Universal central extensions of direct limits of Lie superalgebras

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\textbf{A B S T R A C T}

We show that the universal central extension of a direct limit of perfect Lie superalgebras $L_i$ is (isomorphic to) the direct limit of the universal central extensions of $L_i$. As an application we describe the universal central extensions of some infinite rank Lie superalgebras.

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\textbf{Introduction}

Central extensions appear naturally in the theory of infinite dimensional Lie algebras. For example, they are fundamental for the theory of affine Kac–Moody Lie algebras and extended affine Lie algebras. Centrally extended Lie algebras often have a more interesting representation theory than the original Lie algebra, which makes central extension an interesting topic for applications, e.g., in physics. A convenient way to find “all” of them, is to determine the universal central extension of a given Lie algebra, which exists for perfect Lie algebras (well-known) and superalgebras \cite{N2}.

Direct limit of Lie superalgebras is an important way to construct infinite dimensional Lie superalgebras. Examples include various types of locally finite Lie (super)algebras \cite{BB,DP,PS}, locally extended affine Lie algebras \cite{MY,Nee,N3} and Lie superalgebras graded by locally finite root systems \cite{N1,GN}. These types of Lie algebras and Lie superalgebras have been intensively studied by many

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authors, many more than we have quoted, yet no general results seem to be known about their universal central extensions, besides the paper [S] in which the author studied a rather special case, described in Remark 1.7.

In this paper, we consider the universal central extensions of general direct limits of Lie superalgebras over an arbitrary base superring. We show in Theorem 1.6 that the universal central extension of a direct limit \( \lim_{\to} L_i \) of perfect Lie superalgebras \( L_i \) is canonically isomorphic to the direct limit of the universal central extensions of \( L_i \). This result is new even for the case of Lie algebras. Crucial for its proof is the fact [N2] that one has an endo-functor \( \text{uce} \) on the category of all Lie superalgebras which gives the universal central extension for perfect Lie superalgebras.

As an application, we describe in Section 2 the universal central extensions of some direct limit Lie superalgebras, namely \( \mathfrak{sl}(I; A) \) for \( |I| \geq 5 \) and \( A \) an associative superalgebra (Proposition 2.2, Corollary 2.4), \( \mathfrak{osp}(I; A) \) for \( A \) a commutative associative (Example 2.6), locally finite Lie superalgebras (Example 2.7) and Lie algebras graded by locally finite root systems (Example 2.10). These applications are possible since one knows the universal central extension of the Lie superalgebras over which we take the direct limit.

1. Universal central extensions of direct limits of Lie superalgebras: General results

1.1. Review of universal central extensions of Lie superalgebras

Throughout this section we consider Lie superalgebras \( L \) over a commutative superring \( S \) as defined in [N2]. Thus \( S \) is an associative, unital \( \mathbb{Z}/2\mathbb{Z} \)-graded ring which is commutative in the sense that \( s_1s_2 = (-1)^{|s_1||s_2|} s_2s_1 \) holds for all homogeneous \( s_1 \in S \). Here and in the following \( |s| \) denotes the degree of a homogeneous element. Formulas involving the degree function are supposed to be valid for homogeneous elements – a condition that we will not mention explicitly in the following.

We first describe some facts on central extensions which are needed in the following. Proofs can be found in [N2]. A central extension of \( L \) is an epimorphism \( f : K \to L \) of Lie superalgebras with the property that \( \text{Ker} f \subset Z(K) \), the centre of \( K \). A central extension \( f : K \to L \) is called universal if for any other central extension \( f' : K' \to L \) there exists a unique Lie superalgebra morphism \( g : K \to K' \) such that \( f = f' \circ g \). A universal central extension of \( L \) exists and is then unique up to a unique isomorphism if and only if \( L \) is perfect. To describe a model of a universal central extension of \( L \) one can use the following construction of a Lie superalgebra which is valid for any, not necessarily perfect Lie superalgebra \( L \).

Let \( B = B_l \) be the \( S \)-submodule of the \( S \)-supermodule \( L \otimes S L \) spanned by all elements of type

\[
\begin{align*}
  x \otimes y + (-1)^{|x||y|} y \otimes x, \\
  x_0 \otimes x_0 & \quad \text{for } x_0 \in L_0, \\
  (-1)^{|x||z|} x \otimes [y, z] + (-1)^{|y||x|} y \otimes [z, x] + (-1)^{|z||y|} z \otimes [x, y].
\end{align*}
\]

and put

\[
\text{uce}(L) = (L \otimes_S L)/B \quad \text{and} \quad \langle x, y \rangle = x \otimes y + B \in \text{uce}(L).
\]

The supermodule \( \text{uce}(L) \) becomes a Lie superalgebra over \( S \) with respect to the product

\[
\left[ (l_1, l_2), (l_3, l_4) \right] = \langle [l_1, l_2], [l_3, l_4] \rangle
\]

for \( l_i \in L \). The map

\[
u = u_L : \text{uce}(L) \to L : \langle x, y \rangle \mapsto [x, y]
\]

\hspace{1cm} (1.1)
is a Lie superalgebra morphism with kernel $\text{Ker} \ u \subset z(\text{uce}(L))$. If $L$ is perfect, then $u : \text{uce}(L) \to L$ is a universal central extension of $L$. A morphism of Lie superalgebras $f : L \to M$ gives rise to a morphism of Lie superalgebras

$$\text{uce}(f) : \text{uce}(L) \to \text{uce}(M) : \langle l_1, l_2 \rangle \mapsto \langle f(l_1), f(l_2) \rangle.$$ 

The assignments $L \mapsto \text{uce}(L)$ and $f \mapsto \text{uce}(f)$ define a covariant endo-functor on the category $\text{Lie}_S$ of Lie $S$-superalgebras.

