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Universal central extensions of direct limits of Lie superalgebras ☆

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ABSTRACT

We show that the universal central extension of a direct limit of perfect Lie superalgebras L_i is (isomorphic to) the direct limit of the universal central extensions of L_i. As an application we describe the universal central extensions of some infinite rank Lie superalgebras.

Introduction

Central extensions appear naturally in the theory of infinite dimensional Lie algebras. For example, they are fundamental for the theory of affine Kac–Moody Lie algebras and extended affine Lie algebras. Centrally extended Lie algebras often have a more interesting representation theory than the original Lie algebra, which makes central extension an interesting topic for applications, e.g., in physics. A convenient way to find "all" of them, is to determine the universal central extension of a given Lie algebra, which exists for perfect Lie algebras (well-known) and superalgebras [N2].

Direct limit of Lie superalgebras is an important way to construct infinite dimensional Lie superalgebras. Examples include various types of locally finite Lie (super)algebras [BB,DP,P,PS], locally extended affine Lie algebras [MY,Nee,N3] and Lie superalgebras graded by locally finite root systems [N1,GN]. These types of Lie algebras and Lie superalgebras have been intensively studied by many

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authors, many more than we have quoted, yet no general results seem to be known about their universal central extensions, besides the paper [S] in which the author studied a rather special case, described in Remark 1.7.

In this paper, we consider the universal central extensions of general direct limits of Lie superalgebras over an arbitrary base superring. We show in Theorem 1.6 that the universal central extension of a direct limit $\varinjlim L_i$ of perfect Lie superalgebras L_i is canonically isomorphic to the direct limit of the universal central extensions of L_i. This result is new even for the case of Lie algebras. Crucial for its proof is the fact [N2] that one has an endo-functor $\mathfrak{uce}$ on the category of all Lie superalgebras which gives the universal central extension for perfect Lie superalgebras.

As an application, we describe in Section 2 the universal central extensions of some direct limit Lie superalgebras, namely $\mathfrak{sl}(I; A)$ for $|I| \geqslant 5$ and A an associative superalgebra (Proposition 2.2, Corollary 2.4), $\mathfrak{osp}(I; A)$ for A commutative associative (Example 2.6), locally finite Lie superalgebras (Example 2.7) and Lie algebras graded by locally finite root systems (Example 2.10). These applications are possible since one knows the universal central extension of the Lie superalgebras over which we take the direct limit.

1. Universal central extensions of direct limits of Lie superalgebras: General results

1.1. Review of universal central extensions of Lie superalgebras

Throughout this section we consider Lie superalgebras L over a commutative superring S as defined in [N2]. Thus S is an associative, unital $\mathbb{Z}/2\mathbb{Z}$-graded ring which is commutative in the sense that $s_1 s_2 = (-1)^{|s_1||s_2|} s_2 s_1$ holds for all homogeneous $s_i \in S$. Here and in the following $|s|$ denotes the degree of a homogeneous element. Formulas involving the degree function are supposed to be valid for homogeneous elements – a condition that we will not mention explicitly in the following.

We first describe some facts on central extensions which are needed in the following. Proofs can be found in [N2]. A central extension of L is an epimorphism $f : K \to L$ of Lie superalgebras with the property that $\operatorname{Ker} f \subset \mathfrak{z}(K)$, the centre of K. A central extension $f : K \to L$ is called universal if for any other central extension $f' : K' \to L$ there exists a unique Lie superalgebra morphism $g : K \to K'$ such that $f = f' \circ g$. A universal central extension of L exists and is then unique up to a unique isomorphism if and only if L is perfect. To describe a model of a universal central extension of L one can use the following construction of a Lie superalgebra which is valid for any, not necessarily perfect Lie superalgebra L.

Let $\mathcal{B} = \mathcal{B}_L$ be the S-submodule of the S-supermodule $L \otimes_S L$ spanned by all elements of type

$$x \otimes y + (-1)^{|x||y|} y \otimes x, \quad x_{\bar 0} \otimes x_{\bar 0} \quad \text{for } x_{\bar 0} \in L_{\bar 0},$$
$$(-1)^{|x||z|} x \otimes [y, z] + (-1)^{|y||x|} y \otimes [z, x] + (-1)^{|z||y|} z \otimes [x, y],$$

and put

$$\mathfrak{uce}(L) = (L \otimes_S L)/\mathcal{B} \quad \text{and} \quad \langle x, y\rangle = x \otimes y + \mathcal{B} \in \mathfrak{uce}(L).$$

The supermodule $\mathfrak{uce}(L)$ becomes a Lie superalgebra over S with respect to the product

$$\big[\langle l_1, l_2\rangle, \langle l_3, l_4\rangle\big] = \big\langle [l_1, l_2], [l_3, l_4]\big\rangle$$

for $l_i \in L$. The map

$$\mathfrak{u} = \mathfrak{u}_L : \mathfrak{uce}(L) \to L: \quad \langle x, y\rangle \mapsto [x, y] \tag{1.1}$$

is a Lie superalgebra morphism with kernel $\operatorname{Ker}\mathfrak{u} \subset \mathfrak{z}(\mathfrak{uce}(L))$. If L is perfect, then $\mathfrak{u} : \mathfrak{uce}(L) \to L$ is a universal central extension of L. A morphism of Lie superalgebras $f : L \to M$ gives rise to a morphism of Lie superalgebras

$$\mathfrak{uce}(f) : \mathfrak{uce}(L) \to \mathfrak{uce}(M): \quad \langle l_1, l_2 \rangle \mapsto \big\langle f(l_1), f(l_2) \big\rangle.$$

The assignments $L \mapsto \mathfrak{uce}(L)$ and $f \mapsto \mathfrak{uce}(f)$ define a covariant endo-functor on the category $\mathbf{Lie}_S$ of Lie S-superalgebras.

