§4 Classification of composition algebras

W. R. Hamilton discovered quaternions in the 1840's, and showed they were solutions to the <u>4-square problem</u>: write the product $(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2)$ of two sums of 4 squares as a sum of 4 squares $z_1^2 + z_2^2 + z_3^2 + z_4^2$ for z_i bilinear forms in x and y. (Here $x_1^2 + x_2^2 + x_3^2 + x_4^2$ is the norm $n(x) = n(x_1 1 + x_2 i + x_3 j + x_4 k)$, and the $z = z_1 1 + z_2 i + z_3 j + z_4 k$ is the product $z = x \cdot y$ in the quaternions.) Shortly thereafter, A. Cayley (and independently J. T. Graves) discovered the Cayley numbers, which were solutions to the analogous <u>8-square problem</u>: $n(x)n(y) = n(z)$ for $n(x) = n(x_1 1 + x_2 i + x_3 j + x_4 k + x_5 \ell + x_6 i\ell + x_7 j\ell + x_8 k\ell) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2$.

These results led many people to look for solutions to the next step, the <u>16-square problem</u>. Several erroneous constructions were given, and it was not until 1898 that A. Hurwitz showed that real quadratic forms permitting composition $Q(x \cdot y) = Q(x)Q(y)$ are possible only in dimensions 1, 2, 4, 8.

However, this result did not describe the algebraic structure of the product $x \cdot y$. This was done by A. A. Albert, who showed that in the finite-dimensional case over an arbitrary field the algebras were those obtained by the Cayley-Dickson process: fields, quadratic extensions, quaternion algebras, and Cayley algebras. The constructive proof we shall give is due to N. Jacobson, and is valid without restriction on dimension.

4.1 (Hurwitz Theorem) A composition algebra over a field Φ is either

(0) a purely inseparable extension field Ω of Φ of exponent 2 and

characteristic 2, with identity involution

(I) the base field Φ, with identity involution

(II) a quadratic extension $\mathbb{C} = \Phi e + \Phi e^*$ $(e + e^* = 1)$

(III) a quaternion algebra Q with standard involution

(IV) a Cayley algebra $\mathfrak{C}$ with standard involution.

Proof. The radical R of the bilinear form $n(x,y)$ is an ideal (see 2.11). By nondegeneracy of the quadratic form, $n(z) \neq 0$ for $z \in R$; such z have inverses $z^{-1} = n(z)^{-1}\bar{z}$, so either $R = 0$ or R contains invertible elements and thus is all of A.

We first get rid of the case $R = A$, where the polarized norm form $n(x,y)$ is identically zero. In this case $x \to n(x)$ is a ring homomorphism of A into Φ, since $n(xy) = n(x)n(y)$ and $n(x+y) = n(x) + n(x,y) + n(y) = n(x) + n(y)$, and it is an isomorphism since $n(z) = 0$ implies $z = 0$ by nondegeneracy. Thus A is a field Ω containing $\Phi 1$. Since $t(x) = n(x,1)$ vanishes identically we have $2 = t(1) = 0$, so Ω has characteristic 2, hence $x + \bar{x} = t(x) = 0$ implies $\bar{x} = x$ and the involution is the identity. Then $x^2 = x\bar{x} = n(x)1 \in \Phi 1$ so Ω is purely inseparable of exponent 2. This is case (0).

From now on we assume $R = 0$, so $n(x,y)$ is nondegenerate. We will show that if B is any proper finite-dimensional non-isotropic subalgebra (containing 1) there is $\ell \in B$ with $\ell^2 = \mu 1$ such that $\mathbb{C}(B,\mu) \subset A$. Indeed, we have $A = B \oplus B^\perp$ since B is finite-dimensional and non-isotropic. Since B is assumed proper, $B^\perp \neq 0$, and since n remains nondegenerate on $B^\perp$ we can find $\ell \in B^\perp$ with $n(\ell) = -\mu \neq 0$. We have $B\ell \subset B^\perp$ (hence $B \cap B\ell = 0$ by non-isotropy) since $n(B,B\ell) = n(\bar{B}B,\ell)$ (by 2.9) $\subset n(B,\ell) = 0$ as $\bar{B} = t(B)1 - B \subset B$ (B contains 1) and B is a subalgebra. In particular,

$t(B\ell) = n(B\ell, 1) = 0$ implies the elements of $B\ell$ are skew. By 3.9 we have $\mathbb{C}(B, \mu) = B \oplus B\ell \subset A$ constructed by the Cayley-Dickson process.

As before (only now everything taken place inside an algebra A given at the start) we can begin with $B_1 = \Phi 1$. If $B_1 = A$ we have Case I. If $B_1 \neq A$ but B_1 is non-isotropic we have $B_2 = \mathbb{C}(B_1, \mu_1) = \Phi 1 \oplus \Phi i \subset A$ by the foregoing. If B_1 is isotropic (i.e. the characteristic is 2) there is $e \in A$ with $t(e) = n(e,1) = 1$ by nondegeneracy of $n(x,y)$, so $B_2 = \Phi 1 \oplus \Phi e$ is a subalgebra ($e^2 = t(e)e - n(e)1 \in B_2$) and is non-isotropic (if $n(z,1) = n(z,e) = 0$ for $z = \alpha 1 + \beta e$ then $\beta n(1,e) = \alpha n(1,e) = 0$, $\alpha = \beta = 0$ as $n(1,1) = n(e,e) = 0$). If $B_2 = A$ we have case II. If $B_2 \neq A$ we have $B_3 = \mathbb{C}(B_2, \mu_2) \subset A$. B_3 is associative but not commutative because the involution on B_2 is nontrivial. Either $B_3 = A$, and we have case III, or $B_3 \neq A$ and we have $B_4 = \mathbb{C}(B_3, \mu_3) \subset A$. B_4 is alternative but not associative. We must have $B_4 = A$ or else $A \supset \mathbb{C}(B_4, \mu_4)$ whereas $\mathbb{C}(B_4, \mu_4)$ is not alternative since B_4 is not associative. ∎

