Chapter II

Composition Algebras

§1 Algebras of degree 2

In this chapter we construct and investigate the most important (in some sense the "only") alternative algebras which are not associative, the Cayley algebras. The properties of these algebras will indicate and illuminate the properties of general alternative algebras.

We say a unital nonassociative algebra A over Φ is of degree 2 over Φ if every element $x \in A$ satisfies an equation of degree 2

(1.1) $$x^2 - t(x)x + n(x)1 = 0 \qquad (t(x),\ n(x) \in \Phi)$$

(1.2) $$t(1) = 2, \quad n(1) = 1$$

where the trace t is a linear function and the norm n a quadratic function of x. (Such algebras are usually called quadratic algebras, but for ulterior motives we shall refrain from using this term.)

These hypotheses guarantee A is strictly of degree 2, i.e. every scalar extension A_Ω remains degree 2 over Ω. Indeed, the linearization process applied to (1.1) yields $\{(x+y)^2 - x^2 - y^2\} - \{t(x+y)(x+y) - t(x)x - t(y)y\} + \{n(x+y) - n(x) - n(y)\} = 0$ or

(1.1') $$x \circ y - t(x)y - t(y)x + n(x,y)1 = 0$$

(using the hypothesis $t(x+y) = t(x) + t(y)$ of linearity of t; here $n(x,y)$ is bilinear by the quadratic nature of n). Once (1.1) and its linearization (1.1') hold for the elements of A, they hold for all elements $x = \Sigma\, \omega_i a_i$ in A_Ω ($\omega_i \in \Omega$, $a_i \in A$):

$$x^2 = \Sigma\, \omega_i^2 a_i^2 + \sum_{i<j} \omega_i \omega_j\, a_i \circ a_j$$

$$= \Sigma\, \omega_i^2\{t(a_i)a_i - n(a_i)1\} + \sum_{i\neq j} \omega_i\omega_j\{t(a_i)a_j + t(a_j)a_i - n(a_i,a_j)1\}$$

$$= \sum_{i,j} \{\Sigma\, \omega_j t(a_j)\}\omega_i a_i - \{\Sigma\,\omega_i^2\, n(a_i) + \omega_i\omega_j n(a_i,a_j)\}1$$

$$= t_\Omega(x)x - n_\Omega(x)1$$

for $t_\Omega(x) = \Sigma\, \omega_j t(a_j)$ the linear extension of t on A to 1st on $\Omega \otimes A = A_\Omega$, and $n_\Omega(x) = \Sigma\, \omega_i^2 n(a_i) + \sum_{i\neq j} \omega_i\omega_j n(a_i,a_j)$ the quadratic extension of n on A to A_Ω.

1.3 Remark. Being of degree 2 depends very much on the ring of scalars. If $\Phi \supset \Phi_o$ then any Φ-algebra is by restriction also a Φ_o-algebra, but in general it won't be of degree 2 over Φ_o if it is degree 2 over Φ. The trouble is that for A to be quadratic over Φ_o all $t(x), n(x)$ must be scalars in Φ_o, not just in Φ.

1.4 Remark. The conditions (1.2) are not automatic consequences of (1.1). For example, if Φ does not act faithfully on A and $\varepsilon \in \Phi$ satisfies $\varepsilon 1 = 0$ (hence $\varepsilon A = 0$) then $t'(x) = (1+\varepsilon)t(x)$, $n'(x) = (1+\varepsilon)n(x)$ will still be linear and quadratic and satisfy (1.1), but not (1.2) since $t'(1) = 2 + 2\varepsilon$, $n'(1) = 1 + \varepsilon$. But (1.2) can fail even if Φ acts faithfully: if $A = \Phi 1$ then (1.1) holds for $t'(\lambda) = 2\lambda + \varepsilon\lambda$, $n'(\lambda) = \lambda^2 + \varepsilon\lambda^2$ but $t'(1) = 2 + \varepsilon$, $n'(1) = 1 + \varepsilon$. Here ε can be arbitrary, and Φ can even be a field. From our point of view, a t and n which don't satisfy (1.2) are not really a trace and norm for A.

1.5 Remark. If we do not assume t, n are linear, quadratic then A_Ω need not remain of degree 2. For example, if Φ is the field $\mathbb{Z}_2$ then any Boolean (associative) ring is an algebra over $\mathbb{Z}_2$ satisfying $x^2 = x$ for all x, i.e.

$$x^2 - t(x)x + n(x)1 = 0 \qquad \text{for } t(x) \equiv 1,\ n(x) \equiv 0 .$$

Clearly t is not linear in the case, and if Ω is a proper extension field of Φ (containing $\omega \neq 1,0$) then A_Ω is no longer of degree 2: if $1,a,b \in A$ are independent then: $x = a + \omega b \in A_\Omega$ is not of degree 2 (x^2 is not linearly dependent upon 1 and x) because $x^2 = a^2 + \omega(ab+ba) + \omega^2 b^2 = a + \omega^2 b = x + (\omega^2-\omega)b$ and $(\omega^2-\omega)b$ is independent of 1, x (i.e. of 1, a).

