§7. Representations and modules

A module for an associative algebra A consists of a space M together with a homomorphism of A into the algebra End(M) of linear transformations on M. The advantage of representing an algebra as an algebra of transformations is that it replaces the abstract product $x \cdot y$ in A by the concrete and familiar composition $S \circ T$ of transformations. By analyzing the structure of these representations we can gain penetrating information about the structure of the original algebra.

Clearly there can be no similar program for nonassociative algebras: in representing A as an associative algebra of linear transformations we neglect everything that makes A nonassociative. Representations, which are central to the associative theory, play almost no role in the nonassociative theory.* All that remain in general are birepresentations, which play but a modest role in the associative theory; even in the nonassociative theory they serve more as a convenience than as an essential tool.

The motivating example of a bimodule is the regular bimodule, which consists of the algebra together with its action on itself. This leads to the notion of a multiplication algebra. If A is a subalgebra of an alternative algebra E (E for "extension"), the <u>multiplication algebra of A on E</u> is the associative algebra

$$M_E(A)$$

of linear transformations on E generated by all multiplications L_x, R_x for $x \in A$. (Since in general $L_xL_y \neq L_{xy}$, $R_xR_y \neq R_{yx}$, $L_xR_y \neq R_yL_x$ in

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* This does not apply to Lie algebras, where again the representation theory is decisive.

the nonassociative case we must take all sums of products of such L_x, R_y. To indicate that it contains more than just the L_x and R_y, it is often called the <u>multiplication envelope</u> of A on E). When E = A we speak simply of the <u>multiplication algebra</u>

$$M(A)$$

of A. If M is any subspace of E invariant under the multiplications of A (for example, any two-sided ideal in E) we can restrict these transformations to M and obtain an algebra

$$M_E(A|M)$$

of operators acting on M. (We include E in the notation to indicate all multiplying is taking place within the enveloping algebra E). Since restriction of operators is always a homomorphism, we have restriction homomorphisms $M_E(A) \to M_E(A)|M = M_E(A|M)$. All $M_E(A|M)$ are thus homomorphic images of $M_E(A)$.

Sometimes we will need to consider the <u>multiplication ideal</u>

$$M(A;E)$$

which is the ideal in M(E) generated by $M_E(A)$; it thus consists of all multiplications having at least one factor from A.

Often we are trying to prove that if A is nilpotent then so is M(A), so we definitely do not want M(A) to contain the identity. However, at other times it is convenient to have the identity around. For those occasions we introduce the <u>unital multiplication algebra</u> $\hat{M}(A)$, which is the ordinary multiplication algebra with the identity thrown in,

$$\hat{M}(A) = \Phi I + M(A) .$$

Similarly we can form $\hat{M}_E(A|M)$. These are unital associative algebras of operators. As an application, the ideal generated by a subset $X \subseteq A$ can be written succinctly as

$$\hat{I}(X) = \hat{M}(A)X .$$

$M(A)X$ is an ideal too, but if $I \notin M(A)$ this ideal need not contain X; it is the ideal "strictly" generated by X,

$$I(X) = M(A)X .$$

Knowledge of these multiplication algebras tells us something about how A multiplies, hence something about the algebra structure.

7.1 Example. The subspaces of A invariant under the operators in $M(A)$ are precisely the two-sided ideals of A. ∎

We would like to characterize these abstractly and therefore intrinsically (so the action of A on M can be described without dragging in E as an intermediary). Now the multiplication algebra is generated by operators L_x, R_x $(x \in A)$ satisfying certain relations. One obvious relation is $L_{\alpha x+\beta y} = \alpha L_x + \beta L_y$, i.e. linearity of the map $x \to L_x$, and similarly for $x \to R_x$. Unlike the situation in associative algebras, $x \to L_x$ and $y \to R_y$ needn't be homomorphisms or anti-homomorphisms. What relations do they satisfy? Alternativity of multiplication of A on M is reflected in the alternating nature $[x,x,m] = [x,m,x] = [m,x,x] = 0$, $[x,y,m] = - [x,m,y] = [m,x,y]$ of the associator. (Notice that the first three won't suffice - because of the distinguished roles of $m \in M$ and $x,y \in A$, they only provide information about interchanging variables from A, not about interchanging a variable from A with one from M). As operator relations on m these state $L_{x^2} - L_x^2 = [R_x, L_x] = R_x^2 - R_{x^2} = 0$ and $L_{xy} - L_x L_y = [L_x, R_y] = R_y R_x - R_{xy}$.

