§5. Triality and Local Triality

The transformations which preserve the quadratic norm form of a composition algebra are closely connected to the algebra structure. On the one hand, they can be built up from multiplication operators plus the standard involution, on the other they are precisely the autotopies and anti-autotopies of the algebra. Analogously, the transformations which are Lie similarities relative to the norm form are sums of multiplications and can be identified with the diffeotopies of the algebra.

Similarities and Multiplications

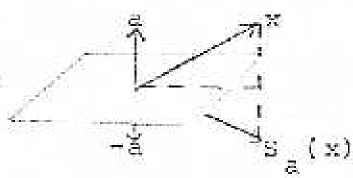
Recall that if Q is any quadratic form on a vector space X over a field Φ, the **similarity group** $S(Q)$ of Q consists of all **similarities**, that is, all bijective transformations S on X which preserve the quadratic form up to a scalar: $Q(Sx) = \sigma Q(x)$ where $\sigma \in \Phi$ is the **multiplier** of S. Those transformations with multiplier 1 comprise the **orthogonal group** $O(Q)$ (or group of **isometries** of Q). If the bilinear form $Q(x,y)$ is nondegenerate and X is finite-dimensional, the orthogonal group is generated by the **symmetries**

$$S_a(x) = x - Q(a)^{-1}Q(a,x) \tag{5.1}$$

determined by non-isotropic vectors $a \in X$ $(Q(a) \neq 0)$. Those isometries (orthogonal transformations) which are products of an even number of symmetries constitute the subgroup $O^+(Q)$ of **proper** isometries (or **rotations**) while products of an odd number of symmetries constitute the **improper** isometries (or **reflections**) $O^-(Q)$.

In characteristic $\neq 2$ propriety and impropriety have a simple interpretation in terms of determinants: $T \in O^+(Q)$ is proper iff $\det T = +1$, and $T \in O^-(Q)$ is improper iff $\det T = -1$. This follows because each symmetry S_a has determinant -1; indeed, S_a is just reflection in the hyperplane $a^\perp = \{x \in X \mid Q(a,x) = 0\}$, fixing the hyperplane and sending a to $-a$ (thus acting as $+I$ on the hyperplane and as $-I$ on the 1-dimensional space Φa).

Symmetry S_a:



hyperplane $a^\perp$

For the rest of this section we will be interested in the case where $Q = n$ is the quadratic norm form of a composition algebra. In this case the composition formula

$$n(xy) = n(x)n(y)$$

shows that left multiplication L_x by an invertible element x is a norm-similarity with multiplier $\sigma = n(x)$, also that right multiplication R_y is a similarity with multiplier $\sigma = n(y)$. In particular (or directly from $n(\alpha x) = \alpha^2 n(x)$), scalar multiplications are norm-similarities. Thus anything in the multiplication group of the algebra A is a norm-similarity,

$$GM(A) \subset S(n). \tag{5.2}$$

The relation $n(x^*) = n(x)$ (by 1.00) shows the standard involution $*$ is an isometry

$$(5.3) \qquad * \in O(n).$$

(Warning: the n in $O(n)$ and $S(n)$ stands for the norm form, not an integer equal to the dimension of X).

The multiplications $GM(A)$ and the involution $*$ are enough to generate all similarities.

5.4 (Similarity Theorem) If n is the norm form of a composition algebra, every norm similarity $S \in S(n)$ has a unique decomposition

$$S = L_x T$$

where $x = S(1)$ and $T \in O(n)$ is an isometry fixing 1. For an ordinary composition algebra the group of norm similarities coincides with the multiplication group and standard involution,

$$S(n) = GM(A) \cup GM(A)^*.$$

Any norm similarity $S \in S(n)$ may be written as

$$(5.5) \qquad S^+ = L_x U_{y_1} \ldots U_{y_k} \quad \text{or} \quad S^- = L_x U_{y_1} \ldots U_{y_k} *$$