Notice that A is <u>forced</u> to be finite-dimensional (of dimension 1, 2, 4, or 8) except in the degenerate case (0).

It is impossible to overstress the fact that (excluding the inseparable field extensions) A COMPOSITION ALGEBRA HAS DIMENSION 1, 2, 4, OR 8. These are magic numbers, and should be dutifully worshipped. For example, by a topological result of Bott and Milnor any finite-dimensional (nonassociative) division algebra over the reals has dimension 1, 2, 4, or 8 (though it need not be related to the particular division algebras $\mathbb{R}$, $\mathbb{C}$, Q, $\mathfrak{C}$). These magic numbers are ingredients in the construction of the Freudenthal Magic Square of Lie theory.

We will call the algebras of Type I-IV the ordinary composition algebras. A purely inseparable extension of Type 0 will be called an extraordinary composition algebra; note that it occurs only in characteristic 2, and is the only composition algebra which can be infinite-dimensional over Φ. The ordinary composition algebras have as their *-centers just Φ, but the extraordinary algebra has *-center Ω. Furthermore, Type 0 becomes Type I when considered over the field Ω rather than Φ; in some sense it is unusual only because we considered it over the "wrong" field.

We can extract a lot more information from the proof of the theorem. Once more (compare 3.14)

4.2 Corollary. The bilinear norm form $n(x,y) = t(x\bar{y})$ of an extraordinary composition algebra (including Type I in characteristic 2) vanishes identically, and $x \to n(x)$ is a ring isomorphism of Ω into Φ. Otherwise the bilinear norm form of an ordinary composition algebra is nondegenerate. ∎

4.3 Corollary. If A is an ordinary composition algebra over Φ, then for any nonisotropic unital subalgebra of dimension $\dim B = \frac{1}{2} \dim A$ the algebra A has the form $A = \mathbb{C}(B,\mu) = B \oplus B\ell$. Here ℓ may be chosen arbitrarily in $B^{\perp}$, and $\mu = \ell^2$. ∎

Thus a Cayley algebra can be built out of any of its quaternion subalgebras by the Cayley-Dickson process.

4.4 (Simplicity Theorem) Any composition algebra over a field Φ is

*-simple, and all are simple except for the case of a split 2-dimensional composition algebra $\Phi e + \Phi e^*$.

Proof. Note that *-simplicity is easy to prove: the Cayley-Dickson Formula (3.3) shows that if C is a proper *-ideal in B then $C + C\ell$ is a proper *-ideal in $\mathbb{C}(B,\mu)$, so that if at any stage of the Cayley-Dickson process there were a proper *-ideal there would remain one at the final 8-dimensional stage, whereas a Cayley algebra hasn't even got proper one-sided ideals by 3.15.

The inseparable extensions Ω (including $\Omega = \Phi$) are trivially simple, and by 3.16 the Cayley algebras are too. If A of dimension 2 or 4 has a proper ideal C then $n(C) = 0$ since $C \neq A$ has no invertible elements, yet $t(C) \supset t(C\bar{A}) = n(C,A) \neq 0$ by 2.7 since $C \neq 0$ and $n(x,y)$ is nondegenerate. Thus there is a proper idempotent e in C: $t(e) = 1$ but $n(e) = 0$. If A has dimension 2 then $e + e^* = t(e) = 1$ and $ee^* = e^*e = n(e) = 0$ shows $A = \Phi e + \Phi e^*$ is a direct sum of two copies of Φ. If A has dimension 4 then $A \supset B = \Phi e \oplus \Phi e^*$, so $A = \mathbb{C}(B,\mu) = B + Bj$ for any $j \perp B$ by 4.3. Then $e \in C$ implies ej and $je = e^*j$ lie in C, which contradicts isotropy of C: $n(ej, e^*j) = \mu n(e,e^*) = \mu \neq 0$. Therefore no such C exists in dimension 4. ∎

The proof of the Hurwitz theorem makes it clear that the norm form and the unit determine the structure of the composition algebra. This leads to a basic criterion for isomorphism.

4.5 (Isomorphism Theorem for Composition Algebras) Two composition algebras over a field Φ are isomorphic iff their norm forms are equivalent.