Similar to Lie algebras, a central extension of a Lie $S$-superalgebra $L$ can be constructed by using a 2-cocycle $\tau : L \times L \to C$. Here $C$ is an $S$-supermodule, $\tau$ is $S$-bilinear of degree 0 whence $\tau(L_\alpha, L_\beta) \subset C_{\alpha + \beta}$ for $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$, alternating in the sense that $\tau(x, y) + (-1)^{|x||y|}\tau(y, x) = 0 = \tau(x_0, x_0)$ for $x_0 \in L_0$, and satisfies

$$(-1)^{|x||z|}\tau(x, [y, z]) + (-1)^{|y||x|}\tau(y, [x, z]) + (-1)^{|z||y|}\tau(z, [x, y]) = 0.$$

Equivalently, a 2-cocycle is a map $\tau : L \times L \to C$ such that $L \oplus C$ is a Lie superalgebra with respect to the grading $(L \oplus C)_\alpha = L_\alpha \oplus C_\alpha$ and product $[l_1 \oplus c_1, l_2 \oplus c_2] = [l_1, l_2]_L \oplus \tau(x_1, x_2)$ where $[\ldots]_L$ is the product of $L$. In this case, the canonical projection $L \oplus C \to L$ is a central extension.

1.2. Review of direct limits

We recall some notions regarding direct limits. Let $(I, \leq)$ be a directed set, which will be fixed throughout this section. A directed system is a family $(L_i : i \in I)$ in $\text{Lie}_S$ together with Lie superalgebra morphisms $f_{ji} : L_i \to L_j$ for every pair $(i, j)$ with $i \leq j$ such that $f_{ii} = \text{Id}_{L_i}$ and $f_{ki} = f_{kj} \circ f_{ji}$ for $i \leq j \leq k$. A direct limit of the directed system $(L_i, f_{ji})$ is a Lie superalgebra $L$ together with Lie superalgebra morphisms $\psi_i : L_i \to L$ satisfying $\psi_i = \psi_j \circ f_{ji}$, and for any other such pair $(Y, \psi_i)$, i.e., $\psi_i = \psi_j \circ f_{ji}$ for $i \leq j$, there exists a unique morphism $\psi : L \to Y$ such that the following diagram commutes.

![Diagram](image)

The usual construction of a direct limit of modules shows that a direct limit of Lie superalgebras exists in $\text{Lie}_S$ and is unique, up to a unique isomorphism. We can therefore speak of “the” direct limit, and follow the usual abuse of notation and denote a direct limit of $(L_i, f_{ji})$ by $\lim L_i$. We will call $\psi_i$ the canonical maps.

Let $(K_i, g_{ji})$ and $(L_i, f_{ji})$ be two directed systems of Lie superalgebras, both indexed by the directed set $I$. A morphism from $(K_i, g_{ji})$ to $(L_i, f_{ji})$ is a family $(h_i : i \in I)$ of Lie superalgebra morphisms $h_i : K_i \to L_i$ such that for all pairs $(i, j)$ with $i \leq j$ the diagram

![Diagram](image)
Thus we have a commutative diagram

\[
\begin{array}{ccc}
L_i & \xrightarrow{f_{ji}} & L_j \\
\varphi_i & \downarrow & \varphi_j \\
\lim L_i & \cong & \lim uce(L_i)
\end{array}
\quad
\begin{array}{ccc}
uce(L_i) & \xrightarrow{\hat{f}_{ji}} & uce(L_j) \\
\varphi_i & \downarrow & \varphi_j \\
\lim uce(L_i) & \cong & \lim uce(L_i)
\end{array}
\]

(1.2)

Let \( u_i : uce(L_i) \rightarrow L_i \) be the Lie superalgebra morphism of (1.1). By construction of the maps \( \hat{f}_{ji} \), we have a commutative diagram

\[
\begin{array}{ccc}
uce(L_i) & \xrightarrow{\hat{f}_{ji}} & uce(L_j) \\
\downarrow u_i & & \downarrow u_j \\
L_i & \xrightarrow{f_{ji}} & L_j
\end{array}
\]

(1.3)

for \( i \leq j \). In other words, the family \( (u_i, i \in I) \) is a morphism from the directed system \( (uce(L_i), \hat{f}_{ji}) \) to the directed system \( (L_i, f_{ji}) \), and therefore gives rise to a morphism

\[
\lim u_i : \lim uce(L_i) \rightarrow \lim L_i.
\]

(1.4)

**Lemma 1.4.** In the setting of 1.3, the map (1.4) has central kernel, and is a central extension if all \( L_i \) are perfect.

**Proof.** To prove that \( v := \lim u_i \) has central kernel, let \( x \in \text{Ker} v \). Thus \( x = \varphi_j(x_j) \) for some \( x_j \in uce(L_j) \) and \( 0 = v(x) = \varphi_j(u_j(x_j)) \) in \( L = \lim L_i \). Hence there exists \( k \geq j \) such that \( f_{kj}(u_j(x_j)) = 0 \in L_k \). Note \( \varphi_k(f_{kj}(u_j(x_j))) = 0 \in L \). For any \( y \in L \), we have to show that \( [x, y] = 0 \) in \( L \). We have \( y = \varphi_p(y_p) \) for some \( y_p \in uce(L_p) \). For the above \( k, p \in I \) there exists \( q \in I \) such that \( q \geq k \geq j \) and \( q \geq p \). Thus \( f_{qj}(u_j(x_j)) = (f_{kj} \circ f_{kj})(u_j(x_j)) = 0 \in L_q \). The commutative diagram (1.3) for \( j \leq q \) now implies \( \hat{f}_{qj}(x_j) \in \text{Ker} u_q \subset \text{Ker}(uce(L_q)) \). So we have \( [\hat{f}_{qj}(x_j), \hat{f}_{qp}(y_p)]_{uce(L_q)} = 0 \in uce(L_q) \) and hence

\[
[x, y]_{\lim uce(L_i)} = [\varphi_j(x_j), \varphi_p(y_p)]_{\lim uce(L_i)} = \varphi_q([\hat{f}_{qj}(x_j), \hat{f}_{qp}(y_p)]_{uce(L_q)}) = 0.
\]
Thus $\ker v \subset \lim \ucce(L_i)$. If all $L_i$ are perfect, every $u_i$ is surjective, and hence so is $v$, proving that $v$ is a central extension. □

