Transformations Groups of the Andersson-Perlman Cone

Erhard Neher[1]

Department of Mathematics, University of Ottawa, Ottawa, Canada, K1N 6N5
email: neher@uottawa.ca

Abstract. An Andersson-Perlman cone is a certain subcone $\Omega(\mathcal{K})$ of the symmetric cone Ω of a Euclidean Jordan algebra. We exhibit a subgroup of the automorphism group of Ω which operates transitively on $\Omega(\mathcal{K})$ and show that $\Omega(\mathcal{K})$ is a simply-connected submanifold of Ω.

1. Introduction. Andersson-Perlman cones in the setting of Euclidean Jordan algebras (henceforth abbreviated as AP cones) were introduced by H. Massam and the author in [MN] as a generalization of certain cones defined by the statisticians S. A. Andersson and M. D. Perlman for real symmetric matrices [AP]. All mathematical results in [AP] were generalized in [MN] to the setting of Euclidean Jordan algebras, except the existence of transitive transformation groups which play a predominant role in the development in [AP]. In fact, the paper [MN] stresses a different, perhaps more direct approach to the description of Andersson-Perlman cones by employing Peirce decompositions and Frobenius transformations.

In this note we show that one can also generalize the results of [AP] on transitive groups to the framework of Andersson-Perlman cones in Euclidean Jordan algebras. Our interest in these groups is explained in the following remarks. An Andersson-Perlman cone is a subcone $\Omega(\mathcal{K})$ of the cone Ω of an Euclidean Jordan algebra V defined in terms of a complete orthogonal system $\mathcal{E} = (e_1, \dots, e_n)$ of idempotents of V and a ring $\mathcal{K}$ of subsets of $I = \{1, \dots, n\}$. (For our purposes it is of advantage to give $\mathcal{E}$ here a different meaning than the one used in [MN]; the exact difference is explained in **7.** below). If Ω_i denotes the symmetric cone of the Peirce-1-space $V(e_i, 1)$ of e_i then always

$$\Omega_1 \oplus \Omega_2 \oplus \cdots \oplus \Omega_n \subset \Omega(\mathcal{K}) \subset \Omega,$$

and both upper and lower bounds can be obtained by varying $\mathcal{K}$. Thus, one may consider $\Omega(\mathcal{K})$ as an interpolation between Ω and $\Omega_1 \oplus \Omega_2 \oplus \cdots \oplus \Omega_n$. In the same spirit, the transitive transformation group T (denoted $T_{\mathcal{E},\preceq}$ in the paper) of $\Omega(\mathcal{K})$ interpolates various well-known subgroups of the automorphism group $G(\Omega) = \{g \in \mathrm{GL}(V); g\Omega = \Omega\}$ of Ω. In general, T is a semidirect product of a unipotent subgroup N of $G(\Omega)$ (denoted $N_{\mathcal{E},\prec}$ in the paper) and the real reductive group

$$M_{\mathcal{E}} = \{g \in G(\Omega); g\Omega_i = \Omega_i\} = P(\Omega_1 \oplus \Omega_2 \oplus \cdots \oplus \Omega_n) \cdot K_{\mathcal{E}} \tag{1}$$

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where $K_{\mathcal{E}} = \{f \in \mathrm{Aut}\, V; fe_i = e_i \text{ for } 1 \leq i \leq n\}$. Observe that (1) is the Cartan decomposition of $M_{\mathcal{E}}$. We always have

$$M_{\mathcal{E}} \subset T = M_{\mathcal{E}} \cdot N \subset G(\Omega), \tag{2}$$

and both bounds are attained. For example, if $\Omega(\mathcal{K}) = \Omega$ and $\mathcal{E}$ is a Jordan frame then N is the so-called strict triangular subgroup [FK], while if $\mathcal{E} = \{e\} (n = 1)$ then also $\Omega(\mathcal{K}) = \Omega$, $N = \{\mathrm{Id}\}$ and $M_{\mathcal{E}} = G(\Omega)$. In this case, (1) is just the standard Cartan decomposition of $G(\Omega)$.

2. Notation and review. Our basic reference for Jordan algebras is [FK]. Some of the results and notations used are summarized below.

Throughout, V denotes an Euclidean Jordan algebra with identity element e, left multiplication $L(u)$ defined by $L(u)v = uv (u, v \in V)$ and quadratic representation P given by $P(u)v = 2u(uv) - u^2 v$. The linearization of P is

$$\{u\, v\, w\} := P(u, w)v := P(u + w)v - P(u)v - P(w)v = 2u(vw) + 2w(uv) - 2(uw)v$$

for $(u, v, w \in V)$. The Jordan triple system left multiplication $L(u, v)$ (denoted $u \square v$ in [FK]) is given by

$$L(u, v) = 2(L(uv) + [L(u), L(v)]),$$

and hence $L(u, v)w = P(u, w)v$. For any endomorphism φ of V, φ^* is the adjoint of φ with respect to the positive definite trace form of V.

We will use the term "Lie group" and "Lie subgroup" as defined in [B]. In particular, any Lie subgroup of a Lie group is closed and has the induced topology. Closed subgroups of a Lie group are always Lie subgroups in a unique way.