Proof. Suppose $F:A \to \tilde{A}$ is an isomorphism of composition algebras. We claim $\tilde{n}(Fx) = n(x)$, so F is automatically an equivalence of quadratic forms. (This is really a special case of the result that the generic norm and minimum polynomial of an algebra are invariant under isomorphisms.) We have $x^2 - t(x)x + n(x)1 = 0$ in A, so applying F yields $\tilde{x}^2 - t(x)\tilde{x} + n(x)\tilde{1} = 0$ for all $\tilde{x} = F(x)$ in $\tilde{A}$. But we already know $\tilde{x}^2 - \tilde{t}(\tilde{x})\tilde{x} + \tilde{n}(\tilde{x})\tilde{1} = 0$ in $\tilde{A}$, so $[\tilde{t}(\tilde{x}) - t(x)]\tilde{x} = [\tilde{n}(\tilde{x}) - n(x)]\tilde{1}$ for all $\tilde{x}$. If $\tilde{x} \notin \Phi\tilde{1}$, independence gives $\tilde{t}(\tilde{x}) - t(x) = \tilde{n}(\tilde{x}) - n(x) = 0$, while if $\tilde{x} = F(x) \in \Phi\tilde{1}$ then (as F is bijective) $x = \alpha 1 \in \Phi 1$ trivially has $\tilde{t}(\tilde{x}) = t(x) = 2\alpha$, $\tilde{n}(\tilde{x}) = n(x) = \alpha^2$. Thus in all cases $\tilde{n}(Fx) = n(x)$, and F is an equivalence.

Now suppose the quadratic forms n, $\tilde{n}$ of A, $\tilde{A}$ are equivalent under a bijection $F: A \to \tilde{A}$, $\tilde{n}(F(x)) = n(x)$. First consider the degenerate case of an inseparable field extension $A = \Omega$, where $n(x,y) \equiv 0$. Then $\tilde{n}(x,y) \equiv 0$ too and $\tilde{A} = \tilde{\Omega}$ is also an inseparable field. In this case F must already be an algebra isomorphism: it preserves products, $F(xy) = F(x)F(y)$ because $\tilde{n}(F(xy)) = n(xy) = n(x)n(y) = \tilde{n}(Fx)\tilde{n}(Fy) = \tilde{n}(Fx\cdot Fy)$, and by Corollary 4.2 $\tilde{n}(\tilde{z}) = \tilde{n}(\tilde{w})$ forces $\tilde{z} = \tilde{w}$ in $\tilde{\Omega}$.

From now on we assume $n(x,y)$ is nondegenerate. In general F itself will not be an algebra isomorphism, but we will build an isomorphism along with composition subalgebras as before. We start off with a rather pedestrian isomorphism $F_1: \mathbb{C}_1 = \Phi 1 \to \tilde{\mathbb{C}}_1 = \Phi\tilde{1}$. In characteristic $\neq 2$ this foothold is enough, but in characteristic 2 the subalgebra $\mathbb{C}_1$ is totally isotropic and (as when building composition algebras) won't do for our induction step. We choose $u \in \mathbb{C}_1$ with $n(u,1) = 1$ by nondegeneracy, $n(u) = -\mu_1$; we can always assume $F1 = \tilde{1}$ (if necessary, replace F by its translate $F_o(x) = F(vx)$ where $F(v) = \tilde{1}$; then $n(v) = \tilde{n}(Fv) = \tilde{n}(\tilde{1}) = 1$ implies $\tilde{n}(F_o x) = \tilde{n}(F(vx)) = n(vx) = n(v)n(x) = n(x)$ so F_o is again an isometry but now $F_o(1) = F(v) = \tilde{1}$). Then $\tilde{u} = Fu$ also has $\tilde{n}(\tilde{u},\tilde{1}) = \tilde{n}(Fu,F1) = n(u,1) = 1$, $\tilde{n}(\tilde{u}) = \tilde{n}(Fu) = n(u) = -\mu_1$, and $F_2(\alpha 1+\beta u) = \alpha\tilde{1} + \beta\tilde{u}$ defines an isomorphism F_2: $\mathbb{C}_2 = \Phi 1 + \Phi u \to \tilde{\mathbb{C}}_2 = \Phi\tilde{1} + \Phi\tilde{u}$ of (now non-isotropic) subalgebras. Indeed, all that is necessary for F_2 to be an isomorphism is that $F_2(u^2) = F_2(u)^2$, which follows from $u^2 = t(u)u - n(u)1 = u + \mu_1 1$ and $\tilde{u}^2 = \tilde{t}(\tilde{u})\tilde{u} - \tilde{n}(\tilde{u})\tilde{1} = \tilde{u} + \mu_1\tilde{1}$.

No matter what the characteristic, once we have an isomorphism F_i: $\mathbb{C}_i \to \tilde{\mathbb{C}}_i$ of proper <u>non-isotropic</u> subalgebras we can enlarge F_i. Indeed, since $\mathbb{C}_i^{\perp}$ is also non-isotropic (by nondegeneracy of n) we can find $\ell \in \mathbb{C}_i^{\perp}$ with $n(\ell_i) = -\mu_i \neq 0$ and build $\mathbb{C}_{i+1} = \mathbb{C}_i + \mathbb{C}_i\ell$ as before. By Witt's Theorem (which is applicable even in characteristic 2 since $n(x,y)$ is nondegenerate; see Part I) the fact that n, $\tilde{n}$ are equivalent and F_i an isometry from $\mathbb{C}_i$ to $\tilde{\mathbb{C}}_i$ implies $\mathbb{C}_i^{\perp}$ and $\tilde{\mathbb{C}}_i^{\perp}$ are also isometric, so corresponding to ℓ_i we can find $\tilde{\ell}_i \in \tilde{\mathbb{C}}_i^{\perp}$ with $\tilde{n}(\tilde{\ell}_i) = -\mu_i \neq 0$ and build $\tilde{\mathbb{C}}_{i+1} = \tilde{\mathbb{C}}_i + \tilde{\mathbb{C}}_i\tilde{\ell}_i$. But then by the Necessity Proposition 3.8, multiplication in $\mathbb{C}_{i+1}$, $\tilde{\mathbb{C}}_{i+1}$ is given by the Cayley-Dickson formula. Hence $F_{i+1}(c_i+d_i\ell_i) = F_i(c) + F_i(d)\tilde{\ell}_i$ defines an algebra isomorphism $\mathbb{C}_{i+1} \to \tilde{\mathbb{C}}_{i+1}$ (using $F_i(\bar{c}_i) = \overline{F_i(c)}$ since F_i is actually a $*$-isomorphism). This is the essential point — once ℓ and μ are given, the multiplication follows. Thus we can keep building up larger and larger isomorphic non-isotropic subalgebras until eventually (by finite-dimensionality when $n(x,y) \neq 0$) we have an isomorphism $\mathbb{C} \to \tilde{\mathbb{C}}$. ∎