1.5. We continue with the setting of 1.3, but assume that every $L_i$ is perfect. Then $L = \lim L_i$ is perfect too and therefore has a universal central extension $u : \ucce(L) \to L$. Our goal is to prove that the central extension (1.4) is a universal central extension of $L$. By the construction of $L$, the canonical maps $\phi_i : L_i \to L$ are Lie superalgebra morphisms. We therefore get a unique Lie superalgebra morphism $\hat{\phi}_i : \ucce(L_i) \to \ucce(L)$ such that the following diagram commutes

$$
\begin{array}{ccc}
\ucce(L_i) & \xrightarrow{\hat{\phi}_i} & \ucce(L) \\
\downarrow u_i & & \downarrow u \\
L_i & \xrightarrow{\phi_i} & L
\end{array}
$$

(1.5)

where $u_i$ and $u$ are universal central extensions of $L_i$ and $L$ respectively. Applying the covariant functor $\ucce$ to the left commutative diagram in (1.2) shows that $\hat{\phi}_i = \ucce(\phi_i) = \ucce(\phi_j \circ f_{ji}) = \ucce(\phi_j) \circ \ucce(f_{ji}) = \hat{\phi}_j \circ \hat{f}_{ji}$. Thus the outer triangle in the diagram below commutes. Hence, by the universal property of $\lim \ucce(L_i)$, there exists a unique Lie superalgebra morphism $\psi : \lim \ucce(L_i) \to \ucce(L)$ such that all triangles commute.

$$
\begin{array}{ccc}
\ucce(L_i) & \xrightarrow{\hat{\phi}_i} & \ucce(L_j) \\
\downarrow \hat{f}_{ji} & & \downarrow \psi \\
\lim \ucce(L_i) & \xrightarrow{\psi} & \ucce(L_j) \\
\downarrow \phi_i & & \downarrow \hat{\phi}_j \\
\ucce(L) & \xrightarrow{\psi} & \ucce(L)
\end{array}
$$

(1.6)

For the next theorem we define $H_2(L)$ for a perfect Lie superalgebra $L$ as the kernel of $u : \ucce(L) \to L$. In case $L$ is a perfect Lie algebra over a ring $S$ it is known that $H_2(L)$ is the second homology group of $L$ with trivial coefficients.

**Theorem 1.6.** Assume that all Lie superalgebras $L_i$ are perfect. Then the map

$$
\varphi : \lim \ucce(L_i) \to \ucce(\lim L_i)
$$

of (1.6) is an isomorphism of Lie superalgebras, and hence $\lim u_i : \lim \ucce(L_i) \to \lim L_i$ is a universal central extension. In particular, $\lim u_i$ induces an isomorphism

$$
\lim H_2(L_i) \cong H_2(\lim L_i).
$$

(1.7)

**Proof.** We have already noted that $L$ is perfect and therefore has a universal central extension $u : \ucce(L) \to L$. By Lemma 1.4 we know that $v = \lim u_i : \lim \ucce(L_i) \to \lim L_i$ is a central extension. Thus the universal property of $\ucce(L)$ implies that there exists a unique Lie superalgebra morphism $\psi : \ucce(L) \to \lim \ucce(L_i)$ such that the following diagram commutes.
We claim that $\psi \circ \phi = \text{Id}_{\lim uce(L_i)}$ and $\phi \circ \psi = \text{Id}_{uce(L)}$. For the proof of these two equations, the following diagram may be helpful.

By the universal property of $\lim uce(L_i)$, in order to show $\psi \circ \phi = \text{Id}_{\lim uce(L_i)}$, we only need to check $(\psi \circ \phi) \circ \tilde{\phi}_i = \tilde{\phi}_i$. Since $\psi \circ \tilde{\phi}_i = \tilde{\phi}_i$, we are left to check $\psi \circ \tilde{\phi}_i = \psi_i$ and this is true by the observation $v \circ \psi \circ \tilde{\phi}_i = u \circ \tilde{\phi}_i = \phi_i \circ u_i = v \circ \tilde{\phi}_i$ and the uniqueness in [N2, Prop. 1.13]. For the proof of $\phi \circ \psi = \text{Id}_{uce(L)}$ it is in view of the universal property of $uce(L)$ enough to verify $u \circ (\phi \circ \psi) = u$. Since $u = v \circ \psi$, we are left to check $u \circ \phi = v$ and this follows from $u \circ \phi \circ \tilde{\phi}_i = u \circ \tilde{\phi}_i = \phi_i \circ u_i = v \circ \tilde{\phi}_i$.

For the proof of (1.7) it suffices to note that $(\text{Ker} u_i, f_{ji}|_{\text{Ker} u_i})$ is a directed system and that

$$0 \to \text{Ker} u_i \to uce(L_i) \to L_i \to 0$$

is exact for every $i \in I$. The claim then follows from the fact that direct limits preserve exact sequences. \Box

**Remark 1.7.** Theorem 1.6 is proven in [S, App.] for the case $I = \mathbb{N}$ with the natural order and a directed system of Lie algebras over algebraically closed fields of characteristic zero $L_0 \xrightarrow{f_0} L_1 \xrightarrow{f_1} \cdots$ satisfying the condition that all $f_i$ are monomorphisms with $f_i(\mathfrak{z}(L_i)) \subset \mathfrak{z}(L_{i+1})$.

