Gelfand-Kirillov dimension and local finiteness of Jordan superpairs covered by grids and their associated Lie superalgebras

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Summary. In this paper we show that a Lie superalgebra L graded by a 3-graded irreducible root system has Gelfand-Kirillov dimension equal to the Gelfand-Kirillov dimension of its coordinate superalgebra A, and that L is locally finite if and only A is so. Since these Lie superalgebras are coverings of Tits-Kantor-Koecher superalgebras of Jordan superpairs covered by a connected grid, we obtain our theorem by combining two other results. Firstly, we study the transfer of the Gelfand-Kirillov dimension and of local finiteness between these Lie superalgebras and their associated Jordan superpairs, and secondly, we prove the analogous result for Jordan superpairs: the Gelfand-Kirillov dimension of a Jordan superpair V covered by a connected grid coincides with the Gelfand-Kirillov dimension of its coordinate superalgebra A, and V is locally finite if and only if A is so.

Introduction

The Gelfand-Kirillov dimension is an interesting invariant for any algebraic structure. For a nonassociative superalgebra A over an arbitrary field k it is defined as follows. For any subspace B of A and any $n \in \mathbb{N}$ we put $B^{(1)} = B$, $B^{(n)} = \sum_{i+j=n} B^{(i)} B^{(j)}$, $B^{[n]} = \sum_{1 \le i \le n} B^{(i)}$. Then the Gelfand-Kirillov dimension of A is defined as

$$\operatorname{GKdim} A = \sup_B \left(\limsup_n \frac{\ln(\dim B^{[n]})}{\ln n} \right), \tag{1}$$

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where the supremum is taken over all finite dimensional subspaces B of A. The Gelfand-Kirillov dimension has been widely considered for Lie algebras. Concerning Jordan algebras the main advances in this area are the investigation of Jordan algebras of Gelfand-Kirillov dimension one by Martinez-Zelmanov ([**14**]) based on earlier work of Martinez ([**13**]) and the recent classification of graded simple Jordan superalgebras of growth one by Kac-Martinez-Zelmanov ([**8**]). The motivation for the paper [**8**] is a conjecture on the structure of $\mathbb{Z}$-graded simple Lie superalgebras, and it confirms this conjecture in the special case that the Lie superalgebra is the Tits-Kantor-Koecher superalgebra of a Jordan superalgebra.

In this paper we study both varieties of superalgebras mentioned above, Jordan and Lie. Our preference is with Jordan structures, and we will use the superversion of the fundamental Tits-Kantor-Koecher construction to translate our results from "Jordan to Lie". For Lie algebraists, Jordan superpairs over a field of characteristic $\neq 2, 3$ can be introduced as follows: We call a Lie superalgebra L over k *Jordan 3-graded* if

(i) $L = L_1 \oplus L_0 \oplus L_{-1}$ is a so-called short $\mathbb{Z}$-grading, i.e., $[L_i, L_j] \subset L_{i+j}$ for $i, j \in \{0, \pm 1\}$, and

(ii) $L_0 = [L_1, L_{-1}]$.

Here the L_i are $\mathbb{Z}_2$-graded subspaces and $[.,.]$ denotes the product of L. In this case, we have well-defined trilinear maps on $V = (L_1, L_{-1})$, namely

$$L_\sigma \times L_{-\sigma} \times L_\sigma \to L_\sigma : (x, y, z) \mapsto [[x, y], z] =: \{x\, y\, z\}$$

for $\sigma = \pm 1$, satisfying two basic identities, one of which is

$$\{x\, y\, z\} = (-1)^{|x||y| + |x||z| + |y||z|} \{z\, y\, x\}$$

for homogeneous x, y, z of degree $|x|, |y|$ and $|z|$ respectively. Taking these two identities as axioms one arrives at the definition of a *Jordan superpair* (see [**11**] or [**16**, 3] for the supersetting). There are in general many Jordan 3-graded Lie superalgebras with a given Jordan superpair, but all are central extensions of a "minimal" one, the so-called Tits-Kantor-Koecher superalgebra $\mathfrak{K}(V)$ of the Jordan superpair V. Thus, the importance of the Tits-Kantor-Koecher construction lies in the fact that every abstractly defined Jordan superpair $V = (V^+, V^-)$ arises in this way from a Jordan 3-graded Lie superalgebra.