4.6 Remark: If Φ is a field of characteristic $\neq 2$ the involution, trace, and norm of any degree 2 algebra are completely determined by the algebra structure, therefore are preserved by any algebra isomorphism. Indeed, $x^* = x \Leftrightarrow 2x = t(x)1 \Leftrightarrow x \in \Phi 1$ (characteristic $\neq 2$) and $x^* = -x \Leftrightarrow t(x) = 0 \Leftrightarrow x^2 \in \Phi 1$ but $x \notin \Phi 1$ or $x = 0$. Thus the symmetric and skew elements are defined algebraically, hence are preserved under any isomorphism, so the whole involution is too: $x = x_+ + x_-$ has $f(x^*) = f(x_+ - x_-) = f(x_+) - f(x_-) = \{f(x_+) + f(x_-)\}^* = f(x^*)$. If f preserves * it also preserves $t(x) = x + x^*$ and $n(x) = xx^*$. ∎

As an immediate consequence of the theorem we find that changing the parameter μ by a norm doesn't change the algebra $\mathbb{C}(B,\mu)$.

4.7 Lemma. If $n(a) \in \Phi$ is invertible then $b \oplus (ac)\ell \overset{F}{\to} b \oplus c\ell$ is an isomorphism of $\mathbb{C}(B,\mu)$ with $\mathbb{C}(B,\mu n(a))$. Thus the isomorphism class of $\mathbb{C}(B,\mu)$ depends only on B and the coset of μ modulo the norm subgroup of Φ, i.e. the image of μ in $\Phi/n(B)$.

Proof. Clearly F is an isometry: $n(b + (ac)\ell) = n(b) - \mu n(ac) = n(b) - \mu n(a)n(c) = \tilde{n}(b + c\ell)$. Thus by the Isomorphism Theorem $\mathbb{C}(B,\mu)$ and $\mathbb{C}(B,\mu n(a))$ are isomorphic.

In this case we can actually show directly F is an isomorphism, over an arbitrary ring Φ: $F((b_1 + ac_1\ell)(b_2 + ac_2\ell)) = F((b_1b_2 + \mu\bar{c}_2\bar{a}ac_1) + a(c_2b_1 + c_1\bar{b}_2)\ell) = (b_1b_2 + \mu n(a)\bar{c}_2c_1) + (c_2b_1 + c_1\bar{b}_2)\ell = (b_1 + c_1\ell)\tilde{\cdot}(b_2 + c_2\ell) = F(b_1 + ac_1\ell)\tilde{\cdot}F(b_2 + ac_2\ell)$. ∎

As another consequence we can establish the fact, alluded to in Chapter I, that all isotopes of a Cayley algebra are isomorphic. By the

Isomorphism Theorem it suffices if they have equivalent norm forms. But in the Isotope Formula 1.14 we saw that the isotope $\mathbb{C}^{(u,v)}$ had norm form $n^{(u,v)}(x) = n(uv)n(x)$, so the bijection $L_{uv}: \mathbb{C}^{(u,v)} \to \mathbb{C}$ satisfies $n(L_{uv}x) = n(uv \cdot x) = n(uv)n(x) = n^{(u,v)}(x)$. Thus L_{uv} is an equivalence and the isotope is isomorphic to $\mathbb{C}$.

4.8 (Isotopy Theorem for Cayley Algebras) All isotopes $\mathbb{C}^{(u,v)}$ of a Cayley algebra $\mathbb{C}$ are isomorphic to $\mathbb{C}$. ∎

As soon as the norm form represents zero, the whole composition algebra dissolves into a very simple form.

4.9 (Splitting Equivalence Theorem) The following conditions are equivalent for a composition algebra A over a field Φ.

(i) A is not a division algebra

(ii) A has zero divisors

(iii) the norm form represents zero, $n(x) = 0$ for some $x \neq 0$

(iv) A contains a proper idempotent $e \neq 1, 0$.

Proof. By Corollary 2.8 to the Inverse Theorem, (i) and (iii) are equivalent; clearly (ii) implies (i) (zero divisors $xy = 0$ destroy injectivity of L_x), and (iii) implies (ii) since if $n(x) = 0$ for $x \neq 0$ then $x\bar{x} = 0$ where $\bar{x} \neq 0$ (recall $\bar{\bar{x}} = x$!).