We denote the symmetric cone of V by $\Omega = \Omega(V)$. This is an open convex cone which is homogeneous with respect to the group $G(\Omega) = \{g \in \mathrm{GL}(V); g\Omega = \Omega\}$, the automorphism group of Ω. The group $G(\Omega)$ is a Lie subgroup of $\mathrm{GL}_{\mathbb{R}}(V)$. Its identity component will be denoted by G. Moreover, $G(\Omega)$ is an open subgroup of the structure group of V, defined as the group of all invertible endomorphisms g of V with the property

$$P(gx) = gP(x)g^* \tag{1}$$

for all $x \in V$, or, equivalently,

$$gL(u, v)g^{-1} = L(gu, g^{*-1}v) \tag{1'}$$

for all $u, v \in V$ ([FK; III.5 and VIII.2]). The Lie algebra $\mathfrak{G}(V)$ of the structure group of V coincides with the Lie algebra of $G(\Omega)$. It consists of all endomorphisms X of V satisfying for all $u, v \in V$

$$[X, L(u, v)] = L(Xu, v) - L(u, X^*v) \tag{2}$$

([FK; VIII.2.6]). The group of automorphisms of V will be denoted $\mathrm{Aut}\, V$. For any $g \in G(\Omega)$ one knows ([FK; III.5] and [FK; VIII.2.4]):

$$ge = e \;\Leftrightarrow\; gg^* = Id \;\Leftrightarrow\; g \in \mathrm{Aut}\, V \tag{3}$$

In particular, $\operatorname{Aut} V$ is a maximal compact subgroup of $G(\Omega)$.

Following [FK] we denote the Peirce spaces of an idempotent $c \in V$ by $V(c,i) = \{v \in V;\, cv = iv\}, i \in \{0, \frac{1}{2}, 1\}$. The Peirce decomposition of an arbitrary $y \in V$ is written in the form $y = y_1 + y_{12} + y_0$ where $y_i \in V(c,i)$ for $i = 0, 1$ and $y_{12} \in V(c, \frac{1}{2})$. The symmetric cone of the Euclidean Jordan algebra $V(c,1)$ will be denoted Ω_c. For an idempotent c and $z \in V(c, \frac{1}{2})$ the Frobenius transformation on V is defined as $\tau_c(z) = \exp(L(z,c))) \in G$. It is straightforward to check that $\tau_c : V(c, \frac{1}{2}) \to G$ is a homomorphism, thus $\tau_c(z + z') = \tau_c(z)\tau_c(z')$ and $\tau_c(-z) = \tau_c(z)^{-1}$. If $x = x_1 + x_{12} + x_0$ is the Peirce decomposition of $x \in V$ with respect to c then

$$\begin{aligned} \tau_c(z)x &= x_1 \oplus 2zx_1 + x_{12} \oplus 2(e-c)[z(zx_1) + zx_{12}] + x_0 \\ &= x_1 \oplus 2zx_1 + x_{12} \oplus P(z)x_1 + 2(e-c)(zx_{12}) + x_0. \end{aligned} \tag{4}$$

The adjoint of the Frobenius transformation operates as follows [MN; 2.7]:

$$\tau_c(z)^* x = (x_1 + 2c(zx_{12}) + P(z)x_0) \oplus (x_{12} + 2zx_0) \oplus x_0. \tag{5}$$

Throughout, we fix a complete orthogonal system $\mathcal{E} = (e_1, \ldots, e_n)$ of (arbitrary) idempotents of V. Thus, $e_i e_j = \delta_{ij} e_i$ and $e_1 + \cdots + e_n = e$. We denote by V_{ij}, $1 \le i, j \le n$, the Peirce spaces of $\mathcal{E}$ [FK IV.2] and define, for $1 \le i < n$, subspaces

$$V^{(i)} := \oplus_{k=i+1}^{n} V_{ik} = V(e_i, \tfrac{1}{2}) \cap V(e_{i+1} + \ldots + e_n, \tfrac{1}{2}).$$

For $x \in V$ we let $x = \sum_{i \le j} x_{ij}, x_{ij} \in V_{ij}$, be the Peirce decomposition of $x \in V$. We abbreviate $\tau_i = \tau_{e_i}$ and $\Omega_i = \Omega_{e_i} = \Omega(V_{ii})$, $1 \le i \le n$. By [MN; 2.8] the map

$$F : V^{(1)} \times \cdots \times V^{(n-1)} \times \Omega_1 \times \cdots \times \Omega_n \quad \to \quad \Omega$$

given by

$$\begin{aligned} F(z_1, \cdots, z_{n-1}, y_1, \cdots y_n) :&= \tau_1(z_1) \cdots \tau_{n-1}(z_{n-1})(y_1 \oplus \cdots \oplus y_n) \\ &= \tau_1(z_1)y_1 + \tau_2(z_2)y_2 + \cdots + \tau_{n-1}(z_{n-1})y_{n-1} + y_n \end{aligned}$$

is a bijection. Even more, we have:

3. Proposition. *The map F is a diffeomorphism.*

Proof. It follows from the definition of the Frobenius transformation that F is differentiable. Since both manifolds have the same dimension, it suffices to show that the tangent map $T_\zeta F$ of F in a point $\zeta = (z_1, \cdots, z_{n-1}, y_1, \cdots y_n) \in M := V^{(1)} \times \cdots \times V^{(n-1)} \times \Omega_1 \times \cdots \times \Omega_n$ is injective. For $n = 2$ and $(u_1, v_1, v_2) \in V_{12} \times V_{11} \times V_{22} = T_\zeta M$, the tangent space of M at ζ, we have

$$T_\zeta F(u_1, v_1, v_2) = v_1 \oplus 2(u_1 y_1 + z_1 v_1) \oplus P(z_1)v_1 + \{z_1\, y_1\, u_1\} + v_2.$$

Hence, if $T_\zeta F(u_1, v_1, v_2) = 0$ we obtain $v_1 = 0$, then $u_1 = 0$ because $4y_1^{-1}(y_1 u_1) = u_1$ by [MN; (2.6.7)] and finally $v_2 = 0$. In general, if $w = (u_1, \cdots, u_{n-1}, v_1, \cdots v_n) \in V^{(1)} \times$

$\cdots \times V^{(n-1)} \times V_{11} \times \cdots \times V_{nn} = T_\zeta M$ lies in the kernel of $T_\zeta F$ then, since $\tau_2(z_2)y_2 + \cdots + \tau_{n-1}(z_{n-1})y_{n-1} + y_n \in V(e_1, 0)$, it follows by considering the V_{11}- and $V^{(1)}$-component of $T_\zeta F w$ that $v_1 = 0 = u_1$, but then $w = 0$ by induction.