The interplay between Jordan superpairs and Jordan 3-graded Lie superalgebras has been very fruitful, and there are many papers where this is used, for example in Zelmanov's fundamental paper on the classification of finite gradings of simple Lie algebras [**21**], or see [**19**], [**5**], [**4**] and [**7**] for more recent examples. Generalizing a result of Martinez for Jordan 3-graded Lie algebras ([**13**, Thm. 3.2]) to the supercase we prove:

Transfer Proposition. (2.10) *Let $L = L_1 \oplus [L_1, L_{-1}] \oplus L_{-1}$ be a Jordan 3-graded Lie superalgebra over a field of characteristic different from 2. Then the Gelfand-Kirillov dimension of L and the Gelfand-Kirillov dimension of its associated Jordan superpair $V = (L_1, L_{-1})$ coincide.*

As the class of Jordan superpairs for which we want to study the Gelfand-Kirillov dimension, we have chosen Jordan superpairs covered by a connected grid. This is at present the only class of arbitrary dimensional Jordan superpairs for which one has a structure theory ([**6**], [**16**]). Two features of this class of Jordan superpairs are important for the following:

(1) There exists a 3-graded irreducible locally finite root system $R = R_1 \dot{\cup} R_0 \dot{\cup} R_{-1}$, the root system associated to the connected grid covering the Jordan superpair V, such that V has a grading $V = \bigoplus_{\alpha \in R_1} V_\alpha$. Locally finite root systems, simply called root systems in the following, are the direct limits of finite root systems, and therefore include not only the usual finite root systems but also the canonical infinite rank analogues of the finite root systems (see [**12**] for an exposition of this theory including a classification). Imposing the condition of a 3-grading excludes $R = \mathrm{E}_8, \mathrm{F}_4$ and G_2.

(2) One can associate a supercoordinate system $\mathcal{C}$ to V which together with R in (1) above completely determines V. This supercoordinate system always consists of a unital superalgebra A, called *coordinate superalgebra*, which is either Jordan ($R = \mathrm{A}_1, \mathrm{B}_2$) or alternative in the other cases (even associative for $R = \mathrm{A}_I$, $|I| \geq 3$, and $R = \mathrm{C}_I$,$|I| \geq 4$, and supercommutative associative for $R = \mathrm{B}_I, |I| \geq 3$, $R = \mathrm{D}_I, \mathrm{E}_6$ and E_7). If R is simply laced then $\mathcal{C} = A$ but in the other cases some additional data, e.g. an involution of A, are part of $\mathcal{C}$. We can now state our main result.

Theorem A. (2.9) *Let V be a Jordan superpair covered by a connected grid and let A be the associated coordinate superalgebra A. Then the Gelfand-Kirillov dimensions of V and of A coincide.*

Combining now the Theorem A above and the Transfer Proposition determines the Gelfand-Kirillov dimension of the Jordan 3-graded Lie superalgebras whose associated Jordan superpairs are covered by a connected grid. In [**7**, 2.9] we have given a characterization of these Lie superalgebras which does not use Jordan theory: they are exactly the Lie superalgebras graded by a 3-graded irreducible root system. Lie superalgebras graded by a root system are a superversion of the concept of a Lie algebra graded by a root system. These Lie algebras were introduced and classified by Berman-Moody [**2**] in the simply-laced cases $\neq \mathrm{A}_1$ and for the other cases by Benkart-Zelmanov [**1**]. Our description for the case of 3-graded root systems is the superversion of a result from [**19**]. We define the coordinate superalgebra A associated to L as the coordinate superalgebra of the Jordan superpair V. Summarizing the above, we now have:

Corollary A. (2.11) *Let L be a Lie superalgebra over a field of characteristic $\neq 2, 3$ graded by a 3-graded root system. Let V be the associated Jordan superpair which, as we know, is covered by a connected grid and let A be the coordinate superalgebra of L. Then L and A have the same Gelfand-Kirillov dimension.*

The last part of this paper is devoted to the study of local finiteness of the superstructures mentioned above. Recall that a nonassociative algebra over a field k is *locally finite* if every finitely generated subalgebra is finite dimensional, and this definition makes also sense for superalgebras and other algebraic superstructures like Jordan superpairs. We prove the analogues of Theorem A and Corollary A above:

Theorem B. (3.7) *Let V be a Jordan superpair covered by a connected grid and let A be the associated coordinate superalgebra. Then V is locally finite if and only if A is so.*

Corollary B. (3.16) *Let L be a Lie superalgebra over a field of characteristic $\neq 2, 3$ graded by a 3-graded root system, and let A be the coordinate superalgebra of L. Then L is locally finite if and only if A is so.*

Theorems A and B and their corollaries are very similar, not only in their statements but also in the method of their proofs. Moreover, under the additional assumption that the base field has characteristic $\neq 2$, Theorem B actually becomes a special case of Theorem A. Indeed, we prove that in this case local finiteness is equivalent to Gelfand-Kirillov dimension equal to 0 for the varieties of associative, Lie, unital alternative or Jordan superalgebras and also Jordan superpairs (3.9, 3.14, 3.15.

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1. Review: definitions and general results.

1.1. Superalgebras. Unless specified otherwise, all algebraic structures are defined over an arbitrary base field k, and *everything is $\mathbb{Z}_2$-graded in the natural sense*. In particular this is so for vector spaces, called *superspaces*, subalgebras of superalgebras, etc. For a superspace $M = M_{\bar{0}} \oplus M_{\bar{1}}$ we will denote by $M_{\bar{0}}$ and $M_{\bar{1}}$ the even – respectively odd – part of M. For $m \in M_\mu$, $\mu \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, we put $|m| = \mu$ the degree (or parity) of m. We denote by G the Grassmann algebra over k in a countable number of generators. It is $\mathbb{Z}_2$-graded, $G = G_{\bar{0}} \oplus G_{\bar{1}}$ where $G_{\bar{0}}$ and $G_{\bar{1}}$ are spanned by monomials in an even – respectively odd – number of generators. The term "algebra" or "superalgebra" without further specification will always mean an arbitrary nonassociative, i.e., not necessarily associative, algebra or superalgebra over k.

Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a superalgebra. The *Grassmann envelope* $G(A) = (G_{\bar{0}} \otimes A_{\bar{0}}) \oplus (G_{\bar{1}} \otimes A_{\bar{1}}) \subset G \otimes A$ is an algebra with respect to the product $(g_a \otimes a)(g_b \otimes b) = g_a g_b \otimes ab$ for homogeneous elements $g_a, g_b \in G$, $a, b \in A$ satisfying $|g_a| = |a|$ and $|g_b| = |b|$. One can then define varieties of superalgebras by requiring that the Grassmann envelopes lie in a specific variety of algebra. For example, A is an alternative or associative or supercommutative or Lie superalgebra if and only if its Grassmann envelope $G(A)$ is, respectively, an alternative, associative, commutative or Lie algebra. For example, supercommutativity simply means $ab = (-1)^{|a||b|} ba$ for homogeneous $a, b \in A$. For a superspace M the Grassmann envelope $G(M)$ is defined as for superalgebras.