The orthogonal group coincides with the standard involution together with all isometric multiplications: any proper or improper isometry may be written as

$$(5.6) \qquad \begin{aligned} T^+ &= n(b)^{-1} U_{b_1} \ldots U_{b_k} && (T^+ \in O^+(n)) \\ T^- &= -n(b)^{-1} U_{b_1} \ldots U_{b_k}^* && (T^- \in O^-(n)) \end{aligned}$$

Proof. Such a decomposition $A = L_x T$ is clearly unique, since if $S = L_x T$ then $S(1) = L_x T(1) = L_x(1) = x$ determines x and $T = L_x^{-1} S$ determines T. Such a decomposition exists since $x = S(1)$ is invertible (by 1.17: $n(x) = n(S1) = \sigma n(1) = \sigma \neq 0$), hence L_x is invertible (by the Inverse Theorem I.4.2), consequently $T = L_x^{-1} S$ is also a norm similarity which fixes 1 ($T(1) = L_x^{-1} S(1) = L_x^{-1}(x) = 1$) and therefore is isometric (in general the multiplier τ of T is $n(T1)$ since $n(T1) = \tau n(1) = \tau$; or directly, $n(Ty) = n(L_{x^{-1}} Sy) = n(x^{-1})n(Sy) = n(x)^{-1} \sigma n(y) = \sigma^{-1} \sigma n(y) = n(y)$ shows T is isometric).

So far all we needed was a composition algebra. If we have an ordinary composition algebra then the bilinear form $n(x,y)$ is nondegenerate and the space is finite-dimensional (dimension 1,2,4, or 8); in this case we know each isometry T breaks up into a product of symmetries, $T = S_{a_1} \ldots S_{a_k}$ $(n(a_i) \neq 0)$. The real key to the relation between similarities and multiplications is the fact that the symmetries S_a are intimately connected with the U-operators and the involution:

$$x^* = -S_1(x) \tag{5.7}$$

$$S_a = n(a)^{-1} U_a S_1 \tag{5.8}$$

$$U_a = n(a) S_a S_1. \tag{5.9}$$

(5.7) holds because $x^* = t(x)1 - x = n(1)^{-1} n(1,x)1 - x = -S_1(x)$ by definition (5.1) of symmetries, likewise for (5.8) $S_a(x) = x - n(a)n(a,x)a = -n(a)^{-1} \{n(a,x)a - n(a)x\} = -n(a)^{-1} U_a x^*$ (by the U-formula I.19b); (5.9) then follows since $S_1^2 = I$ (in fact $S_a^2 = I$ for any symmetry - see Ex. 5.3). These relations show once more how the algebraic structure (in this case the involution and U-operators) of a composition algebra are determined by the norm form n and its symmetries.

From (5.8) we have $S_a = n(a)^{-1} U_a S_1$; since * is an involution we have $S_1 U_a = U_{a^*} S_1$ $(S_1 U_a(x) = -U_a x^* = -(U_{a^*} x)^* = S_1 U_{a^*}(x))$, so we can move all the factors S_1 to the right in $T = S_{a_1} \ldots S_{a_k}$ $= n(a_1^{-1}) \cdots n(a_k^{-1})\, U_{a_1} S_1 \cdots U_{a_k} S_1$ to get $T = n(b)^{-1} U_{b_1} \ldots U_{b_k} S_1^k$ ($b_i = a_i$ or a_i^*, $b = a_1(\ldots a_k)$). If $T \in O^+(n)$, k is even and $S_1^k = I$; if $T \in O^-(n)$, k is odd and $S_1^k = S_1 = -*$ by (5.7). This gives the representation $T^+ = n(b)^{-1} U_{b_1} \ldots U_{b_k}$ and $T^- = -n(b)^{-1} U_{b_1} \ldots U_{b_k} *$ as required in (5.6)

We can absorb a scalar $\beta = \pm n(b)^{-1}$ into the L_x to represent $S = L_x T$ as $S^+ = L_{\beta x} U_{b_1} \ldots U_{b_k}$ or $S^- = L_{\beta x} U_{b_1} \ldots U_{b_k} *$ as in (5.5). Thus $S(n) \subset GM(A) \cup GM(A)*$, and by (5.2), (5.3) we know $GM(A) \cup GM(A)* \subset S(N)$. ∎

