Chapter I

Alternative Algebras

§1 The variety of alternative algebras

Recall that a linear algebra A over a unital, commutative, associative ring of scalars Φ consists of a unital Φ-module (usually denoted by the same symbol A) together with a Φ-bilinear multiplication on A, which is denoted by $x\cdot y$ or simply xy. Multiplication need not be associative. A more old-fashioned way of saying that multiplication is bilinear is to say it is left and right distributive

$$x\cdot(y+z) = x\cdot y + x\cdot z, \qquad (y+z)\cdot x = y\cdot x + z\cdot x$$

and commutes with scalars

$$\alpha(x\cdot y) = (\alpha x)\cdot y = x\cdot(\alpha y) .$$

The fact that multiplication need not associate means we must be scrupulous in inserting parentheses: $(xy)z$ is not the same as $x(yz)$.

An algebra is <u>left alternative</u> if it satisfies the <u>left alternative law</u>

$$(1.1) \qquad x^2y = x(xy)$$

(where we abbreviate xx by x^2)

for all elements x,y. Similarly, an algebra is <u>right alternative</u> if it satisfies the <u>right alternative law</u>

$$(1.2) \qquad yx^2 = (yx)x .$$

An <u>alternative</u> algebra is one which is both left and right alternative ("two-sided" alternative).

The defining identities (1.1) and (1.2) are linear in y but quadratic in the variable x. THE BASIC INSTINCT OF A NONASSOCIATIVE ALGEBRAIST WHEN FACED WITH AN IDENTITY IS: LINEARIZE! As indicated in connection with the Nagata-Higman Theorem in Part 1, linearization usually requires certain hypotheses (such as that Φ be a field with enough elements). However, QUADRATIC IDENTITIES CAN ALWAYS BE LINEARIZED.

Linearization of a quadratic function f(x) consists merely in forming the polarized form $f(x+y) - f(x) - f(y)$. In the present situation it works as follows. For any x,y,z in an alternative algebra A we have $(x+z)^2y = (x+z)\{(x+z)y\}$ as well as $x^2y = x(xy)$ and $z^2y = z(zy)$. Subtracting the last two from the first one (using bilinearity of multiplication) yields

$$(xz+zx)y = x(zy) + z(xy) \,. \tag{1.1'}$$

It is essential that you practice until you can "read off" this linearization directly from (1.1). What is involved is replacing one x at a time by a z in all possible ways.

A similar linearization of (1.2) yields

$$y(xz+zx) = (yx)z + (yz)x \,. \tag{1.2'}$$

These linearized formulas need not be memorized; you need only memorize the basic formulas (1.1), (1.2), and the process of linearization.

The linearized formulas give relations among three arbitrary variables. We can obtain less complicated formulas, involving fewer variables, if we specialize certain of the arguments. For example, we could see that happens when we set $z = x$:

$$(xx+xx)y = x(xy) + x(xy) .$$

This is just $2\, x^2 y = 2\, x(xy)$, twice what we started with. In general one gets no new information if one starts with a homogeneous identity, linearizes x to x and z, then sets $z = x$. To get new information, try setting $y = x$:

$$(xz+zx)x = x(zx) + z(xx) .$$

Now from (1.2) we already know $(zx)x = z(xx)$, so subtracting this gives the <u>middle alternative law</u> (better known as the <u>flexible law</u>)

(1.3) $$(xz)x = x(zx) .$$

Any algebra satisfying the flexible law is called a <u>flexible</u> algebra; we have just shown that an alternative algebra is automatically flexible. Like the other alternative laws, the flexible law is a weak form of the associative law $(xy)z = x(yz)$, but it is also a weak form of commutativity; any associative or commutative algebra is automatically flexible.

A very important concept is that of the associator $[x,y,z]$ of three elements of a nonassociative algebra, defined by

$$[x,y,z] = (xy)z - x(yz) .$$

The associator measures how far x,y,z are from associating, since it is just the difference between the two ways $(xy)z$ and $x(yz)$ of associating x,y,z in the given order. An algebra is <u>associative</u> if

$(xy)z = x(yz)$ for all x,y,z, which is equivalent to the condition that all associators vanish.

In the same way the commutator

$$[x,y] = xy - yx$$

measures how far x and y are from commuting: $xy = yx$ iff $[x,y] = 0$. An algebra is commutative if $xy = yx$ for all x,y; this is equivalent to the condition that all commutators vanish.

The associator or commutator allows us to express a difference of two terms by means of one symbol. The alternative laws may be succinctly expressed in terms of associators as

(1.1a) $$[x,x,y] = 0 \quad \text{(left alternativity)}$$

(1.2a) $$[y,x,x] = 0 \quad \text{(right alternativity)}$$

(1.3a) $$[x,y,x] = 0 . \quad \text{(flexibility)}$$

This just means the associator $[x,y,z]$ is an alternating function of its arguments. (Recall that a multilinear function $f(x_1,\ldots,x_n)$ is alternating if it vanishes whenever two of its arguments coincide. As we saw in the section on polynomial identities in Part 1, this implies f changes sign under permutation of its arguments, and conversely if A has no 2-torsion then a function which changes sign under permutations is necessarily alternating.)

