§2 Constructions

We can turn alternative Φ-algebras into a category by taking morphisms to be homomorphisms of linear algebras in the usual sense, i.e. linear maps $A \xrightarrow{F} \tilde{A}$ which also preserve multiplication

$$F(xy) = F(x)F(y) .$$

We have corresponding notions of <u>monomorphism</u>, <u>epimorphism</u>, <u>isomorphism</u>, <u>endomorphism</u>, <u>automorphism</u>. The <u>identity automorphism</u> on A is denoted by I_A or I. We write $A \cong \tilde{A}$ to indicate A is isomorphic to $\tilde{A}$. An <u>involution</u> * is an anti-automorphism of period 2, $(xy)^* = y^* x^*$ and $x^{**} = x$. A <u>derivation</u> of A is a linear map from A to itself satisfying

$$D(xy) = D(x)y + xD(y) .$$

We have the usual notions of <u>subalgebra</u>, <u>left ideal</u>, <u>right ideal</u>, and (two-sided) <u>ideal</u>. We will abbreviate the fact that B is an ideal in A by

$$B \lhd A .$$

The <u>image</u> Im $F = F(A)$ of A under a homomorphism $A \xrightarrow{F} \tilde{A}$ is a subalgebra of $\tilde{A}$, and the <u>kernel</u> Ker $F = F^{-1}(0) = \{x \in A | F(x) = 0\}$ is an ideal of A. A and 0 are trivially ideals of A; a <u>proper ideal</u> is an ideal $B \neq 0, A$. Each subalgebra of an alternative algebra is again alternative, since it inherits the alternative law from its parent algebra. The same is true of the <u>quotient algebra</u> $\bar{A} = A/B$ of A by an ideal B (which consists of all cosets $\bar{x} = x + B$ with product $\bar{x}\bar{y} = \overline{xy}$). We have the canonical projection epimorphism $A \xrightarrow{\pi} A/B$ with kernel B, and for any homomorphism F we have a canonical isomorphism

$$A/\mathrm{Ker}\ F \cong \mathrm{Im}\ F\ .$$

The <u>First Isomorphism Theorem</u> says that under this isomorphism there is a 1-1 correspondence between subalgebras $\tilde{B}$ (resp. left, right, or two-sided ideals) in the image and those B in A which contain Ker F, given by $\tilde{B} = F(B)$ and $B = F^{-1}(\tilde{B})$. The <u>Second Isomorphism Theorem</u> says

$$B/B \cap C \cong (B + C)/C \qquad (C \triangleleft A,\ B \subset A \text{ a subalgebra})$$

and the <u>Third Isomorphism Theorem</u>

$$(A/C)/(B/C) \cong A/B \qquad (B, C \triangleleft A \text{ with } C \subset B)\ .$$

All these theorems are proved exactly as in the associative case, indeed are valid in any linear algebra. In general, any theorem whose proof never involves products of 3 elements is independent of associativity.

A <u>trivial</u> algebra is one whose multiplication is completely trivial, $xy = 0$ for all x and y. (These are sometimes called <u>zero algebras</u>, but this could cause confusion with <u>the</u> zero algebra 0). Clearly any module can be given the structure of a trivial algebra, and any trivial algebra is alternative (even associative, and what's more commutative). An algebra is <u>simple</u> if it has no proper ideals and is not trivial. These are to be regarded as basic building blocks, since they cannot be built up from an ideal B and its quotient A/B. Our general goal in structure theory is to describe these simple algebras and show how a suitably well-behaved algebra can be built up out of simple algebras.

Construction 1. Direct sums.

A useful way of building new algebras out of old is by means of the direct sum. The direct sum

$$A = A_1 \boxplus A_2$$

of two algebras A_1, A_2 is defined to be the ordinary direct sum $A_1 \oplus A_2$ as module, and with multiplication

$$(a_1 \oplus a_2)(b_1 \oplus b_2) = a_1b_1 \oplus a_2b_2 .$$

Thus A_1, A_2 are ideals in the direct sum, and any product of an element of A_1 with one from A_2 is zero: A_1 and A_2 are put together "orthogonally". Notice that multiplication in the direct sum is completely determined by multiplication in the pieces. This is important in using direct sums to tear an algebra down: if we have decomposed an algebra into a direct sum, the structure of the algebra is completely determined by the structure of the individual pieces.