In analogy with the case of the orthogonal group, we can define the **proper similarities** $S^+(n)$ to be those similarities $S = L_x T$ whose unique orthogonal part T is proper ($T \in O^+(n)$ with $T(1) = 1$), and the **improper similarities** $S^-(n)$ to be those with improper orthogonal parts ($T \in O^-(n)$ with $T(1) = 1$). [Note that by this convention when $A = \Phi 1$ has characteristic $\neq 2$ we have $S^+(n)$ = the scalar multiplications and $S^-(n) = \emptyset$, since when $T \in O(n)$ fixes 1 it fixes everything and $T = I \notin O^-(n)$; yet $O^+(n) = \{I\}$ and $O^-(n) = \{-I\}$, so $O^-(n) \neq S^-(n) \cap O(n)$. But who cares about dimension 1, anyway?]

With this notation (5.5) becomes

(5.10) $\qquad S^+(n) = GM(A), \quad S^-(n) = GM(A)* \qquad (\dim A > 1).$

Clearly (5.5) says $S^+(n) \subset GM(A)$, $S^-(n) \subset GM(A)*$ where

$S(n) = S^+(n) \cup S^-(n) = GM(A) \cup GM(A)^*$; when dim A > 1 the sets $GM(A)$ and $GM(A)^*$ are disjoint ($* \notin GM(A)$: if A is noncommutative this follows from Lemma 5.14, while if A is commutative $GM(A) = L_A^{\cdot}$ so $* = L_x \Rightarrow x = L_x 1 = 1^* = 1 \Rightarrow * = I \Rightarrow \dim A = 1$) therefore $S^+(n)$ must be all of $GM(A)$ and $S^-(n)$ all of $GM(A)^*$ in order that their union $S^+(n) \cup S^-(n)$ be all of $GM(A) \cup GM(A)^*$.

With more effort we could prove

$$\begin{aligned} O^+(n) &= O(n) \cap S^+(n) = O(n) \cap GM(A) \\ O^-(n) &= O(n) \cap S^-(n) = O(n) \cap GM(A)^*, \end{aligned} \qquad (\dim A > 1)$$

but instead we relegate this to a Problem Set (VIII.5.1).

Similarities and Autotopies

So far we have characterized the norm similarities in terms of the algebra multiplications and the standard involution. Next we relate these to the autotopies of the composition algebra.

Recall that an invertible linear transformation S is an **autotopy** of A if there exist invertible S', S" with

$$S(xy) = S'(x)S''(y) \tag{5.11}$$

for all $x, y \in A$, and an **anti-autotopy** if

$$S(xy) = S''(y)S'(x). \tag{5.12}$$

5.13 (Autotopy Theorem) The norm similarities of an ordinary composition algebra A are precisely the autotopies and anti-autotopies,

$$S(n) = \mathrm{Autop}(A) \cup \mathrm{Anti\text{-}autop}(A)$$

For noncommutative composition algebras (dimension 4 or 8)

$$S^{+}(n) = \mathrm{Autop}(A)$$

$$S^{-}(n) = \text{Anti-autop}(A)$$

In the expression (5.11) or (5.12) for a similarity S, the transformations S', S'' are proper or improper according as S is.

Proof. (Compare Problem Set VI.1.2) The left and middle Moufang formulas $L_a(xy) = a\{x(aa^{-1}y)\} = \{axa\}\{a^{-1}y\} = U_a x \cdot L_{a^{-1}} y$ and $U_a(xy) = a(xy)a = \{ax\}\{ya\} = L_a x \cdot R_a y$ show L_a, U_a are autotopies of A. Since any product of autotopies is again an autotopy, (5.5) shows $S^{+}(n) \subset \mathrm{Autop}(A)$. Since the standard involution $*$ is an anti-autotopy, $(xy)^* = y^* \cdot x^*$, and the product of an autotopy and an anti-autotopy is again an anti-autotopy, (5.5) shows $S^{-}(n) \subset \mathrm{Antiautop}(A)$. Thus always $S(n) \subset \mathrm{Autop}(A) \cup \mathrm{Antiautop}(A)$.

