§3. The Cayley-Dickson process

The Cayley-Dickson process, as its name suggests, is due to A. A. Albert. It is a doubling process for building new algebras out of old ones. Given a scalar $\mu \in \Phi$ and a unital algebra B with scalar involution $b \to \bar{b}$ (so $b\bar{b} = \bar{b}b = n(b)$ and $t(b) = b + \bar{b}$ lie in $\Phi 1$), we will build a new algebra

$$\mathbb{C}(B,\mu) = B \oplus B\ell \tag{3.1}$$

from two copies of B as linear space, with new involution

$$(b+c\ell)^* = \bar{b} - c\ell \tag{3.2}$$

and new multiplication given by the <u>Cayley-Dickson formula</u>

$$(b_1+c_1\ell)(b_2+c_2\ell) = (b_1b_2+\mu\bar{c}_2c_1) + (c_2b_1+c_1\bar{b}_2)\ell \ . \tag{3.3}$$

This eminently forgettable formula can be broken down into bite-sized pieces. Besides the fact that B is imbedded as a subalgebra with its usual multiplication we have

$$b\ell = \ell\bar{b} \tag{3.4}$$

$$b(c\ell) = (cb)\ell \tag{3.5}$$

$$(c\ell)b = (c\bar{b})\ell \tag{3.6}$$

$$(b\ell)(c\ell) = \mu\bar{c}b \ . \tag{3.7}$$

Some helpful mnemonic devices: in (3.5) notice that to multiply b against $c\ell$ you slip the b in <u>behind</u> the c, and also in (3.6) the b gets put between the c and the ℓ, but in moving past the ℓ it gets conjugated as in (3.4). (WARNING: many authors use $B \oplus \ell B$ instead, which turns all these formulas around.)

The multiplication formula is forced upon us: we <u>must</u> define multiplication by (3.3) if we want to have $\ell^2 = \mu 1$ and $b\ell = \ell\bar{b}$.

3.8 (Necessity Proposition) If B is a unital subalgebra with scalar involution of a unital alternative algebra A, and ℓ an element of A satisfying $\ell^2 = \mu 1$ and $b\ell = \ell\bar{b}$ for all $b \in B$, then $B + B\ell$ is a subalgebra of A whose multiplication is given by the Cayley-Dickson formula.

Proof. Since (3.4) is one of our assumptions, we have $(b\ell)(c\ell) = (\ell\bar{b})(c\ell) = \ell(\bar{b}c)\ell$ (Middle Moufang) $= \ell\{\ell(\overline{\bar{b}c})\} = \ell^2(\bar{c}b) = \mu\,\bar{c}b$, so (3.7) is forced upon us. From alternativity $(cb)\ell + (c\ell)b = c(b\ell+\ell b)$ we see $(c\ell)b = c\{(b+\bar{b})\ell\} - (cb)\ell$ (by (3.4)) $= \{c(b+\bar{b}) - cb\}\ell$ (since $b + \bar{b} \in \Phi 1$) $= (c\bar{b})\ell$, so (3.6) too is forced. If A had an involution with $\bar{\ell} = -\ell$ this would imply (3.5), but since we are not assuming an involution we argue dually $b(c\ell) = b(\ell\bar{c}) = -\ell(b\bar{c}) + (b\ell+\ell b)\bar{c} = -\ell(b\bar{c}) + \{\ell(b+\bar{b})\}\bar{c} = \ell\{-b\bar{c} + (b+\bar{b})\bar{c}\} = \ell(\bar{b}\bar{c}) = \ell(\overline{cb}) = (bc)\ell$.

If μ is cancellable ($\mu x = 0 \Rightarrow x = 0$) we could also derive (3.5) - (3.6) from the Moufang formulas: for example, $\mu\{b(c\ell)\} = \ell(\ell\{b(\ell\bar{c})\} = \ell(\{\ell b\ell\}\bar{c}) = \ell(\{\ell^2\bar{b}\}\bar{c}) = \mu\ell(\bar{b}\bar{c}) = \mu(cb)\ell$ and $\mu\{(c\ell)b\} = (\{(c\ell)b\}\ell)\ell = (c\{\ell b\ell\})\ell = \mu(c\bar{b})\ell$.