This approach to defining varieties of superalgebras also works for Jordan superalgebras in case $\frac{1}{2} \in k$, because then Jordan (super)algebras can be defined via a bilinear product. However, since several of our results do not need the assumption char $k \neq 2$, we will use the approach to Jordan superalgebras via quadratic maps ([**9**], [**16**]). For those readers who are happily willing to assume $\frac{1}{2} \in k$, the relations between the bilinear product and the quadratic product $U = (U_{\bar{0}}, U(.,.))$ are given by the formulas

$$U_{\bar{0}}(a_{\bar{0}})b = 2a_{\bar{0}}(a_{\bar{0}}b) - a_{\bar{0}}^2 b \tag{1}$$

$$(-1)^{|c||d|}U(b,d)c = 2(b(cd) + (bc)d - (-1)^{|c||d|}(bd)c) =: \{b\ c\ d\} \tag{2}$$

where $a_{\bar{0}} \in A_{\bar{0}}$ and $b, c, d \in A$ are homogeneous. Note that $U_{\bar{0}}$ is a quadratic map in the usual sense and that we do not have a $U(a_{\bar{1}})$ for odd $a_{\bar{1}}$.

A *superextension of* k is a supercommutative associative unital superalgebra over k. For example, G is a superextension of k. A basic recipe to create superalgebras in a given variety is to take an algebra B in the variety and a superextension S of k, and form the superalgebra $S \otimes B$. Its product is given by

$$(s_1 \otimes b_1) \cdot (s_2 \otimes b_2) = (s_1 s_2) \otimes (b_1 b_2). \tag{3}$$

We will call $S \otimes B$ the *S-extension of B*.

1.2. Jordan superpairs. We now come to the main object of this paper: Jordan superpairs over k. Suppose we have a pair $V = (V^+, V^-)$ of superspaces together with a pair $Q = (Q^+, Q^-)$ of quadratic maps $Q^\sigma \colon V^\sigma \to \mathrm{Hom}_k(V^{-\sigma}, V^\sigma)$, $\sigma = \pm$. By definition ([**16**, 2.8]), we therefore have supersymmetric k-bilinear maps $Q^\sigma(.,.) \colon V^\sigma \times V^\sigma \to \mathrm{Hom}_k(V^{-\sigma}, V^\sigma)$ of degree 0 and k-quadratic maps $Q^\sigma_{\bar{0}} \colon V^\sigma_{\bar{0}} \to \mathrm{Hom}_k(V^{-\sigma}, V^\sigma)_{\bar{0}}$ which are related by $Q^\sigma(u,w) = Q^\sigma_{\bar{0}}(u+w) - Q^\sigma_{\bar{0}}(u) - Q^\sigma_{\bar{0}}(w)$ for $u, w \in V^\sigma_{\bar{0}}$. Since $2Q^\sigma_{\bar{0}}(u) = Q^\sigma(u,u)$ the maps $Q^\sigma_{\bar{0}}$ are determined by Q^σ in case char$k \neq 2$. These quadratic maps induce canonical G-quadratic maps $\widetilde{Q}^\sigma \colon G(V^\sigma) \to \mathrm{Hom}_G(G(V^{-\sigma}), G(V^\sigma))$, and one calls V a *Jordan superpair* if its *Grassmann envelope* $G(V) = (G(V^+), G(V^-))$ together with $\widetilde{Q} = (\widetilde{Q}^+, \widetilde{Q}^-)$ is a Jordan pair in the usual sense ([**16**, 3.2]). We will follow common practice in Jordan theory and leave out the superscripts σ if no confusion can arise.

It is sometimes easier to define a Jordan superpair via the supertriple products, which are k-trilinear maps $\{...\} \colon V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$ related to the maps Q by $\{u\,v\,w\} = (-1)^{|v||w|}Q(u,w)v$ (in the introduction we have explained how these maps arise in the setting of Jordan 3-graded Lie superalgebras). Subpairs of Jordan superpairs are defined in the obvious way ([**16**, 3.3]).

For a Jordan superpair $V = (V^+, V^-)$ we define $\dim V = \dim V^+ + \dim V^-$, where by dim we mean $\dim_k$.

1.3. Jordan supertriple systems. Jordan superpairs are closely related to the so-called Jordan supertriple systems (or Jordan supertriples for short) which have the sometimes useful advantage of being defined on a single vector space (or k-module in the general setting). In the classical theory this is well-known and can for example be found in [**11**, §1] (for the supercase see [**16**, 3.9]).