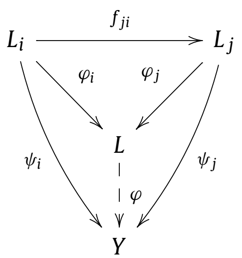
Similar to Lie algebras, a central extension of a Lie S-superalgebra L can be constructed by using a 2-cocycle $\tau : L \times L \to C$. Here C is an S-supermodule, τ is S-bilinear of degree 0 whence $\tau(L_\alpha, L_\beta) \subset C_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$, alternating in the sense that $\tau(x, y) + (-1)^{|x||y|}\tau(y, x) = 0 = \tau(x_{\bar 0}, x_{\bar 0})$ for $x_{\bar 0} \in L_{\bar 0}$, and satisfies

$$(-1)^{|x||z|}\tau\big(x, [y, z]\big) + (-1)^{|y||x|}\tau\big(y, [z, x]\big) + (-1)^{|z||y|}\tau\big(z, [x, y]\big) = 0.$$

Equivalently, a 2-cocycle is a map $\tau : L \times L \to C$ such that $L \oplus C$ is a Lie superalgebra with respect to the grading $(L \oplus C)_\alpha = L_\alpha \oplus C_\alpha$ and product $[l_1 \oplus c_1, l_2 \oplus c_2] = [l_1, l_2]_L \oplus \tau(x_1, x_2)$ where $[\,.\,,.\,]_L$ is the product of L. In this case, the canonical projection $L \oplus C \to L$ is a central extension.

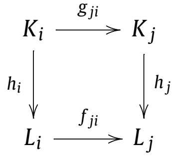
1.2. Review of direct limits

We recall some notions regarding direct limits. Let $(I, \leqslant)$ be a directed set, which will be fixed throughout this section. A *directed system* is a family $(L_i\colon i \in I)$ in $\mathbf{Lie}_S$ together with Lie superalgebra morphisms $f_{ji} : L_i \to L_j$ for every pair (i, j) with $i \leqslant j$ such that $f_{ii} = \mathrm{Id}_{L_i}$ and $f_{ki} = f_{kj} \circ f_{ji}$ for $i \leqslant j \leqslant k$. A *direct limit* of the directed system (L_i, f_{ji}) is a Lie superalgebra L together with Lie superalgebra morphisms $\varphi_i : L_i \to L$ satisfying $\varphi_i = \varphi_j \circ f_{ji}$, and for any other such pair (Y, ψ_i), i.e., $\psi_i = \psi_j \circ f_{ji}$ for $i \leqslant j$, there exists a unique morphism $\varphi : L \to Y$ such that the following diagram commutes.



The usual construction of a direct limit of modules shows that a direct limit of Lie superalgebras exists in $\mathbf{Lie}_S$ and is unique, up to a unique isomorphism. We can therefore speak of "the" direct limit, and follow the usual abuse of notation and denote a direct limit of (L_i, f_{ji}) by $\varinjlim L_i$. We will call φ_i the *canonical maps*.

Let (K_i, g_{ji}) and (L_i, f_{ji}) be two directed systems of Lie superalgebras, both indexed by the directed set I. A *morphism* from (K_i, g_{ji}) to (L_i, f_{ji}) is a family $(h_i\colon i \in I)$ of Lie superalgebra morphisms $h_i : K_i \to L_i$ such that for all pairs (i, j) with $i \leqslant j$ the diagram



commutes. A morphism from (K_i, g_{ji}) to (L_i, f_{ji}) gives rise to a unique Lie superalgebra morphism

$$h = \varinjlim h_i : \varinjlim K_i \to \varinjlim L_i$$

such that $h \circ \varphi_i = \psi_i \circ h_i$ for all $i \in I$, where $\varphi_i : K_i \to \varinjlim K_i$ and $\psi_i : L_i \to \varinjlim L_i$ are the canonical maps. Since direct limits preserve exact sequences [Bo, II, §6.2, Prop. 3], it follows that h is injective (resp. surjective) if all h_i are injective (resp. surjective).

1.3. Let (L_i, f_{ji}) be a directed system of Lie superalgebras in $\mathbf{Lie}_S$ and let $\varinjlim L_i$ be its direct limit with canonical maps $\varphi_i : L_i \to \varinjlim L_i$. Since $\mathfrak{uce}$ is a covariant functor, it is immediate that $(\mathfrak{uce}(L_i), \mathfrak{uce}(f_{ji}))$ is also a directed system of Lie superalgebras. We abbreviate $\mathfrak{uce}(f_{ji})$ by $\hat{f}_{ji}$, and let $\tilde{\varphi}_i : \mathfrak{uce}(L_i) \to \varinjlim \mathfrak{uce}(L_i)$ be the canonical maps into the direct limit of $(\mathfrak{uce}(L_i), \hat{f}_{ji})$.

$$\begin{array}{ccc} L_i & \xrightarrow{f_{ji}} & L_j \\ & {\scriptstyle\varphi_i}\searrow \quad \swarrow{\scriptstyle\varphi_j} & \\ & \varinjlim L_i & \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathfrak{uce}(L_i) & \xrightarrow{\hat{f}_{ji}} & \mathfrak{uce}(L_j) \\ & {\scriptstyle\tilde{\varphi}_i}\searrow \quad \swarrow{\scriptstyle\tilde{\varphi}_j} & \\ & \varinjlim \mathfrak{uce}(L_i) & \end{array} \tag{1.2}$$

Let $\mathfrak{u}_i : \mathfrak{uce}(L_i) \to L_i$ be the Lie superalgebra morphism of (1.1). By construction of the maps $\hat{f}_{ji}$, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{uce}(L_i) & \xrightarrow{\hat{f}_{ji}} & \mathfrak{uce}(L_j) \\ {\scriptstyle\mathfrak{u}_i}\downarrow & & \downarrow{\scriptstyle\mathfrak{u}_j} \\ L_i & \xrightarrow{f_{ji}} & L_j \end{array} \tag{1.3}$$

for $i \leqslant j$. In other words, the family $(\mathfrak{u}_i, i \in I)$ is a morphism from the directed system $(\mathfrak{uce}(L_i), \hat{f}_{ji})$ to the directed system (L_i, f_{ji}), and therefore gives rise to a morphism

$$\varinjlim \mathfrak{u}_i : \varinjlim \mathfrak{uce}(L_i) \to \varinjlim L_i. \tag{1.4}$$

Lemma 1.4. *In the setting of 1.3, the map (1.4) has central kernel, and is a central extension if all L_i are perfect.*