Thus the various pieces (3.4) - (3.7) of the Cayley-Dickson formula are forced upon us. In particular, $B + B\ell$ is a subalgebra. ∎

3.9 Corollary. If B is a unital subalgebra with scalar involution of a unital alternative algebra A and $\ell \in A$ an element satisfying $\ell^2 = \mu 1$ and $b\ell = \ell\bar{b}$, then the map $b \oplus c\ell \to b + c\ell$ is a homomorphism $\mathbb{C}(B,\mu)$ onto the subalgebra $B + B\ell$. When A has an involution extending that on B with $\ell^* = -\ell$, this homomorphism is a $*$-homomorphism. When $B \cap B\ell = 0$ and μ is cancellable, the map is an isomorphism.

Proof. Since both $\mathbb{C}(B,\mu)$ and $B + B\ell$ have multiplication given by the Cayley-Dickson formula, the map is a homomorphism. If $B \cap B\ell = 0$ then $b + c\ell = 0 \Rightarrow b = c\ell = 0$, and if μ is cancellable $c\ell = 0 \Rightarrow \mu c = (c\ell)\ell = 0 \Rightarrow c = 0$, so in these cases the map is injective. If an involution $*$ on A satisfies $\ell^* = -\ell$ and $b^* = \bar{b}$, then $b\ell = \ell\bar{b} \Rightarrow b\ell = -\ell^* b^* = -(b\ell)^* \Rightarrow (b\ell)^* = -b\ell$, so the involution on A satisfies (3.2). In this case the map preserves the involution. $\square$

We say an algebra $\mathbb{C}(B,\mu)$ (alternative or not) is obtained from B by the <u>Cayley-Dickson process</u>. Note that this process doubles dimension.

Returning to $\mathbb{C}(B,\mu)$ in general, let us verify that $*$ is again a scalar involution. Clearly it is linear of period 2. It is an anti-homomorphism since for $x_i = b_i + c_i\ell$ we have $\overline{x_1x_2} = \overline{(b_1b_2 + \mu\bar{c}_2c_1)} - (c_2b_1 + c_1\bar{b}_2)\ell = (\bar{b}_2\bar{b}_1 + \mu\bar{c}_1c_2) - (c_1\bar{b}_2 + c_2b_1)\ell = (\bar{b}_2 - c_2\ell)(\bar{b}_1 - c_1\ell) = \bar{x}_2\bar{x}_1$ by (3.3). Furthermore, the involution is scalar since

$$t(x) = t(b) \qquad (x = b + c\ell) \tag{3.10}$$
$$n(x) = n(b) - \mu n(c) .$$

The trace is easy, while for the norm we have $x\bar{x} = (b + c\ell)(\bar{b} - c\ell) = b\bar{b} - \mu\bar{c}c = n(b) - \mu n(c)$.

Notice that since it possesses a scalar involution, $\mathbb{C}(B,\mu)$ is of degree 2.

From the expression for the norm on $\mathbb{C}(B,\mu)$, we see that it inherits nondegeneracy from B:

3.11 (Nondegeneracy Criterion) If μ is cancellable then $n(x,y)$ is nondegenerate as a bilinear form on $\mathbb{C}(B,\mu)$ if it is nondegenerate on B. If in addition $n(b) \neq \mu n(c)$ whenever $n(b), n(c) \neq 0$ but $n(b,B) = n(c,B) = 0$, then n is nondegenerate as a quadratic form on $\mathbb{C}(B,\mu)$ if it is nondegenerate on B.

Proof. If $z = b + c\ell$ has $n(z,x) = 0$ for all $x = a + d\ell$ then from linearized (3.10) $n(b,a) - \mu n(c,d) = 0$ for all a,d; setting successively $d = 0$, $a = 0$ we see $n(b,B) = \mu n(c,B) = 0$. If $n(x,y)$ is nondegenerate on B and μ is cancellable, this yields $b = c = 0$ and $z = 0$. If we only assume the quadratic form is nondegenerate on B, then when $n(z) = n(z,x) = 0$ for all x we have $n(b) = \mu n(c)$ yet $n(b,B) = n(c,B) = 0$, hence $n(b), n(c) \neq 0$ by nondegeneracy on B, contrary to our hypothesis $n(b) \neq \mu n(c)$. ∎

3.12 Remark. If Φ is a field, the condition $n(b) \neq \mu n(c)$ is equivalent to the condition that $\mu \neq n(b)n(c)^{-1} = n(bc^{-1})$ not be a norm, $\mu \notin n(B)$. ∎

To see what sort of algebras the Cayley-Dickson process leads to, we prove

3.13 (Criterion) If B is a unital algebra with scalar involution then

(1) $\mathbb{C}(B,\mu)$ is commutative iff B is commutative with trivial

involution; (2) $\mathbb{C}(B,\mu)$ is associative iff B is commutative and associative; (3) $\mathbb{C}(B,\mu)$ is alternative iff B is associative.