Notice in this example Φ has only two elements, and A has zero divisors. If Φ were any bigger, or A were free of zero divisors, then A would have been strictly degree 2 because of

1.6 (Degree 2 Criteria) Let A be a unital algebra over a field Φ such that every element $x \in A$ satisfies an equation of degree 2,

$$x^2 + \alpha x + \beta 1 = 0 \qquad (\alpha, \beta \in \Phi, \text{ depending on } x) .$$

Then A will be of degree 2 if either of

(i) Φ contains more than 2 elements

(ii) A is alternative and contains no zero divisors.

In this case $t(x) = -\alpha$, $n(x) = \beta$ if $x \notin \Phi 1$, and $t(\lambda 1) = 2\lambda$, $n(x) = \lambda^2$ for $x = \lambda 1 \in \Phi 1$.

Proof. If $x \notin \Phi 1$ then the α, β satisfying $x^2 + \alpha x + \beta 1 = 0$ are uniquely determined by x, so t(x) and n(x) are well-defined by the above formula, and it is just a question of whether t is linear (if it is, $n(x)1 = t(x)x - x^2$ is automatically quadratic).

Certainly $t(\lambda x) = \lambda t(x)$: this is trivial if $\lambda = 0$ or $x \in \Phi 1$, while otherwise $\lambda^2\{x^2 - t(x)x + n(x)1\} = 0 = (\lambda x)^2 - t(\lambda x)\lambda x + n(\lambda x)1$ implies $\lambda\{\lambda t(x) - t(\lambda x)\}x = \{\lambda^2 n(x) - n(\lambda x)\}1$; since x, 1 are independent if $x \notin \Phi 1$ both coefficients must vanish, and since $\lambda \neq 0$ we see $t(\lambda x) = \lambda t(x)$, $n(\lambda x) = \lambda^2 n(x)$. It remains only to show $t(x+y) = t(x) + t(y)$.

First consider the case where 1, x, y are dependent; i.e. they lie in a 2-dimensional subspace $\Phi 1 + \Phi z$. It will suffice if t is linear on this subspace, $t(\alpha 1 + \beta z) = \alpha t(1) + \beta t(z)$ for all α, β; clearly we need only consider $\alpha, \beta \neq 0$, and dividing by α (recall $t(\lambda x) = \lambda t(x)$) reduces the problem to showing $t(1+w) = t(1) + t(w)$ for any w. Here we may assume $w \notin \Phi 1$, so $t(w+1)(w+1) - n(w+1)1 = (w+1)^2 = w^2 + 2w + 1 = t(w)w - n(w)1 + 2w + 1$ implies (equating coefficients of w) $t(w+1) = t(w) + 2 = t(w) + t(1)$.

Now assume 1, x, y are independent. Then $(x+\lambda y)^2 - x^2 - (\lambda y)^2 = \lambda(xy+yx) = \lambda\{(x+y)^2 - x^2 - y^2\}$ and the formulas for squares give $\{t(x+\lambda y)(x+\lambda y) - n(x+\lambda y)1\} - \{t(x)x - n(x)1\} - \{t(\lambda y)\lambda y - n(\lambda y)1\} = \lambda\{t(x+y)(x+y) - n(x+y)1\} - \lambda\{t(x)x - n(x)1\} - \lambda\{t(y)y - n(y)1\}$. By independence we can equate coefficients of x and of y to get

$$t(x+\lambda y) - t(x) = \lambda t(x+y) - \lambda t(x) \qquad t(x+\lambda y) - \lambda^2 t(y) = \lambda t(x+y) - \lambda t(y).$$

So far we haven't used (i) or (ii). If we assume $|\Phi| > 2$ we can choose $\lambda \neq 0, 1$ in Φ; dividing the second relation by λ and subtracting from the first gives

$$(\lambda - 1)\{t(x+y) - t(x) - t(y)\} = 0$$

and we can cancel $\lambda - 1 \neq 0$ to get $t(x+y) = t(x) + t(y)$.