**Remark 1.8.** We note that for Lie algebras the formula (1.7) is not new. Indeed, by [Wei, Cor. 7.3.6] the homology groups of a Lie algebra can be interpreted as torsion groups for the universal enveloping algebra and it is known that torsion commutes with direct limits, see e.g., [Wei, Cor. 2.6.17]. The proof presented here is more direct and works in the super setting as well.

For the next corollary we recall that a perfect Lie superalgebra is called centrally closed if $u : uce(L) \to L$ is an isomorphism.

**Corollary 1.9.** If $(L_i, f_{ji})$ is a directed system of perfect and centrally closed Lie superalgebras, then $\lim L_i$ is perfect and centrally closed.

### 2. Examples: Universal central extensions of some infinite rank Lie superalgebras

In this section we will consider some examples of universal central extensions of direct limit Lie superalgebras, mainly those which are direct limits of some of the classical Lie superalgebras. In order to use the known results on their universal central extensions, we will in this section assume that all
Lie superalgebras are defined over a commutative, associative, unital ring \( k \), rather than an arbitrary base superring as in Section 1.

**Example 2.1 (Special linear Lie superalgebra \( \mathfrak{sl}(I; A) \) for \( A \) an associative superalgebra).** Let \( I = I_\emptyset \cup I_1 \) be a superset, i.e., a partitioned set. Let \( A \) be a unital associative, but not necessarily commutative \( k \)-superalgebra. We denote by \( \text{Mat}(I; A) \) the associative \( k \)-superalgebra whose underlying module consists of \( |I| \times |I| \)-finitary matrices with entries from \( A \) (only finitely many non-zero entries) and \( \mathbb{Z}_2 \)-grading given by \([E_{ij}(a)] = |i| + |j| + |a|\). Here \( E_{ij}(a) \in \text{Mat}(I; A) \) has entry \( a \) at the position \((ij)\) and 0 elsewhere. The product of \( \text{Mat}(I; A) \) is the usual matrix multiplication. Clearly \( \text{Mat}(I; A) \) only depends on the cardinality of \( I \) (and of course on \( A \)). For a finite \( I \) we put

\[
\text{Mat}(m, n; A) := \text{Mat}(I; A) \quad \text{if } |I_\emptyset| = m \text{ and } |I_1| = n.
\]

A matrix \( x \in \text{Mat}(m, n; A) \) written as

\[
x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \\ m & n \end{bmatrix}
\]

is then even (resp. odd) if \( x_1 \) and \( x_4 \) are matrices with even (resp. odd) entries and \( x_2 \) and \( x_3 \) are matrices with odd (resp. even) entries.

We let \( \mathfrak{gl}(I; A) \) be the Lie superalgebra associated to the associative superalgebra \( \text{Mat}(I; A) \). Its product is \([x, y] = xy - (-1)^{|x||y|}yx\). We assume \(|I| \geq 3\) and then have that the Lie superalgebra

\[
\mathfrak{sl}(I; A) := \left[ \mathfrak{gl}(I; A), \mathfrak{gl}(I; A) \right] \quad \text{is perfect.}
\]

Indeed, the canonical matrix units \( E_{ij}(a) \) for \( i, j \in I \) and \( a \in A \) satisfy the relations

\[
a \mapsto E_{ij}(a) \quad \text{is } k\text{-linear};
\]

\[
[E_{ij}(a), E_{pq}(b)] = \delta_{jp} E_{iq}(ab) - (-1)^{|E_{ij}(a)||E_{pq}(b)|} \delta_{iq} E_{pj}(ba).
\]

In particular, relation (2.4) implies that for distinct \( i, j, l \in I \) we have \( E_{ij}(a) = [E_{il}(a), E_{lj}(1)] \in \mathfrak{sl}(I; A) \) (1) = [\mathfrak{sl}(I; A), \mathfrak{sl}(I; A)]. Moreover, there are two types of diagonal elements in \( \mathfrak{sl}(I; A) \) – those which are products of off-diagonal elements and therefore lie in \( \mathfrak{sl}(I; A)^{(1)} \) by what we just proved, and those which are commutators of diagonal elements, more precisely, elements of type \([E_{ii}(a), E_{jl}(b)] = E_{ii}([a, b])\) for \([a, b] = ab - (-1)^{|a||b|}ba\). To show that \( \mathfrak{sl}(I; A) \) is perfect, it is therefore enough to prove that \( E_{ii}([a, b]) \in \mathfrak{sl}(I; A)^{(1)} \). But the latter is a consequence of the following identity, in which again \( i, j, l \) are distinct elements in \( I \):

\[
E_{ii}([a, b]) = [E_{ij}(a), E_{lj}(b)] + (-1)^{|E_{ij}(a)||E_{lj}(b)|} [E_{ji}(b), E_{lj}(a)] + (-1)^{|E_{ij}(a)||E_{lj}(b)| + |E_{ji}(b)||E_{lj}(a)|} [E_{il}(a), E_{lj}(b)].
\]

We also observe that the calculation above implies that

\[
\mathfrak{sl}(I; A) \text{ is generated by } E_{ij}(a), \quad i \neq j \in I, \quad a \in A.
\]

Moreover, we also have

\[
\text{if } I_1 = \emptyset \text{ or } A_1 = \{0\} \text{ then } \mathfrak{sl}(I; A) = \{x \in \text{Mat}(I; A) : \text{str}(x) \in [A, A] \}.
\]
Here the (super)trace str of a matrix $x = (x_{ij}) \in \text{Mat}(I; A)$ is given by $\text{str}(x) = \sum_{i \in I_0} x_{ii} - \sum_{i \in I_1} x_{ii}$, and $[A, A]$ is the span of all commutators $[a_1, a_2]$, $a_i \in A$.