Proof. From the Cayley-Dickson formula the left and right multiplications by an element $x = b + c\ell$ in $\mathbb{C}(B,\mu)$ have matrices

$$L_x = \begin{pmatrix} L_b & \mu R_c{}^* \\ L_c{}^* & R_b \end{pmatrix} \qquad R_x = \begin{pmatrix} R_b & \mu L_{\bar c} \\ L_c & R_{\bar b} \end{pmatrix}$$

relative to the decomposition $\mathbb{C}(B,\mu) = B \oplus B\ell$: $L_x b_2 = (bb_2) \oplus (c\bar b_2)\ell = \{L_b(b_2)\} \oplus \{L_c{}^*(b_2)\}\ell$, $L_x(c_2\ell) = \mu\bar c_2 c \oplus (c_2 b)\ell = \{\mu R_c{}^*(c_2)\} \oplus \{R_b(c_2)\}\ell$, $R_x b_1 = (b_1 b) \oplus (cb_1)\ell = \{R_b(b_1)\} \oplus \{L_c(b_1)\}\ell$, $R_x(c_1\ell) = \mu\bar c c_1 \oplus (c_1\bar b)\ell = \{\mu L_{\bar c}(c_1)\} \oplus \{R_{\bar b}(c_1)\}\ell$. We will deal mainly with the R_x, since they have a slightly simpler form.

Commutativity of $\mathbb{C}$ means $L_x = R_x$ for all x, and this is equivalent to $L_b = R_b$ and (setting $c = 1$, $L_c = I$) $* = I$, i.e. that B is commutative with identity involution.

Associativity of $\mathbb{C}$ means $R_{x_2}R_{x_1} = R_{x_1 x_2}$, i.e.

$$\begin{pmatrix} R_{b_2}R_{b_1} + L_{\bar c_2}L_{c_1} & \mu\{R_{b_2}L_{\bar c_1} + L_{\bar c_2}R_{\bar b_1}\} \\ L_{c_2}R_{b_1} + R_{\bar b_2}L_{c_1} & \mu L_{c_2}L_{\bar c_1} + R_{\bar b_2}R_{\bar b_1} \end{pmatrix} = \begin{pmatrix} R_{b_1 b_2} + R_{\bar c_2 c_1} & \mu\{L_{\bar b_1 \bar c_2} + L_{b_2 \bar c_1}\} \\ L_{c_2 b_1} + L_{c_1 \bar b_2} & \mu R_{\bar b_2 \bar b_1} + \mu R_{\bar c_1 c_2} \end{pmatrix}.$$

Setting $c_1 = c_2 = 0$ shows $R_{b_2}R_{b_1} = R_{b_1 b_2}$, i.e. B is associative; setting $b_2 = c_1 = 0$ and $c_2 = 1$ shows $R_{b_1} = L_{b_1}$ (lower left entry), i.e. B is commutative. Conversely, if B is commutative associative then $R_b R_c = R_{cb}$,

$L_b L_c = L_{bc}$, $L_b = R_b$ show $R_{x_2} R_{x_1} = R_{x_1 x_2}$.

Alternativity of $\mathbb{C}$ means $R_x^2 = R_{x^2}$ (the involution then gives left alternativity), or equivalently $R_x R_{\bar{x}} = R_{x\bar{x}}$ (since $x+\bar{x} = t(x) \in \Phi 1$),

$$\begin{pmatrix} R_b R_{\bar{b}} - \mu L_{\bar{c}} L_c & \mu\{-R_b L_{\bar{c}} + L_{\bar{c}} R_b\} \\ L_c R_{\bar{b}} - R_{\bar{b}} L_c & -\mu L_c L_{\bar{c}} + R_{\bar{b}} R_b \end{pmatrix} = (n(b) - \mu n(c)) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

From $[L_c, R_{\bar{b}}] = 0$ we see B is associative. Conversely, if B is associative the off-diagonals vanish and $R_b R_{\bar{b}} - \mu L_{\bar{c}} L_c = R_{\bar{b}b} - \mu L_{\bar{c}c} = \{n(b) - \mu n(c)\}I = R_{\bar{b}} R_b - \mu L_c L_{\bar{c}}$, proving $R_x R_{\bar{x}} = R_{x\bar{x}}$. ∎

Thus you must start with an associative algebra in order to obtain an alternative one by the Cayley-Dickson construction; once you have built an alternative but not associative algebra you can go no further, anything further will not be alternative.