4. Lemma. a) *For $z_{ij} \in V_{ij}$, $i \neq j$, and $x_{mn} \in V_{mn}$ the Frobenius transformation $\tau_i(z_{ij})$ operates as follows*

$$\tau_i(z_{ij})(x_{mn}) - x_{mn} = \begin{cases} 2x_{ii}z_{ij} \oplus P(z_{ij})x_{ii} \in V_{ij} \oplus V_{jj} & \text{for } m = n = i \\ 2e_j(z_{ij}x_{ij}) \in V_{jj} & \text{for } \{m,n\} = \{i,j\} \\ 2z_{ij}x_{ik} \in V_{jk} & \text{for } \{m,n\} = \{i,k\},\ i,j,k \neq \\ 0 & \text{for } i \notin \{m,n\} \end{cases} \tag{1}$$

(b) *For $z_{ij} \in V_{ij}$ and $z_{kl} \in V_{kl}$ we have the following commutation formulas*:

$$\tau_i(z_{ij})\tau_k(z_{kl}) = \tau_k(z_{kl})\,\tau_i(z_{ij}) \quad i \notin \{j,k,l\} \text{ and } k \notin \{l,i,j\}, \tag{2}$$

$$\tau_i(z_{ij})\tau_k(z_{ki}) = \tau_k(z_{ki} + 2z_{ij}z_{ki})\tau_i(z_{ij}) \quad |\{i,j,k\}| = 3, \tag{3}$$

$$\tau_i(z_{ij})\tau_j(z_{jl}) = \tau_j(z_{jl})\,\tau_i(z_{ij} - 2z_{ij}z_{jl}) \quad |\{i,j,l\}| = 3. \tag{4}$$

Proof. a) is immediate from (2.4). The formulas in b) can be checked by using (1) and a case-by-case analysis. An alternative proof for (2) and (3) goes as follows. Since $\tau_c(z) = \exp(L(z,c))$ we have for any invertible endomorphism g of V

$$g\tau_k(z_{kl})g^{-1} = \exp(gL(z_{kl}, e_k)g^{-1}). \tag{5}$$

By (2.1′)

$$\tau_i(z_{ij})L(z_{kl}, e_k)\tau_i^{-1}(z_{ij}) = L(\tau_i(z_{ij})z_{kl}, \tau_i^{*-1}(z_{ij})e_k)$$

where $\tau_i(z_{ij})z_{kl} = z_{kl} + \delta_{li}2z_{ij}z_{kl}$ by (1) and $\tau_i(z_{ij})^{*-1}e_k = \tau_i(-z_{ij})^*e_k = e_k$ by (2.5). This, together with (5) for $g = \tau_i(z_{ij})$ implies (2) and (3). One can prove (4) in a similar fashion:

$$\tau_j(z_{jl})^{-1}\tau_i(z_{ij})\tau_j(z_{ij}) = \exp L(\tau_j(-z_{jl})z_{ij}, \tau_j(z_{jl})^*e_i) = \exp L(z_{ij} - 2z_{ij}z_{jl}, e_i).$$

5. Transformation groups of Ω defined by $\mathcal{E}$. We define

$$\begin{aligned} \Omega_1 \oplus \Omega_2 \oplus \cdots \oplus \Omega_n &= \omega_1 +_2 + \cdots + \omega_n; \omega_i \in \Omega_i, 1 \leq i \leq n\} \subset \Omega, \\ A_\mathcal{E} &= P(\Omega_1 \oplus \Omega_2 \oplus \cdots \oplus \Omega_n) = \exp L(V_{11} \oplus V_{22} \oplus \cdots \oplus V_{nn}), \\ K_\mathcal{E} &= \{f \in \operatorname{Aut} V\,; fe_i = e_i\,, 1 \leq i \leq n\}, \\ M_\mathcal{E} &= \{m \in G(\Omega)\,; mV_{ii} \subset V_{ii}\,, 1 \leq i \leq n\}. \end{aligned}$$

The second equality in the definition of $A_\mathcal{E}$ follows from $P(\exp x) = \exp L(2x)$, see [FK; II.3.4], and $\Omega = \exp V$, see the proof of [FK; III.2.1]. Clearly, $K_\mathcal{E}$ and $M_\mathcal{E}$ are Lie subgroups of $G(\Omega)$.

Theorem. a) $M_{\mathcal{E}} = \{g \in G(\Omega); gV_{ij} = V_{ij}$ *for all* $i,j\} = \{g \in G(\Omega); gL(e_i)g^{-1} = L(e_i)$ *for* $1 \le i \le n\}$.

b) $M_{\mathcal{E}}$ *operates transitively on* $\Omega_1 \oplus \Omega_2 \oplus \cdots \oplus \Omega_n \subset \Omega$. *More precisely,* $A_{\mathcal{E}} \subset M_{\mathcal{E}}$ *and for every* $\omega \in \Omega_1 \oplus \Omega_2 \oplus \cdots \oplus \Omega_n$ *there exists a unique* $a \in A_{\mathcal{E}}$ *such that* $\omega = a(e)$.

c) $K_{\mathcal{E}}$ *is a subgroup of* $M_{\mathcal{E}}$ *satisfying*

$$K_{\mathcal{E}} = M_{\mathcal{E}} \cap \operatorname{Aut} V = \{m \in M; mm^* = \operatorname{Id}\}. \tag{1}$$

d) *Any* $m \in M_{\mathcal{E}}$ *can be uniquely written in the form* $m = ak$ *where* $a \in A_{\mathcal{E}}$ *and* $k \in K_{\mathcal{E}}$. *Thus, we have a decomposition*

$$M_{\mathcal{E}} = A_{\mathcal{E}} \cdot K_{\mathcal{E}} \approx (V_{11} \oplus V_{22} \oplus \cdots \oplus V_{nn}) \times K_{\mathcal{E}} \quad \textit{(diffeomorphism).} \tag{2}$$