Conversely if S is an autotopy of A, $S(xy) = S'(x)S''(y)$ for all $x, y \in A$, then by Schafer's Isotopy Theorem I.5.9 S is an isomorphism of A onto an isotope $A^{(u,v)}$ $(u = S''(1)^{-1}$, $v = S'(1)^{-1})$. By (4.00) any isomorphism of composition algebras preserves norms, and by (1.20) the isotope $A^{(u,v)}$ is again a composition algebra with norm $n^{(u,v)} = n(uv)n$, so $n(x) = n^{(u,v)}(Sx) = n(uv)n(Sx)$ implies $n(Sx) = n(uv)^{-1}n(x)$ and S is a norm-similarity. Thus

$$\mathrm{Autop}(A) \subset S(n).$$

S is an anti-autotopy iff $S \circ *$ is an autotopy, so $\mathrm{Antiautop}(A) = \mathrm{Autop}(A)^* \subset S(n)^* = S(n)$. This shows $\mathrm{Autop}(A) \cup \mathrm{Antiautop}(A) \subset S(n)$ so

$$S(n) = \mathrm{Autop}(A) \cup \mathrm{Antiautop}(A).$$

If A is commutative then autotopies and anti-autotopies are the same thing, Autop(A) = Antiautop(A), so we can't hope to distinguish between $S^+(n)$ and $S^-(n)$ by these. If, however, A is noncommutative (i.e. quaternion or Cayley) we claim $S^+(n) = \text{Autop}(A)$ (and therefore $S^-(n) = S^+(n)^* = \text{Autop}(A)^* = \text{Antiautop}(A)$). We know $S^+(n) \subset \text{Autop}(A)$ and $\text{Autop}(A) \subset S(n)$; all we need is that every autotopy S is proper. But if it were improper, $S \in S^-(n) \subset$ Antiautop, it would be an antiautopy and an autotopy at the same time. This is impossible because of the

5.14 Lemma. If $\mathbb{C}(B,\mu)$ is a noncommutative composition algebra (thus a quaternion or Cayley algebra of dimension 4 or 8) then $\text{Autop}(\mathbb{C}) \cap \text{Antiautop}(\mathbb{C}) = \emptyset$.

Proof. If T were both an autotopy and an antiautotopy, then T and $T\circ *$ would be autotopies and hence $T^{-1}\circ(T\circ *) = *$ would be autotopy, i.e. an isomorphism $\mathbb{C} \to \mathbb{C}^{(u,v)}$: $y^*x^* = (xy)^* = x^*u\cdot vy^*$ for all x,y. Replacing x,y by x^*,y^* shows

$$y\cdot x = xu\cdot vy.$$

Setting $x = y = 1$ shows $uv = 1$, so u and v are inverses. Setting $y = u$ shows $ux = xu$, then replacing y by uy yields

$$uy\cdot x = xu\cdot y = ux\cdot y$$

Write $u = b + c\ell$ for $b,c \in B$. Then for $x = a \in B$, $y = \ell$ we have $0 = uy\cdot x - ux\cdot y = \{u\ell\}a - \{ua\}\ell = \{b\ell + \mu c\}a - \{ba + (ca^*)\ell\}\ell = \mu c(a-a^*) + b(a^*-a)\ell$ implies $b = c = 0$ since there are invertible $a - a^* \in B$ when dim B = 2 or 4 (by 0,000). But then $u = 0$ contradicts invertibility of u, and no such u,v exist. ∎

Thus $S^+(n) = \mathrm{Autop}(A)$ and $S^-(n) = \mathrm{Antiautop}(A)$.