This leads us to define an (alternative) A-bimodule to be a space M together with a birepresentation of A on M, a pair of linear maps $x \to \ell_x$ and $x \to r_x$ of A into $\mathrm{End}_\Phi(M)$ satisfying the relations

$$(7.2) \qquad \ell_{x^2} = \ell_x^2, \quad r_{x^2} = r_x^2$$

$$(7.3) \qquad \ell_{xy} - \ell_x \ell_y = [\ell_x, r_y] = r_y r_x - r_{xy} .$$

(Notice that in an associative birepresentation, the three terms in (7.3) would not only be equal but equal to zero). Equivalently, a bimodule is a space M together with two bilinear products $A \times M \to M$, $M \times A \to M$ satisfying

$$(7.2)' \quad x^2 \cdot m = x \cdot (x \cdot m), \quad m \cdot x^2 = (m \cdot x) \cdot x$$

$$(7.3)' \quad xy \cdot m - x \cdot (y \cdot m) = x \cdot (m \cdot y) - (x \cdot m) \cdot y = (m \cdot x) \cdot y - m \cdot xy .$$

for all $x,y \in A$, $m \in M$.

To show that our axioms do characterize multiplication algebras, we would like to show an abstract birepresentation ℓ_x, r_x comes from concrete multiplications L_x, R_x in some algebra E. To find E, we take the algebra and the bimodule and glue them together.

7.4 (Bimodule Theorem) If M is an A-bimodule, the split null extension

$$E = A \oplus M$$

with multiplication

$$(a \oplus m)(b \oplus n) = ab \oplus (\ell_a n + r_b m)$$

is an alternative algebra containing A as a subspace, M as a trivial ideal, and such that the restrictions to M of left and right multiplications by $a \in A$ coincide with the given birepresentation: $L_a|_M = \ell_a$, $R_a|_M = r_a$.

Proof. That A is a subalgebra of E, M a trivial ideal ($M^2 = 0$), with multiplication of A on M given by ℓ_a, r_a – all this is obvious from the definition of multiplication in E. The only thing to show is that the bimodule condition (7.2), (7.3) force alternativity of E.

We content ourselves with checking left alternativity. In expanding $[a \oplus m, a \oplus m, b \oplus n]$ we may forget about terms with two or more factors from M, since M is a trivial ideal, or with all three factors from A, since A is an alternative subalgebra. This leaves only the terms $[a,a,n] + [a,m,b] + [m,a,b]$ with one factor from M. By definition $[a,a,n] = a^2n - a(an) = \{\ell_{a^2} - \ell_a\ell_a\}n = 0$ from (7.2), $[a,m,b] + [m,a,b] = (am)b - a(mb) + (ma)b - m(ab) = \{-[\ell_a, r_b] + r_b r_a - r_{ab}\}m = 0$ from (7.3). Thus $[a \oplus m, a \oplus m, b \oplus n] = 0$ and E is left alternative. ∎

E is called the <u>split null extension</u> of A <u>by</u> M since firstly it is an <u>extension</u> of A by M ($0 \to M \xrightarrow{i} E \xrightarrow{\pi} A \to 0$ is an exact sequence), secondly it is <u>split</u> (A is a semidirect summand of E, there is a subalgebra of E isomorphic to A under π), and thirdly it is <u>null</u> (we have extended by a null or trivial algebra M).

In consequence of this theorem bimodules, birepresentations, and multiplication algebras are essentially equivalent concepts.