Proof. To prove that $\mathfrak{v} := \varinjlim \mathfrak{u}_i$ has central kernel, let $x \in \operatorname{Ker} \mathfrak{v}$. Thus $x = \tilde{\varphi}_j(x_j)$ for some $x_j \in \mathfrak{uce}(L_j)$ and $0 = \mathfrak{v}(x) = \varphi_j(\mathfrak{u}_j(x_j))$ in $L = \varinjlim L_i$. Hence there exists $k \geqslant j$ such that $f_{kj}(\mathfrak{u}_j(x_j)) = 0 \in L_k$. Note $\varphi_k(f_{kj}(\mathfrak{u}_j(x_j))) = 0 \in L$. For any $y \in L$, we have to show that $[x, y] = 0$ in L. We have $y = \tilde{\varphi}_p(y_p)$ for some $y_p \in \mathfrak{uce}(L_p)$. For the above $k, p \in I$ there exists $q \in I$ such that $q \geqslant k \geqslant j$ and $q \geqslant p$. Thus $f_{qj}(\mathfrak{u}_j(x_j)) = (f_{qk} \circ f_{kj})(\mathfrak{u}_j(x_j)) = 0 \in L_q$. The commutative diagram (1.3) for $j \leqslant q$ now implies $\hat{f}_{qj}(x_j) \in \operatorname{Ker} \mathfrak{u}_q \subset \mathfrak{z}(\mathfrak{uce}(L_q))$. So we have $[\hat{f}_{qj}(x_j), \hat{f}_{qp}(y_p)]_{\mathfrak{uce}(L_q)} = 0 \in \mathfrak{uce}(L_q)$ and hence

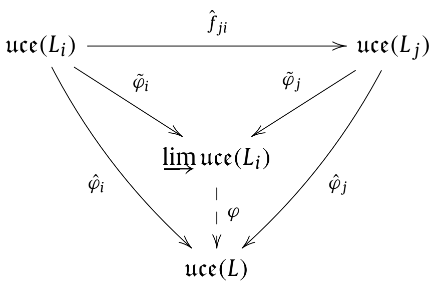
$$\begin{aligned} [x, y]_{\varinjlim \mathfrak{uce}(L_i)} &= \big[\tilde{\varphi}_j(x_j), \tilde{\varphi}_p(y_p)\big]_{\varinjlim \mathfrak{uce}(L_i)} \\ &= \tilde{\varphi}_q\big(\big[\hat{f}_{qj}(x_j), \hat{f}_{qp}(y_p)\big]_{\mathfrak{uce}(L_q)}\big) = 0. \end{aligned}$$

Thus $\operatorname{Ker}\mathfrak{v} \subset \mathfrak{z}(\varinjlim \mathfrak{uce}(L_i))$. If all L_i are perfect, every $\mathfrak{u}_i$ is surjective, and hence so is $\mathfrak{v}$, proving that $\mathfrak{v}$ is a central extension. □

1.5. We continue with the setting of 1.3, but assume that every L_i is perfect. Then $L = \varinjlim L_i$ is perfect too and therefore has a universal central extension $\mathfrak{u} : \mathfrak{uce}(L) \to L$. Our goal is to prove that the central extension (1.4) is a universal central extension of L. By the construction of L, the canonical maps $\varphi_i : L_i \to L$ are Lie superalgebra morphisms. We therefore get a unique Lie superalgebra morphism $\hat{\varphi}_i : \mathfrak{uce}(L_i) \to \mathfrak{uce}(L)$ such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{uce}(L_i) & \xrightarrow{\hat{\varphi}_i} & \mathfrak{uce}(L) \\ {\scriptstyle \mathfrak{u}_i}\downarrow & & \downarrow{\scriptstyle \mathfrak{u}} \\ L_i & \xrightarrow{\varphi_i} & L \end{array} \tag{1.5}$$

where $\mathfrak{u}_i$ and $\mathfrak{u}$ are universal central extensions of L_i and L respectively. Applying the covariant functor $\mathfrak{uce}$ to the left commutative diagram in (1.2) shows that $\hat{\varphi}_i = \mathfrak{uce}(\varphi_i) = \mathfrak{uce}(\varphi_j \circ f_{ji}) = \mathfrak{uce}(\varphi_j) \circ \mathfrak{uce}(f_{ji}) = \hat{\varphi}_j \circ \hat{f}_{ji}$. Thus the outer triangle in the diagram below commutes. Hence, by the universal property of $\varinjlim \mathfrak{uce}(L_i)$, there exists a unique Lie superalgebra morphism $\varphi : \varinjlim \mathfrak{uce}(L_i) \to \mathfrak{uce}(L)$ such that all triangles commute.



(1.6)

For the next theorem we define $\mathrm{H}_2(L)$ for a perfect Lie superalgebra L as the kernel of $\mathfrak{u} : \mathfrak{uce}(L) \to L$. In case L is a perfect Lie algebra over a ring S it is known that $\mathrm{H}_2(L)$ is the second homology group of L with trivial coefficients.

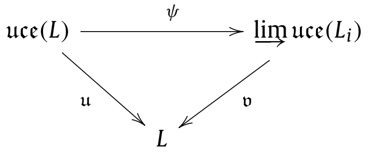
Theorem 1.6. *Assume that all Lie superalgebras L_i are perfect. Then the map*

$$\varphi : \varinjlim \mathfrak{uce}(L_i) \to \mathfrak{uce}(\varinjlim L_i)$$

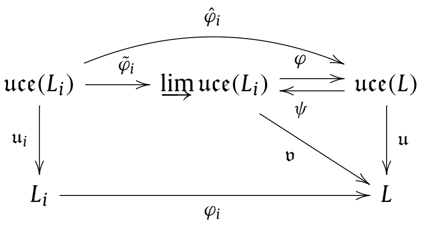
of (1.6) is an isomorphism of Lie superalgebras, and hence $\varinjlim \mathfrak{u}_i : \varinjlim \mathfrak{uce}(L_i) \to \varinjlim L_i$ is a universal central extension. In particular, $\varinjlim \mathfrak{u}_i$ induces an isomorphism

$$\varinjlim \mathrm{H}_2(L_i) \cong \mathrm{H}_2(\varinjlim L_i). \tag{1.7}$$

Proof. We have already noted that L is perfect and therefore has a universal central extension $\mathfrak{u} : \mathfrak{uce}(L) \to L$. By Lemma 1.4 we know that $\mathfrak{v} = \varinjlim \mathfrak{u}_i : \varinjlim \mathfrak{uce}(L_i) \to \varinjlim L_i$ is a central extension. Thus the universal property of $\mathfrak{uce}(L)$ implies that there exists a unique Lie superalgebra morphism $\psi : \mathfrak{uce}(L) \to \varinjlim \mathfrak{uce}(L_i)$ such that the following diagram commutes.