If $e \neq 1, 0$ is idempotent then $e \neq \alpha 1$, so $0 = e^2 - t(e)e + n(e)1 = [1 - t(e)]e + n(e)1$ implies $t(e) = 1$, $n(e) = 0$, and n represents zero nontrivially. Thus (iv) $\Rightarrow$ (iii). Conversely, suppose $n(x) = 0$ for $x \neq 0$. Then A is not an inseparable field, so by 4.2 the bilinear norm form is nondegenerate, $n(x,A) \neq 0$, and we can find y with $n(x,\bar{y}) = 1$. Thus $e = xy$ has trace $t(e) = t(xy) = n(x,\bar{y}) = 1$ (see 2.1) and norm $n(e) = n(xy) = n(x)n(y) = 0$, so that $e^2 = t(e)e - n(e)1 = e$ is idempotent. Clearly $e \neq 1$ since $n(e) \neq n(1) = 1$, and $e \neq 0$ since $t(e) \neq t(0) = 0$. Thus (iii) $\Longrightarrow$ (iv). ∎

4.10 (Idempotent Criterion) An element e in a composition algebra over a field is a proper idempotent iff $t(e) = 1$, $n(e) = 0$. ∎

We say a composition algebra over an arbitrary ring of scalars is <u>split</u> if it contains a proper idempotent $e \neq 1, 0$. By the above, a composition algebra over a field Φ is either split or a division algebra (this isn't true for general Φ - indeed $\Phi 1$ already may be neither a division algebra nor contain proper idempotents).

It is very important that COMPOSITION ALGEBRAS OVER A FIELD COME IN TWO KINDS, DIVISION ALGEBRAS OR SPLIT ALGEBRAS (according as n does not or does represent zero). Another important fact is that ALL SPLIT COMPOSITION ALGEBRAS OF A GIVEN DIMENSION LOOK ALIKE.

4.11 (Split Isomorphism Theorem) Any two split composition algebras of the same dimension over a field Φ are isomorphic.

Proof. We will show that the norm form of a split algebra necessarily has maximal Witt index. Since any two quadratic forms of the same (even) dimension having maximal Witt index are equivalent, and since equivalence of norm forms implies isomorphism of algebras by the Isomorphism Theorem, this will establish our result.

If A is split it contains an idempotent e with $t(e) = 1$, $n(e) = 0$, so $e + e^* = 1$ and $e^*e = 0$. Thus $A = A1 \subset Ae + Ae^*$ is a sum of two subspaces; this sum is direct because $ae = be^*$ implies $ae = ae^2 = (ae)e = (be^*)e = b(e^*e) = 0$. (This is a particular case of the Peirce Decomposition relative to two orthogonal idempotents, which we will discuss in more detail in Chapter VII.) Thus we have decomposed A into a direct sum $A = Ae \oplus Ae^*$ of totally isotropic subspaces (note $n(Ae) = n(A)n(e) = 0$, $n(Ae^*) = n(A)n(e^*) = n(A)n(e) = 0$) so that by definition it has maximal Witt index. ∎

Thus if you've seen one split algebra, you've seen them all. One way of building a split algebra is to take $\mathbb{C}(B,\mu)$ for $\mu = 1$ (whether B is split or not) because for any $b \neq 0$ the element $x = b + b\ell \neq 0$ has $n(x) = n(b) - \mu n(b) = 0$. Therefore we obtain the following (exhaustive) list of split composition algebras over a field Φ.

<u>Dimension 1.</u> A one-dimensional composition algebra $\Phi 1$ is never split according to our definition.

<u>Dimension 2.</u> If the characteristic $\neq 2$ the split two-dimensional composition algebras look like $\mathbb{C}(\Phi,1) = \Phi 1 + \Phi i$ with $i^2 = 1$. Then $\mathbb{C} = \Phi e \oplus \Phi e^*$ is a <u>direct sum of two copies of</u> Φ <u>with exchange involution</u>, where $e = \frac{1}{2}(1 + i)$, $e^* = \frac{1}{2}(1 - i)$. If the characteristic $= 2$ one takes $\mathbb{C}'(\Phi,0) = \Phi 1 + \Phi u$ where $t(u) = 1$, $n(u) = 0$ ($\mu = 0$ rather than $\mu = 1$) so again u is a proper idempotent and $\mathbb{C}' = \Phi u \oplus \Phi u^*$ consists of two copies of Φ.

In $A = \Phi e_1 \oplus \Phi e_2$ the trace and norm of an element $x = \alpha_1 e_1 + \alpha_2 e_2$ are $t(x) = \alpha_1 + \alpha_2$ and $n(x) = \alpha_1\alpha_2$. Notice that $n(x_1 + x_2) = n(x_1)n(x_2)$

is the product of norms $n_i(\alpha e_i) = \alpha$ on Φe_i.