Proof. We abbreviate $A = A_{\mathcal{E}}, K = K_{\mathcal{E}}$ and $M = M_{\mathcal{E}}$.

a) Let $m \in M$. Since m is invertible, we have $mV_{ii} = V_{ii}$. For $i \ne j$ and $z_{ij} \in V_{ij}$ we have $z_{ij} = \{e_i\, z_{ij}\, e_j\}$ and hence, by (2.2') and the Peirce multiplication rules,

$$mz_{ij} = m\{e_i\, z_{ij}\, e_j\} = \{me_i\, m^{*-1}z_{ij}\, me_j\} \in \{V_{ii}\, V\, V_{jj}\} \subset V_{ij}\,,$$

whence the first equality in a). The second is then immediate since the Peirce spaces V_{ij} are the joint eigenspaces of the commuting endomorphisms $L(e_i), 1 \le i \le n$.

b) Let $\omega = \omega_1 \oplus \cdots \oplus \omega_n \in \Omega_1 \oplus \cdots \oplus \Omega_n$. Then, by the Peirce multiplication rules, $P(\omega)V_{ii} = P(\omega_i)V_{ii} \subset V_{ii}$ and hence $A \subset M$. Let $\sqrt{\omega} = \sqrt{\omega_1} \oplus \cdots \sqrt{\omega_n}$ where $\sqrt{\omega_i} \in \Omega_i$ is the unique square root in Ω_i of ω_i. Then $P(\sqrt{\omega}) \in A$ and $P(\sqrt{\omega})e = \omega$. If there exist $a, a' \in A$ with $ae = a'e$ and $a = P(x)$, $a' = P(x')$ for $x, x' \in \Omega_1 \oplus \cdots \oplus \Omega_n$ we get $x^2 = P(x)e = P(y)e = y^2$, thus $x = y$ by the uniqueness of the square root on Ω, and $a = a'$. Since $g\bar{\Omega} = \bar{\Omega}$ for any $g \in G(\Omega)$, we have $m\Omega_i = m(\bar{\Omega} \cap V_{ii}) \subset \bar{\Omega} \cap V_{ii} = \Omega_i$ for every $m \in M$. Therefore $M(\Omega_1 \oplus \cdots \oplus \Omega_n) \subset \Omega_1 \oplus \cdots \oplus \Omega_n$.

c) For any $m \in M \cap \operatorname{Aut} V$ we have $m|V_{ii} \in \operatorname{Aut} V_{ii}$ and hence $me_i = e_i$. Conversely, any $f \in K \subset \operatorname{Aut} V \subset G(\Omega)$ has the property $fV_{ii} = fV(e_i, 1) = V(fe_i, 1) = V_{ii}$ and thus lies in $M \cap \operatorname{Aut} V$. The equality $M_{\mathcal{E}} \cap \operatorname{Aut} V = \{m \in M; mm^* = Id\}$ then follows from (2.3).

d) For $m \in M$ there exists a unique $a \in A$ such that $me = ae$, i.e., $k = a^{-1}m \in \operatorname{Aut} V \cap M = K$ in view of (2.3) and c). (2) follows from the fact that exp is a diffeomorphism.

Remarks. 1) Let $\operatorname{Str}(V)$ be the structure group of V. Since $\operatorname{Str}(V) = \operatorname{Str}(V)^*$, it is the group of real points of a reductive algebraic group, and $G(\Omega) \subset \operatorname{Str}(V)$ is a finite covering of the (topological) identity component $\operatorname{Str}(V)^0$. More generally, $\operatorname{Str}(V)_{\mathcal{E}} := \{g \in \operatorname{Str}(V); mV_{ij} = V_{ij}$ for all $i,j\}$ is invariant under * and hence the group of real points of a reductive algebraic group. Since $\operatorname{Str}(V)^0_{\mathcal{E}} \subset M_{\mathcal{E}} \subset \operatorname{Str}(V)_{\mathcal{E}}$ it follows that $M_{\mathcal{E}}$ is a real reductive group in the sense of [W; 2.1]. The decomposition (2) is the Cartan decomposition of $M_{\mathcal{E}}$ in the sense of [W; 2.1.8]. In particular, $K_{\mathcal{E}}$ is a maximal compact subgroup of $M_{\mathcal{E}}$.

2) If $\mathcal{E} = \{e\}$ then (2) specializes to the well-known Cartan decomposition $G(\Omega) = P(\Omega) \cdot \operatorname{Aut} V$ ([BK; XI Satz 4.5]). The corresponding decomposition of the Lie algebra

Lie$G(\Omega) = \mathfrak{g}(V)$ is the Cartan decomposition $\mathfrak{g}(V) = L(V) \oplus \operatorname{Der} V$. If $\mathcal{E}$ is a Jordan frame, i.e., every e_i is primitive: $V_{ii} = \mathbb{R}e_i$, $A_{\mathcal{E}}$ is an abelian group and coincides with the group A of [FK; VI.3, p. 112]. In this case $\mathfrak{a} = L(V_{11} \oplus V_{22} \oplus \cdots \oplus V_{nn})$ is a maximal abelian subspace of $L(V) \subset \mathfrak{g}(V)$ so that $M_{\mathcal{E}}$ coincides with the group M of [W; 2.2.4].