A more common but less precise way of decomposing an algebra is into a semi-direct sum

$$A = A_o \oplus B$$

where B is an ideal but A_o merely a subalgebra, and A is their direct sum as a module. Here multiplication is $(a_1 \oplus b_1)(a_2 \oplus b_2) = a_1a_2 \oplus (a_1b_2 + b_1a_2 + b_1b_2)$, so it depends not only on the products in the pieces A_o and B but also on how they are put together (how A_o multiplies with B). We will be very careful to distinguish between direct sum <u>as module</u> and direct sum <u>as algebra</u>, denoting them by different symbols $\oplus$ and $\boxplus$.

We can generalize this to more than one summand. The <u>direct sum</u> $A = \boxplus_i A_i$ of an arbitrary collection of algebras is just the direct sum $\oplus_i A_i$ as module, with componentwise multiplication $(\boxplus_i a_i)(\boxplus_i b_i) = \boxplus_i a_i b_i$. Less useful in algebra is the <u>direct product</u> $A = \pi_i A_i$, which as module is the direct product of the modules A_i, and which again has componentwise multiplication. The direct sum $\boxplus A_i$ can be identified with that subalgebra of the direct product πA_i consisting of those elements πa_i (formally, functions $f: I \to \cup A_i$ with $f(i) = a_i \in A_i$) such that $a_i = 0$ for all but a finite number of indices.

As is customary, we make no distinction between the <u>external direct sum</u> (where we start with the A_i, not a priori contained in any larger algebra, and construct an enveloping algebra $A = \boxplus A_i$) and the <u>internal direct sum</u> (where we start with A and decompose it into a direct sum $A = \boxplus A_i$ of ideals A_i in A).

We should, of course, make sure direct sums or products of alternative algebras are still alternative. It suffices to do the direct product, since it contains the direct sum as subalgebra, and subalgebras inherit alternativity. If $x = \pi x_i$, $y = \pi y_i$ are elements of $A = \pi A_i$, then $x^2 y$ and $x(xy)$ are the same because they both have exactly the same ith components $x_i^2 y_i = x_i(x_i y_i)$ for all indices i. (Similarly A is right alternative, $yx^2 = (yx)x$.)

If $F_i: A_i \to \tilde{A}_i$ are homomorphisms it is immediate that $F_1 \boxplus F_2: A_1 \boxplus A_2 \to \tilde{A}_1 \boxplus \tilde{A}_2$ is a homomorphism of direct sums.

Construction 2. Adjunction of a unit.

Another construction which does not take us out of the class of alternative algebras is adjunction of a unit. In any linear algebra we have the usual notions of *left*, *right*, and (two-sided) *unit* elements. (These are usually called *identity* elements, but we want to reserve the term identity to refer to a law or identical relation.) If A has a left unit e and a right unit f, ex = x = xf for all x ∈ A, then e = ef = f is a two-sided unit. The unit element is unique if it exists; it will always be denoted by 1: 1x = x1 = x for all x. An algebra is *unital* if it has a unit.

If a unit doesn't exist, we can create one: we form the *unital hull*

(example of semi-direct sum)

$$\hat{A} = \Phi 1 \oplus A$$

of A, whose module structure is a direct sum of A with a copy $\Phi 1$ of Φ and whose multiplication is given in the natural way by

$$(\alpha 1 \oplus x)(\beta 1 \oplus y) = \alpha\beta 1 \oplus (\alpha y+\beta x+xy) .$$

$\hat{A}$ has unit $1 = 1 \oplus 0$, contains A as subalgebra, and is still alternative: for example, we still have left alternativity

$$[a,a,b] = [\alpha 1+x,\alpha 1+x,\beta 1+y] = [x,x,y] = 0 \quad (\alpha,\beta \in \Phi, x,y \in A)$$

since A itself is left alternative, and any associator involving 1 collapses in any linear algebra. (For example, $(1\cdot a)\cdot b - 1\cdot(a\cdot b) = a\cdot b - a\cdot b = 0$.)

Any homomorphism $A_1 \overset{F}{\to} A_2$ extends uniquely to a homomorphism $\hat{A}_1 \overset{\hat{F}}{\to} \hat{A}_2$ of unital algebras by $\hat{F}(\alpha 1_1 + a_1) = \alpha 1_2 + F(a_1)$.