We are now in a position to build composition algebras. We begin with B_1 being just the ring $\Phi 1$ of scalars with identity involution. Next we form a two dimensional $B_2 = \mathbb{C}(B_1, \mu_1) = \Phi 1 + \Phi i$, which will be commutative associative since B_1 is commutative associative with identity involution; if Φ has characteristic $\neq 2$, $2 \neq 0$, the involution $\alpha + \beta i \to \alpha - \beta i$ is not the identity.

In characteristic 2 the usual process applied to any algebra with identity involution will still have identity involution. To break out of this cycle we must modify the process slightly. We form

$$B_2' = \Phi 1 + \Phi u \qquad u^2 - u + \mu_1 1 = 0$$

so that B_2 is a commutative associative algebra with nontrivial involution determined by

$$u + \bar{u} = 1 .$$

Indeed, the only non-trivial product of basis elements 1, u is u^2, and $\bar{u}^2 = (1-u)^2 = 1 - 2u + u^2 = 1 - u - \mu_1 1 = \bar{u} - \mu_1 1 = \overline{u - \mu_1 1} = \overline{u^2}$. Note $t(u) = 1$, $n(u) = -\mu_1$. We will still call this algebra $\mathbb{C}(B_1, \mu_2)$ and pretend it is obtained by the Cayley-Dickson process. Remember: <u>the second stage of the Cayley-Dickson process is defined differently in characteristic 2</u>.

Once we have arrived at a second-stage algebra B_2 of dimension 2 with nontrivial scalar involution, we can form a 4-dimensional $B_3 = \mathbb{C}(B_2, \mu_2) = B_2 \oplus B_2 j$ which will be associative but not commutative. These are precisely the <u>quaternion algebras</u> over Φ.

From B_3 we construct an 8-dimensional algebra $B_4 = \mathbb{C}(B_3, \mu_3)$ which will be alternative but (since B_3 is not commutative) not associative. Such an 8-dimensional algebra is called a <u>Cayley algebra</u> (alias <u>Cayley-Dickson</u>, <u>Cayley-Graves</u>, <u>Albert-Dickson</u>, <u>octaves</u>, you name it; in analogy with quaternions, they are often called <u>octonions</u>). Thus Cayley algebras are those obtained from quaternion algebras by the Cayley-Dickson process.

At this point the construction stops, for further algebras will not be alternative.

These algebras

$$B_1 = \Phi 1 \quad B_2 = \mathbb{C}(B_1, \mu_1) \quad B_3 = \mathbb{C}(B_2, \mu_2) \quad B_4 = \mathbb{C}(B_3, \mu_3)$$

of dimensions 1, 2, 4, 8 are the basic algebras obtained by the Cayley-

Dickson process, and so we will call them the <u>Cayley-Dickson process algebras</u>. If we had let the general construction run on in Characteristic 2 (rather than modifying it) we would simply have gotten larger and larger commutative, associative algebras with identity involution such that $x^2 \in \Phi 1$ for all x. If Φ is a field, this means we have a purely inseparable extension of exponent 2.

As alternative algebras with scalar involution, the Cayley-Dickson process algebras are of degree 2 and their norm forms permit composition (see (2.4)). To be composition algebras according to our definition, the norms must be nondegenerate. In the case of B_1, $n(\alpha,\beta) = 2\alpha\beta$ is nondegenerate iff Φ has no 2-torsion, and $n(\alpha) = \alpha^2$ is nondegenerate iff Φ has no elements with $\alpha^2 = 2\alpha = 0$. If Φ has no 2-torsion and we take μ_1, μ_2, μ_3 to be cancellable, then by the Nondegeneracy Criterion 3.11 the quadratic extension B_2, the quaternion algebra B_3, and the Cayley algebra B_4 will all have nondegenerate forms $n(x,y)$, and so are composition algebras. If Φ has characteristic 2 the form $n(x,y)$ on the modified $\mathbb{C}_1'(\Phi,\mu_1) = \Phi 1 + \Phi u$ is nondegenerate (being hyperbolic, $n(\alpha 1+\beta u, \alpha' 1+\beta' u) = \alpha\beta' + \alpha'\beta$ if $2 = 0$), so again B_2, B_3, B_4 are composition algebras.