6. Transformation groups of Ω defined by $\mathcal{E}$ and a partial order. We let $\preceq$ be a partial order on $I = \{1, \ldots, n\}$ which is weaker than the canonical order: $i \preceq j \Rightarrow i \leq j$. We put $i \prec j \Leftrightarrow i \preceq j, i \neq j$ and define

$$
\begin{aligned}
e_{\langle i\rangle} &= \textstyle\sum_{k \prec i} e_k\,, & \tau_{\langle i\rangle} &= \tau_{e_{\langle i\rangle}}\,,\\
V_{\langle i]} &= \oplus_{k \prec i} V_{ki} = V(e_{\langle i\rangle}, \tfrac{1}{2}) \cap V(e_i, \tfrac{1}{2})\,, & V^{(i\prec)} &= \oplus_{i \prec j} V_{ij}\,,\\
V_{ij\prec} &= (\oplus_{j \prec l} V_{il}) \oplus (\oplus_{i<k\leq l} V_{kl})\,, (1 \leq i \leq j \leq n), & V_{ij\preceq} &= V_{ij} \oplus V_{ij\prec}.
\end{aligned}
$$

Thus, $V^{(i\prec)} = V^{(i)}$ in case $\preceq$ coincides with the canonical order. We will consider the following subgroups of $G(\Omega)$:

$$
\begin{aligned}
N_{\mathcal{E},\prec} &= \{u \in G(\Omega); (u - \mathrm{Id})V_{ij} \subset V_{ij\prec} \text{ for all } i \leq j\},\\
T_{\mathcal{E},\preceq} &= \{t \in G(\Omega)\,;\, tV_{ij} \subset V_{ij\preceq} \text{ for all } i \leq j\}.
\end{aligned}
$$

Theorem. a) *The group $N_{\mathcal{E},\prec}$ is a unipotent simply-connected Lie subgroup of $T_{\mathcal{E},\preceq}$ and has the descriptions*

$$
\begin{aligned}
N_{\mathcal{E},\prec} &= \{\tau_1(z_1) \cdots \tau_{n-1}(z_{n-1})\,;\, z_i \in V^{(i\prec)}, 1 \leq i < n\} && (1)\\
&= \{\tau_{\langle n\rangle}(z_n) \ \cdots \ \tau_{\langle 2\rangle}(z_2)\,;\, z_i \in V_{\langle i]}\,, 1 < i \leq n\}. && (2)
\end{aligned}
$$

The Lie algebra of $N_{\mathcal{E},\prec}$ is

$$\mathfrak{n}_{\mathcal{E},\prec} = \oplus_{i=1}^{n-1} \{L(z_i\,, e_i)\,;\, z_i \in V^{(i\prec)}\} = \oplus_{i\prec j} L(V_{ij}\,, e_i).$$

b) *The group $M_{\mathcal{E}} \subset T_{\mathcal{E},\preceq}$ normalizes $N_{\mathcal{E},\prec}$, and $T_{\mathcal{E},\preceq}$ is a semidirect product*: $T_{\mathcal{E},\preceq} = M_{\mathcal{E}} \cdot N_{\mathcal{E},\prec}$.

c) $K_{\mathcal{E}} = T_{\mathcal{E}} \cap \operatorname{Aut} V = \{g \in T_{\mathcal{E}}; ge = e\} = \{g \in T_{\mathcal{E}}; gg^* = \mathrm{Id}\}$.

Proof. For easier notation we abbreviate $K = K_{\mathcal{E}}$, $M = M_{\mathcal{E}}$, $N = N_{\mathcal{E},\prec}$ and $T = T_{\mathcal{E},\preceq}$.

a) Any $u \in N$ is of the form $u = \mathrm{Id} + n$ with n nilpotent, i.e., u is unipotent. Transitivity of $\prec$ implies that $\mathfrak{n} = \{n \in \operatorname{End} V; nV_{ij} \subset V_{ij\prec} \text{ for all } i \leq j\}$ is a nilpotent subalgebra of $\operatorname{End} V$. Therefore, $u^{-1} = \mathrm{Id} + \sum_{i\geq 1} (-n)^i$ shows that N is closed under taking inverses. Similarly, N is also closed under products and therefore a subgroup of $G(\Omega)$. It is a closed subgroup of $G(\Omega)$ and therefore a Lie subgroup of $G(\Omega)$. It follows from (1) that N is simply-connected (This is not so surprising since, by [B; §9.5, Cor. 2 of Prop. 18], any unipotent group is simply-connected.) We are therefore left with proving (1) and (2).

Proof of (1): For any $i \prec j$ we have $\tau_i(z_{ij}) \in N$ by (4.1). Since $\tau_i(\sum_{j \succ i} z_{ij}) = \prod_{j \succ i} \tau_i(z_{ij})$, we also have $\{\tau_1(z_1) \cdots \tau_{n-1}(z_{n-1})\,;\, z_i \in V^{(i\prec)}\} \subset N$. Conversely, let $u \in N$. By definition, there exist unique $z_1 \in V^{(1\prec)}$ and $v_0 \in V(e_1, 0)$ such that $ue_1 = e_1 + z_1 + v_0$. Observe that $u^* x_{11} = x_{11}$ for all $x_{11} \in V_{11}$ since $(u - \mathrm{Id})V \subset V_{11}^{\perp}$. Hence, by (2.4) and the Peirce multiplication rules,

$$\begin{aligned} ux_{11} &= uP(e_1)x_{11} = P(ue_1)u^{*-1}x_{11} = P(e_1 + z_1 + v_0)x_{11} \\ &= x_{11} \oplus \{e_1\, x_{11}\, z_1\} \oplus P(z_1)x_{11} = x_{11} \oplus 2x_{11}z_1 \oplus P(z_1)x_{11}. \end{aligned}$$