A subspace $N \subset M$ is a <u>sub-bimodule</u> if it is invariant under the birepresentation, i.e. under all L_a and R_a:

$$A \cdot N + N \cdot A \subset N.$$

If N is a sub-bimodule we can form the <u>quotient</u> or factor bimodule M/N, which is just the ordinary quotient of linear spaces together with the induced birepresentation; that is, the bimodule action of A on the cosets $\bar{m} = m + N$ is

$$a \cdot \bar{m} = \overline{a \cdot m}\ ,\ \bar{m} \cdot a = \overline{m \cdot a}\ .$$

If M_i are bimodules we can form their <u>direct sum</u> $\oplus_i M_i$, which is the ordinary direct sum of linear spaces together with the obvious birepresentation $\ell(x) = \oplus_i \ell_i(x)$, $r(x) = \oplus_i r_i(x)$, (thus the bimodule action is just componentwise action $a \cdot \{\oplus m_i\} = \oplus(am_i)$, $\{\oplus m_i\} \cdot a = \oplus(m_i \cdot a)$).
We can turn any Φ-module M into a <u>trivial</u> A-bimodule for any A by taking the trivial birepresentation $\ell = r = 0$: $a \cdot m = m \cdot a = 0$ for all $a \in A$, $m \in M$.

A bimodule M is <u>irreducible</u> (or <u>simple</u>) if it contains no proper sub-bimodules $N \neq 0$, and M is not a trivial bimodule. M is <u>indecomposable</u> if it cannot be written as a direct sum $M = M_1 \oplus M_2$ of proper sub-bimodules.

It is completely reducible (or semisimple) if it can be written as a direct sum $M = \oplus_i M_i$ of irreducible modules M_i.

A homomorphism $M \overset{F}{\to} \tilde{M}$ of A-bimodules is a linear map which preserves the bimodule operations

$$F(a\cdot m) = a\cdot F(m), \; F(m\cdot a) = F(m)\cdot a \; .$$

The kernel Ker F and image Im F are sub-bimodules of M and $\tilde{M}$ respectively. As always we can talk about epimorphisms, monomorphisms, and isomorphisms. For example, the canonical projection $M \overset{\pi}{\to} M/N$ is an epimorphism, and we have a canonical isomorphism $M/\text{Ker } F \cong \text{Im } F$ for any homomorphism F. These concepts for bimodules can be reduced to the ordinary concepts for algebras: $M \overset{F}{\to} \tilde{M}$ is a homomorphism of A-bimodules iff $E = A \oplus M \xrightarrow{1\oplus F} A \oplus \tilde{M} = \tilde{E}$ is a homomorphism of split null extensions since $(1\oplus F)\{(a\oplus m)(b\oplus n)\} - \{(1\oplus F)(a\oplus m)\}\{(1\oplus F)(b\oplus n)\} = ab \oplus F(a\cdot n+m\cdot b) - \{a\oplus F(m)\}\{b\oplus F(n)\} = 0 \oplus \{F(a\cdot n) - a\cdot F(n) + F(m\cdot b) - F(m)\cdot b\}$. F is an epimorphism etc. iff $1 \oplus F$ is, the kernel of F is the same as that for $1 \oplus F$, the image of F is $\text{Im } F = \tilde{M} \cap \text{Im } (1 \oplus F)$, and so forth.

Practically anything that makes sense for algebras also makes sense for bimodules. For instance, if M is an A-bimodule a derivation of A in M is a linear map $A \overset{D}{\to} M$ satisfying the formal analogue of the condition for a derivation of an algebra,

$$D(ab) = D(a)\cdot b + a\cdot D(b) \qquad (a,b \in A)$$

though this time D(a), D(b) lie in M and the product on the right is the bimodule product. The notion of derivation into a bimodule can be reduced to the ordinary notion of derivation of an algebra: D is a derivation of A in M iff the map $\tilde{D}(a\oplus m) = 0 \oplus D(a)$ is a derivation of the split null extension $E = A \oplus M$, since $\tilde{D}\{(a\oplus m)(b\oplus n)\} - \{\tilde{D}(a\oplus m)\}(b\oplus n) - (a\oplus m)\{\tilde{D}(b\oplus n)\} = \tilde{D}(ab\oplus\{an+mb\}) - (0\oplus D(a))(b\oplus n) - (a\oplus m)(0\oplus D(b)) = 0 \oplus \{D(ab) - D(a)b - aD(b)\}$. We denote the collection of derivations $A \overset{D}{\to} M$ by