3.14 (Composition Proposition) If Φ has no 2-torsion or all 2-torsion, then any algebra of dimension 2, 4 or 8 obtained by the Cayley-Dickson process by means of cancellable parameters μ_1, μ_2, μ_3 is a composition algebra with nondegenerate norm bilinear form $n(x,y)$; degeneracy of $n(x,y)$ is possible only in dimension 1. If in addition Φ has no nilpotent elements, then the norm form n is nondegenerate in all cases and all the algebras are composition algebras. □

In the next section we will see that conversely, over a field, all composition algebras are obtained by the Cayley-Dickson process.

In case Φ has no 2-torsion, the algebras we have built look like the following.

Dimension 1: Base ring

$B_1 = \Phi 1 = \Phi$ is a commutative associative algebra with identity involution $\bar{1} = 1$ and norm $n(\alpha_o) = \alpha_o^2$.

Dimension 2: Quadratic extension

$B_2 = \Phi 1 \oplus \Phi i$ is a commutative associative algebra will nontrivial involution $\bar{i} = -i$, multiplication $i^2 = \mu_1 1$, and norm $n(\alpha_o 1 + \alpha_1 i) = \alpha_o^2 - \alpha_1^2 \mu_1$.

Dimension 4: Quaternion algebra

$B_3 = \Phi 1 \oplus \Phi i \oplus \Phi j \oplus \Phi k$ is a non-commutative associative algebra with involution $\bar{i} = -i$, $\bar{j} = -j$, $\bar{k} = -k$, norm

$$n(\alpha_o 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k) = \alpha_o^2 - \alpha_1^2 \mu_1 - \alpha_2^2 \mu_2 + \alpha_3^2 \mu_1 \mu_2 ,$$

and multiplication

$$i^2 = \mu_1 1 \quad j^2 = \mu_2 1 \quad k^2 = -\mu_1 \mu_2 1$$

$$ij = k = -ji \quad jk = -\mu_2 i = -kj \quad ki = -\mu_1 j = -ik .$$

Dimension 8: Cayley algebra

$B_4 = \Phi 1 \oplus \Phi i \oplus \Phi j \oplus \Phi k \oplus \Phi \ell \oplus \Phi i\ell \oplus \Phi j\ell \oplus \Phi k\ell$ is an alternative but not associative algebra with involution $\bar{i} = -i, \bar{j} = -j, \bar{k} = -k, \bar{\ell} = -\ell, \overline{(i\ell)} =$

$-i\ell$, $\overline{(j\ell)} = -j\ell$, $\overline{(k\ell)} = -k\ell$, norm

$$n(\alpha_o 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k + \alpha_4 \ell + \alpha_5 i\ell + \alpha_6 j\ell + \alpha_7 k\ell)$$

$$= \alpha_o^2 - \alpha_1^2\mu_1 - \alpha_2^2\mu_2 + \alpha_3^2\mu_1\mu_2 - \alpha_4^2\mu_3 + \alpha_3^2\mu_1\mu_3 + \alpha_6^2\mu_2\mu_3 - \alpha_7^2\mu_1\mu_2\mu_3$$

and multiplication

$i^2 = \mu_1 1 \quad j^2 = \mu_2 1 \quad k^2 = -\mu_1\mu_2 1 \quad \ell^2 = \mu_3 1 \quad (i\ell)^2 = -\mu_1\mu_3 1 \quad (j\ell)^2 = -\mu_2\mu_3$

$(k\ell)^2 = \mu_1\mu_2\mu_3 1 \quad ij = k = -ji \quad jk = -\mu_2 i = -kj \quad ki = -\mu_1 j = -ik$

$\ell i = -i\ell \quad \ell j = -j\ell \quad \ell k = -k\ell$

$i(i\ell) = \mu_1\ell = -(i\ell)i \quad j(j\ell) = \mu_2\ell = -(j\ell)j \quad k(k\ell) = -\mu_1\mu_2\ell = -(k\ell)k$

$i(j\ell) = -k\ell = -(j\ell)i \quad j(k\ell) = \mu_2 i\ell = -(k\ell)j \quad k(i\ell) = \mu_1 j\ell = -(i\ell)k$

$j(i\ell) = k\ell = -(i\ell)j \quad k(j\ell) = -\mu_2 i\ell = -(j\ell)k \quad i(k\ell) = -\mu_1 j\ell = -(k\ell)i$

$(i\ell)(j\ell) = \mu_3 k = -(j\ell)(i\ell) \quad (j\ell)(k\ell) = -\mu_2\mu_3 i = -(k\ell)(j\ell) \quad (k\ell)(i\ell) =$

$-\mu_1\mu_3 j = -(i\ell)(k\ell)$

$(i\ell)\ell = \mu_3 i = -\ell(i\ell) \quad (j\ell)\ell = \mu_3 j = -\ell(j\ell) \quad (k\ell)\ell = \mu_3 k = -\ell(k\ell)$.