2.1 Remark: The fact that any commutator or associator involving 1 is zero in a linear algebra,

$$[1,a] = [a,1] = 0$$

$$[1,a,b] = [a,1,b] = [a,b,1] = 0 ,$$

is trivial but worth remembering. ∎

It is frequently convenient to have a unit around (as we saw in Part 1), so we often pass without comment from A to $\hat{A}$. Note that A is an ideal in $\hat{A}$, so that a product $(\alpha 1+a)b$ makes sense as an element of A (even though $\alpha 1+a$ only exists in $\hat{A}$).

Construction 3. Scalar extension.

Still another construction we can perform is scalar extension. This is frequently useful in ensuring we have "enough" or "the right kind of" scalars. If Ω is an extension of Φ (usually thought of as a ring of scalars containing Φ as a unital subring, $\Omega \supset \Phi$, although everything works if Ω is merely a unital Φ-algebra, containing a homomorphic rather than isomorphic copy $\Phi 1$ of Φ), we can form the scalar extension

$$A_\Omega = \Omega \otimes_\Phi A .$$

As a module this is just the usual tensor product of Φ-modules, with multiplication defined in the natural way on a spanning set by

$$(\omega \otimes x)(\mu \otimes y) = \omega\mu \otimes xy \ .$$

(Since this is Φ-bilinear, it automatically extends to a Φ-bilinear product on all of $\Omega \otimes A$). Such an extension of an alternative Φ-algebra is now an alternative Ω-algebra: for example, left alternativity is

$$[a,a,b] = [\Sigma\omega_i \otimes x_i, \ \Sigma\omega_j \otimes x_j, \ \Sigma\mu_k \otimes y_k]$$

$$= \Sigma[\omega_i \otimes x_i, \ \omega_j \otimes x_j, \ \mu_k \otimes y_k]$$

$$= \Sigma_i \ \omega_i^2\mu_k \otimes [x_i,x_i,y_k] + \Sigma_{i>j} \ \omega_i\omega_j\mu_k \otimes \{[x_i,x_j,y_k]$$

$$+ [x_j,x_i,y_k]\}$$

$$= 0$$

by $[x,x,y] = [x,z,y] + [z,x,y] = 0$. Roughly, A_Ω consists of Ω-linear combinations $\Sigma\omega_i a_i$ of elements of A, multiplied in the natural way.

If $A \overset{F}{\to} \tilde{A}$ is a homomorphism of Φ-algebras, the ordinary linear extension $F_\Omega = 1 \otimes F$ is a homomorphism $A_\Omega \xrightarrow{F_\Omega} \tilde{A}_\Omega$ of Ω-algebras: $F_\Omega(x \cdot y) = F_\Omega((\Sigma\omega_i \otimes a_i)(\Sigma\mu_j \otimes b_j)) = F_\Omega(\Sigma\omega_i\mu_j \otimes a_i b_j) = \Sigma \ \omega_i\mu_j \otimes F(a_i b_j) = \Sigma \ \omega_i\mu_j \otimes F(a_i)F(b_j) = (\Sigma\omega_i \otimes F(a_i))(\Sigma\mu_j \otimes F(b_j)) = F_\Omega(x)F_\Omega(y)$.

Suppose that A is an algebra over Ω; then it is also an algebra over any unital subring Φ of Ω. We denote these algebra structures by A/Ω and A/Φ (there can be no confusion of this with some sort of quotient). The Ω-algebra A/Ω should not be confused with the Ω-algebra $A_\Omega = \Omega \otimes_\Phi A$, which is the scalar extension of A as Φ-algebra. The underlying space of A/Ω is just A, whereas that of A_Ω is the larger space $\Omega \otimes A$.

These scalar extensions are the only type of tensor products that will be of interest to us; in general the tensor product of two alternative algebras will not be alternative.

2.2 Remark. The tensor product $A\otimes_\Phi B$ of two unital alternative algebras A, B over a field Φ will be again alternative only if (i) both A and B are associative, or (ii) one of A or B is commutative associative (scalar extension of Φ).