<u>Dimension 4</u>. We obtain a split four-dimensional algebra $\mathbb{C}(\Phi;1,1) = \{\Phi e + \Phi e^*\} \oplus \{\Phi e + \Phi e^*\}j = \Phi e_{11} + \Phi e_{22} + \Phi e_{12} + \Phi e_{21}$ which is isomorphic <u>to the algebra $M_2(\Phi)$ of 2×2 matrices over Φ with standard involution</u> $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \to \begin{pmatrix}\delta & -\beta\\ -\gamma & \alpha\end{pmatrix}$. Here $e_{11} = e, e_{22} = e^*, e_{12} = ej, e_{21} = e^*j$ act like matrix units. This can be checked directly, or note that we have an imbedding $\Phi e \oplus \Phi e^* \to \Phi e_{11} + \Phi e_{22}$ by $\alpha e + \delta e^* \to \begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}$; since $j = e_{12} + e_{21} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ satisfies $jb = b^*j$ for all $b \in \Phi e_{11} + \Phi e_{22}$ and $j^2 = 1$, by 3.9 we have an isomorphism $\mathbb{C}(\Phi e \oplus \Phi e^*, 1) \to M_2(\Phi)$ via $(\alpha e + \delta e^*) + (\beta e + \gamma e^*)j \to (\alpha e_{11} + \delta e_{22}) + (\beta e_{11} + \gamma e_{22})(e_{12} + e_{21}) = \alpha e_{11} + \delta e_{22} + \beta e_{12} + \gamma e_{21} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$.

It might be well to stop for a moment and consider the involution in the (split) quaternion algebra $M_2(\Phi)$ in more detail. Here

$$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}^* = \begin{pmatrix}\alpha_{22} & -\alpha_{12}\\ -\alpha_{21} & \alpha_{11}\end{pmatrix}$$

Note that this is very different from the transpose involution $x \to x^t$ or any isotope $x \to cx^t c^{-1}$ thereof. The essential fact is that all traces $t(x) = x + x^*$ are scalars in $\Phi 1$. No other matrix algebra $M_n(\Phi)$, $n > 2$, can carry such an involution. Indeed, you will recall from the associative theory (the Cartan-Brauer-Hua Theorem, for example) that these quaternion algebras with standard involution are the unique exceptions to many general statements about associative algebras with involution.

The norm and trace are the usual ones for matrices $x = \begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$,

$n(x) = \det x = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$, $t(x) = \text{trace } x = \alpha_{11} + \alpha_{22}$.

Dimension 8. The unique eight-dimensional split Cayley algebra $\mathbb{C}(\Phi,1,1,1)$ is obtained as $M_2(\Phi) + M_2(\Phi)\ell = \{\Phi e_{11} + \Phi e_{22} + \Phi e_{12}^{(1)} + \Phi e_{21}^{(1)}\} + \{\Phi e_{12}^{(2)} + \Phi e_{12}^{(3)} + \Phi e_{21}^{(2)} + \Phi e_{21}^{(3)}\}$ for $e_{12}^{(1)} = e_{12}$, $e_{21}^{(1)} = e_{21}$, $e_{12}^{(2)} = e_{21}\ell$, $e_{21}^{(2)} = -e_{12}\ell$, $e_{12}^{(3)} = e_{11}\ell$, $e_{21}^{(3)} = e_{22}\ell$ where $e_{12}^{(k)}$, $e_{21}^{(k)}$ again act like matrix units e_{12}, e_{21}, but we have in addition products $e_{12}^{(k)}e_{12}^{(\ell)} = 0$ if $k \neq \ell$ and $e_{12}^{(k)}e_{12}^{(k+1)} = e_{21}^{(k+2)}$, $e_{21}^{(k+1)}e_{21}^{(k)} = e_{12}^{(k+2)}$ (indices mod 3). (More about these "Cayley matrix units" in Section VII.5).

Still another way of looking at the split Cayley algebra is to think of it as the Zorn vector matrix algebra consisting of all matrices

$$A = \begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix} \qquad (\alpha, \beta \in \Phi;\ x, y \in \Phi^{(3)})$$

by means of the isomorphism $\alpha e_{11} + \beta e_{22} + \xi_1 e_{12}^{(1)} + \xi_2 e_{12}^{(2)} + \xi_3 e_{12}^{(3)} + \eta_1 e_{21}^{(1)} + \eta_2 e_{21}^{(2)} + \eta_3 e_{21}^{(3)} \to \begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix}$ for $x = (\xi_1, \xi_2, \xi_3), y = (\eta_1, \eta_2, \eta_3)$. Here the multiplication is

$$A_1 A_2 = \begin{pmatrix} \alpha_1\alpha_2 + x_1 \cdot y_2 & \alpha_1 x_2 + x_1\beta_2 + y_1 \times y_2 \\ y_1\alpha_2 + \beta_1 y_2 + x_1 \times x_2 & \beta_1\beta_2 + y_1 \cdot x_2 \end{pmatrix}$$

where $x \cdot y$ is the ordinary inner product $(\xi_1, \xi_2, \xi_3)\cdot(\eta_1, \eta_2, \eta_3) = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$ on $\Phi^{(3)}$ and $x \times y$ the ordinary vector cross product $(\xi_1, \xi_2, \xi_3) \times (\eta_1, \eta_2, \eta_3) = (\xi_2\eta_3 - \xi_3\eta_2, \xi_3\eta_1 - \xi_1\eta_3, \xi_1\eta_2 - \xi_2\eta_3)$ (which may be easier to recall if you write $x = (\xi_1, \xi_2, \xi_3) = \xi_1 i + \xi_2 j + \xi_3 k$ with rules

$i \times j = k$, $j \times k = i$, $k \times i = j$ cyclically, with anti-commutativity $i \times i = j \times j = k \times k = 0$, $j \times i = -i \times j$, $k \times j = -j \times k$, $i \times k = -k \times i$).