The results in characteristic 2 are (even?) less memorable, since in place of 1,i we have 1,u with $u^2 = u + \mu_2 1$. Multiplication in the Cayley algebra can be summarized in a multiplication table for the products xy:

$x \backslash y$	1	i	j	k	ℓ	$i\ell$	$j\ell$	$k\ell$
1	1	i	j	k	ℓ	$i\ell$	$j\ell$	$k\ell$
i	i	$\mu_1 1$	k	$\mu_1 j$	$i\ell$	$\mu_1 \ell$	$-k\ell$	$-\mu_1 j\ell$
j	j	$-k$	$\mu_2 1$	$-\mu_2 i$	$j\ell$	$k\ell$	$\mu_2 \ell$	$\mu_2 i\ell$
k	k	$-\mu_1 j$	$\mu_2 i$	$-\mu_1\mu_2 1$	$k\ell$	$\mu_1 j\ell$	$-\mu_2 i\ell$	$-\mu_1\mu_2 \ell$
ℓ	ℓ	$-i\ell$	$-j\ell$	$-k\ell$	$\mu_3 1$	$-\mu_3 i$	$-\mu_3 j$	$-\mu_3 k$
$i\ell$	$i\ell$	$-\mu_1 \ell$	$-k\ell$	$-\mu_1 j\ell$	$\mu_3 i$	$-\mu_1\mu_3 1$	$\mu_3 k$	$\mu_1\mu_3 j$
$j\ell$	$j\ell$	$k\ell$	$-\mu_2 \ell$	$\mu_2 i\ell$	$\mu_3 j$	$-\mu_3 k$	$-\mu_2\mu_3 1$	$-\mu_2\mu_3 i$
$k\ell$	$k\ell$	$\mu_1 j\ell$	$-\mu_2 i\ell$	$\mu_1\mu_2 \ell$	$\mu_3 k$	$-\mu_1\mu_3 j$	$\mu_2\mu_3 i$	$\mu_1\mu_2\mu_3 1$

Staring at this table usually is NOT the best way to understand the structure of a Cayley algebra.

If $\Phi = \mathbb{R}$ is the field of real numbers and $\mu_1 = \mu_2 = \mu_3 = -1$ then the quadratic extension B_2 is simply the field $\mathbb{C}$ of complex numbers with conjugation as involution, B_3 is the division algebra Q of (ordinary) quaternions with standard involution, and B_4 is the algebra $\mathcal{C}$ of <u>Cayley numbers</u>; it is an alternative division algebra. It is often helpful to think of the Cayley-Dickson process algebras as generalizations of

$$\mathbb{R}, \mathbb{C}, Q, \mathcal{C} .$$

We have previously referred to the following result in connection with one-sided and quadratic ideals:

3.15 (One-Sided Ideal Proposition) A Cayley algebra (split or not) over a field Φ contains no proper one-sided ideals.

Proof. Because of the symmetry resulting from the involution, it suffices to show there are no proper left ideals B in a Cayley algebra $\mathbb{C} = \mathbb{C}(A,\mu)$ (A a quaternion algebra). If B is nonzero it contains an element $c + b\ell$ for $b \neq 0$ (if $c + 0\ell \in B$ then also $\ell(c + 0\ell) = \ell c = \bar{c}\ell \in B$ for $\bar{c} \neq 0$); since it is a left ideal, B also contains $z\{(xy)(c+b\ell) - x[y(c+b\ell)]\} = z\{(xy)c - x(yc)\} + \{[b(xy) - (by)x]z\}\ell = \{b[x,y]z\}\ell$ for all $x,y,z \in A$. Thus B contains $\{b[A,A]A\}\ell = \{bA\}\ell$ ($[A,A]A$ is an ideal in the simple quaternion algebra A). So far we have left-multiplied B only by A. If we now left-multiply by $A\ell$ we see B contains $A\ell\{(bA)\ell\} = \mu(\bar{A}\bar{b})A = A\bar{b}A$. Again by simplicity, $\bar{b} \neq 0$ implies B contains A, therefore $\ell \cdot A = \bar{A}\ell = A\ell$ as well, and $B = A + A\ell = \mathbb{C}$. Thus as soon as $B \neq 0$ we have $B = \mathbb{C}$. ∎

3.16 Corollary. A Cayley algebra over a field is a simple alternative algebra of dimension 8. ∎

By means of the Inverse Criterion 2.7 for composition algebras we can decide when the Cayley-Dickson process yields a division algebra.