Proof. If $A \otimes B$ is left alternative then for $a_i \in A$, $b_i \in B$

$$0 = [a_1\otimes b_1, a_2\otimes b_2, a_3\otimes b_3] + [a_2\otimes b_2, a_1\otimes b_1, a_3\otimes b_3]$$

$$= (a_1a_2)a_3\otimes(b_1b_2)b_3 - a_1(a_2a_3)\otimes b_1(b_2b_3) + (a_2a_1)a_3\otimes(b_2b_1)b_3$$

$$- a_2(a_1a_3)\otimes b_2(b_1b_3)$$

$$= [a_1,a_2,a_3]\otimes(b_1b_2)b_3 + a_1(a_2a_3)\otimes[b_1,b_2,b_3] + [a_2,a_1,a_3]\otimes(b_2b_1)b_3$$

$$+ a_2(a_1a_3)\otimes[b_2,b_1,b_3]$$

$$= [a_1,a_2,a_3]\otimes[b_1,b_2]b_3 + (a_1(a_2a_3)-a_2(a_1a_3))\otimes[b_1,b_2,b_3]$$

Suppose neither (i) nor (ii) holds, so A (say) is not associative and B is either not associative or not commutative. By non-associativity of A we can find $[a_1^o, a_2^o, a_3^o] \neq 0$. If B is not commutative we can find $[b_1^o, b_2^o] \neq 0$, hence $[a_1^o, a_2^o, a_3^o] \otimes [b_1^o, b_2^o] \neq 0$ since we are tensoring vector spaces, and this contradicts the above with $b_3^o = 1$. If B is commutative but not associative, $[b_1^o, b_2^o, b_3^o] \neq 0$ then the above collapses to $0 = (a_1(a_2a_3) - a_2(a_1a_3)) \otimes [b_1^o, b_2^o, b_3^o]$ and $a_1(a_2a_3) = a_2(a_1a_3)$

for all $a_i \in A$. In particular, for $a_3 = 1$ we see $a_1a_2 = a_2a_1$ and A is commutative. But then $0 = a_1^0(a_2^0a_3^0) - a_2^0(a_1^0a_3^0) = (a_2^0a_3^0)a_1^0 - a_2^0(a_3^0a_1^0) = [a_2^0, a_3^0, a_1^0] = [a_1^0, a_2^0, a_3^0]$ by alternativity, contradicting our choice of a_i^0. ◻

Construction 4. The opposite algebra.

The opposite algebra A^{op} of any linear algebra is that algebra which has the same linear structure as A, but the reverse or opposite multiplication

$$x\cdot_{op} y = y\cdot x .$$

Associators and commutators in the opposite algebra are expressed as

$$[x,y]^{op} = [y,x] = -[x,y]$$

$$[x,y,z]^{op} = -[z,y,x]$$

since $(x\cdot_{op}y)\cdot_{op}z - x\cdot_{op}(y\cdot_{op}z) = (yx)\cdot_{op}z - x\cdot_{op}(zy) = z(yx) - (zy)x$. This shows that A^{op} is commutative or associative or alternative if A is. In general, since multiplication in A^{op} is reversed, whenever A satisfies a collection of identities which is left-right symmetric then A^{op} will satisfy the same identities. This is not true for asymmetric axioms: if A is only left alternative, A^{op} is only right alternative.

The importance of the opposite algebra is that it leads to a notion of duality for alternative algebras: whenever a general statement about alternative algebras is true, so is the dual statement obtained by everywhere interchanging "left" and "right" and reversing all multiplications. The reason for this is that if A is any alternative algebra, the truth of the general statement for A^{op} is equivalent to the truth of its dual for A.

As an example, we will state and prove theorems about left ideals, but will feel free to make use of the dual theorems for right ideals. Or we may state a theorem (with symmetric hypotheses) about all one-sided ideals, but prove it only for left ideals. Duality allows us to apply the magic words "similarly", "by symmetry", "dually", etc. in proofs.

Construction 5. Free algebras.

In keeping with a general terminology for algebraic systems, we define the subalgebra $\Phi[X]$ of an alternative Φ-algebra A generated by a subset X to be the smallest Φ-subalgebra of A containing X. Abstractly, $\Phi[X]$ is the intersection of all subalgebras of A which contain X; concretely, $\Phi[X]$ consists of all finite sums of finite products $x_{i_1} \ldots x_{i_n}$ (with some distribution of parentheses) of the generators $x_i \in X$. Thus $\Phi[X]$ consists of the "alternative polynomials" in the elements x_i. We say a subset X generates or is a set of generators for an algebra A if $A = \Phi[X]$. (Thus X always generates $\Phi[X]$!)