In this notation the norm and trace take the suggestive form

$$t(A) = \alpha + \beta = \mathrm{tr}\,(A), \quad n(A) = \alpha\beta + n(x,y) = \alpha\beta - x \cdot y = \det A.$$

4.12 Remark. In the split characteristic $\neq 2$ case the Cayley-Dickson formula (3.3) can be straightened out. If $\ell^2 = 1$ then $e_1 = \frac{1}{2}(1 - \ell)$, $e_2 = \frac{1}{2}(1 + \ell)$ are idempotents and we can write

$$(4.13) \quad \mathbb{C} = Be_1 \oplus Be_2 = \Phi e_1 \oplus \Phi e_2 \oplus B_o e_1 \oplus B_o e_2$$

instead of $\mathbb{C} = B1 \oplus B\ell$, where B_o are the trace zero elements and multiplication is given by

$$(4.14) \qquad \begin{aligned} (be_i)(ce_i) &= (b \times c)e_j \\ & \qquad\qquad (i \neq j,\ b,c \in B_o) \\ (be_i)(ce_j) &= (b \cdot c)e_j = \tfrac{1}{2}t(bc)e_j \end{aligned}$$

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$= -\frac{1}{2}n(b,c) = -\frac{1}{2}t(b\bar{c}) = \frac{1}{2}t(bc)$

where $\quad b \cdot c = \frac{1}{2}(bc + cb)$ and $b \times c = \frac{1}{2}(bc - cb)$. Indeed $(be_1)(ce_1) = \frac{1}{4}(b - b\ell)(c - c\ell) = \frac{1}{4}\{(bc + \bar{c}b) - (b\bar{c} + cb)\ell\} = \frac{1}{4}\{(bc - cb) + (bc - cb)\ell\}$ and $(be_1)(ce_2) = \frac{1}{4}(b - b\ell)(c + c\ell) = \frac{1}{4}\{(bc - \bar{c}b) + (cb - b\bar{c})\ell\} = \frac{1}{4}\{(bc + cb) + (bc + cb)\ell\}$. The formulas for $(be_2)(ce_2)$ and $(be_2)(ce_1)$ follow by replacing ℓ by $-\ell$. ∎

Although all split algebras of a given dimension look alike, the same is by no means true of division algebras. The classification of division algebras depends very much on arithmetic properties of the base field. We can say

(i) there are no composition division algebras of dimension > 1 over an algebraically closed field

(ii) there are no composition division algebras of dimension > 2 over a finite field

(iii) the only composition division algebra of dimension 2 over the real field $\mathbb{R}$ is the field $\mathbb{C} = \mathbb{C}(\mathbb{R},-1)$ of complex numbers; the only quaternion division algebra over the reals is the algebra of ordinary quaternions $\mathbb{C}(\mathbb{R},-1,-1)$; and the only Cayley division algebra over the reals is the algebra of ordinary Cayley numbers $\mathbb{C}(\mathbb{R},-1,-1,-1)$.

The algebraically closed case (i) is trivial since any non-constant form $n(x_1,\ldots,x_n)$ for $n > 1$ has nontrivial zeros in an algebraically closed field; similarly for a finite field Φ the norm form $n(x_1,\ldots,x_n)$ has more variables $n = 4, 8$ than its degree 2, hence (by Artin-Chevally; see Part 1) has a nontrivial zero. For the real case $\mathbb{R}$ we know that if we ever take $\mu = 1$ in the Cayley-Dickson process, $C(B,\mu)$ will be split; but that leaves us only $\mu = -1$ each time, because every real μ can be written $\mu = \pm\alpha^2 = \pm n(\alpha)$ (according as μ is positive a negative), so $\mathbb{C}(B,\mu) \cong \mathbb{C}(B,\pm 1)$ by Lemma 4.6.

Exercise

4.1 In the proof of the Simplicity Theorem 4.4, use *-simplicity of A to show that if C is a proper ideal in A then $A = C \boxplus C^*$. In dimension 2 show $C = \Phi c$ where $c^2 = \gamma c$ for $\gamma \neq 0$; conclude $C = \Phi e$.

4.2 Give an example of an isometry $A \xrightarrow{F} \tilde{A}$ of 4-dimensional composition algebras such that $F(1) = 1$ but F is not an isomorphism. Is this possible in dimension 2?

4.3 Give an example of an isomorphism $B \xrightarrow{F} \tilde{B}$ of 4-dimensional unital subalgebras (isotropic, of course!) which cannot be extended to an isomorphism $A \to \tilde{A}$ of 8-dimensional composition algebras. Show that nevertheless A and $\tilde{A}$ must be isomorphic.

4.4 Describe all proper nilpotent elements in a composition algebra (Nilpotent Criterion).

#8. Problem Set: Wright's Theorem on Absolute-Valued Algebras

An absolute value on an algebra A over the field $\mathbb{R}$ of real numbers is a real-valued function $|x|$ on A satisfying

(i) $|x| > 0$ for $x \neq 0$

(ii) $|\alpha x| = |\alpha||x|$ for $\alpha \in \mathbb{R}$

(iii) $|x+y| \leq |x| + |y|$

(iv) $|xy| = |x||y|$.

The last relation shows A cannot have any zero divisors.