In the expression (5.11) we know $S' = R_u S$ and $S'' = L_v S$ by Schafer's Isotopy Theorem I.5.9 again, so S', S'' are again autotopies. In (5.12) the operator $T = *\circ S$ is an autotopy $T(xy) = S(xy)^* = S'(x)^* S''(y)^* = T'(x)\,T''(y)$, so $T' = *\circ S'$, $T'' = *\circ S''$ are autotopies, hence S', S'' are anti-autotopies. ∎

5.15 Remark. The S', S'' in (5.11) need not be isometries if S is. However, if Φ is closed under square roots (e.g. if it is algebraically closed) we can scale them up so they are isometric: if S', S'' have multipliers $\sigma' = n(S'1)$, $\sigma'' = n(S''1)$ then $\sigma'\sigma'' = n(S'1)n(S''1) = n(S'1\cdot S''1) = n(S(1\cdot 1)) = n(S1) = 1$ (S is isometric!), so if $\sigma' = \tau^2$ for some $\tau\in\Phi$ we still have $S(xy) = T'(x)T''(y)$ for $T' = \tau^{-1}S'$, $T'' = \tau S''$ with multipliers $n(T'1) = \tau^{-2}n(S'1) = \sigma'^{-1}\sigma' = 1$ and $n(T''1) = \tau^2 n(S''1) = \sigma'\sigma'' = 1$, i.e. T' and T'' are both isometric ∎

The fact that norm similarities correspond to autotopies means they come in triples.

5.16 (Principle of Triality) If $S\in S^-(n)$ is a proper norm similarity of an ordinary composition algebra A, there are proper similarities S', S'' with

$$S(xy) = S'(x)S''(y) \qquad (x,y\in A),$$

while if $S\in S^-(n)$ is an improper norm similarity there are improper similarities S', S'' with

$$S(xy) = S''(y)S'(x) \qquad (x,y\in A). \quad ∎$$

Local Triality

We can develop an analogous theory of local triality. A linear transformation W is a **Lie similarity** (or **semi-alternating**) relative to a quadratic form Q if $Q(Wx,x) = \omega Q(x)$ for all x, where $\omega \in \Phi$ is some fixed **multiplier**; W is **alternating** if $\omega = 0$, i.e. $Q(Wx,x) = 0$. This implies W is skew, but as usual in characteristic 2 skew does not quite imply alternating. We denote by $LS(Q)$ and $A(Q)$ the Lie algebras of Lie similarities and alternating transformations. If $Q(x,y)$ is nondegenerate and finite-dimensional over a field, the alternating transformations are spanned by the

$$S_{a,b}(x) = Q(x,a)b - Q(x,b)a.$$

There is no trouble here with propriety or impropriety.

Once more we are interested in the case where $Q = n$ is the norm form of composition algebra A. In this case the Lie multiplications are Lie similarities

$$(5.17) \qquad LM(A) \subset LS(n)$$

since by 1.16 the generators L_a and R_a of $LM(A)$ are Lie similarities with multiplier $t(a)$: $n(ax,x) = t(a)n(x) = n(xa,x)$. Note that the multiplications by traceless elements $t(a) = 0$ are actually alternating,

$$(5.17)' \qquad LM(A_0) \subset A(n).$$

The alternating transformations $S_{a,b}$ can be obtained from multiplications $V_{a,b}$ by

$$(5.18) \qquad S_{a,b} = n(a,b)I - V_{a,b^*}$$

where $V_{a,b^*}(x) = U_{a,x}(b^*) = n(a,b)x + n(x,b)a - n(a,x)b$ (linearizing 1.196) $= n(a,b)x - S_{a,b}(x)$. This proves all $V_{a,b}$ are Lie similarities; in fact, though not at first glance obvious, $V_{a,b}$ belongs to the Lie multiplication algebra:

(5.19) $\qquad V_{a,b} = L_{ab} + R_{ba} - [L_a, R_b] \in LM(A)$

since $V_{a,b}(x) = (ab)x + (xb)a = (ab)x - (bx)a + x(ba) + b(xa)$.

Once more, the Lie norm similarities are precisely the Lie multiplications of the composition algebra.