In view of (2.4) this shows $ux_{11} = \tau_1(z_1)x_{11}$. Let $\tilde{u} = \tau_1(z_1)^{-1}u \in N$ and put $c = e - e_1$. Since $V' := V(c, 1) = V(e_1, 0) = \oplus_{2 \le k \le l \le n} V_{kl}$ it follows that $\tilde{u}$ leaves V' invariant. Because $\tilde{u}\bar{\Omega} = \bar{\Omega}$ and $\Omega_c = \bar{\Omega} \cap V(c, 1)$ we see that $\tilde{u}|V'$ lies in the corresponding subgroup N' of $G(\Omega_c)$ defined with respect to $\mathcal{E} \cap V(c, 1) = (e_2, \ldots, e_n)$ and the restriction of $\preceq$ to $\{2, \ldots, n\}$. By induction, $\tilde{u}|V' = \tau_2(z_2) \cdots \tau_{n-1}(z_{n-1})|V'$ for suitable $z_i \in V^{(i\prec)}$ (= Id if $n = 2$). Then

$$\widehat{u} := (\tau_2(z_2) \cdots \tau_{n-1}(z_{n-1}))^{-1}\tilde{u} = \tau_{n-1}(-z_{n-1}) \cdots \tau_2(-z_2)\tilde{u} \in N$$

has the property $\widehat{u}x_{ii} = x_{ii}$ for all $1 \le i \le n$. Thus, $\widehat{u} = M \cap N = \{\mathrm{Id}\}$.

Proof of (2): We have for $k \prec i$

$$\tau_{\langle i \rangle}(z_{ki}) = \exp L(z_{ki}, e_{\langle i \rangle}) = \exp L(z_{ki}, e_k) = \tau_k(z_{ki}), \tag{4}$$

and hence for $z_i \in V_{\langle i]}$

$$\tau_{\langle i \rangle}(z_i) = \prod_{k \prec i} \tau_{\langle i \rangle}(z_{ki}) = \prod_{k \prec i} \tau_k(z_{ki}).$$

This shows that

$$N' := \{\tau_{\langle n \rangle}(z_n) \ \cdots \ \tau_{\langle 2 \rangle}(z_2)\,;\, z_i \in V_{\langle i]}, 1 < i \le n\} \subset N.$$

By (4), N' contains the canonical generators of N. Hence $N' = N$ if N' is a subgroup of N. To prove this, it suffices to show that for $j < l$ and $i \prec j, k \prec l$ we have $\tau_{\langle j \rangle}(z_{ij})\, \tau_{\langle l \rangle}(z_{kl}) \in N'$. Since $|\{i, j, l\}| = 3$ and $\tau_{\langle j \rangle}(z_{ij})\, \tau_{\langle l \rangle}(z_{kl}) = \tau_i(z_{ij})\, \tau_k(z_{kl})$ there are two cases to be considered: if $k = i$ or $k \notin \{i, j, l\}$ then, by (4.2), $\tau_i(z_{ij})\, \tau_k(z_{kl}) = \tau_k(z_{kl})\tau_i(z_{ij}) = \tau_{\langle l \rangle}(z_{kl})\tau_{\langle j \rangle}(z_{ij}) \in N'$, while for $k = j$ we have, by (4.4) and (4)

$$\begin{aligned} \tau_i(z_{ij})\tau_j(z_{jl}) &= \tau_j(z_{jl})\tau_i(z_{ij} - 2z_{ij}z_{jl}) = \tau_{\langle l \rangle}(z_{jl})\tau_{\langle l \rangle}(-2z_{ij}z_{jl})\tau_{\langle j \rangle}(z_{ij}) \\ &= \tau_{\langle l \rangle}(z_{jl} - 2z_{ij}z_{jl})\, \tau_{\langle j \rangle}(z_{ij}) \in N'. \end{aligned}$$

This finishes the proof of (2).

Since $\tau_i(z_i) = \exp L(z_i, e_i)$ we have $\mathfrak{n}' := \sum_{i=1}^{n} L(V^{(i\prec)}, e_i) \subset \mathfrak{n} := \mathrm{Lie} N_{\mathcal{E},\prec}$ by (1). That the sum is direct follows from $L(z_i\,, e_i)e_j = \delta_{ij} z_i$. To conclude $\mathfrak{n}' = \mathfrak{n}$ it is sufficient to prove that $\mathfrak{n}'$ is a subalgebra. Indeed, the Lie subgroup N' of N corresponding to $\mathfrak{n}'$ contains $\tau_i(V^{(i\prec)})$, hence $N' = N$ by (1) and therefore $\mathfrak{n}' = \mathfrak{n}$. That $\mathfrak{n}'$ is a subalgebra of $\mathfrak{n}$ follows from the following calculations. Let $z_i \in V^{(i)}, w_j \in V^{(j)}$. If $i = j$ then, by (2.2),

$$[L(z_i\,, e_i), L(w_i\,, e_i)] = L(\{z_i\, e_i\, w_i\}, e_i) - L(w_i, \{e_i\, z_i\, e_i\}) = 0$$

since $\{e_i\, z_i\, e_i\} = 0$, $\{z_i\, e_i\, w_i\} \in V(e_i, 0)$ and $L(V(e_i, 0), V(e_i, 1)) = 0$. If $i < j$ then $w_j \in V(e_i, 0)$ and so $\{z_i\, e_i\, w_j\} = 0$. Hence, (2.2) shows

$$[L(z_i\,, e_i), L(w_j\,, e_j)] = -L(w_j\,, \{e_i\, z_i\, e_j\}).$$

Here $\{e_i\, z_i\, e_j\} = z_{ij} \in V_{ij}$ and so $\{e_i\, e_i\, z_{ij}\} = z_{ij}$. A second application of (2.2) then yields

$$-L(w_j, z_{ij}) = [L(e_i, e_i), L(w_j, z_{ij})] = -[L(w_j, z_{ij}), L(e_i, e_i)] = -L(\{w_j z_{ij} e_i\}, e_i)$$

where $\{w_j z_{ij} e_i\} = \sum_{j\prec k} \{w_{jk}\, z_{ij}\, e_i\}$. Each term $\{w_{jk}\, z_{ij}\, e_i\} \in V_{ik}$ with $i \prec j \prec k$ since $z_{ij} = 0$ unless $i \prec j$. This proves $[L(z_i\,, e_i), L(w_j, e_j)] \in L(V^{(i\prec)}, e_i)$.