3.17 (Division Algebra Construction) If B is a composition algebra over a field then $\mathbb{C}(B,\mu)$ is a division algebra iff B is a division algebra and $\mu \notin n(B)$ is not a norm.

Proof. The condition $\mathbb{C}(B,\mu)$ be a division algebra is that its norm not represent zero, $n(x) = n(b) - \mu n(c) \neq 0$ for $x = b + c\ell \neq 0$. Clearly this implies $n(b) \neq 0$ for $b \neq 0$ (take $c = 0$), so B must be a division algebra, and $n(b) \neq \mu$ (take $c = 1$), so μ is not a norm.

Conversely, if these two conditions are met then $n(b) - \mu n(c) \neq 0$ if $c = 0$ but $b \neq 0$ (since $n(b) \neq 0$) and $n(b) - \mu n(c) \neq 0$ if $c \neq 0$ (since $\mu \neq n(b)n(c)^{-1} = n(bc^{-1})$), so $n(x) \neq 0$ for $x \neq 0$. ∎

On several occasions we will need invertible elements of various forms.

3.18 Lemma. Any Cayley-Dickson process algebra of dimension ≥ 2 contains cancellable skew elements $x-\bar{x}$, and in dimension ≥ 4 contains cancellable commutators $[x,y]$. If Φ is a field all these elements are invertible. If Φ is a field with more than 2 elements, the algebra has a basis of invertible elements.

Proof. When Φ has no 2-torsion, the element $i-\bar{i} = 2i$ in B_2 is cancellable since μ_1 is: $ix = 0 \Rightarrow \mu_1 x = i(ix) = 0 \Rightarrow x = 0$. In characteristic 2, the element $u-\bar{u} = u+\bar{u} = 1$ is invertible in B_2'.

In characteristic $\neq 2$ the commutator $[i,j] = ij - ji = 2ij = 2k$ in B_3 is cancellable since $n(k) = + \mu_1\mu_2$ is. In characteristic 2 $[u,j] = uj - ju = (u-\bar{u})j = (u+\bar{u})j = j$ is cancellable.

If Φ is a field then x cancellable $\Rightarrow n(x) \neq 0 \Rightarrow n(x)$ is invertible $\Rightarrow x$ is invertible by the Inverse Criterion 2.7.

If B has a basis of invertible $\{b_i\}$ then $\mathbb{C}(B,\mu)$ has a basis of invertible elements $\{b_i\}$ and $\{b_i\ell\}$, and we start off with $B_1 = \phi 1$ having invertible basis $\{1\}$. (In characteristic 2, $B_2' = \phi 1 + \phi u$ for $u^2 - u + \nu 1 = 0$ has invertible basis $\{1, \lambda 1+u\}$ as long as $0 \neq n(\lambda 1+u) = \lambda^2 + \lambda t(u) + n(u) = \lambda^2 + \lambda + \nu$; since Φ has at least 3 elements and $\lambda^2 + \lambda + \nu = 0$ at most 2 roots, there is at least one non-root λ.) ∎

Let us also observe that the symmetric elements of a Cayley-Dickson process algebra are just those in $\Phi 1$, except when the characteristic is 2 and the dimension is 4 or 8. This is clear from (3.2) in characteristic $\neq 2$, and also in characteristic 2 for dimension 1. In characteristic 2, dimension 2 the only element $x = \beta u + \alpha \bar{u}$ which is symmetric, $x = \bar{x} = \beta u + \alpha \bar{u}$, is $x = \alpha u + \alpha \bar{u} = \alpha 1$.

Clearly everything in a Cayley-Dickson process algebra of dimension 1, 2, 4 associates and in dimension 1, 2 everything commutes. The unclear cases are clarified by

3.19 (Commuting and Associating Criterion) An element of a quaternion algebra commutes with everything iff it is a scalar in $\Phi 1$. An element of a Cayley algebra commutes with everything or associates with everything iff it is a scalar in $\Phi 1$.