1. If A is a composition division algebra over $\mathbb{R}$, show $|x| = \sqrt{n(x)}$ is a well-defined absolute value on A.

   We want to establish the converse, that every absolute-valued (nonassociative) division algebra is a composition algebra with $|x| = \sqrt{n(x)}$. What we must do is show $n(x) = |x|^2$ is a nondegenerate quadratic form on A permitting composition. It certainly permits composition, $n(xy) = |xy|^2 = |x|^2|y|^2 = n(x)n(y)$ by (iv), and also $n(\alpha x) = \alpha^2 n(x)$ by (ii). The whole difficulty resides in showing n is quadratic, i.e. $n(x,y)$ is bilinear.

2. (Jordan-von Neumann Characterization of Inner Product Spaces) Show a function $f(x)$ on a Φ-module ($\frac{1}{2} \in \Phi$) which satisfies $f(\alpha x) = \alpha^2 f(x)$, $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ necessarily has the form $f(x) = g(x,x)$ where $g(x,y) = \frac{1}{4}\{f(x+y) - f(x-y)\}$ is a symmetric $\mathbb{Z}$-bilinear function. If $\Phi = \mathbb{R}$ and $f(x+y) \leq f(x) + f(y)$, show g is $\mathbb{R}$-bilinear.

3. Conclude that a normed linear space (or Banach space) is an inner product space (or Hilbert space) iff it satisfies the parallelogram

law $||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2$. Conclude further a normed linear space X is an inner product space iff every 2-dimensional subspace X_o is. Observe that a 2-dimensional normed space is an inner product space iff its unit sphere $S = \{x \mid ||x|| = 1\}$ is an ellipse $\{x \mid g(x,x) = 1\}$, g a symmetric bilinear form.

4. Show that in a normed division algebra $|x+y|^2 + |x-y|^2 \geq 4$ if $|x| = |y| = 1$.

5. Show that if $|\ |$ is a norm on a 2-dimensional real vector space satisfying $|x+y|^2 + |x-y|^2 \geq 4$ for x,y in the unit sphere S, and $||\ ||$ a norm with unit sphere an ellipse E ($||x||^2 = g(x,x)$) such that E lies inside S with intersection $E \cap S$ consisting of more than 2 points, then necessarily E = S and $|\ | = ||\ ||$ is given by the inner product g.

6. Let $|\ |$ be a norm on a 2-dimensional space with unit sphere S. Show that if E is an ellipse $\{x \mid g(x,x) = 1\} = \{x \mid q(x) = 1\}$ (q quadratic form) contained inside S and having maximal area among such ellipses, then E meets S in at least 4 points.

7. Prove the <u>Day-Schoenberg Theorem</u>: If a normed linear space satisfies $|x+y|^2 + |x-y|^2 \geq 4$ for $|x| = |y| = 1$, it is an inner product space.

8. Prove <u>Wright's Theorem</u>: An absolute valued real division algebra is a composition algebra, hence is either $\mathbb{R}$, $\mathbb{C}$, Q, or $\mathfrak{C}$ under $|x| = \sqrt{n(x)}$.

#9. Problem Set: Albert's Theorem on Absolute-Valued Algebras

By techniques of linear algebra (rather than geometry) one can establish the finite-dimensional version of Wright's Theorem.

1. Show that if L_x satisfies an equation of degree 2 for each x in a unital algebra A, then A is left alternative.

   Thus we will try to prove L_x is degree 2 when A is absolute-valued (a similar argument on R_x will give right alternativity). This leads us to investigate the minimum polynomial of L_x.

2. For a linear transformation T on a finite-dimensional space over $\mathbb{R}$ let $\sigma(T)$, the <u>spectrum</u> of T, consist of the characteristic roots λ of T (roots of $\det|\lambda I - T| = 0$); $\sigma(T)$ is a subset of $\mathbb{C}$. Show $\sigma(p(T)) = p(\sigma(T))$ for any polynomial $p(\lambda)$.

3. Show that if X is a finite-dimensional normed space and T is a bounded linear transformation, $|Tx| \leq |T|\ |x|$ for all x, then $|\sigma(T)| \leq |T|$ (i.e. the spectrum is contained in a disc of radius $|T|$).

4. If T, T^{-1} are bounded and $|T^{-1}| = |T|^{-1}$ (for example, if $|Tx| = \tau|x|$ for <u>all</u> x then $|T| = \tau$ and $|T^{-1}| = \tau^{-1}$) show $|\sigma(T)| = |T|$ (i.e. $\sigma(T)$ lies on the circle of radius $|T|$).

5. If A is an absolute-valued finite-dimensional unital algebra, show that any L_x has at most two (conjugate) characteristic roots.

6. Show that if $S = I - Z$ for Z nilpotent has $|Sx| = |x|$ for all x, then $Z = 0$ and $S = I$.

7. Show that any L_x for x in a unital finite-dimensional absolute-valued algebra has degree 2.

8. Show every finite-dimensional absolute-valued algebra has a unital isotope.

9. Prove <u>Albert's Theorem</u>: Every finite-dimensional absolute-valued algebra over $\mathbb{R}$ is an isotope of $\mathbb{R}$, $\mathbb{C}$, Q, or $\mathcal{K}$ (and every unital one is itself one of $\mathbb{R}$, $\mathbb{C}$, Q, or $\mathcal{K}$).