For the convenience of the reader we shortly review the basic construction without however giving all details. Let T be a superspace together with a superquadratic map $P\colon T \to \operatorname{End}_k T$. As in the definition of Jordan superpairs, one then has a quadratic map $\widetilde{P}$ on the Grassmann envelope $\widetilde{P}\colon G(T) \to \operatorname{End}_G(G(T))$ and one defines T to be a *Jordan supertriple* if $G(T)$ together with $\widetilde{P}$ is a Jordan triple in the classical sense. In particular, we then have a trilinear map $\{\ldots\}\colon T \times T \times T \to T$ given by $\{u\,v\,w\} = (-1)^{|v||w|}P(u,w)v$. This so-called supertriple product is supersymmetric in the outer variables,

$$\{u\,v\,w\} = (-1)^{|u||v|+|u||w|+|v||w|}\{w\,v\,u\} \tag{1}$$

for homogeneous $u, v, w \in T$. It also satisfies the superversion of the so-called 5-linear identity (JP15), [**11**, 2.1].

If k has characteristic $\neq 2, 3$ then Jordan supertriples can be defined by (1) and this 5-linear identity. In our set-up, a unital Jordan superalgebra is the same as a Jordan supertriple containing an invertible element ([**16**, 3.11]).

To explain the connection between Jordan superpairs and Jordan supertriples let us first define an *involution* of a Jordan superpair V to be a homomorphism $\eta = (\eta^+, \eta^-)\colon V \to V^{\mathrm{op}}$ from V to the opposite Jordan superpair V^{op} satisfying $\eta^{-\sigma} \circ \eta^\sigma = \mathrm{Id}$ on V^σ, [**11**, 1.5] for Jordan pairs or [**16**, 3.9] for superpairs. Any Jordan supertriple T gives rise to a Jordan superpair $V(T) = (T, T)$ with involution $(\mathrm{Id}, \mathrm{Id})$. Conversely, if V is a Jordan superpair with an involution η one can define a Jordan supertriple T_η on V^+ by $P_{\bar 0}(u_{\bar 0})v = Q_{\bar 0}(u_{\bar 0})\eta^+(v)$ and $P(u,v) = Q(u,w)\eta^+(v)$ for $u, v, w \in V^+$. The associated Jordan superpair $V(T_\eta)$ is then canonically isomorphic to V. In this way, Jordan superpairs with involutions are the "same" as Jordan triple systems. Subtriples of Jordan supertriples are defined in the obvious way.

1.4. Jordan superalgebras associated to alternative superalgebras. An important source of examples for Jordan superalgebras, hence Jordan supertriples and Jordan pairs, are the so-called special Jordan superalgebras, i.e., the subalgebras of $A^{(+)}$ for A an associative superalgebra. We will need a slight generalization of this class of Jordan superalgebras, which is associated to alternative superalgebras.

Thus, let A be an alternative unital superalgebra. For $a_{\bar 0} \in A_{\bar 0}$ and homogeneous $a, b, c \in A$ define

$$U_{\bar 0}(a_{\bar 0})b = a_{\bar 0} b a_{\bar 0} \quad \text{and} \quad \{a\,b\,c\} = a(bc) + (-1)^{|a||b|+|a||c|+|b||c|}c(ba). \tag{1}$$

Then A together with the operations (1) form a unital Jordan superalgebra, denoted $A^{(+)}$. Indeed, this can either be seen by considering the Grassmann envelopes of A and $A^{(+)}$ (or by observing that $A^{(+)}$ is the McCrimmon-Meyberg algebra of the collinear pair e, f in $\mathbb{M}_{12}(A)$, see 1.6).

1.5. Jordan superpairs covered by grids. A grid $\mathcal{G}$ in a Jordan superpair V is a family of idempotents $g \in V_{\bar{0}}$ satisfying certain properties which we are going to explain now.