$$\mathrm{Der}(A,M) .$$

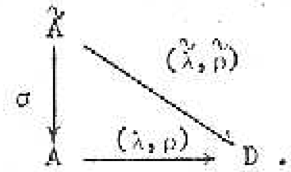
We can abstract even further the notion of birepresentation $A \xrightarrow{(\ell,r)} D = \mathrm{End}_\Phi(M)$. The relevant fact is that D is an associative algebra; its particular nature as an algebra of transformations is irrelevant. This leads us to define a <u>bispecialization</u> of an alternative algebra A in an arbitrary associative algebra D to be a pair (λ,ρ) of linear maps from A to D satisfying

$$(7.5) \qquad \lambda(x^2) = \lambda(x)^2, \ \rho(x^2) = \rho(x)^2$$

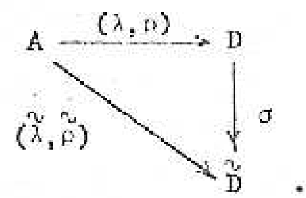
$$(7.6) \qquad \lambda(xy) - \lambda(x)\lambda(y) = [\lambda(x),\rho(y)] = \rho(y)\rho(x) - \rho(xy) .$$

Birepresentations are just bispecializations in $\mathrm{End}_\Phi(M)$, with $\ell_x = \lambda(x)$, $r_x = \rho(x)$. On the other hand, any associative algebra can be identified with an algebra of linear transformations, so a bispecialization is "essentially" equivalent to a birepresentation. Bispecializations are just abstract birepresentations.

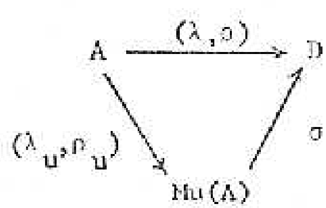
Composing bispecializations with homomorphisms provides a convenient method of generating new bispecializations. For example, if $A \xrightarrow{(\lambda,\rho)} D$ is a bispecialization and $\tilde{A} \overset{\sigma}{\to} A$ a homomorphism of alternative algebras, the composites $\tilde{\lambda} = \lambda\circ\sigma$ and $\tilde{\rho} = \rho\circ\sigma$ provide a bispecialization $\tilde{A} \xrightarrow{(\tilde{\lambda},\tilde{\rho})} D$



For example, any bispecialization or birepresentation on a quotient A/B lifts to one on A. On the other end, if $D \xrightarrow{\sigma} \tilde{D}$ is a homomorphism of associative algebras the composites $\tilde{\lambda} = \sigma \circ \lambda$, $\tilde{\rho} = \sigma \circ \rho$ form a bispecialization $A \xrightarrow{(\tilde{\lambda}, \tilde{\rho})} \tilde{D}$



One advantage of this abstractness is that we can define a universal gadget for bispecializations. A <u>universal multiplication algebra</u> $Mu(A)$ consists of an associative algebra (again denoted $Mu(A)$) and a <u>universal bispecialization</u> (λ_u, ρ_u) of A in $Mu(A)$, which is universal in the sense that any other bispecialization (λ, ρ) of A in D can be factored uniquely through the universal one



where σ is a homomorphism of associative algebras. The condition that σ be unique is just the condition that the $\lambda_u(x)$, $\rho_u(x)$ generate $Mu(A)$.