5.20 (Lie Similarity Theorem) If n is the norm form of a composition algebra A, every Lie similarity $W \in LS(n)$ has a unique decomposition

$$W = L_z + Z$$

where $z = W1$ and $Z \in A(n)$ kills 1. If A is ordinary we can write

$$W = L_x + \Sigma V_{x_i, y_i}$$

so that the Lie similarities coincide with the Lie multiplications

$$LS(n) = LM(A)$$

Proof. If $z = W1$ then $A = W - L_z$ is still a Lie similarity but now has $Z1 = W1 - a = 0$, which forces A to actually be alternating since in general the multiplier of W is $\omega = \omega\, n(1) = n(W1, 1)$.

If A is ordinary then $n(x,y)$ is nondegenerate and A is finite-dimensional, so any alternating A is a sum $\Sigma S_{a_i, b_i}$; by (5.18) this means $A = \Sigma\{n(a_i, b_i) - V_{a_i, b_i^*}\} = \alpha I + \Sigma V_{x_i, y_i}$, and we can absorb $\alpha I = L_{\alpha 1}$ into L_z to get $W = L_x + \Sigma V_{x_i, y_i}$ ($x = z + \alpha 1$, $x_i = a_i$, $y_i = -b_i^*$).

This shows $LS(n) \subset LM(A)$ by (5.19), and by (5.17) we already knew $LM(A) \subset LS(n)$, so $LS(n) = LM(A)$. ∎

In turn, these Lie similarities or Lie multiplications coincide with the **diffeotopies** (or **local isotopies**), those transformations W for which there exist linear W', W'' with

$$W(xy) = W'(x)y + xW''(y) \tag{5.21}$$

for all $x, y \in A$.

5.22 (Diffeotopy Theorem) If n is the norm form of an ordinary composition algebra A, the Lie norm similarities are precisely the diffeotopies of A:

$$LS(n) = LM(A) = \text{Diffeotop}(A).$$

Proof. To show the Lie algebra $LM(A)$ is contained in the Lie algebra Diffeotop(A), it suffices if the generators L_a, R_a of $LM(A)$ are diffeotopies. This follows directly from alternativity: $L_a(xy) = a(xy) = (ax+xa)y - x(ay) = V_a(x)\cdot y - x\cdot L_a(y)$ and $R_a(xy) = (xy)a = x(ya+ay) - (xa)y = -R_a(x)\cdot y + x\cdot V_a(y)$. Thus $LS(n) = LM(A) \subset$ Diffeotop(A).

In proving every diffeotopy W is a Lie similarity, we have no characterization of diffeotopies as derivations of A into an isotope (the way autotopies were isomorphisms of A onto an isotope), so we will have to start from scratch. Let $W(1) = w$, $W'(1) = w'$, $W''(1) = w''$. Setting consecutively $y = 1$, $x = 1$, $x = y = 1$ in (5.21) yields $W(x) = W'(x) + xw''$, $W(y) = w'y + W''(y)$, $w = w' + w''$:

$$W' = W - R_{w''},\quad W'' = W - L_{w'},\quad w = w' + w''. \tag{5.23}$$

Using these expressions for W' and W'', (5.21) becomes

$$W(xy) = W(x)y + xW(y) - (xw'')y - x(w'y).$$

Thus $0 = W(x^2) - xoWx + xw''x + xw'x = W\{t(x)x - n(x)1\}$
$- \{t(x)Wx + t(Wx)x - n(Wx,x)1\} + U_x w$ (by (1.1), (1.1)', (5.23))
$= -n(x)w - t(Wx)x + n(Wx,x)1 + n(x,w^*)x - n(x)w^* = \{n(Wx,x)$
$- t(w)n(x)\}1 + \{n(x,w^*) - t(Wx)\}x$. If $x \notin \Phi 1$ is independent of 1 we can equate coefficients to see $n(Wx,x) = t(w)n(x)$, while if $x = \alpha 1 \in \Phi 1$ then $Wx = \alpha w = wx$ so trivially $n(Wx,x) = n(wx,x)$ $= t(w)n(x)$. Thus $n(Wx,x) = t(w)n(x)$ for all x, W is a Lie similarity with multiplier $t(w)$, and $\mathrm{Diffeotop}(A) \subset LS(n)$. ∎

Since the W', W'' in (5.21) are diffeotopies by (5.23) and (5.22), they are also Lie similarities, so once more Lie similarities come in triples.