We claim that $\psi \circ \varphi = \mathrm{Id}_{\varinjlim \mathfrak{uce}(L_i)}$ and $\varphi \circ \psi = \mathrm{Id}_{\mathfrak{uce}(L)}$. For the proof of these two equations, the following diagram may be helpful.



By the universal property of $\varinjlim \mathfrak{uce}(L_i)$, in order to show $\psi \circ \varphi = \mathrm{Id}_{\varinjlim \mathfrak{uce}(L_i)}$, we only need to check $(\psi \circ \varphi) \circ \tilde{\varphi}_i = \tilde{\varphi}_i$. Since $\varphi \circ \tilde{\varphi}_i = \hat{\varphi}_i$, we are left to check $\psi \circ \hat{\varphi}_i = \tilde{\varphi}_i$ and this is true by the observation $\mathfrak{v} \circ \psi \circ \hat{\varphi}_i = \mathfrak{u} \circ \hat{\varphi}_i = \varphi_i \circ \mathfrak{u}_i = \mathfrak{v} \circ \tilde{\varphi}_i$ and the uniqueness in [N2, Prop. 1.13]. For the proof of $\varphi \circ \psi = \mathrm{Id}_{\mathfrak{uce}(L)}$ it is in view of the universal property of $\mathfrak{uce}(L)$ enough to verify $\mathfrak{u} \circ (\varphi \circ \psi) = \mathfrak{u}$. Since $\mathfrak{u} = \mathfrak{v} \circ \psi$, we are left to check $\mathfrak{u} \circ \varphi = \mathfrak{v}$ and this follows from $\mathfrak{u} \circ \varphi \circ \tilde{\varphi}_i = \mathfrak{u} \circ \hat{\varphi}_i = \varphi_i \circ \mathfrak{u}_i = \mathfrak{v} \circ \tilde{\varphi}_i$.

For the proof of (1.7) it suffices to note that $(\mathrm{Ker}\,\mathfrak{u}_i, \hat{f}_{ji}|_{\mathrm{Ker}\,\mathfrak{u}_i})$ is a directed system and that

$$0 \to \mathrm{Ker}\,\mathfrak{u}_i \to \mathfrak{uce}(L_i) \to L_i \to 0$$

is exact for every $i \in I$. The claim then follows from the fact that direct limits preserve exact sequences. □

Remark 1.7. Theorem 1.6 is proven in [S, App.] for the case $I = \mathbb{N}$ with the natural order and a directed system of Lie algebras over algebraically closed fields of characteristic zero $L_0 \xrightarrow{f_0} L_1 \xrightarrow{f_1} \cdots$ satisfying the condition that all f_i are monomorphisms with $f_i(\mathfrak{z}(L_i)) \subset \mathfrak{z}(L_{i+1})$.

Remark 1.8. We note that for Lie algebras the formula (1.7) is not new. Indeed, by [Wei, Cor. 7.3.6] the homology groups of a Lie algebra can be interpreted as torsion groups for the universal enveloping algebra and it is known that torsion commutes with direct limits, see e.g., [Wei, Cor. 2.6.17]. The proof presented here is more direct and works in the super setting as well.

For the next corollary we recall that a perfect Lie superalgebra is called *centrally closed* if $\mathfrak{u} : \mathfrak{uce}(L) \to L$ is an isomorphism.

Corollary 1.9. *If (L_i, f_{ji}) is a directed system of perfect and centrally closed Lie superalgebras, then $\varinjlim L_i$ is perfect and centrally closed.*

2. Examples: Universal central extensions of some infinite rank Lie superalgebras

In this section we will consider some examples of universal central extensions of direct limit Lie superalgebras, mainly those which are direct limits of some of the classical Lie superalgebras. In order to use the known results on their universal central extensions, we will in this section assume that all

Lie superalgebras are defined over a commutative, associative, unital ring k, rather than an arbitrary base superring as in Section 1.

Example 2.1 *(Special linear Lie superalgebra $\mathfrak{sl}(I;A)$ for A an associative superalgebra).* Let $I = I_{\bar 0} \cup I_{\bar 1}$ be a superset, i.e., a partitioned set. Let A be a unital associative, but not necessarily commutative k-superalgebra. We denote by $\mathrm{Mat}(I;A)$ the associative k-superalgebra whose underlying module consists of $|I| \times |I|$-finitary matrices with entries from A (only finitely many non-zero entries) and $\mathbb{Z}_2$-grading given by $|E_{ij}(a)| = |i| + |j| + |a|$. Here $E_{ij}(a) \in \mathrm{Mat}(I;A)$ has entry a at the position (ij) and 0 elsewhere. The product of $\mathrm{Mat}(I;A)$ is the usual matrix multiplication. Clearly $\mathrm{Mat}(I;A)$ only depends on the cardinality of I (and of course on A). For a finite I we put

$$\mathrm{Mat}(m,n;A) := \mathrm{Mat}(I;A) \quad \text{if } |I_{\bar 0}| = m \text{ and } |I_{\bar 1}| = n.$$

A matrix $x \in \mathrm{Mat}(m,n;A)$ written as

$$x = \begin{array}{cc} \left[\begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array}\right] & \begin{array}{c} m \\ n \end{array} \\ \begin{array}{cc} m & n \end{array} & \end{array} \tag{2.1}$$

is then even (resp. odd) if x_1 and x_4 are matrices with even (resp. odd) entries and x_2 and x_3 are matrices with odd (resp. even) entries.