Suppose now $\Phi = \mathbb{Z}_2$ but that A is alternative without zero divisors. In this case $x^2 + \alpha x + \beta 1 = 0$ implies $\beta \neq 0$ unless $x \in \Phi 1$ (if $\beta = 0$ then $x^2 + \alpha x = x(x+\alpha) = 0$ forces $x = 0$ or $x = -\alpha 1$ if there are no zero divisors). If we try to repeat the argument for (1.1') we get only

$$(1.7)\quad x\circ y = \{t(x+y)-t(x)\}x + \{t(x+y)-t(y)\}y + \{n(x+y)-n(x)-n(y)\}1 .$$

If the commutator $[x,y] \neq 0$ then commuting the relation with x gives $[x,x\circ y] = \{t(x+y)-t(y)\}[x,y]$. On the other hand, $[x,x\circ y] = -[y,x^2]$ (linearizing $[x,x^2] = 0$, or using flexibility directly $x(xy+yx) - (xy+yx)x = x(xy) - (yx)x = x^2y - [x,x,y] - [y,x,x] - yx^2 = [x^2,y]$) if A is flexible, and $-[y,x^2] = -t(x)[y,x] = t(x)[x,y]$. Identifying coefficients of $[x,y]$ gives $t(x+y) - t(y) = t(x)$.

So assume $[x,y] = 0$. Since $\Phi = \mathbb{Z}_2$ we are in characteristic 2, so $[x,y] = 0$ is equivalent to $x\circ y = 0$. Because 1, x, y are independent the coefficients in (1.7) must all vanish,

$$t(x+y) = t(x) = t(y) .$$

We will show these traces are all zero, so $t(x+y) = t(x) + t(y)$ holds in a rather trivial fashion. (However, the proof is not trivial!) If the traces were not all zero, we could scale them up (recall $t(\lambda z) = \lambda t(z)$) so they are all 1. Then $t(xy)xy - n(xy)1 = (xy)^2 = x^2y^2$ (by alternativity and the fact that x commutes with y and so also with y^2: $(xy)(yx) = xy^2x = x(xy^2) = x^2y^2$) $= \{x-n(x)1\}\{y-n(y)1\}$ implies

$$\{t(xy)-1\}xy + n(x)y + n(y)x - \{n(xy)+n(x)n(y)\}1 = 0 .$$

As we remarked above, $n(x)$ and $n(y)$ are nonzero; since 1, x, y are independent the coefficient of xy cannot be zero, $t(xy) \neq 1$, and $xy = \alpha x + \beta y + \gamma 1$ for nonzero α, β. In particular, xy cannot be dependent on x and 1 alone. But then $0 = x\,(x \circ y) = x \circ xy = \{t(x+xy) - t(xy)\}xy + \{t(x+xy)-1\}x - \{n(x+xy)-n(x)-n(xy)\}1$ (using (1.7)) is possible only when the coefficients vanish, $t(xy) = t(x+xy) = 1$, contradicting $t(xy) \neq 1$. Thus $t(x+y) = t(x) = t(y) = 1$ is impossible. □

We can introduce a <u>standard involution</u>

$$x^* = t(x)1 - x \tag{1.8}$$

in any degree 2 algebra. The map $x \to x^*$ is clearly linear of period 2, $x^{**} = \{t(x)1-x\}^* = t(x)1^* - x^* = t(x)1 - x^* = x$ since $1^* = t(1)1 - 1 = 2 - 1 = 1$ by (1.2). Thus $*$ is a linear involution; however, it is not generally an involution of the algebra structure.

The degree 2 equation (1.1) becomes $n(x)1 = t(x)x - x^2 = x\{t(x)1 - x\} = xx^*$, so

$$\begin{aligned} x + x^* &= t(x)1 \\ xx^* &= x^*x = n(x)1\,. \end{aligned} \qquad \text{(Trace and Norm Formulas)} \tag{1.9}$$

We can run the trace and norm formulas backwards to construct degree 2 algebras. An involution $*$ on a unital algebra over Φ is a <u>scalar involution</u> if all <u>norms</u> $n(x) = xx^*$ and all <u>traces</u> $t(x) = x + x^*$ lie in $\Phi 1$. In this case t, n are automatically linear and quadratic, and $n(x) = xx^* = x\{t(x)-x\} = t(x)x - x^2$ becomes

$$x^2 - t(x)x + n(x) = 0 .$$

This is (1.1), and (1.2) comes from $1 + 1^* = 1 + 1 = 2$, $11^* = 11 = 1$. We have shown

1.10 Proposition. If A has a scalar involution * then A is of degree 2 over $\Phi 1$, and the standard involution associated with A is just *. ∎

WARNING: Since we are not assuming Φ acts faithfully on A we cannot identify Φ with $\Phi 1$; in general $n(x), t(x) \in \Phi 1$ can't be lifted to quadratic and linear $\tilde{n}(x)$, $\tilde{t}(x) \in \Phi$ satisfying $\tilde{n}(x)1 = n(x)$, $\tilde{t}(x)1 = t(x)$, so A need not be degree 2 over Φ. ∎

In the case of an alternative degree 2 algebra, the U-operator takes on a particularly simple form in terms of the standard involution:

(1.11) $$U_x y = n(x,y^*)x - n(x)y^*$$

(1.12) $$U_x y^* = n(x,y)x - n(x)y \qquad \text{(U-formulas)}$$

(1.13) $$U_x x^* = n(x)x \quad .$$

It suffices to prove (1.12) (for (1.13) recall $n(x,x) = 2n(x)$): $U_x y^* = xy^*x = (xy^* + yx^*)x - y(x^*x)$ (by alternativity) $= n(x,y)x - n(x)y$ (linearizing $xx^* = x^*x = n(x)1$).