As in the classical theory ([**11**, 5.4]) every idempotent $c = (c^+, c^-) \in V_{\bar{0}}$ gives rise to a *Peirce decomposition* $V = V_2(c) \oplus V_1(c) \oplus V_0(c)$. A family $\mathcal{C}$ of idempotents is called a *cog* if for two distinct $c, c' \in \mathcal{C}$ we have exactly one of the following possibilities:

(i) $c \in V_0(c')$ or, equivalently, $c' \in V_0(c)$ (one says that c and c' are *orthogonal*);

(ii) $c \in V_1(c')$ and $c' \in V_1(c)$ (one calls c and c' *collinear*);

(iii) $c \in V_2(c')$ and $c' \in V_1(c)$ (one says that c' *governs* c);

(iv) $c \in V_1(c')$ and $c' \in V_2(c)$, i.e., c governs c'.

For any family $\mathcal{C}$ of idempotents we have simultaneous Peirce spaces

$$V_I(\mathcal{C}) = \bigcap_{c \in \mathcal{C}} V_{i(c)}(c), \quad I = (i(c))_{c \in \mathcal{C}}$$

a family of numbers in $\{0, 1, 2\}$. The sum of all $V_I(\mathcal{C})$ is direct but in general not all of V. In case of a cog $\mathcal{C}$, every $c \in \mathcal{C}$ lies in a certain $V_I(\mathcal{C})$ and one calls $\mathcal{C}$ a covering cog if V is the sum of the simultaneous Peirce spaces $V_I(\mathcal{C})$ with $\mathcal{C} \cap V_I(\mathcal{C}) \neq \emptyset$.

It turns out that every cog $\mathcal{C}$ can be enlarged to a so-called closed cog which has the same simultaneous Peirce spaces [**17**, 4.11]. These closed cogs can be defined in terms of closure properties with respect to forming idempotents ([**17**, 4.1]) or, equivalently, with the help of (locally finite) root systems ([**18**], [**12**]). We will review the latter definition. A *3-grading* of a root system R is a partition $R = R_1 \dot{\cup} R_0 \dot{\cup} R_{-1}$ such that $(R_i + R_j) \cap R \subset R_{i+j}$ for $i, j = 0, \pm 1$ and $(R_1 + R_{-1}) \cap R_0 = R_0$. Then a cog $\mathcal{C}$ is defined to be *closed* if there exist a 3-graded root system $R = R_1 \dot{\cup} R_0 \dot{\cup} R_{-1}$ and a bijection $R_1 \to \mathcal{C} : \alpha \mapsto c_\alpha$ such that $c_\alpha \in V_{\langle \alpha, \beta^\vee \rangle}(c_\beta)$ for all $\alpha, \beta \in R_1$ where $\langle \alpha, \beta^\vee \rangle$ denotes the Cartan integer in R. In this case we abbreviate $V_\alpha = V_I(\mathcal{C})$ if $c_\alpha \in V_I(\mathcal{C})$, and then have the simultaneous Peirce decomposition

$$V = \bigoplus_{\alpha \in R_1} V_\alpha \,. \tag{1}$$

Finally, a covering closed cog $\mathcal{G}$ is called a *covering grid*, and in this case V is said to be covered by $\mathcal{G}$. (One can also define grids in general, see [**16**, 4.3].) A covering grid whose associated 3-graded root system R is irreducible is called *connected*.

Every locally finite root system R is a direct sum of irreducible locally finite root systems $R^{(i)}$, $i \in I$. If R is 3-graded, every irreducible component $R^{(i)}$ is 3-graded too. Suppose V is covered by a grid $\mathcal{G}$ with associated 3-graded root system R. Corresponding to the decomposition $R = \bigcup_{i \in I} R^{(i)}$ in irreducible components $R^{(i)}$ is the decomposition of V in a direct sum of ideals

$$V = \bigoplus_{i \in I} V^{(i)}, \quad V^{(i)} = \bigoplus_{\alpha \in R^{(i)}} V_\alpha \tag{2}$$

where each ideal $V^{(i)}$ is now covered by the connected grid $\mathcal{G}^{(i)} = \mathcal{G} \cap V^{(i)}$. The decomposition allows one to reduce questions on V to the case where $\mathcal{G}$ is connected, see for example 2.2 and 3.5.