We let $\mathfrak{gl}(I;A)$ be the Lie superalgebra associated to the associative superalgebra $\mathrm{Mat}(I;A)$. Its product is $[x,y] = xy - (-1)^{|x||y|}yx$. We assume $|I| \geqslant 3$ and then have that the Lie superalgebra

$$\mathfrak{sl}(I;A) := \big[\mathfrak{gl}(I;A), \mathfrak{gl}(I;A)\big] \quad \textit{is perfect.} \tag{2.2}$$

Indeed, the canonical matrix units $E_{ij}(a)$ for $i,j \in I$ and $a \in A$ satisfy the relations

$$a \mapsto E_{ij}(a) \quad \text{is } k\text{-linear;} \tag{2.3}$$

$$\big[E_{ij}(a), E_{pq}(b)\big] = \delta_{jp} E_{iq}(ab) - (-1)^{|E_{ij}(a)||E_{pq}(b)|} \delta_{iq} E_{pj}(ba). \tag{2.4}$$

In particular, relation (2.4) implies that for distinct $i,j,l \in I$ we have $E_{ij}(a) = [E_{il}(a), E_{lj}(1)] \in \mathfrak{sl}(I;A)^{(1)} = [\mathfrak{sl}(I;A), \mathfrak{sl}(I;A)]$. Moreover, there are two types of diagonal elements in $\mathfrak{sl}(I;A)$ – those which are products of off-diagonal elements and therefore lie in $\mathfrak{sl}(I;A)^{(1)}$ by what we just proved, and those which are commutators of diagonal elements, more precisely, elements of type $[E_{ii}(a), E_{ii}(b)] = E_{ii}([a,b])$ for $[a,b] = ab - (-1)^{|a||b|}ba$. To show that $\mathfrak{sl}(I;A)$ is perfect, it is therefore enough to prove that $E_{ii}([a,b]) \in \mathfrak{sl}(I;A)^{(1)}$. But the latter is a consequence of the following identity, in which again i, j, l are distinct elements in I:

$$\begin{aligned} E_{ii}\big([a,b]\big) = {} & \big[E_{ij}(a), E_{ji}(b)\big] + (-1)^{|E_{ij}(a)||E_{ji}(b)|}\big[E_{jl}(b), E_{lj}(a)\big] \\ & + (-1)^{|E_{ij}(a)||E_{ji}(b)| + |E_{jl}(b)||E_{lj}(a)|}\big[E_{li}(a), E_{il}(b)\big]. \end{aligned}$$

We also observe that the calculation above implies that

$$\mathfrak{sl}(I;A) \textit{ is generated by } E_{ij}(a),\ i \neq j \in I,\ a \in A.$$

Moreover, we also have

$$\textit{if } I_{\bar 1} = \emptyset \textit{ or } A_{\bar 1} = \{0\} \quad \textit{then } \mathfrak{sl}(I;A) = \big\{x \in \mathrm{Mat}(I;A)\colon \mathrm{str}(x) \in [A,A]\big\}. \tag{2.5}$$

Here the (super)trace str of a matrix $x = (x_{ij}) \in \mathrm{Mat}(I; A)$ is given by $\mathrm{str}(x) = \sum_{i\in I_{\bar 0}} x_{ii} - \sum_{i\in I_{\bar 1}} x_{ii}$, and $[A, A]$ is the span of all commutators $[a_1, a_2]$, $a_i \in A$.

For I as above we define the (linear) *Steinberg Lie superalgebra* $\mathfrak{st}(I; A)$ as the Lie k-superalgebra presented by generators $e_{ij}(a)$ with $i, j \in I, i \neq j$, $a \in A$ and relations (2.3)–(2.4) with $E_{ij}(a)$ replaced by $e_{ij}(a)$. We then have a canonical Lie superalgebra epimorphism

$$\mathfrak{v}_I : \mathfrak{st}(I; A) \to \mathfrak{sl}(I; A), \quad e_{ij}(a) \mapsto E_{ij}(a).$$

If $|I_{\bar 0}| = m$ and $|I_{\bar 1}| = n$, we put $\mathfrak{st}(m, n; A) = \mathfrak{st}(I; A)$, $\mathfrak{sl}(m, n; A) = \mathfrak{sl}(I; A)$ and $\mathfrak{v}_{mn} : \mathfrak{st}(m, n; A) \to \mathfrak{sl}(m, n; A)$ for v_I.

We also need the first cyclic homology group $\mathrm{HC}_1(A)$. To define it, we use

$$\langle\langle A, A\rangle\rangle = (A \otimes_k A)/\mathcal{H}$$

where $\mathcal{H}$ is the span of all elements of type

$$a\otimes b + (-1)^{|a||b|} b\otimes a, \quad a_{\bar 0}\otimes a_{\bar 0} \quad \text{for } a_{\bar 0} \in A_{\bar 0},$$
$$(-1)^{|a||c|} a\otimes bc + (-1)^{|b||a|} b\otimes ca + (-1)^{|c||b|} c\otimes ab$$

for $a, b, c \in A$. We abbreviate $\langle\langle a, b\rangle\rangle = a\otimes b + \mathcal{H}$. Observe that there is a well-defined commutator map

$$\mathfrak{c} : \langle\langle A, A\rangle\rangle \to A, \quad \langle\langle a, b\rangle\rangle \mapsto [a, b].$$

We put

$$\mathrm{HC}_1(A) = \mathrm{Ker}\,\mathfrak{c} = \left\{ \sum_i \langle\langle a_i, b_i\rangle\rangle : \sum_i [a_i, b_i] = 0 \right\}.$$

We will use the following assumption:

$$\mathfrak{v}_F \textit{ for } 5 \leqslant |F| < \infty \textit{ is a universal central extension with } \mathrm{Ker}\,\mathfrak{v}_F \cong \mathrm{HC}_1(A). \tag{2.6}$$

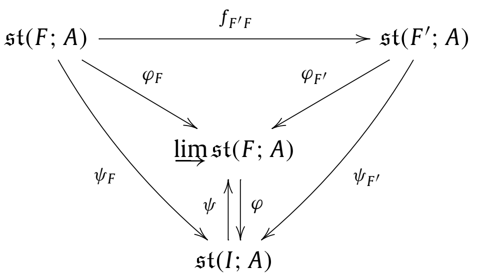
The assumption (2.6) is true in any one of the following situations:

(a) $n = 0$, A an algebra [KL] or a superalgebra [CG],
(b) A an algebra [MP1,IK1].

Refs. [CG] and [IK1] assume that k is a commutative ring containing $\frac{1}{2}$ and that the underlying module of A is free with a basis containing the identity element of A. We note that (2.6) is not true for $m + n \leqslant 4$, see the papers [CG,G,GS,SCG] which deal with the case $3 \leqslant m + n \leqslant 4$.

Proposition 2.2. *Assume* (2.6) *holds and I is a* (*possibly infinite*) *set with* $|I| \geqslant 5$. *Then* $\mathfrak{v}_I : \mathfrak{st}(I; A) \to \mathfrak{sl}(I; A)$ *is a universal central extension with kernel isomorphic to* $\mathrm{HC}_1(A)$.