Proof. If $z = b + cj$ is in the center of $\mathbb{Q} = \mathbb{C}(B_2, \mu_2)$ then $0 = [a,z] = [a,b] + (ca - c\bar{a})j$ for all $a \in B_2$ implies $c(a - \bar{a}) = 0$; since there are invertible elements in B_2 of the form $a - \bar{a}$ by the Lemma 3.18, this implies $c = 0$ and $z = b$. Then $0 = [z,j] = (b - \bar{b})j$ implies $b = \bar{b}$ is a scalar in B_2, $b = \beta 1$, and $z = \beta 1$.

If $z = b + c\ell$ commutes with everything in $\mathbb{C} = \mathbb{C}(B_3, \mu_3)$ then $0 = [a,z] = [a,b] + \{c(a - \bar{a})\}\ell$ shows $[a,b] = c(a - \bar{a}) = 0$ for all $a \in B_3$. By the previous result, if b lies in the center of B_3 we have $b = \beta 1$, and again since invertible $a - \bar{a}$ exist in B_3 we have $c = 0$, so $z = \beta 1$.

If z associates with everything in $\mathbb{C}$ then $0 = [a,d,z] = [a,d,b] + \{c(ad) - (cd)a\}\ell = \{c[a,d]\}\ell$ implies $c[a,d] = 0$ for all a,d in the associative algebra $Q = B_3$. Once more by the Lemma 3.18, B_3 has invertible commutators $[a,d]$, so c must be zero. Then $0 = [a,z,\ell] = [a,b,\ell] = (ab-ba)\ell$ for all a implies $b = \beta 1$ lies in the center of Q, so once more $z = \beta 1$. ∎

3.20 Corollary. If an element of a Cayley-Dickson process algebra commutes with everything, it also associates with everything. ∎

3.21 Corollary. A Cayley-Dickson process algebra is either associative, or else every element which associates with everything also commutes with everything. ∎

These properties will turn out to be general properties of alternative algebras. Commutativity will force associativity (except in characteristic 3; see Section III.4), and for simple not-associative algebras nucleus and center will coincide (see III.1). This is important for the general structure theory (see Appendix II).

Exercise

3.1 Find the Cayley-Dickson formula for multiplication in $B + \ell B$ if $\ell^2 = \mu 1$, $\ell b = \bar{b}\ell$.

3.2 Try to define a general "modified Cayley-Dickson process" $\mathbb{C}(B,\nu) = Bu \oplus B\bar{u}$ with $u + \bar{u} = 1$, $u^2 - u + \nu 1 = 0$.

3.3 In the "modified" construction $B_2' = \Phi 1 \oplus \Phi u$ in arbitrary characteristic, $u^2 - u + \nu 1 = 0$, show $n(x,y)$ is nondegenerate iff $1 + 4\nu$ is cancellable. Conclude in characteristic 2 it is nondegenerate no matter what ν is chosen.

3.4 If $\frac{1}{2} \in \Phi$ show $u = \frac{1}{2}(1 + i)$ in $B_1 = \mathbb{C}(\Phi,\mu_1) = \Phi 1 \oplus \Phi i$ has $u + \bar{u} = 1$, $u^2 - u + \nu_1 1 = 0$. What is $1 + 4\nu_1$?

3.5 If B is a simple algebra with involution which is associative but not commutative, and $\mu \in \Phi$ is invertible, show the algebra $\mathbb{C}(B,\mu)$ defined as in (3.1) with the Cayley-Dickson formula (3.3) has no proper ideals, in particular is simple. (Notice B is not assumed unital nor the involution scalar, so $\mathbb{C}(B,\mu)$ need not be alternative. Also the element ℓ need not exist in $\mathbb{C}$, so $b\ell$ cannot be interpreted as a product of b with ℓ.)

3.6 If B is unital but the involution not necessarily scalar, show for cancellable μ that $\mathbb{C}(B,\mu)$ is commutative iff B is commutative and $* = 1$ (examine commutators); show $\mathbb{C}$ is associative iff B is commutative associative (examine associators); show $\mathbb{C}$ is alternative iff (i) B is alternative, (ii) all $n(b) = b\bar{b} = \bar{b}b$ commute with B, (iii) all $b + t(b)$ associate with B (examine associators). Show from (iii) that if B has no 3-torsion then B is associative, so $*$ is a central involution . Thus to get an alternative $\mathbb{C}$ the involution has to be at least central anyway; regarding B as an algebra over its center Ω, this means * is scalar over Ω.

3.7 Give an example of a 2-dimensional Cayley-Dickson process algebra over the field $\mathbb{Z}_2$ which has no basis of invertible elements. Can you give a 4-dimensional example?