorphic images of the regular and Cayley-Dickson exchange bimodules $\mathrm{ex}(\mathbb{C})$ and $\mathrm{ex}(\mathbb{C}\ell)$. The list of images is

I. $\mathbb{C} = \Phi 1$: $\mathrm{ex}(\mathbb{C})$, $\mathrm{reg}_{\pm}(\mathbb{C})$

IIa. $\mathbb{C} = \mathbb{C}(\Phi, \mu_1)$ division: $\mathrm{ex}(\mathbb{C})$, $\mathrm{reg}_{+}(\mathbb{C})$; $\mathrm{ex}(\mathbb{C}j)$, $\mathrm{cay}_{+}(\mathbb{C})$

IIb. $\mathbb{C} = \mathbb{C}(\Phi, \mu_1)$ split: $\mathrm{ex}(\mathbb{C}e_1) \cong \mathrm{ex}(\mathbb{C}e_2) \cong \mathrm{reg}_{+}(\mathbb{C})$; $\mathrm{ex}((e_1\mathbb{C})j)$, $\mathrm{ex}((e_2\mathbb{C})j)$, $\mathrm{cay}_{\pm}(e_1\mathbb{C})$, $\mathrm{cay}_{\pm}(e_2\mathbb{C})$

IIIa. $\mathbb{C} = \mathbb{C}(\Phi, \mu_1, \mu_2)$ division: $\mathrm{ex}(\mathbb{C})$, $\mathrm{reg}_{\pm}(\mathbb{C})$; $\mathrm{ex}(\mathbb{C}\ell)$, $\mathrm{cay}_{\pm}(\mathbb{C})$

IIIb. $\mathbb{C} = \mathbb{C}(\Phi, \mu_1, \mu_2)$ split: $\mathrm{ex}(\mathbb{C}e_1) \cong \mathrm{ex}(\mathbb{C}e_2) \cong \mathrm{reg}_{+}(\mathbb{C})$, $\mathrm{reg}_{-}(\mathbb{C})$; $\mathrm{ex}((e_1\mathbb{C})\ell)$, $\mathrm{ex}((e_2\mathbb{C})\ell)$, $\mathrm{cay}_{\pm}(e_1\mathbb{C})$, $\mathrm{cay}_{\pm}(e_2\mathbb{C})$

IV. $\mathbb{C} = \mathbb{C}(\Phi, \mu_1, \mu_2, \mu_3)$: $\mathrm{ex}(\mathbb{C})$, $\mathrm{reg}_{\pm}(\mathbb{C})$.

The $\mathrm{ex}(\mathbb{C})$ and $\mathrm{ex}(\mathbb{C}\ell)$ are never *-irreducible, and are completely *-reducible only in characteristic $\neq 2$.

Proof. From the ordinary Bimodule Theorem 7.1 we know M is a (direct) sum of homomorphic images M_i of regular or Cayley-Dickson bimodules $\mathbb{C}$ or $\mathbb{C}\ell$. By the Universal Property 8.4 of the exchange bimodule, the homomorphisms $\mathbb{C}s \to M_i \xrightarrow{in} M$ extend to *-homomorphisms $\mathrm{ex}(\mathbb{C}s) \to M$. Thus M is a sum of *-homomorphic images of $ex(\mathbb{C})$ and $ex(\mathbb{C}\ell)$.

It remains to list the images. The regular bimodule is irreducible when $\mathbb{C}$ is simple, i.e. in all cases but IIb; when $\mathbb{C}$ is irreducible we know by 8.7 the only nonzero images of $ex(\mathbb{C})$ are $ex(\mathbb{C})$ and $reg_{\pm}(\mathbb{C})$ (recall by 8.2 the only involutions on $\mathbb{C}$ are $\pm$ the standard involution in dimensions 1, 4, 8, and in dimension 2 are all equivalent to the standard involution). In IIb we have $\mathbb{C} = \Phi e_1 \boxplus \Phi e_2$ and $ex(\mathbb{C}) = ex(\Phi e_1) \oplus ex(\Phi e_2)$ for $ex(\Phi e_i) \cong reg_{+}(\mathbb{C})$ by 8.5, which by 8.7 are irreducible since there are no involutions f on Φe_i according to 8.2. This classifies the regular images.

Turning to the Cayley-Dickson images, we know cay $\mathbb{C} = \mathbb{C}\ell$ is irreducible when $\mathbb{C}$ has no right ideals, i.e. I, IIa, IIIa. When $\mathbb{C}\ell$ is irreducible, by 8.7 the only nonzero * images are $ex(\mathbb{C}\ell)$ and $cay_{\pm}(\mathbb{C})$ (by 8.2, the only involutions on $\mathbb{C}\ell$ are $\pm$ the standard involution in these cases). In the split cases IIb and IIIb we have $\mathbb{C}\ell = (e_1\mathbb{C})\ell \oplus (e_2\mathbb{C})\ell$ for $(e_i\mathbb{C})\ell$ irreducible, so $ex(\mathbb{C}\ell) = ex((e_1\mathbb{C})\ell) \oplus ex((e_2\mathbb{C})\ell)$. The * images of $ex(\mathbb{C}\ell)$ are thus sums of *-images of $ex((e_i\mathbb{C})\ell)$, which by 8.7 and 8.2 are again either $ex((e_i\mathbb{C})\ell)$ or $cay_{\pm}(e_i\mathbb{C})$. ∎

Exercises

8.1 Verify directly that 1 is an involution on cay($\mathbb{C}$).

8.2 Let $t(m) = m + \bar{m}$ in any $*$-bimodule. If m is a $*$-commuter, show $t(\{m\})$ is a $*$-submodule, while if m is a commuter and A is a composition algebra with nontrivial involution then $t(\{m\})$ generates $\{m, \bar{m}\}$.

8.3 Verify directly that if m is a commuter then $(a,b) \to am + \bar{b}\bar{m}$ is a $*$-homomorphism ex($\mathbb{C}$) $\to \{m, \bar{m}\}$, and if m is a $*$-commuter then $(a,b) \to am + b\bar{m}$ is a $*$-homomorphism ex($\mathbb{C}\ell$) $\to \{m, \bar{m}\}$.

8.4 In Proposition 8.2 construct infinitely many involutions on the 2-dimensional module reg($\mathbb{C}$) when $\Phi = \mathbb{R}$. Similarly construct infinitely many involutions on cay($\mathbb{C}$) when $\mathbb{C}$ is split of dimension 2 or 4.

8.5 Verify $F(ma) = F(m)\cdot a$ directly in 8.4, in $\mathrm{ex}(M)/t_{\bar{f}}(M) \cong M_{-f}$, in $t_{\bar{f}}(M) \cong M_f$.

8.6 Prove that the exchange bimodules ex($\mathbb{C}$) and ex($\mathbb{C}\ell$) are definitely not completely $*$-reducible in characteristic 2.