Proof. This can be proven by adapting the proof of (2.6) to our setting. Instead we prefer to give a proof based on Theorem 1.6. This is possible since, denoting by $\mathcal{F}$ the set of finite subsets of I ordered by inclusion, the Lie superalgebra $\mathfrak{sl}(I; A)$ is indeed a direct limit: $\mathfrak{sl}(I; A) = \bigcup_{F\in\mathcal{F}} \mathfrak{sl}(F; A) \cong \varinjlim_{F\in\mathcal{F}} \mathfrak{sl}(F; A)$. Hence $\mathfrak{uce}(\mathfrak{sl}(I; A)) \cong \varinjlim_{F\in\mathcal{F}} \mathfrak{uce}(\mathfrak{sl}(F; A)) \cong \varinjlim_{F\in\mathcal{F}} \mathfrak{st}(F; A)$. Thus we need to show that $\varinjlim_{F\in\mathcal{F}} \mathfrak{st}(F; A) \cong \mathfrak{st}(I; A)$. This follows from the diagram below, where ψ_F is given by sending a generator $e_{ij}(a) \in \mathfrak{st}(F; A)$ to $e_{ij}(a) \in \mathfrak{st}(I; A)$. The existence of φ then follows from the definition of a direct limit, applied to $(\psi_F;\ F \in \mathcal{F})$. The families $(e_{ij}(a) \in \mathfrak{st}(F; A){:}\ F \in \mathcal{F})$ give rise to elements $\mathfrak{e}_{ij}(a) \in \varinjlim_{F\in\mathcal{F}} \mathfrak{st}(F; A)$ satisfying the relations (2.3)–(2.4), whence the existence of the map ψ sending $e_{ij}(a) \in \mathfrak{st}(I; A)$ to $\mathfrak{e}_{ij}(a)$.



It is immediate that φ and ψ are inverses of each other, and that $\operatorname{Ker}\mathfrak{v}_I \cong \mathrm{HC}_1(A)$. □

Example 2.3 *($\mathfrak{sl}(I; A)$ for A an associative commutative superalgebra).* Let A be a unital associative and commutative k-superalgebra, thus $[A, A] = 0$. Therefore the descriptions of $\mathfrak{sl}(I; A)$ and $\mathrm{HC}_1(A)$ simplify to

$$\mathfrak{sl}(I; A) = \{x \in \mathfrak{gl}(I; A){:}\ \operatorname{str}(x) = 0\} \cong \mathfrak{sl}(I; k) \otimes_k A,$$
$$\mathrm{HC}_1(A) = \langle\langle A, A\rangle\rangle.$$

Moreover, the universal central extension $\mathfrak{st}(I; A)$ can be described via a 2-cocycle as follows. The Lie superalgebra $\mathfrak{sl}(m, n; A)$ has a central 2-cocycle τ_{mn} with values in $\mathrm{HC}_1(A)$:

$$\tau_{mn}(x, y) = \sum_{1\leqslant i\leqslant m,\, 1\leqslant j\leqslant m+n} \langle\langle x_{ij}, y_{ji}\rangle\rangle - \sum_{m+1\leqslant i\leqslant m+n,\, 1\leqslant j\leqslant m+n} \langle\langle x_{ij}, y_{ji}\rangle\rangle$$

for $x = (x_{ij}),\ y = (y_{ij}) \in \mathfrak{sl}(m, n; A)$. We let $\mathfrak{sl}(m, n, A) \oplus \mathrm{HC}_1(A)$ be the corresponding Lie superalgebra, and view it as a central extension of $\mathfrak{sl}(m, n; A)$ by projecting onto the first factor. From now on we suppose $m + n \geqslant 5$ and that the map

$$h_{mn} : \mathfrak{uce}\big(\mathfrak{sl}(m, n; A)\big) \to \mathfrak{sl}(m, n; A) \oplus \mathrm{HC}_1(A), \quad h_{mn}\langle x, y\rangle = [x, y] \oplus \tau_{mn}(x, y)$$

is an isomorphism of central extensions:

$$\mathfrak{uce}\big(\mathfrak{sl}(m, n; A)\big) \cong \mathfrak{sl}(m, n; A) \oplus \mathrm{HC}_1(A) \quad \textit{as central extensions.} \tag{2.7}$$

The assumption (2.7) is true in any one of the following situations:

(a) $n = 0$, A an algebra [KL] or a superalgebra [CG],
(b) A an algebra [MP1,IK1].

Let now $(\mathfrak{sl}(m_i, n_i, A), f_{ji})$ be a directed system of Lie superalgebras with $m_i + n_i \geqslant 5$. The transition maps $f_{ji} : \mathfrak{sl}(m_i, n_i, A) \to \mathfrak{sl}(m_j, n_j, A)$ lift uniquely to Lie superalgebra morphisms $\hat{f}_{ji} : \mathfrak{uce}(\mathfrak{sl}(m_i, n_i, A)) \to \mathfrak{uce}(\mathfrak{sl}(m_j, n_j, A))$. Hence, we get a directed system $(\mathfrak{sl}(m_i, n_i, A) \oplus \mathrm{HC}_1(A))_{i\in I}$

with transition maps $h_{m_j n_j} \circ \hat{f}_{ji} \circ h_{m_i n_i}^{-1} = f_{ji} \oplus g_{ji}$ where the map $g_{ji} : \mathrm{HC}_1(A) \to \mathrm{HC}_1(A)$ is given by $g_{ji}(\tau_{m_i n_i}(x, y)) = \tau_{m_j n_j}(f_{ji}(x), f_{ji}(y))$ for $x, y \in \mathfrak{sl}(m_i, n_i, A)$. We now define a central 2-cocycle τ_I for the direct limit Lie superalgebra

$$\mathfrak{sl}(I; A) = \varinjlim \mathfrak{sl}(m_i, n_i, A)$$

with values in $\mathrm{HC}_1(A)$. Let $x, y \in \mathfrak{sl}(I; A)$. Thus $x = \varphi_p(x_p)$ and $y = \varphi_q(y_q)$ for some $p, q \in I$, $x_p \in \mathfrak{sl}(m_p, n_p, A)$ and $y_q \in \mathfrak{sl}(m_q, n_q, A)$. Here φ_p, φ_q are the canonical maps for $\mathfrak{sl}(I; A)$. There exists $k \in I$ such that $k \geqslant p$, $k \geqslant q$ and $f_{kp}(x_p), f_{kq}(y_q) \in \mathfrak{sl}(m_k, n_k, A)$. Then the cocycle τ_I for $\mathfrak{sl}(I; A)$ is given by

$$\tau_I(x, y) = \tau_{m_k n_k}\big(f_{kp}(x_p), f_{kq}(y_q)\big). \tag{2.8}$$

We now get from Proposition 2.2 that $\mathfrak{st}(I; A) \cong \mathfrak{uce}(\mathfrak{sl}(I; A)) \cong \mathfrak{sl}(I; A) \oplus \mathrm{HC}_1(A)$, where the 2-cocycle τ_I is given explicitly by (2.8). Summarizing the above, we have proven the following.