The classification of connected grids ([**17**]), or equivalently 3-graded root systems, shows that there are the following seven types of connected grids (for a definition see e.g. [**16**, 5]; I, J and K are arbitrary sets):

(i) rectangular grid $\mathcal{R}(J,K), 1 \le |J| \le |K|$, (R, R_1) is the rectangular grading $\mathrm{A}_I^{J,K}$ where $J \dot\cup K = I \dot\cup \{0\}$ for some element $0 \notin I$ and R is a root system of type A and rank $|I|$;

(ii) hermitian grid $\mathcal{H}(I)$, $2 \le |I|$, (R, R_1) is the hermitian grading of $R = \mathrm{C}_I$;

(iii) even quadratic form grid $\mathcal{Q}_e(I)$, $3 \le |I|$, (R, R_1) is the even quadratic form grading of $R = \mathrm{D}_{I \dot\cup \{0\}}$;

(iv) odd quadratic form grid $\mathcal{Q}_o(I)$, $2 \le |I|$, (R, R_1) is the odd quadratic form grading of $R = \mathrm{B}_{I \dot\cup \{0\}}$;

(v) alternating grid $\mathcal{A}(I)$, $5 \le |I|$, (R, R_1) is the alternating grading of $R = \mathrm{D}_I$;

(vi) Bi-Cayley grid $\mathcal{B}$, $R = \mathrm{E}_6$;

(vii) Albert grid $\mathcal{A}$, $R = \mathrm{E}_7$.

Corresponding to each type of covering grid is a coordinatization theorem which describes the corresponding Jordan superpair up to isomorphism. These so-called standard examples will be described in 1.7 below.

1.6. McCrimmon-Meyberg superalgebras and supercoordinate systems. Let V be a Jordan superpair over k covered by a connected standard grid $\mathcal{G}$ with associated 3-graded root system (R, R_1). For the further development the concept of a McCrimmon-Meyberg superalgebra [**16**, 4.2] is important. This is an alternative superalgebra defined for every collinear pair (g_α, g_β) on $V_\alpha^+ = V_2^+(g_\alpha) \cap V_1^+(g_\beta)$ by the product formula $ab = \{\{a\, g_\alpha^-\, g_\beta^+\}\, g_\beta^-\, b\}$. Modulo isomorphisms and taking the opposite algebra, the McCrimmon-Meyberg superalgebra does not depend on the chosen collinear pair g_α, g_β (see [**6**, 1.4]).

We will associate to V a *supercoordinate system* $\mathcal{C}$. Its definition depends on the type of R. However, for a simply-laced R of rank $R \ge 2$, equivalently $\mathcal{G}$ is an ortho-collinear family with $|\mathcal{G}| \ge 2$, we have the following uniform description

$$\mathcal{C} = \textit{McCrimmon-Meyberg superalgebra of some collinear pair } g_\alpha, g_\beta \in \mathcal{G}. \qquad (1)$$

This superalgebra is associative for $\operatorname{rank} R \ge 3$ and even associative supercommutative, i.e., a superextension of k, for R of type D or E. For non-simply-laced root systems, $\mathcal{C}$ will have more structure and will be defined in the review of the coordinatization theorems below 1.7.

1.7. Standard examples. The coordinatization theorems of [**16**, §5] described in (a) – (i) below can be summarized by saying that a Jordan superpair V is covered by a grid $\mathcal{G}$ if and only if V is isomorphic to a standard example $V(\mathcal{G}, \mathcal{C})$ depending on $\mathcal{G}$ and a supercoordinate system $\mathcal{C}$.