For $I$ as above we define the (linear) Steinberg Lie superalgebra $\mathfrak{sl}(I; A)$ as the Lie $k$-superalgebra presented by generators $e_{ij}(a)$ with $i, j \in I$, $i \neq j$, $a \in A$ and relations (2.3)–(2.4) with $E_{ij}(a)$ replaced by $e_{ij}(a)$. We then have a canonical Lie superalgebra epimorphism

$$\nu_I : \mathfrak{sl}(I; A) \to \mathfrak{sl}(I; A), \quad e_{ij}(a) \mapsto E_{ij}(a).$$

If $|I_0| = m$ and $|I_1| = n$, we put $\mathfrak{sl}(m, n; A) = \mathfrak{sl}(I; A)$, $\mathfrak{sl}(m, n; A) = \mathfrak{sl}(I; A)$ and $\nu_{mn} : \mathfrak{sl}(m, n; A) \to \mathfrak{sl}(m, n; A)$ for $\nu_I$.

We also need the first cyclic homology group $\text{HC}_1(A)$. To define it, we use

$$\langle A, A \rangle = (A \otimes_k A) / \mathcal{H}$$

where $\mathcal{H}$ is the span of all elements of type

$$a \otimes b + (-1)^{|a||b|} b \otimes a, \quad a_\circ \otimes a_\circ \quad \text{for } a_\circ \in A_\circ,$$

$$( -1)^{|a||c|} a \otimes bc + (-1)^{|b||a|} b \otimes ca + (-1)^{|c||b|} c \otimes ab$$

for $a, b, c \in A$. We abbreviate $\langle a, b \rangle = a \otimes b + \mathcal{H}$. Observe that there is a well-defined commutator map

$$c : \langle A, A \rangle \to A, \quad \langle a, b \rangle \mapsto [a, b].$$

We put

$$\text{HC}_1(A) = \text{Ker} c = \left\{ \sum_i \langle a_i, b_i \rangle : \sum_i [a_i, b_i] = 0 \right\}.$$

We will use the following assumption:

$$\nu_F \text{ for } 5 \leq |F| < \infty \text{ is a universal central extension with } \text{Ker} \nu_F \cong \text{HC}_1(A). \tag{2.6}$$

The assumption (2.6) is true in any one of the following situations:

(a) $n = 0$, $A$ an algebra [KL] or a superalgebra [CG],

(b) $A$ an algebra [MP1,IK1].

Refs. [CG] and [IK1] assume that $k$ is a commutative ring containing $\frac{1}{2}$ and that the underlying module of $A$ is free with a basis containing the identity element of $A$. We note that (2.6) is not true for $m + n \leq 4$, see the papers [CG,G,GS,SCG] which deal with the case $3 \leq m + n \leq 4$.

**Proposition 2.2.** Assume (2.6) holds and $I$ is a (possibly infinite) set with $|I| \geq 5$. Then $\nu_I : \mathfrak{sl}(I; A) \to \mathfrak{sl}(I; A)$ is a universal central extension with kernel isomorphic to $\text{HC}_1(A)$.

**Proof.** This can be proven by adapting the proof of (2.6) to our setting. Instead we prefer to give a proof based on Theorem 1.6. This is possible since, denoting by $\mathcal{F}$ the set of finite subsets of $I$ ordered by inclusion, the Lie superalgebra $\mathfrak{sl}(I; A)$ is indeed a direct limit: $\mathfrak{sl}(I; A) = \bigcup_{F \in \mathcal{F}} \mathfrak{sl}(F; A) \cong \lim_{F \in \mathcal{F}} \mathfrak{sl}(F; A)$. Hence $\text{uce}(\mathfrak{sl}(I; A)) \cong \lim_{F \in \mathcal{F}} \text{uce}(\mathfrak{sl}(F; A)) \cong \lim_{F \in \mathcal{F}} \mathfrak{sl}(F; A)$. Thus we need to show...
that \( \lim_{F \in \mathcal{F}} \mathfrak{sl}(F; A) \cong \mathfrak{sl}(I; A) \). This follows from the diagram below, where \( \psi_F \) is given by sending a generator \( e_{ij}(a) \in \mathfrak{sl}(F; A) \) to \( e_{ij}(a) \in \mathfrak{sl}(I; A) \). The existence of \( \psi \) then follows from the definition of a direct limit, applied to \((\psi_F; F \in \mathcal{F})\). The families \( (e_{ij}(a) \in \mathfrak{sl}(F; A); F \in \mathcal{F}) \) give rise to elements \( e_{ij}(a) \in \lim_{F \in \mathcal{F}} \mathfrak{sl}(F; A) \) satisfying the relations (2.3)-(2.4), whence the existence of the map \( \psi \) sending \( e_{ij}(a) \in \mathfrak{sl}(I; A) \) to \( e_{ij}(a) \).

\[
\begin{array}{ccc}
\mathfrak{sl}(F; A) & \xrightarrow{f_{F'}} & \mathfrak{sl}(F'; A) \\
\psi_F & & \psi_{F'} \\
\varphi & & \varphi' \\
\lim \mathfrak{sl}(F; A) & \xrightarrow{\psi} & \mathfrak{sl}(I; A)
\end{array}
\]

It is immediate that \( \varphi \) and \( \psi \) are inverses of each other, and that \( \text{Ker} \varphi \cong \text{HC}_1(A) \). \( \square \)