Such a $Mu(A)$ always exists: take $Mu(A) = U/K$ where U is the free associative algebra generated by elements $\ell(x)$, $r(x)$ (one for each $x \in A$) and K the ideal generated by the relations $\ell(\alpha x) = \alpha\ell(x)$, $\ell(x+y) = \ell(x) + \ell(y)$, $r(\alpha x) = \alpha r(x)$, $r(x+y) = r(x) + r(y)$, $\ell(x^2) = \ell(x)^2$, $r(x^2) = r(x)^2$, $\ell(xy) - \ell(x)\ell(y) = [\ell(x), r(y)] = r(y)r(x) - r(xy)$ for all $x, y \in A$, $\alpha \in \Phi$. Then $\lambda_u(x) = \ell(x) + K$, $\rho_u(x) = r(x) + K$ constitute a bispecialization (by definition of K), which is universal since any $\lambda, \rho\colon A \to D$ induce an algebra homomorphism $U \xrightarrow{S} D$ by $S(\ell(x)) = \lambda(x)$, $S(r(x)) = \rho(x)$, with $K \subset \operatorname{Ker} S$ since S takes the defining relations for K into the defining relations for (λ, ρ) to be a bispecialization so S passes to $U/K \xrightarrow{\sigma} D$. (This is a very general method of constructing universal gadgets: form a free object and divide out by the relations you need. Compare with the definition of the free alternative algebra.)

Since bispecializations, birepresentations, and bimodules are equivalent, the universal multiplication algebra ought to correspond to some universal bimodule. If we compose the universal bispecialization $A \xrightarrow{(\lambda_u, \rho_u)} Mu(A) \subset \widehat{Mu(A)} = M$ with the left regular representation $M \xrightarrow{L} \operatorname{End}_\Phi(M)$ we get a bispecialization $A \xrightarrow{(\lambda, \rho)} \operatorname{End}_\Phi(M)$, by $\lambda(x) = L_{\lambda_u(x)}$, $\rho(x) = L_{\rho_u(x)}$. This is called the <u>universal birepresentation</u> and M the <u>universal bimodule</u> for A. Note that M is just $\widehat{Mu(A)} = \Phi 1 + Mu(A)$ as linear space, with bimodule operations

$$a \cdot m = \lambda_u(a)m \qquad m \cdot a = \rho_u(a)m \ .$$

The universal property for M is not quite what you might expect: it is not true that all bimodules are homomorphic images of M, because bimodules can get arbitrarily big, but it is true that every cyclic bimodule (generated by one element) is an image of M. Namely, we map $M = \widehat{Mu(A)}$ 1 cyclic module $\hat{M}(A)m$ generated by m by $M = \widehat{Mu(A)} \to \hat{M}(A) \to \hat{M}(A)m$. Thus M is at least universal for cyclic bimodules.

The reason for introducing the universal gadget is that information about all possible bispecializations, birepresentations, and bimodules is locked up inside this one object. Further, the birepresentation theory of A is equivalent to the associative representation theory of Mu(A) (i.e. to the possible homomorphisms σ). This explains why all the concepts of ordinary associative module theory carried over to theory of alternative bimodules.

As an example of universal gadgetry we show that if a collection of elements generate A, their multiplications generate any multiplication algebra of A.

7.7 (Generation Lemma). If $\{x_i\}_{i \in I}$ generate the algebra A then the elements $\{\lambda(x_i), \rho(x_i)\}_{i \in I}$ generate Mu(A).

Proof. Let M_o be the subalgebra of M generated by the $\lambda(x_i)$ and $\rho(x_i)$. If we can prove all $\lambda(x), \rho(x)$ for all $x \in A$ belong to M_o, then M_o will be all of M. Now (7.6) can be rewritten to say

$$\lambda(xy) = \lambda(x)\lambda(y) + [\lambda(x), \rho(y)]$$

$$\rho(xy) = \rho(y)\rho(x) + [\lambda(x), \rho(y)]$$

These show that if $x, y \in A_o = \{z \in A \mid \lambda(z), \rho(z) \in M_o\}$, so that $\lambda(x)$, $\rho(x)$, $\lambda(y)$, $\rho(y) \in M_o$, then also $\lambda(xy), \rho(xy) \in M_o$ and $xy \in A_o$. Trivially

A_o is a linear subspace, so it is actually a subalgebra. As A_o contains the generating set $\{x_i\}$ by definition of M_o, A_o must be all of A, and consequently all $\lambda(x),\rho(x)$ belong to M_o. ∎

As a corollary, if the $\{x_i\}$ generate universally they generate any particular multiplication algebra.