b) It follows from Theorem 4.a) that $M \subset T$. Moreover, M normalizes N since for $m \in M$ and $u \in N$ we have

$$(mum^{-1} - \mathrm{Id})V_{ij} = m(u - \mathrm{Id})m^{-1}V_{ij} = m(u - \mathrm{Id})V_{ij} \subset mV_{ij\prec} = V_{ij\prec}.$$

Because $M \cap N = \{\mathrm{Id}\}$ it is clear that $MN = \{mn; m \in M, n \in N\} \subset T$ is a semidirect product. To prove the other inclusion, let $t \in T$. We will construct inductively an $n \in N$ such that $nt \in M$. Assuming that $tV_{jj} = V_{jj}$ for $1 \le j < i$ we will find $n_i \in N$ such that $n_i t V_{jj} = V_{jj}$ for $1 \le j \le i$. Let $te_i = x_{ii} + x_{i\prec} + b$ where b is an element of

$$B = \oplus_{i<k\le l\le n} V_{kl} = V(e_{i+1} + \cdots + e_n, 1) \subset V(e_i, 0).$$

We claim that $x_{ii} \in \Omega_i$. Indeed, $te = te_1 + \cdots + te_i + \cdots + te_n = x_{11} + \cdots + x_{ii} + x_{i\prec} + \tilde{b}$ for suitable $x_{jj} \in V_{jj}$ and $\tilde{b} \in B$, and therefore $x_{ii} = P(e_i)te \in P(e_i)\Omega = \Omega_i$ by [MN; 3.2]. For any $z \in V^{(i\prec)}$ we have $\tau_i(z)te_i = x_{ii} \oplus 2zx_{ii} + x_{i\prec} \oplus b'$ for a suitable $b' \in B$. Since $x_{ii} \in \Omega_i$ is invertible in V_{ii}, we can find $z' \in V^{(i\prec)}$ such that $2z'x_{ii} + x_{i\prec} = 0$. Thus, replacing t by $\tau_i(z')t$, we can assume $te_i = x_{ii} + b'$ and, by (2.4), still have $tV_{jj} \subset V_{jj}$ for $j < i$. Let

$$C = (\oplus_{i<l\le n} V_{il}) \oplus (\oplus_{i<k\le l} V_{kl}) = (\oplus_{i<l\le n} V_{il}) \oplus B.$$

Since $t^{-1}C \subset C$ we have $t^{*-1}V_{ii} \subset D := C^\perp = V_{ii} \oplus (\oplus_{1\le k<i, k\le l} V_{kl})$, the orthogonal complement of C with respect to the trace form. Because of $P(B)D = 0 = \{V_{ii}\, D\, B\}$ it now follows for arbitrary $v_{ii} \in V_{ii}$

$$\begin{aligned} tv_{ii} &= tP(e_i)v_{ii} = P(te_i)t^{*-1}v_{ii} \in P(x_{ii} + b')D \\ &= P(x_{ii})D + P(b')D + \{x_{ii}\, D\, b'\} = P(x_{ii})D = V_{ii}, \end{aligned}$$

which completes the induction process.

c) With respect to a suitable orthonormal basis of V, any $g \in T$ is represented by an upper triangular block matrix whose block structure is determined by the Peirce spaces V_{ij}. If such a g is also orthogonal, the matrix is in fact a diagonal block matrix. It follows that $ge_i \in V_{ii}$ is an idempotent of the same rank as e_i and hence $ge_i = e_i$. Thus $T \cap \operatorname{Aut} V \subset K$, and the other inclusion is obvious. The remaining equalities then follow from (2.3).

Remarks. 1) Since $N_{\mathcal{E},\prec}$ is unipotent it does not contain any non-trivial compact subgroup, and thus $K_{\mathcal{E}}$ is also a maximal compact subgroup of $T_{\mathcal{E},\preceq}$, see the remark in **5.**

2) The map

$$V^{(1\prec)} \times \cdots \times V^{(n-1\prec)} \to N_{\mathcal{E}} : (z_1, \ldots, z_{n-1}) \mapsto \tau_1(z_1) \cdots \tau_{n-1}(z_{n-1})$$

is in fact a diffeomorphism. Indeed, that the map is a bijection follows from (1) and Proposition 3. As a product of exponentials, it is obviously differentiable. That its inverse is differentiable too, can be shown inductively, following the method of the proof of (1). Of course, since N is nilpotent this is also a special case of a general result on canonical coordinates of solvable Lie groups ([B; §9.6, Prop. 20]).

3) If $\preceq$ is the *minimal order*, i.e., $i \preceq j \Leftrightarrow i = j$, we have $N_{\mathcal{E},\prec} = \{\mathrm{Id}\}$ and $T_{\mathcal{E},\preceq} = M_{\mathcal{E}}$. For example, this is the case if $\mathcal{E} = \{e\}$. On the other extreme, if $\mathcal{E}$ is a Jordan frame and $\preceq$ is the canonical order, the group $N_{\mathcal{E},\prec}$ coincides with the so-called *strict triangular subgroup* N of [FK; VI.3]. By (3) it is also the group N of [W; 2.1.8]. In this case, $A_{\mathcal{E}} \cdot N_{\mathcal{E},\prec}$ is a subgroup of $T_{\mathcal{E},\prec}$, the so-called *triangular subgroup* T of [FK; VI.3].