One useful fact about generating sets is that if two homomorphisms agree on a generating set they agree everywhere: the set $\{a|F(a)=G(a)\}$ where F and G agree is a subalgebra, so if it contains a generating set X it contains all of $\Phi[X] = A$.

If X is just a set, not contained a priori in any alternative algebra, we can still form an alternative algebra F[X] "freely generated" by X, analogous to the free associative algebra on a set X.

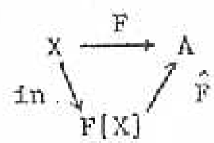
More generally, for any set X we can construct the <u>free nonassociative algebra</u> $F_o[X]$ on the set X ("algebra" always being understood as "Φ-algebra"). We recursively define the <u>monomials</u> of degree n in the variables $x \in X$, taking the monomials of degree 1 to be the elements of X, those of degree 2 to be all symbols (xy) for $x,y \in X$, and in general those of degree n to be all symbols (pq) where p,q are monomials of degrees $\partial p, \partial q \geq 1$ with $\partial p + \partial q = n$. (For example, in degree 3 we get all $(x(yz))$ and $((xy)z)$.) Roughly, the monomials consist of all formal products $x_{i_1} \cdots x_{i_n}$ for $x_i \in X$ with some distribution of parentheses. We take $F_o[X]$ to be the free Φ-module with basis all monomials, where multiplication is defined on basis elements p,q by $p \cdot q = (pq)$. The mapping $B \times B \to B$ on the basis B extends uniquely to a bilinear multiplication on all of $F_o[X]$.

$F_o[X]$ has the universal property that any set-theoretic map $X \xrightarrow{F} A$ of X into a linear algebra A extends uniquely to a homomorphism $F_o[X] \xrightarrow{\hat{F}} A$. Indeed, to define a homomorphism $\hat{F}$ on $F_o[X]$ it suffices to define $\hat{F}$ on the basis B satisfying $\hat{F}(p \cdot q) = \hat{F}(p) \cdot \hat{F}(q)$ and then extend by linearity. We do this inductively, defining $\hat{F}(x) = F(x)$ in degree 1 and $\hat{F}((pq)) = \hat{F}(p) \cdot \hat{F}(q)$ in degree n if $\partial p + \partial q = n$. Less formally, $\hat{F}(\Sigma \alpha_I x_{i_1} \cdots x_{i_n}) = \Sigma \alpha_I F(x_{i_1}) \cdots F(x_{i_n})$ where we distribute parentheses in $x_{i_1} \cdots x_{i_n}$ in some fashion and in the same fashion in $F(x_{i_k}) \cdots F(x_{i_n})$.

To get the <u>free alternative algebra</u> $F[X]$ on the set X we simply form $F[X] = F_o[X]/K$ where K is the ideal generated by all elements of the form $[a,a,b]$ or $[b,a,a]$ for a,b arbitrary elements of $F_o[X]$. In the quotient $F[X]$ we have the relations $[\bar{a}, \bar{a}, \bar{b}] = [\bar{b}, \bar{a}, \bar{a}] = \bar{0}$ for all cosets $\bar{a} = a + K$ since $[\bar{a}, \bar{a}, \bar{b}] = \overline{[a, a, b]}$ and $[a, a, b] \in K$. We have taken the free nonassociative algebra on X and divided out by the relations necessary to make it alternative (namely the left and right alternative laws).

The freedom or universality of the free algebra consists of the following:

2.3 (Universal Property of the Free Algebra) Any set-theoretic map $X \xrightarrow{F} A$ of X into an alternative algebra A extends uniquely to a homomorphism $F[X] \xrightarrow{\hat{F}} A$