**Example 2.3 (\( \mathfrak{sl}(I; A) \) for \( A \) an associative commutative superalgebra).** Let \( A \) be a unital associative and commutative \( k \)-superalgebra, thus \([A, A] = 0\). Therefore the descriptions of \( \mathfrak{sl}(I; A) \) and \( \text{HC}_1(A) \) simplify to

\[
\mathfrak{sl}(I; A) = \big\{ x \in \mathfrak{gl}(I; A): \text{str}(x) = 0 \big\} \cong \mathfrak{sl}(I; k) \otimes_k A,
\]

\[
\text{HC}_1(A) = \langle [A, A] \rangle.
\]

Moreover, the universal central extension \( \mathfrak{sl}(I; A) \) can be described via a 2-cocycle as follows. The Lie superalgebra \( \mathfrak{sl}(m, n; A) \) has a central 2-cocycle \( \tau_{mn} \) with values in \( \text{HC}_1(A) \):

\[
\tau_{mn}(x, y) = \sum_{1 \leq i \leq m, 1 \leq j \leq m+n} \langle x_{ij}, y_{ji} \rangle - \sum_{m+1 \leq i \leq m+n, 1 \leq j \leq m+n} \langle x_{ij}, y_{ji} \rangle
\]

for \( x = (x_{ij}), y = (y_{ij}) \in \mathfrak{sl}(m, n; A) \). We let \( \mathfrak{sl}(m, n, A) \oplus \text{HC}_1(A) \) be the corresponding Lie superalgebra, and view it as a central extension of \( \mathfrak{sl}(m, n; A) \) by projecting onto the first factor. From now on we suppose \( m+n \geq 5 \) and that the map

\[
h_{mn}: \text{ucc}(\mathfrak{sl}(m, n; A)) \to \mathfrak{sl}(m, n; A) \oplus \text{HC}_1(A), \quad h_{mn}(x, y) = [x, y] \oplus \tau_{mn}(x, y)
\]

is an isomorphism of central extensions:

\[
\text{ucc}(\mathfrak{sl}(m, n; A)) \cong \mathfrak{sl}(m, n; A) \oplus \text{HC}_1(A) \quad \text{as central extensions.} \tag{2.7}
\]

The assumption (2.7) is true in any one of the following situations:

(a) \( n = 0 \), \( A \) an algebra [KL] or a superalgebra [CG],
(b) \( A \) an algebra [MP1,JK1].

Let now \( (\mathfrak{sl}(m_i, n_i, A), f_{ji}) \) be a directed system of Lie superalgebras with \( m_i + n_i \geq 5 \). The transition maps \( f_{ji} : \mathfrak{sl}(m_i, n_i, A) \to \mathfrak{sl}(m_j, n_j, A) \) lift uniquely to Lie superalgebra morphisms \( \hat{f}_{ji} : \text{ucc}(\mathfrak{sl}(m_i, n_i, A)) \to \text{ucc}(\mathfrak{sl}(m_j, n_j, A)) \). Hence, we get a directed system \( (\mathfrak{sl}(m_i, n_i, A) \oplus \text{HC}_1(A))_{i \in I} \).
with transition maps \( h_{m, n_1} \circ f_{ji} \circ h^{-1}_{m, n_1} = f_{ji} \oplus g_{ji} \) where the map \( g_{ji} : \text{HC}_1(A) \to \text{HC}_1(A) \) is given by \( g_{ji}(\tau_{m,n}(x,y)) = \tau_{m,n}(f_{ji}(x), f_{ji}(y)) \) for \( x, y \in \mathfrak{sl}(m_1, n_1, A) \). We now define a central 2-cocycle \( \tau_I \) for the direct limit Lie superalgebra

\[
\mathfrak{sl}(I; A) = \lim_{\rightarrow} \mathfrak{sl}(m_1, n_1, A)
\]

with values in \( \text{HC}_1(A) \). Let \( x, y \in \mathfrak{sl}(I; A) \). Thus \( x = \varphi_p(x_p) \) and \( y = \varphi_q(y_q) \) for some \( p, q \in I, x_p \in \mathfrak{sl}(m_p, n_p, A) \) and \( y_q \in \mathfrak{sl}(m_q, n_q, A) \). Here \( \varphi_p, \varphi_q \) are the canonical maps for \( \mathfrak{sl}(I; A) \). There exists \( k \in I \) such that \( k \geq p, k \geq q \) and \( f_{kp}(x_p), f_{kq}(y_q) \in \mathfrak{sl}(m_k, n_k, A) \). Then the cocycle \( \tau_I \) for \( \mathfrak{sl}(I; A) \) is given by

\[
\tau_I(x, y) = \tau_{m_k, n_k}(f_{kp}(x_p), f_{kq}(y_q)). \tag{2.8}
\]

We now get from Proposition 2.2 that \( \mathfrak{sl}(I; A) \cong \text{uce}(\mathfrak{sl}(I; A)) \cong \mathfrak{sl}(I; A) \oplus \text{HC}_1(A) \), where the 2-cocycle \( \tau_I \) is given explicitly by (2.8). Summarizing the above, we have proven the following.

**Corollary 2.4.** Let \( A \) be a unital associative commutative superalgebra over a commutative ring \( k \). Let \( (I, \preceq) \) be an arbitrary directed set and let \( (\mathfrak{sl}(m_i, n_i, A), f_{ji}) \) be a directed system of Lie superalgebras with \( m_i + n_i \geq 5 \). We suppose (2.7) and denote by \( \mathfrak{sl}(I; A) := \lim_{\rightarrow} \mathfrak{sl}(m_i, n_i, A) \) the corresponding direct limit, which is a perfect Lie superalgebra of possibly infinite rank. Then

\[
\text{uce}(\mathfrak{sl}(I; A)) \cong \mathfrak{sl}(I; A) \oplus \text{HC}_1(A) \tag{2.9}
\]

as central extensions, where the Lie superalgebra structure on the right is given by the 2-cocycle \( \tau_I \) of (2.8).