7. The AP cone ([MN]). An AP cone $\Omega(\mathcal{K}) \subset \Omega$ is defined in terms of an orthogonal system $(c_1, \ldots, c_s)$ of primitive idempotents $c_i \in V$ and a unital ring $\mathcal{K}$, i.e., a set of subsets of $\{1, \ldots, s\}$ which is closed under union and intersection: $K, L \in \mathcal{K} \Rightarrow K \cup L \in \mathcal{K}$ and $K \cap L \in \mathcal{K}$, and which moreover has the property that $\emptyset \in \mathcal{K}$ and $\{1, \ldots, s\} \in \mathcal{K}$. To describe $\Omega(\mathcal{K})$ we need the following notations. For any $K \subset \{1, \ldots, s\}$ and $x \in V$ we put $c_K = \sum_{k \in K} c_k$ and $x_K = P(c_K)x$, the $V(c_K, 1)$-component of x. If $x \in \Omega$ and $K \neq \emptyset$ then $x_K \in P(c_K)\Omega$, and one knows that this is the symmetric cone of the Euclidean Jordan algebra $V(c_K, 1)$. In particular, x_K is invertible in $V(c_K, 1)$. We denote by x_K^{-1} the inverse of x_K in $V(c_K, 1)$ and view x_K^{-1} as an element of V. We note that in general $x_K^{-1} \neq P(c_K)(x^{-1})$. For $K = \emptyset$ we put $c_\emptyset = 0$ and $x_K^{-1} = 0^{-1} = 0$. The *AP cone* $\Omega(\mathcal{K})$ is then defined as the set of all $x \in \Omega$ satisfying

$$x_{K \cup L}^{-1} + x_{K \cap L}^{-1} = x_K^{-1} + x_L^{-1}$$

for all $K, L \in \mathcal{K}$. Equivalent characterizations of $\Omega(\mathcal{K})$ are given in [MN; Thm. 2.4].

The link with the results obtained so far in this paper is property (1) below. To explain it, we recall that $\emptyset \neq K \in \mathcal{K}$ is *join-irreducible* if K is not a union of proper subsets of K belonging to $\mathcal{K}$. Thus, if we put $\langle K \rangle := \cup\{K' \in \mathcal{K}\,;\, K' \subsetneq K\}$ and $[K] := K \setminus \langle K \rangle$ then K is join-irreducible if and only if $[K] \neq \emptyset$. We denote by $\mathcal{J}(\mathcal{K})$ the set of all join-irreducible sets in $\mathcal{K}$. One knows [AP; 2.1] that any $K \in \mathcal{K}$ is partitioned by $\{[L]\,;\, L \in \mathcal{J}(\mathcal{K}) \text{ and } L \subset K\}$. Moreover, by [AP; 2.7], one can always find a *never-decreasing listing of* $\mathcal{J}(\mathcal{K})$, i.e., an enumeration $\mathcal{J}(\mathcal{K}) = (K_1, \ldots, K_n)$ with the property $i < j \Rightarrow K_j \not\subset K_i$. We fix such a listing and define a partial order $\preceq$ on $\{1,\ldots,n\}$ by $i \preceq j \;\Leftrightarrow\; [K_i] \subset K_j$. For $1 \le j \le s$ we put $e_j = \sum_{i \preceq j} c_i$ and obtain in this way an orthogonal system $\mathcal{E} = (e_1,\ldots,e_n)$. After renumbering, we may assume that $\preceq$ is weaker than the canonical order, so that we are in the setting of **6.** Then, by [MN; 2.14], the map

$$F_{\mathcal{K}} \,:\, V^{(1\prec)} \times \cdots \times V^{(n-1\prec)} \times \Omega_1 \times \cdots \times \Omega_n \;\rightarrow\; \Omega(\mathcal{K})$$

given by

$$F_{\mathcal{K}}(z_1,\,\cdots,z_{n-1},y_1,\,\cdots,y_n) = \tau_1(z_1)\,\cdots\,\tau_{n-1}(z_{n-1})(y_1 \oplus\,\cdots\,\oplus y_n)$$

is a bijection. Thus,

$$\Omega(\mathcal{K}) = N_{\mathcal{E},\prec}(\Omega_1 \oplus \cdots \oplus \Omega_n) \tag{1}$$

We transport the obvious manifold structure of $V^{(1\prec)} \times \cdots \times V^{(n-1\prec)} \times \Omega_1 \times \cdots \times \Omega_n$ to $\Omega(\mathcal{K})$ via $F_{\mathcal{K}}$. By Proposition 3, $\Omega(\mathcal{K})$ is then a simply-connected closed submanifold of Ω (with the induced topology). Also, Proposition 3 implies,

$$\Omega(\mathcal{K}) = \Omega \Leftrightarrow \preceq \text{ is the canonical order.} \tag{2}$$

$$\Omega(\mathcal{K}) = \Omega_1 \oplus \cdots \oplus \Omega_n \Leftrightarrow \preceq \text{ is the minimal order.} \tag{3}$$

8. Theorem. *$T_{\mathcal{E},\preceq}$ is a transitive Lie transformation group of $\Omega(\mathcal{K})$. For this operation, the isotropy group of $e \in \Omega(\mathcal{K})$ is $K_{\mathcal{E}}$, and we have an isomorphism of manifolds*

$$\Omega(\mathcal{K}) \approx T_{\mathcal{E},\preceq}/K_{\mathcal{E}}. \tag{1}$$

Proof. For easier notation we abbreviate $K = K_{\mathcal{E}}$, $M = M_{\mathcal{E}}$, $N = N_{\mathcal{E},\prec}$ and $T = T_{\mathcal{E},\preceq}$. By Theorem 5.b, we know that M operates transitively on $\Omega_1 \oplus \cdots \oplus \Omega_n$. Thus, by (7.1), $\Omega(\mathcal{K}) = NMe$. But this implies that both M and N leave $\Omega(\mathcal{K})$ invariant: $N\Omega(\mathcal{K}) = NNMe = \Omega(\mathcal{K})$ and, since M normalizes N, $M\Omega(\mathcal{K}) = MNMe = NMMe = \Omega(\mathcal{K})$. Therefore, T operates transitively on $\Omega(\mathcal{K})$. By Theorem 6.c), the isotropy group of e in T is $K_{\mathcal{E}}$, and hence (1) follows from ([B; §1.7 Prop. 14]).

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