Corollary 2.4. *Let A be a unital associative commutative superalgebra over a commutative ring k. Let $(I, \leqslant)$ be an arbitrary directed set and let $(\mathfrak{sl}(m_i, n_i, A), f_{ji})$ be a directed system of Lie superalgebras with $m_i + n_i \geqslant 5$. We suppose (2.7) and denote by $\mathfrak{sl}(I; A) := \varinjlim \mathfrak{sl}(m_i, n_i, A)$ the corresponding direct limit, which is a perfect Lie superalgebra of possibly infinite rank. Then*

$$\mathfrak{uce}\big(\mathfrak{sl}(I; A)\big) \cong \mathfrak{sl}(I; A) \oplus \mathrm{HC}_1(A) \tag{2.9}$$

as central extensions, where the Lie superalgebra structure on the right is given by the 2-cocycle τ_I of (2.8).

Example 2.5 *($\mathfrak{sl}_J(A)$ for A an associative algebra).* Let A be an associative unital k-algebra over a commutative ring k containing $\frac{1}{2}$, and let J be an arbitrary, possible infinite set with $|J| \geqslant 5$. We denote by $\mathfrak{sl}_J(A)$ the Lie algebra of finitary matrices over A (only finitely many non-zero entries) and with trace in $[A, A]$. Since $\mathfrak{sl}_J(A)$ is the direct limit of the Lie algebras $\mathfrak{sl}_F(A)$ where F runs through the finite subsets of J, Corollary 2.4 implies that $\mathfrak{uce}(\mathfrak{sl}_J(A)) \cong \mathfrak{sl}_J(A) \oplus \mathrm{HC}_1(A)$. This is proven in [KL] for J finite or countable and in [Wel] for arbitrary J, using the theory of root graded Lie algebras.

Example 2.6 *(Ortho-symplectic Lie superalgebra $\mathfrak{osp}(I; A)$ for A an associative commutative superalgebra).* The ortho-symplectic Lie superalgebra $\mathfrak{osp}(m, n; A)$ can be defined in the usual way, see for example [IK1,IK2,MP2]. Since $\mathfrak{osp}(m, n; A)$ is a subalgebra of $\mathfrak{sl}(m, n; A)$, the restriction of the 2-cocycle τ_{mn} of Example 2.3 defines a 2-cocycle of $\mathfrak{osp}(m, n; A)$ with values in $\mathrm{HC}_1(A)$ and thus gives rise to a central extension. We suppose that the map $\mathfrak{uce}(\mathfrak{osp}(m, n; A)) \to \mathfrak{osp}(m, n; A) \oplus \mathrm{HC}_1(A)$, given by $\langle x, y\rangle \mapsto [x, y] \oplus \tau_{mn}(x, y)$ is an isomorphism:

$$\mathfrak{uce}\big(\mathfrak{osp}(m, n; A)\big) \cong \mathfrak{osp}(m, n; A) \oplus \mathrm{HC}_1(A) \quad \textit{as central extensions.} \tag{2.10}$$

Our assumption (2.10) is fulfilled in any one of the following situations:

(a) k a field of characteristic 0 [IK2],
(b) A a commutative algebra [IK1,MP2].

Ref. [MP2] assumes that $m \geqslant 5$, $n \geqslant 10$.

Let now $(\mathfrak{osp}(m_i, n_i; A), f_{ji})$ be a directed system of Lie superalgebras. One shows as in Example 2.3 that there exists a well-defined 2-cocycle τ_I' for the direct limit Lie superalgebra

$$\mathfrak{osp}(I; A) := \varinjlim \mathfrak{osp}(m_i, n_i; A)$$

with values in $\mathrm{HC}_1(A)$. From Theorem 1.6 we then get as in Corollary 2.4

$$\begin{aligned}\mathfrak{uce}\big(\mathfrak{osp}(I;\,A)\big) &\cong \varinjlim \mathfrak{uce}\big(\mathfrak{osp}(m_i, n_i;\,A)\big) \cong \varinjlim\big(\mathfrak{osp}(m_i, n_i;\,A) \oplus \mathrm{HC}_1(A)\big)\\ &\cong \mathfrak{osp}(I;\,A) \oplus \mathrm{HC}_1(A).\end{aligned} \tag{2.11}$$

Example 2.7 *(Locally finite Lie superalgebras).* Classically semisimple locally finite Lie superalgebras over algebraically closed fields of characteristic 0 were introduced and studied in [P], including a classification of the simple infinite dimensional ones which admit a local system of root injections of classical finite dimensional Lie superalgebras. They are all direct limits $L = \varinjlim_i L_i$ of classical simple Lie superalgebras L_i, $i \in \mathbb{N}$ with $f_i : L_i \to L_{i+1}$ being the natural inclusions. Referring the reader to [P] for details, we simple present the classification list. We abbreviate $\mathfrak{sl}(m,n) = \mathfrak{sl}(m,n;k)$ and $\mathfrak{osp}(m,n) = \mathfrak{osp}(m,n;k)$ in the notation of 2.3 and 2.6 respectively. The Lie superalgebra $\mathrm{SP}(m)$ is the subalgebra of $\mathfrak{sl}(m,m)$ which leaves invariant an odd nondegenerate super-antisymmetric bilinear form, and $\mathfrak{sq}(m)$ is the subalgebra of $\mathfrak{sl}(m,m)$ consisting of matrices of the form (2.1) with $x_1 = x_4$, $x_2 = x_3$ and $\mathrm{tr}(x_2) = 0$. With these notations, an infinite dimensional simple Lie superalgebra, which admits a local system of root injections of classical finite dimensional Lie superalgebras L_i, is isomorphic to a Lie superalgebra L in the following table ($i \geqslant 2$, $n \geqslant 0$, $k \geqslant 0$, $r \geqslant 0$, $m \geqslant 2$):