7.8 Corollary. If $\{x_i\}$ generate A then $\{L_{x_i}, R_{x_i}\}$ generate $M_E(A|M)$. ∎

7.9 Corollary. If $\{x_i\}$ generate A then the $\{L_{x_i}, R_{x_i}\}$ generate M(A). ∎

7.10 Corollary. If A is finitely generated so is M(A) and any $M_E(A|M)$. ∎

In practice we will try to avoid the universal multiplication algebra and deal as much as possible with "real" multiplications, because it is more comfortable dealing with honest-to-goodness multiplication operators than with the highly formal "universal" or "ideal" multiplications.

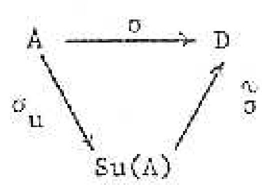
There is an analogous theory for modules (as opposed to bimodules). A left specialization, or simply specialization, of an alternative algebra A in an associative algebra D is a homomorphism $A \xrightarrow{\lambda} D$. Similarly a right specialization or anti-specialization of A in D is an anti-homomorphism $A \xrightarrow{\rho} D$. In case $D = \mathrm{End}_\Phi(M)$ we speak of a left or right representation of A on M, and call M (together with ℓ or r) a left or right module for A.

Any left specialization λ can be turned into a bispecialization by taking $\rho = 0$. Indeed, the relations $\lambda(x^2) = \lambda(x)^2$ and $\lambda(xy) = \lambda(x)\lambda(y)$ follow from the fact that λ is a homomorphism, while the other relations in (7.5), (7.6) all involve a factor ρ and hence vanish trivially.

Conversely, if (λ, ρ) is a bispecialization such that $\rho = 0$ then by (7.6) λ must satisfy $\lambda(xy) - \lambda(x)\lambda(y) = 0$ and is therefore a left specialization. In summary: specializations are precisely those bispecializations (λ, ρ) for which $\rho = 0$. In particular, left modules are precisely those bimodules M such that $m \cdot a = 0$ for all $m \in M$, $a \in A$.

Unlike the associative case, alternative algebras need not possess any modules at all - for example, a simple but not associative algebra has none (since A is not associative λ and ρ cannot be injective, so by simplicity they must be zero). Thus an algebra must be slightly associative to have left or right modules.

A <u>special universal envelope</u> for A is an associative algebra Su(A) together with a universal specialization σ_u of A in σ_u through which all other specializations factor uniquely



where $\hat{\sigma}$ is a homomorphism of associative algebras and the diagram commutes. The uniqeness of $\hat{\sigma}$ is equivalent to the condition that σ_u is surjective, $\sigma_u(A) = Su(A)$. The special universal envelope is easy to construct: $Su(A) = A/\hat{T}([A,A,A])$ with σ_u the canonical projection, where $\hat{T}([A,A,A])$ is the ideal generated by the collection of all associators $[x,y,z]$. Certainly this σ_u is surjective, and any specialization $A \overset{\sigma}{\rightarrow} D$ factors through σ_u because Ker σ is an ideal (if σ is a homomorphism) which contains all associators, $\sigma([x,y,z]) = [\sigma(x), \sigma(y), \sigma(z)] = 0$

(in an associative algebra D, all associators vanish), therefore contains the ideal $\hat{I}([A,A,A])$ they generate. Once σ kill $\hat{I}([A,A,A])$ it factors through $A/\hat{I}([A,A,A])$.