5.24 (Principle of Local Triality) If $W \in LS(n)$ is a Lie similarity relative to the norm of an ordinary composition algebra A, there are Lie similarities W', W'' with

$$W(xy) = W'(x)y + xW''(y) \qquad (x, y \in A).$$

5.25 Remark: If Φ is closed under division by 2, i.e. has characteristic $\neq 2$ (the additive analogue of the multiplicative condition that Φ be closed under square roots), then W', W'' can be chosen alternating if W is. In fact since the multipliers ω', ω'' are negatives by $\omega' + \omega'' = n(W'1,1) + n(W''1,1) = n(W'(1)\cdot 1 + 1\cdot W''(1),1) = n(W1,1)$ $= \omega = 0$ we have $W(xy) = S'(x)y + xS''(y)$ for $S' = W' - \frac{1}{2}\omega' I$, S'' $= W'' - \frac{1}{2}\omega'' I$ where S', S'' now have multipliers $\sigma' = n(W'1,1)$ $- \frac{1}{2}\omega' n(1,1) = \omega' - \frac{1}{2}\omega'\{2n(1)\} = 0$ and $\sigma'' = n(W''1,1) = \omega'' - \frac{1}{2}\omega''\cdot 2 = 0$ ∎

VIII.5.1 Exercises

5.1 For a general quadratic form show the set $S(Q)$ of similarities is a group, and the subset $O(Q)$ of isometries is a normal subgroup with $[[S(Q), S(Q)]] \subset O(Q)$. (i.e. $S(Q)/O(Q)$ is abelian).

5.2 If Φ is closed under square roots, show $\sigma^{-1/2}S$ is isometric for any similarity S, hence that $S(Q) = \dot{\Phi}\, O(Q)$.

5.3 Verify directly that S_a is isometric with $S_a^2 = I$ $(Q(a) \neq 0)$.

5.4 If $Q(x,y)$ is nondegenerate and Φ has characteristic $\neq 2$, show S_a is reflection in the hyperplane $a^{\perp}$ (i.e. fixes all $x \perp a$ and sends a to $-a$).

5.5 Show directly $S_a(xy) = -N(a)^{-1}(ay^*)(x^*a)$ is an anti-autotopy.

5.6 For a general autotopy T, show $n(Tx) = n(T1)n(x)$ by computing $T(xx^*)$.

5.7 Show directly that the set $LS(Q)$ of Lie similarities relative to Q forms a Lie algebra, and the subset $A(Q)$ of alternating transformations forms a Lie ideal with $[LS(Q), SA(Q)] \subset A(Q)$.

5.8 If Φ has characteristic $\neq 2$ show $W - \frac{1}{2}\omega I$ is alternating for any Lie similarity W, hence that $LS(Q) = \Phi I + A(Q)$.

5.9 Verify directly that $S_{a,b}$ is alternating.

5.10 If $Q(x,y)$ is nondegenerate and a,b are orthogonal show $S_{a,b}$ fixes the space $(\Phi a + \Phi b)^{\perp}$ of codimension ≤ 2 and interchanges a and b: $a \rightarrow 2Q(a)b$, $b \rightarrow -2Q(a)a$.

5.11 Prove directly that $V_{a,b}$ is a Lie similarity relative to the norm form with multiplier $2n(a,b^*) = 2t(ab)$. Show $V_{a,b} - V_{b,a}$ is alternating.

5.12 Show directly $S_{a,b}(xy) = x\{(ya^*)b\} - \{a(b^*x)\}y$ is a diffeotopy.

5.13 For a diffeotopy W on a composition algebra with $W1 = 0$, show W is alternating by computing $W(xx^*)$. Use this to show any diffeotopy ($W1 = w \neq 0$) is a Lie similarity.

5.14 When $t(a) = 0$ show $V_a = S_{1,a}$; if $t(b) = 0$ too show $[V_a, V_b] = 2S_{a,b}$. In characteristic $\neq 2$ show the V_a with $t(a) = 0$ generate the $S_{a,b}$ as a Lie algebra; conclude that since the V_a are diffeotopies so are all $S_{a,b}$.