L	L_i	L	L_i
$\mathfrak{sl}(\infty, n)$	$\mathfrak{sl}(i, n)$	$\mathfrak{sl}(\infty, \infty)$	$\mathfrak{sl}(i, i)$
$\mathrm{B}(\infty, 2k)$	$\mathfrak{osp}(2i+1, 2k)$	$\mathrm{B}(\infty, \infty)$	$\mathfrak{osp}(2i+1, 2i)$
$\mathrm{B}(2r+1, \infty)$	$\mathfrak{osp}(2r+1, 2i)$	$\mathrm{C}(\infty)$	$\mathfrak{osp}(2, 2i)$
$\mathrm{D}(\infty, 2k)$	$\mathfrak{osp}(2i, 2k)$	$\mathrm{D}(\infty, \infty)$	$\mathfrak{osp}(2i, 2i)$
$\mathrm{D}(2m, \infty)$	$\mathfrak{osp}(2m, 2i)$	$\mathrm{SP}(\infty)$	$\mathrm{SP}(i)$
$\mathfrak{sq}(\infty)$	$\mathfrak{sq}(i)$		

(2.12)

Corollary 2.8. *The Lie superalgebras L listed in table* (2.12) *are all centrally closed.*

Proof. The Lie superalgebras L_i in table (2.12) are all perfect. In view of Theorem 1.6 it therefore remains to show that they are centrally closed for large i. For L_i of type $\mathfrak{sl}(m,n)$ or $\mathfrak{osp}(m,n)$ this follows from (2.7) and (2.10) since $\mathrm{HC}_1(k) = \{0\}$. For the remaining two types this follows from [IK2, Th. 5.10]. □

Example 2.9 *(Locally finite Lie algebras).* Corollary 2.8 applies in particular to the simple locally finite Lie algebras $\mathfrak{sl}(\infty) = \mathfrak{sl}(\infty, 0)$, $\mathfrak{o}(\infty) = \mathrm{B}(\infty, 0) = \mathrm{D}(\infty, 0)$, $\mathfrak{sp}(\infty) = \mathrm{D}(0, \infty)$ (the only infinite dimensional simple root reductive Lie algebras), studied for example in [BB,DP,PS].

Example 2.10 *(Root-graded Lie algebras).* (a) Let k be a ring in which 2 and 3 are invertible, and let L be a Lie algebra graded by a locally finite reduced root system R as defined in [N1], see also [N3, 5.1]. Thus, L is graded by $\mathcal{Q}(R) = \mathrm{Span}_{\mathbb{Z}}(R)$ with $\mathrm{supp}_{\mathcal{Q}(R)} L = R$, i.e.,

$$L = \bigoplus_{\alpha \in R} L_\alpha, \quad [L_\alpha, L_\beta] \subset L_{\alpha+\beta},$$

satisfies $L_0 = \sum_{0 \neq \alpha \in R} [L_\alpha, L_{-\alpha}]$ and has the property that for $\alpha \neq 0$ there exists an $\mathfrak{sl}_2$-triple $(e_\alpha, h_\alpha, f_\alpha) \in L_\alpha \times L_0 \times L_{-\alpha}$ such that $[h_\alpha, x_\beta] = \langle \beta, \alpha^\vee \rangle x_\beta$ holds for every $\beta \in R$ and $x_\beta \in L_\beta$. By [LN, 3.15], R is a direct limit of finite root systems, say $R = \varinjlim R_i$ where i runs through a directed set $(I, \leqslant)$. The subalgebra

$$L_i = \left(\bigoplus_{0 \neq \alpha \in R_i} L_\alpha \right) \oplus \sum_{0 \neq \alpha \in R_i} [L_\alpha, L_{-\alpha}] \tag{2.13}$$

is graded by the root system R_i, and it is immediate that $L = \varinjlim L_i$.

Any root-graded Lie algebra is perfect, whence $\mathfrak{uce}(L) \cong \varinjlim \mathfrak{uce}(L_i)$ by Theorem 1.6. In fact, something more precise is true. One knows that the universal central extension of a root-graded Lie algebra is again graded by the same locally finite root system, [N3, Prop. 5.4]. Thus the root-graded Lie algebra $\mathfrak{uce}(L)$ is a direct limit of root-graded Lie algebras,

$$\mathfrak{uce}(L) \cong \varinjlim \mathfrak{uce}(L)_i,$$

where $\mathfrak{uce}(L)_i$ is defined in the same way as L_i.

(b) Suppose in the following that k is a field of characteristic zero. If K is a centreless Lie algebra graded by a finite irreducible reduced root system, the group $\mathrm{H}_2(K)$ is known to be the full skew-dihedral homology group $\mathrm{HF}(\mathfrak{a})$, where $\mathfrak{a}$ is the coordinate algebra of K, [ABG, Th. 4.13].

Let now L be a centreless Lie algebra graded by a locally finite irreducible reduced root system R of rank $\geqslant 9$. Then $R = \varinjlim R_i$ where the R_i are finite, irreducible, reduced, have rank $\geqslant 9$ and are of the same type as R, [LN, 8.3]. It is moreover no harm to assume that $R_0 \subseteq R_i$ for some fixed $0 \in I$. It then follows that the root-graded Lie algebras L_i of (2.13) all have the same coordinate algebra $\mathfrak{a}$ (this has also been noted by M. Yousofzadeh in [Y]). Notice that if $f_{ji}(\mathfrak{z}(L_i)) \subset \mathfrak{z}(L_j)$, then $\varinjlim L_i \cong \varinjlim L_i/\mathfrak{z}(L_i)$. Hence

$$\begin{aligned} \mathfrak{uce}(L) &\cong \mathfrak{uce}\big(\varinjlim L_i/\mathfrak{z}(L_i)\big) \cong \varinjlim \mathfrak{uce}\big(L_i/\mathfrak{z}(L_i)\big) \\ &\cong \varinjlim L_i/\mathfrak{z}(L_i) \oplus \mathrm{HF}(\mathfrak{a}) \cong L \oplus \mathrm{HF}(\mathfrak{a}). \end{aligned} \tag{2.14}$$

Special cases of root-graded Lie algebras are the so-called Lie tori, which occur as cores and centreless cores of locally extended affine Lie algebras [MY,Nee]. More generally, the cores of affine reflection Lie algebras are root-graded Lie algebras of possibly infinite rank [N3, 6.4, 6.5].

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