**Example 2.5** (\( \mathfrak{sl}_I(A) \) for \( A \) an associative algebra). Let \( A \) be an associative unital \( k \)-algebra over a commutative ring \( k \) containing \( \frac{1}{2} \), and let \( I \) be an arbitrary, possible infinite set with \( |I| \geq 5 \). We denote by \( \mathfrak{sl}_I(A) \) the Lie algebra of finitary matrices over \( A \) (only finitely many non-zero entries) and with trace in \([A, A]\). Since \( \mathfrak{sl}_I(A) \) is the direct limit of the Lie algebras \( \mathfrak{sl}_F(A) \) where \( F \) runs through the finite subsets of \( I \), Corollary 2.4 implies that \( \text{uce}(\mathfrak{sl}_I(A)) \cong \mathfrak{sl}_I(A) \oplus \text{HC}_1(A) \). This is proven in [KL] for \( I \) finite or countable and in [Wel] for arbitrary \( I \), using the theory of root graded Lie algebras.

**Example 2.6** (Ortho-symplectic Lie superalgebra \( \mathfrak{osp}(I; A) \) for \( A \) an associative commutative superalgebra). The ortho-symplectic Lie superalgebra \( \mathfrak{osp}(m, n; A) \) can be defined in the usual way, see for example [IK1,IK2,MP2]. Since \( \mathfrak{osp}(m, n; A) \) is a subalgebra of \( \mathfrak{sl}(m, n; A) \), the restriction of the 2-cocycle \( \tau_{mn} \) of Example 2.3 defines a 2-cocycle of \( \mathfrak{osp}(m, n; A) \) with values in \( \text{HC}_1(A) \) and thus gives rise to a central extension. We suppose that the map \( \text{uce}(\mathfrak{osp}(m, n; A)) \to \mathfrak{osp}(m, n; A) \oplus \text{HC}_1(A) \), given by \((x, y) \mapsto [x, y] \oplus \tau_{mn}(x, y)\) is an isomorphism:

\[
\text{uce}(\mathfrak{osp}(m, n; A)) \cong \mathfrak{osp}(m, n; A) \oplus \text{HC}_1(A) \quad \text{as central extensions.} \tag{2.10}
\]

Our assumption (2.10) is fulfilled in any one of the following situations:

(a) \( k \) a field of characteristic 0 [IK2],

(b) \( A \) a commutative algebra [IK1,MP2].

Ref. [MP2] assumes that \( m \geq 5, n \geq 10 \).

Let now \( (\mathfrak{osp}(m_i, n_i; A), f_{ji}) \) be a directed system of Lie superalgebras. One shows as in Example 2.3 that there exists a well-defined 2-cocycle \( \tau'_I \) for the direct limit Lie superalgebra

\[
\mathfrak{osp}(I; A) := \lim_{\rightarrow} \mathfrak{osp}(m_i, n_i; A)
\]
with values in $\text{HC}_1(A)$. From Theorem 1.6 we then get as in Corollary 2.4

$$\text{uce}(\text{osp}(I; A)) \cong \lim_{i} \text{uce}(\text{osp}(m_i; n_i; A)) \cong \lim_{i} (\text{osp}(m_i, n_i; A) \oplus \text{HC}_1(A)) \cong \text{osp}(I; A) \oplus \text{HC}_1(A).$$

(2.11)

**Example 2.7 (Locally finite Lie superalgebras).** Classically semisimple locally finite Lie superalgebras over algebraically closed fields of characteristic 0 were introduced and studied in [P], including a classification of the simple infinite dimensional ones which admit a local system of root injections of classical finite dimensional Lie superalgebras. They are all direct limits $L = \lim_{i} L_i$ of classical simple Lie superalgebras $L_i$, $i \in \mathbb{N}$ with $f_i : L_i \to L_{i+1}$ being the natural inclusions. Referring the reader to [P] for details, we simply present the classification list. We abbreviate $\text{sl}(m, n) = \text{sl}(m, n; k)$ and $\text{osp}(m, n) = \text{osp}(m, n; k)$ in the notation of 2.3 and 2.6 respectively. The Lie superalgebra $\text{SP}(m)$ is the subalgebra of $\text{sl}(m, m)$ which leaves invariant an odd nondegenerate super-antisymmetric bilinear form, and $\text{sq}(m)$ is the subalgebra of $\text{sl}(m, m)$ consisting of matrices of the form (2.1) with $x_1 = x_4$, $x_2 = x_3$ and $\text{tr}(x_2) = 0$. With these notations, an infinite dimensional simple Lie superalgebra, which admits a local system of root injections of classical finite dimensional Lie superalgebras $L_i$, is isomorphic to a Lie superalgebra $L$ in the following table ($i \geq 2$, $n \geq 0$, $k \geq 0$, $r \geq 0$, $m \geq 2$):