1.4 (Alternating Theorem) An algebra is alternative iff the associator $[x,y,z]$ is an alternating function of its arguments. ∎

This, of course, is the reason for the name "alternative algebra".

Since associative algebras are defined by the identity $[x,y,z] = 0$, they clearly satisfy (1.1a)-(1.3a), so

1.5 Proposition. Any associative algebra is alternative. ∎

We will see that in some sense alternative algebras are not far removed from associative algebras.

Another convenient way of phrasing identities is to formulate them as operator-identities; if L_x denotes the left multiplication operator

$$L_x(y) = x \cdot y$$

and R_x the right multiplication operator

$$R_x(y) = y \cdot x$$

then the alternative laws take the form

(1.1op) $$L_{x^2} = L_x^2 \quad \text{(left alternativity)}$$

(1.2op) $$R_{x^2} = R_x^2 \quad \text{(right alternativity)}$$

(1.3op) $$R_x L_x = L_x R_x \text{ or } [R_x, L_x] = 0 \quad \text{(flexibility)}$$

This last flexible law allows us to introduce a "two-sided" multiplication operator U_x by

$$U_x y = xyx \quad \text{or} \quad U_x = L_x R_x = R_x L_x ;$$

no parentheses are needed in xyx by flexibility. The operator U_x is quadratic in the variable x (as opposed to L_x and R_x, which are linear in x). Therefore we can linearize in x to get

$$U_{x,y} = U_{x+y} - U_x - U_y = L_x R_y + L_y R_x = R_x L_y + R_y L_x .$$

It will also be convenient to denote the linearization of the square by a circle,

$$x \circ y = (x+y)^2 - x^2 - y^2 = xy + yx .$$

Products $U_x y$, $U_{x,y} z$, x^2, $x \circ y$ will be called Jordan products (for reasons which will become clear in Part III).

Operator notation is not only notationally clearer (there is one less variable), but often conceptually clearer as well: witness the above.

Note, however, that to interpret an element identity as an operator, one variable must appear linearly. If more than one variable appears linearly, the element identity can be interpreted in more than one way as an operator identity. For example, the linearized left alternative law $(xz+zx)y = x(zy) + z(xy)$ can be viewed as an operator identity acting on y

$$L_{x \circ z} = L_{xz+zx} = L_x L_z + L_z L_x$$

(this is just a linearization of (1.lop), so is not surprising), or on z

$$R_y (L_x + R_x) = L_x R_y + R_{xy}$$

(which is surprising), or on x

$$R_y(L_z+R_z) = R_{zy} + L_zR_y$$

(this is the same as the previous relation with x and z interchanged).

This method of passing back and forth between elements and operators, and reinterpreting an operator identity on one variable as an operator identity on another variable, is a standard technique in nonassociative algebras.

Exercises

1.1 Suppose we have an identity $f(x,x,y) = 0$ where $f(x,y,z)$ is some trilinear function from a module M to a module N (for example, $f(x,y,z) = x(yz)$ or $f(x,y,z) = (xy)z - x(yz)$ from A to A). Guess what the linearization of this identity is, then check it by computing out $f(x+z,x+z,y) - f(x,x,y) - f(z,z,y)$.

1.2 Apply the previous exercise to $f(x,y,z) = [x,y,z]$ and the left alternative law $[x,x,y] = 0$. Set $y = x$ in the linearized version. What do you get?

1.3 Prove $[x,x] = 0$ in all linear algebras. Linearize this in terms of commutators; what does the linearization say in terms of elements?

1.4 Linearize the right alternative law (1.2op), subtract the operator identity $R_y(L_x+R_x) = L_xR_y + R_{xy}$. What do you get as an operator identity? Let it act on z and write the result as an element identity in x,y,z. Reinterpret as an operator identity acting on y.

1.5 Linearize the following relations in x: $x\{(zy)x\} = (xz)(yx)$, $yx^2y = (yx)(xy)$, $(xy)\{z(xy)\} = x\{y(zx)y\}$, $[x,y,x] = 0$, $[x^2,y] = x\circ[x,y]$ (where $x\circ y = xy + yx$).

1.6 In an arbitrary linear algebra show $[x,yz] - [x,y]z - y[x,z] = -[x,y,z] + [y,x,z] - [y,z,x]$. Conclude that in an alternative algebra $[x,yz] - [x,y]z - y[x,z] = 3[y,x,z]$.

1.7 In an arbitrary linear algebra show $(x\circ y)z - y(x\circ z) - [x,y,z] = [y,z,x]$.

Exercises (Continued)

1.8 Show $[x, y \circ z] + [y, z \circ x] + [z, x \circ y] = 0$ in any flexible algebra.

1.9 Show $[x, xy, y] = [y, yx, x] = 0$ in an alternative algebra. Linearize to show $[x, xy, z] = [x, y, xz]$ and $[x, yx, z] = [x, y, zx]$.

1.10 Interpret the associative law $(xy)z = x(yz)$ in three different ways as an operator identity.

1.11 Interpret the linearized right alternative law in three different ways as an operator identity. Do the same for the middle alternative law.

1.12 Show that any alternative algebra over Φ spanned by 4 elements is necessarily associative.

1.13 Write the flexible law (1.5) in terms of commutators and conclude anew that any commutative algebra is flexible.