Any isotope of a degree 2 alternative algebra remains degree 2, indeed

1.14 (Isotope Formula) If A is a degree 2 alternative algebra then any isotope $A^{(u,v)}$ is again a degree 2 algebra,

$$x^{2(u,v)} - t^{(u,v)}(x)x + n^{(u,v)}(x)1^{(u,v)} = 0$$

where $t^{(u,v)}(x) = n(x,(uv)^*)$ and $n^{(u,v)}(x) = n(uv)n(x)$.

Proof. In $A^{(u,v)}$ the square is $x^{2(u,v)} = U_x(uv) = n(x,(uv)^*)x - n(x)(uv)^*$ (by the U-formula 1.11). Now by definition of isotope, uv must be invertible and $(uv)^{-1} = 1^{(u,v)}$ is the unit for $A^{(u,v)}$. But $U_{uv}\{n(uv)1^{(u,v)}\} = n(uv)U_{uv}(uv)^{-1} = n(uv)uv = U_{uv}(uv)^*$ by (1.13), so injectivity of U_{uv} implies $(uv)^* = n(uv)1^{(u,v)}$. Making this replacement in our previous formula gives

$$x^{2(u,v)} = n(x,(uv)^*)x - n(x)n(uv)1^{(u,v)} = t^{(u,v)}(x)x - n^{(u,v)}(x)1^{(u,v)}.$$

Clearly the new $t^{(u,v)}$, $n^{(u,v)}$ are linear and quadratic if the old ones were. ▨

Exercise

1.1 If t, n satisfy (1.1) show $t(1)1 = 2$ iff $n(1)1 = 1$. If Φ acts faithfully, conclude $t(1) = 2$ iff $n(1) = 1$.

1.2 If A contains "independent" elements x, 1 such that $\alpha x + \beta 1 = 0$ implies $\alpha = \beta = 0$, show (1.1) implies (1.2).

1.3 Show $t(x) = n(x,1)$ in any degree 2 algebra.

1.4 Show that any 2 elements in an alternative degree 2 algebra generate (unitally) a subalgebra spanned by 4 elements. Write down a multiplication table. Conclude that any 2 elements generate an associative subalgebra.

1.5 Show that any (unital) subalgebra and any homomorphic image of a degree 2 algebra is degree 2.

#6. Problem Set on Degree 2 Algebras

We will give a recipe for constructing all degree 2 algebras, and use it to give an example of an algebra with scalar involution which is not flexible.

1. Given a bilinear alternating product $\times$ on A_o (so $a \times a = 0$) and a bilinear form σ on A_o, define an algebra $A = (A_o, \sigma, \times)$ to be $A = \Phi 1 \oplus A_o$ as linear space, with multiplication

   $$xy = \{\alpha 1 \oplus a\}\{\beta 1 \oplus b\} = \{\alpha\beta + \sigma(a,b)\}1 \oplus \{\alpha b + \beta a + a \times b\} .$$

   Show A is degree 2, and find the trace t and norm n.
2. Show * is an involution on A iff the bilinear form σ is symmetric. (If * is an involution, it is necessarily a scalar involution).
3. Show A is flexible iff $\sigma(a,b)a = \sigma(b,a)a$ and $\sigma(a,b \times a) = \sigma(a \times b, a)$ for all a,b. When Φ is a field this reduces to σ being symmetric and $2\sigma(a, a \times b) = 0$. Deduce that there exist degree 2 algebras with scalar involutions which are not flexible, but not over a field Φ.
4. Show A is left alternative iff $\sigma(a, a \times b) = 0$ and $a \times (a \times b) = \sigma(a,a)b - \sigma(a,b)a$; in this case show A is (two-sided) alternative. Show A is associative iff $\sigma(a \times b, c) = \sigma(a, b \times c)$, $(a \times b) \times c - a \times (b \times c) = \sigma(b,c)a - \sigma(a,b)c$.
5. Show that n permits composition iff $\sigma(a,a)\sigma(b,b) + \sigma(a \times b, a \times b) = \sigma(a,b)^2$, $\sigma(a,b) = \sigma(b,a)$, $2\sigma(a, a \times b) = 0$ for all a,b.
6. Construct an example of an associative degree 2 algebra which does not permit composition and for which * is not an involution (Φ must contain nilpotent elements!)