<table>
<thead>
<tr>
<th>$L$</th>
<th>$L_i$</th>
<th>$L$</th>
<th>$L_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{sl}(\infty, n)$</td>
<td>$\text{sl}(i, n)$</td>
<td>$\text{sl}(\infty, \infty)$</td>
<td>$\text{sl}(i, i)$</td>
</tr>
<tr>
<td>$\text{B}(\infty, 2k)$</td>
<td>$\text{osp}(2i + 1, 2k)$</td>
<td>$\text{B}(\infty, \infty)$</td>
<td>$\text{osp}(2i + 1, 2i)$</td>
</tr>
<tr>
<td>$\text{B}(2r + 1, \infty)$</td>
<td>$\text{osp}(2r + 1, 2i)$</td>
<td>$\text{C}(\infty)$</td>
<td>$\text{osp}(2, 2i)$</td>
</tr>
<tr>
<td>$\text{D}(\infty, 2k)$</td>
<td>$\text{osp}(2i, 2k)$</td>
<td>$\text{D}(\infty, \infty)$</td>
<td>$\text{osp}(2i, 2i)$</td>
</tr>
<tr>
<td>$\text{D}(2m, \infty)$</td>
<td>$\text{osp}(2m, 2i)$</td>
<td>$\text{SP}(\infty)$</td>
<td>$\text{SP}(i)$</td>
</tr>
<tr>
<td>$\text{sq}(\infty)$</td>
<td>$\text{sq}(i)$</td>
<td>$\text{sq}(\infty)$</td>
<td>$\text{sq}(i)$</td>
</tr>
</tbody>
</table>

(2.12)

**Corollary 2.8.** The Lie superalgebras $L_i$ listed in table (2.12) are all centrally closed.

**Proof.** The Lie superalgebras $L_i$ in table (2.12) are all perfect. In view of Theorem 1.6 it therefore remains to show that they are centrally closed for large $i$. For $L_i$ of type $\text{sl}(m, n)$ or $\text{osp}(m, n)$ this follows from (2.7) and (2.10) since $\text{HC}_1(k) = \{0\}$. For the remaining two types this follows from [IK2, Th. 5.10]. □

**Example 2.9 (Locally finite Lie algebras).** Corollary 2.8 applies in particular to the simple locally finite Lie algebras $\text{sl}(\infty) = \text{sl}(\infty, 0)$, $\alpha(\infty) = \text{B}(\infty, 0) = \text{D}(\infty, 0)$, $\text{osp}(\infty) = \text{D}(0, \infty)$ (the only infinite dimensional simple root reductive Lie algebras), studied for example in [BB,DP,PS].

**Example 2.10 (Root-graded Lie algebras). (a)** Let $k$ be a ring in which 2 and 3 are invertible, and let $L$ be a Lie algebra graded by a locally finite reduced root system $R$ as defined in [N1], see also [N3, 5.1]. Thus, $L$ is graded by $\mathcal{Q}(R) = \text{Span}_{\mathbb{C}}(R)$ with $\text{supp}_{\mathcal{Q}(k)} L = R$, i.e.,

$$L = \bigoplus_{\alpha \in R} L_\alpha, \quad [L_\alpha, L_\beta] \subset L_{\alpha+\beta},$$

satisfies $L_0 = \sum_{0 \neq \alpha \in R} [L_\alpha, L_{-\alpha}]$ and has the property that for $\alpha \neq 0$ there exists an $\text{sl}_2$-triple $(e_\alpha, h_\alpha, f_\alpha) \in L_\alpha \times L_0 \times L_{-\alpha}$ such that $[h_\alpha, x_\beta] = (\beta, \alpha^\vee)x_\beta$ holds for every $\beta \in R$ and $x_\beta \in L_\beta$. By [LN, 3.15], $R$ is a direct limit of finite root systems, say $R = \lim_{\leftarrow} R_i$ where $i$ runs through a directed set $(I, \leq)$. The subalgebra
\[ L_i = \left( \bigoplus_{0 \neq \alpha \in R_i} L_\alpha \right) \oplus \sum_{0 \neq \alpha \in R_i} [L_\alpha, L_{-\alpha}] \]  

(2.13)

is graded by the root system \( R_i \), and it is immediate that \( L = \varinjlim L_i \).

Any root-graded Lie algebra is perfect, whence \( uce(L) \cong \varinjlim uce(L_i) \) by Theorem 1.6. In fact, something more precise is true. One knows that the universal central extension of a root-graded Lie algebra is again graded by the same locally finite root system, [N3, Prop. 5.4]. Thus the root-graded Lie algebra \( uce(L) \) is a direct limit of root-graded Lie algebras,

\[ uce(L) \cong \varinjlim uce(L)_i, \]

where \( uce(L)_i \) is defined in the same way as \( L_i \).

(b) Suppose in the following that \( k \) is a field of characteristic zero. If \( K \) is a centreless Lie algebra graded by a finite irreducible reduced root system, the group \( H_2(K) \) is known to be the full dihedral homology group \( HF(\alpha) \), where \( \alpha \) is the coordinate algebra of \( K_i \), [ABG, Th. 4.13].

Let now \( L \) be a centreless Lie algebra graded by a locally finite irreducible reduced root system \( R \) of rank \( \geq 9 \). Then \( R = \varinjlim R_i \) where the \( R_i \) are finite, irreducible, reduced, have rank \( \geq 9 \) and are of the same type as \( R \), [LN, 8.3]. It is moreover no harm to assume that \( R_0 \subseteq R_i \) for some fixed \( i \in I \). Then it follows that the root-graded Lie algebras \( L_i \) of (2.13) all have the same coordinate algebra \( \alpha \) (this has also been noted by M. Yousofzadeh in [Y]). Notice that if \( f_j(\gamma(L_i)) \subset \gamma(L_i) \), then \( \lim L_i \cong \varinjlim \gamma(L_i) \). Hence

\[ uce(L) \cong uce(\varinjlim \gamma(L_i)) \cong \varinjlim uce(\gamma(L_i)) \]

\( \cong \varinjlim \gamma(L_i) \cong HF(\alpha) \cong L \oplus HF(\alpha). \)  

(2.14)

Special cases of root-graded Lie algebras are the so-called Lie tori, which occur as cores and centreless cores of locally extended affine Lie algebras [MY, Nee]. More generally, the cores of affine reflection Lie algebras are root-graded Lie algebras of possibly infinite rank [N3, 6.4, 6.5].

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**References**


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