An algebra A is special if it has an injective specialization σ, that is, a monomorphism into an associative algebra. This just means A is associative. Injectivity of σ forces injectivity of σ_u, so σ_u is bijective and $Su(A) \cong A$. This just means $\hat{I}([A,A,A]) = 0$ or $[A,A,A] = 0$. An algebra which is not special is exceptional; it is purely exceptional if $Su(A) = 0$, which means the ideal $A[AAA]$ generated by the associators is all of A. A purely exceptional algebra has no specializations whatsoever and hence no module theory. For example, we observed that a simple but not-associative algebra is purely exceptional.

If A has unit e we call an A-bimodule M unital if $e \cdot m = m \cdot e = m$ for all $m \in M$ ($\ell_e = r_e = I$), in other words if e acts like a unit on M as well as on A. This is precisely the condition that e remain the unit for the split null extension: M is unital as bimodule iff $E = A \oplus M$ is unital as algebra. In the same way a left module is unital if $e \cdot m = m$ for all M, and similarly for a right module.

We essentially need only consider unital bimodules because of

7.11 (Unital Decomposition of a Bimodule) Any bimodule M for a unital algebra A decomposes into a direct sum

$$M = M_{11} \oplus M_{10} \oplus M_{01} \oplus M_{00}$$

of sub-bimodules, where M_{11} is a unital bimodule, M_{10} a unital left module, M_{01} a unital right module, and M_{00} a trivial module.

Proof. If e is the unit for A we define

$$M_{11} = e\cdot M\cdot e \quad M_{10} = e\cdot M\cdot (1-e) \quad M_{01} = (1-e)\cdot Me \quad M_{00} = (1-e)\cdot M\cdot (1-e)$$

where 1 is the ghost unit for E, i.e. $1 \in \hat{E}$. M decomposes into the direct sum of the M_{ij}: $M = \Sigma M_{ij}$ since any m can be written $m = 1\cdot m\cdot 1 = \{e + (1-e)\}\cdot m\cdot \{e + (1-e)\} = e\cdot m\cdot e + e\cdot m\cdot (1-e) + (1-e)\cdot m\cdot e + (1-e)\cdot m\cdot (1-e) = m_{11} + m_{10} + m_{01} + m_{00}$ for $m_{ij} \in M_{ij}$, and the sum is direct since if $m = m_{11} + m_{10} + m_{01} + m_{00}$ we necessarily have $m_{11} = e\cdot m\cdot e$, $m_{10} = e\cdot m\cdot (1-e)$, $m_{01} = (1-e)\cdot m\cdot e$, $m_{00} = (1-e)\cdot m\cdot (1-e)$ because of the multiplication rules

$$e\cdot m_{ij} = i\, m_{ij} \qquad m_{ij}\cdot e = j\, m_{ij} \qquad (m_{ij} \in M_{ij})$$

from $e\cdot (e\cdot m) = e^2\cdot m = e\cdot m$, $e\cdot \{(1-e)\cdot m\} = \{e(1-e)\}\cdot m = 0$ and similarly on the right.

Thus M_{11} is unital, and it is a sub-bimodule since (for example) $a\cdot m_{11} = (ea)\cdot (m_{11}e)$ (e is unit on A and M_{11}) $= e(am_{11})e \in M_{11}$. M_{00} is trivial since $m_{00}\cdot a = m_{10}\cdot eae = \{(m_{10}e)a\}e = 0$ and the same on the left. The same argument shows $m_{10}\cdot a = 0$. In much the same way $a\cdot m_{10} = e\{a(em_{10})\} = e\{am_{10}\}$ and $0 = a\cdot em_{10}e = \{(ae)m_{10}\}e = \{am_{10}\}e$, so $a\cdot m_{10} \in M_{10}$ and M_{10} is a unital left module. Dually M_{01} is a unital right module. ∎

7.12 Remark. This decomposition is a particular case of the Peirce decomposition relative to an idempotent, which we will consider in Chapter VII.

The bimodule theory for a unital algebra A breaks into three parts:

the trivial bimodules M_{00} are just all possible Φ-modules and bear no relation to the structure of A, hence are of little interest; the unital left and right modules for A are just the associative specializations and anti-specializations of A; while the most important bimodules and birepresentations are the unital ones.