Proof. Since A is a linear algebra, X extends uniquely to a homomorphism $\hat{F}_o\colon F_o[X] \to A$. Since A is alternative we have $\hat{F}_o([a,a,b]) = [\hat{F}_o(a), \hat{F}_o(a), \hat{F}_o(b)] = 0$ and $\hat{F}_o([b,a,a]) = 0$ for all $a, b \in F_o[X]$; therefore all $[a,a,b]$ and $[b,a,a]$ lie in the ideal Ker $\hat{F}_o$, so the ideal K they generate is contained in Ker $\hat{F}_o$, and $\hat{F}_o$ passes uniquely to a homomorphism $\hat{F}$ from the quotient $F[X] = F_o[X]/K$ to A. Uniqueness of $\hat{F}$ follows from the fact that its action on the generating set X is prescribed, $\hat{F}(x) = F(x)$. ∎

In practice this allows us to think of elements of the free algebra as "alternative polynomials" in the variables x. We obtain polynomial functions on an alternative algebra A by specialization of the variables: if $p(x_1, \ldots, x_n)$ is an element of F[X] ($x_i \in X$) and $a_1, \ldots, a_n$

elements of A we can "evaluate $p(x_1,\ldots,x_n)$ at $a_1,\ldots,a_n$" by forming $p(a_1,\ldots,a_n)$. Informally $p(a_1,\ldots,a_n)$ is just obtained by replacing each x_i by a_i, but formally $p(a_1,\ldots,a_n)$ is defined as $\hat{F}(p(x_1,\ldots,x_n))$, where $\hat{F}$: $F[X] \to A$ is the extension of $F: X \to A$ given by $F(x_i) = a_i$. (What F does to the x not appearing in p is immaterial; we can take $F(x) = 0$ if we like.)

In Appendix I we will investigate algebras satisfying a polynomial identity, i.e. for which there exists a nonzero $p(x_1,\ldots,x_n)$ in the free algebra such that $p(a_1,\ldots,a_n) = 0$ identically on A. Since the specializations $f(x_1,\ldots,x_n) \to f(a_1,\ldots,a_n)$ are precisely all homomorphisms $F[x_1,\ldots,x_n] \to A$ (there is a 1-1 correspondence between homomorphisms $F[X] \to A$ and maps $X \to A$), another way of putting it is that $F(p(x_1,\ldots,x_n)) = 0$ for all homomorphisms $F[X] \overset{F}{\to} A$. Note that $p(x,y) = [x,x,y]$ does not count as a polynomial identity; it is certainly satisfied by A, but it is zero as an element of the free algebra.

Free alternative algebras behave badly. Their trouble is that they have <u>too many</u> identities, they are <u>too close</u> to associativity. There are some "would-be" identities $p(x_1,\ldots x_n)$ which don't vanish identically but whose squares do: the free algebra has nilpotent elements! As a consequence it can't possibly be imbedded in a division algebra, which is very different from the situation with free associative algebras.

We confine the misbehavior of free algebras to Appendix III.

Exercises

2.1 Prove the set of automorphisms of a linear algebra A forms a group. Prove the set of derivations of A forms a Lie algebra.

2.2 Prove that if B, C are left, right, or two-sided ideals of a linear algebra, so is their sum B+C. What if B, C are subalgebras?

2.3 Refresh your memory of linear algebra by proving the First, Second, and Third Isomorphism Theorems.

2.4 If $B \lhd A$ and A/B, B are trivial as algebras, is A trivial?

2.5 Formulate and prove a universal property for the direct sum $A_1 \boxplus A_2$.

2.6 Show $(A_1,A_2) \to A_1 \boxplus A_2$, $(F_1,F_2) \to F_1 \boxplus F_2$ defines a functor on suitable categories.

2.7 If B_i are ideals of A such that $A = \oplus B_i$ is their direct sum as module, show $A = \boxplus B_i$ is necessarily their direct sum as algebra.

2.8 If B_i are ideals in A_i, show $B = \boxplus B_i$ is an ideal in $A = \boxplus A_i$ and $\tilde{B} = \pi B_i$ in $\tilde{A} = \pi A_i$. If C is an ideal in B, is it an ideal in a semi-direct sum $A = A_0 \oplus B$? Generally, if $B \lhd A$ it is not true that all B-ideals are A-ideals: $C \lhd B \lhd A \not\Rightarrow C \lhd A$.

2.9 Formulate and prove a universal property for the unital hull $\hat{A}$.

2.10 Show $A \to \hat{A}$, $F \to \hat{F}$ defines a functor from the category of alternative algebras to the category of unital alternative algebras. (What are morphisms in the latter category?)