5.15 Show the S', S'' determined by (5.11) are unique up to a multiple from the nucleus (in the case of a Cayley algebra, up to a scalar). Show the W', W'' in (5.21) are unique up to a translation from the nucleus (in a Cayley algebra, up to a scalar). When W is alternating of characteristic $\neq 2$, show the alternating W', W'' of (5.25) are unique. When T is orthogonal and Φ is closed under square roots, show the orthogonal T', T'' of (5.15) are unique up to sign.

5.16 In characteristic $\neq 2$ and dimension 8, where skew W', W'' are uniquely determined by a skew W, show $W \to W'$ and $W \to W''$ are automorphisms of $A(n)$. Prove a multiplicative analogue when Φ is closed under square roots.

5.17 For arbitrary Φ show the maps $W \to W'$, $W \to W''$ are automorphisms of $LS(n)/\Phi T$ and $S \to S'$, $S \to S''$ of $S(n)/\dot{\Phi}1$ when n is the norm of a Cayley algebra.

5.18 Extend the results of this section as far as possible to degree 2 alternative algebras over an arbitrary ring of scalars.

5.19 Define a trilinear form on A by $\langle x,y,z\rangle = t(x(yz))$. Show $\langle x,y,z\rangle = \langle z,x,y\rangle = \langle y,z,x\rangle = t((xy)z) = n(xy,z^*)$. An ordered triple of alternating linear transformations $\{W_1, W_2, W_3\}$ is *Lie-related* if

$\langle W_1x, y, z\rangle + \langle x, W_2y, z\rangle + \langle x, y, W_3z\rangle = 0$. In this case show $\{W_3, W_1, W_2\}$ and $\{W_2, W_3, W_1\}$ are also Lie-related, as is $\{\hat{W}_2, \hat{W}_1, \hat{W}_3\}$ for $\hat{W} = *\circ W\circ *$. If $\{V_1, V_2, V_3\}$ is Lie related show $\{[V_1, W_1], [V_2, W_2], [V_3, W_3]\}$ is Lie related, as is any $\{\omega_1 I, \omega_2 I, \omega_3 I\}$ ($\omega_i \in \Phi$).

5.20 Show $\{W_1, W_2, W_3\}$ is Lie-related iff $\hat{W}_i(xy) = (W_{i+1}x)y + x(W_{i+2}y)$ (indices mod 3).

5.21 An ordered triple $\{S_1, S_2, S_3\}$ of similarities is **related** if $\langle S_1x, S_2y, S_3z\rangle = \lambda\langle x, y, z\rangle$ for $\lambda \neq 0$ in Φ and all $x, y, z \in A$. Show in this case $\{S_3, S_1, S_2\}$, $\{S_2, S_3, S_1\}$, and $\{\hat{S}_2, \hat{S}_1, \hat{S}_3\}$ are related. If $\{T_1, T_2, T_3\}$ is another related triple show $\{S_1T_1,\ S_2T_2,\ S_3T_3\}$ is related, as is any $\{\sigma_1 I,\ \sigma_2 I,\ \sigma_3 I\}$ ($\sigma_i \neq 0$ in Φ).

5.22 Show $\{S_1, S_2, S_3\}$ is related iff $\hat{S}_i(xy) = \lambda^{-1}\sigma_i(S_{i+1}x)(S_{i+2}y)$ (indices mod 3) (σ_i = multiplier of S_i).

5.23 (Principle of Triality) If $\{S_1, S_2, S_3\}$ is a related triple of similarities then every S_i is proper. Conversely, every proper similarity T_1 determines similarities T_2, T_3 such that $\{T_1, T_2, T_3\}$ is related.

5.24 Without expressing W', W'' in terms of W, show directly from (5.21) that W' is a diffeotopy by replacing x by xy, y by y^{-1}. This argument works in any algebra where the invertible elements are "dense". Use a similar argument on (5.11).

5.25 Show $O^+(n)$ for an ordinary composition algebra is generated by all $U_a' = n(a)^{-1}\,U_a$ $(n(a) \neq 0)$.