We leave to the reader the definition of unital birepresentations, unital bispecializations, unital universal multiplication algebras, etc.

Exercise

1. If B is a subalgebra of A, show $M_A(B)|_B = M(B)$ as algebras of operators. (Note, however, that certain products of multiplications from B can vanish on B but not on all A, i.e. the restriction homomorphism $M_A(B) \to M_A(B)|_B$ need not be an isomorphism).

2. If x is any element of E, show its image $M(A;E)x$ under the operators in the multiplication ideal forms an ideal in the algebra E. Show $\{x | M(A;E)x = 0\}$ is also an ideal in E.

3. Is $M(A;E)A = M_E(A)A$ true in a general linear algebra?

4. If M is an A-bimodule show $A \cdot M + M \cdot A$ and $\{m | A \cdot m = m \cdot A = 0\}$ are sub-bimodules of M.

5. What kind of a bimodule could have no proper sub-bimodules and yet fail to be irreducible? Describe them all.

6. If $D: A \to M$ is a derivation, show $\mathrm{Ker}\, D = \{x \in A | D(x) = 0\}$ is a subalgebra of A. Is Im D a sub-bimodule of M?

7. Show Der (A,M) can be identified with the set of derivations $\tilde{D}$ of $E = A \oplus M$ such that $\tilde{D}(M) = 0$ and $\tilde{D}(A) \subset M$. Do these $\tilde{D}$ form a Lie subalgebra of Der (E)? Do they form a Lie module for Der (A)? Is Der (A) $\subset$ Der (E) in any sense?

8. If M_i are sub-bimodules of M show $\bigcap M_i$ and ΣM_i are too. Is the "cyclic bimodule" $A \cdot m + m \cdot A$ a sub-bimodule of M for all elements $m \in M$? If N is a sub-bimodule of M, is the same true of $A \cdot N$, $N \cdot A$, $A \cdot N + N \cdot A$? If $B \lhd A$ is the annihilator $\mathrm{Ann}_m B = \{m | B \cdot m = m \cdot B\}$ a sub-bimodule of M?

Exercise (Continued)

9. Construct Su(A) by generators and relations in the way we did for Mu(A). Why didn't we define a universal gadget for anti-specializations?

10. Show $\hat{I}([A,A,A]) = A[A,A,A] = [A,A,A]A$.

11. What is the relation between $\check{M}(A)$ and $M(\hat{A})$? Between $\check{M}u(A)$ (the universal unital multiplication algebra) and $Mu(\hat{A})$?

12. Show the special universal envelope Su(A) of the free alternative algebra $A = F[X]$ on a set X is none other than the free associative algebra on X.

#4. Problem Set on Faithfulness

1. The <u>left kernel</u> of a bispecialization (λ,ρ) is $\mathrm{Ker}\ \lambda = \{x \mid \lambda(x) = 0\}$. Show $\mathrm{Ker}\ \lambda$ is an ideal in A. Define right kernel and prove the corresponding result. The <u>kernel</u> of (λ,ρ) is the intersection $\mathrm{Ker}\ (\lambda,\rho) = \mathrm{Ker}\ \lambda \cap \mathrm{Ker}\ \rho$, which is an ideal too.
2. A bispecialization (in particular, a birepresentation) is <u>faithful</u> if $\mathrm{Ker}\ (\lambda,\rho) = 0$. If a bispecialization is not faithful to A, show it will at least be faithful to $\bar{A} = A/\mathrm{Ker}(\lambda,\rho)$.
3. Is the regular birepresentation ever faithful? Always faithful? Conclude that every alternative algebra has a faithful birepresentation. Show the universal bispecialization is always faithful.
4. Describe a recipe for obtaining all bispecializations of A whose kernel contains a given ideal. Convince yourself that an unfaithful A-bispecialization is not <u>really</u> an A-bispecialization, but rather a faithful $\bar{A}$-bispecialization which has been artificially blown up to an A-bispecialization. In this sense we would be justified in restricting our attention to faithful ones.
5. Observe that a birepresentation of a simple algebra is perforce faithful.