2.11 If B is an ideal or subalgebra in A, show it is still an ideal or subalgebra in $\hat{A}$.

2.12 Formulate and prove a universal property for A_Ω .

2.13 Show $A \to A_\Omega$, $F \to F_\Omega$ defines a functor from the category of alternative Φ-algebras to the category of alternative Ω-algebras.

2.14 If B is an ideal in A, show ΩB is an ideal in A_Ω. Why do we say ΩB instead of B_Ω?

2.15 If F is an automorphism (resp. D a derivation) of A, show F_Ω is an automorphism (resp. D_Ω a derivation) of A_Ω. If F is an automorphism of Ω, is $F \otimes I_A$ an automorphism of $\Omega \otimes A = A_\Omega$?

2.16 Show that a homomorphism of a simple algebra must be zero or a monomorphism.

2.17 If F is a homomorphism of any linear algebra, show $F([x,y]) = [F(x),F(y)]$ and $F([x,y,z]) = [F(x),F(y),F(z)]$. If D is a derivation show $D([x,y]) = [D(x),y] + [x,D(y)]$ and $D([x,y,z]) = [D(x),y,z] + [x,D(y),z] + [x,y,D(z)]$.

2.18 Prove that if $A \xrightarrow{F} \tilde{A}$ is a homomorphism of algebras it is also a homomorphism of their opposite algebras, and that the correspondence $A \to A^{op}$, $F \to F$ determines a functor from alternative algebras to alternative algebras.

2.19 Prove that $A \xrightarrow{F} \tilde{A}$ is an anti-homomorphism iff it is a homomorphism $A^{op} \to \tilde{A}$ or $A \to \tilde{A}^{op}$. Show A possesses an anti-automorphism (e.g. an involution) iff $A \cong A^{op}$.

2.20 Prove that $X \to F[X]$, $F \to \hat{F}$ is a functor from sets to alternative algebras.

2.21 If $Y \subset X$ prove that the subalgebra $\Phi[Y] \subset F[X]$ generated by Y is isomorphic to $F[Y]$; loosely $F[Y] \subset F[X]$.

#1. Problem Set on Direct Sums

Suppose A is an arbitrary linear algebra which has a decomposition $A = \boxplus_{i=1}^{n} A_i$ into a finite direct sum of ideals. We investigate the uniqueness of this decomposition.

1. If A is idempotent in the sense that $A^2 = A$, show $A_i^2 = A_i$ too. If the A_i are idempotent and indecomposable (can't themselves be written as direct sums $A_i = A_{i1} \boxplus A_{i2}$) show any other decomposition $A = \boxplus_{i=1}^{m} \tilde{A}_i$ into indecomposables has the same constituents (up to order). Note: this only characterizes uniquely those indecomposable ideals A_i which appear as part of a decomposition of A.
2. Conclude that if $A = \boxplus A_i$ where the A_i are idempotent minimal ideals (contain no smaller ideals except 0), then the A_i are uniquely determined.
3. If $A = \boxplus A_i$ show A is unital iff each A_i is, and $1 = \Sigma e_i$ where e_i is the unit of A_i. In this case show $A_i^2 = A_i$. If the A_i are minimal ideals too, show any ideal B of A has the form $B = \boxplus_{i \in I(B)} A_i$ (a sum of certain of the building blocks A_i). In particular, conclude there are only finitely many ideals.
4. Conclude that if $A = \boxplus A_i$ where A is unital and the A_i minimal, then the A_i are uniquely determined as precisely all minimal ideals of A.
5. Show $A = \boxplus_{i=1}^{n} A_i$ for A_i simple ideals iff $A_i \cong A/M_i$ for M_i maximal ideals with (i): $\bigcap_i M_i = 0$, (ii): $M_i \boxplus \bigcap_{j \neq i} M_j = A$.
6. If $\bigcap_\alpha M_\alpha = 0$ where A has descending chain condition on ideals, show $\bigcap_{i=1}^{m} M_i = 0$ for some finite subcollection. If $\bigcap_{i=1}^{m} M_i = 0$ show there is a subcollection satisfying (i) and (ii).

7. Conclude that a unital algebra is a finite direct sum of simple ideals iff it has d.c.c. on ideals and the intersection $\bigcap M$ of all maximal ideals is zero.