(a) For the rectangular grading of $R = \mathrm{A}_1$ with $|J| = |K| = 1$, we have $|R_1| = 1$ and $\mathcal{G}$ just consists of a single idempotent $\mathcal{G} = \{g\}$ which covers V in the sense that $V = V_2(g)$. Any such Jordan superpair is isomorphic to the superpair $\mathbb{J} = (J, J)$ of a unital Jordan superalgebra J over k. In this case $\mathcal{C} = J$.

(b) The standard examples for the remaining rectangular grids $\mathcal{R}(J, K)$, $|J| + |K| \geq 3$, are the rectangular matrix superpairs $\mathbb{M}_{JK}(A) = (\mathrm{Mat}(J, K; A), \mathrm{Mat}(K, J; A))$, where $\mathrm{Mat}(J, K; A)$ denotes the $J \times K$-matrices with finitely many non-zero entries from the unital superalgebra A which is alternative in case $R = \mathrm{A}_2$, i.e., $|J| + |K| = 3$, and associative otherwise. In the alternative case the product is described in [**16**, 5.4]. In the associative case $\mathbb{M}_{JK}(A)$ is a special Jordan superpair canonically imbedded in $(\mathrm{Mat}(J \cup K; A), \mathrm{Mat}(J \cup K; A))$, hence has Jordan supertriple product

$$\{u\, v\, w\} = uvw + (-1)^{|u||v|+|u||w|+|v||w|} wvu.$$

The $\mathbb{Z}_2$-grading of $\mathrm{Mat}(J, K; A)$ – respectively, of $\mathrm{Mat}(K, J; A)$ – is the one induced from A: $\mathrm{Mat}(J, K; A)_\mu = \mathrm{Mat}(J, K; A_\mu)$ for $\mu \in \mathbb{Z}_2$. Here $\mathcal{C} = A$.

(c) The Jordan superpairs covered by a hermitian grid $\mathcal{H}(I)$, $|I| = 2$, are exactly the $\mathbb{J} = (J, J)$ where J is a Jordan superalgebra with two strongly connected supplementary idempotents giving rise to a Peirce decomposition $\mathfrak{P}$ of J in the form $\mathfrak{P} : J = J_{11} \oplus J_{12} \oplus J_{22}$. In this case, the supercoordinate system of V is $\mathcal{C} = (J, \mathfrak{P})$.

(d) Examples of Jordan superpairs covered by hermitian grids $\mathcal{H}(I)$ are the hermitian Jordan superpairs $\mathbb{H}_I(A, A_0, \pi) = (\mathrm{H}_I(A, A_0, \pi), \mathrm{H}_I(A, A_0, \pi))$, where $\mathrm{H}_I(A, A_0, \pi) = \{x = (x_{ij}) \in \mathrm{Mat}(I, I; A) : x = x^{\pi \mathrm{T}},\ \text{all } x_{ii} \in A_0\}$, A is an alternative superalgebra which is associative for $|I| \geq 4$ and π is a nuclear involution with ample subspace A_0 ([**16**, 5.10]). We have $A_0 \subset \mathrm{H}(A, \pi) = \{a \in A : a^\pi = a\}$ and this is an equality if $\frac{1}{2} \in S$. For an associative A these are special Jordan superpairs and in the alternative case the product is described in [**16**, 5.11]. The $\mathbb{Z}_2$-grading of $\mathrm{H}_I(A, A_0, \pi)$ is induced from the $\mathbb{Z}_2$-grading of A (see (b)). Conversely, any Jordan superpair covered by a hermitian grid $\mathcal{H}(I)$, $|I| \geq 3$, is isomorphic to some hermitian matrix superpair $\mathbb{H}_I(A, A_0, \pi)$ as soon as the extreme radical of V vanishes (this is always the case if char $k \neq 2$). In the following we always assume this additional assumption when we consider Jordan superpairs covered by a hermitian grid. We put $\mathcal{C} = (A, A_0, \pi)$.

(e) For a superextension A of k and a set $I \neq \emptyset$ we denote by $H(I, A)$ the free A-module with even basis $\{h_{\pm i} : i \in I\}$ equipped with the hyperbolic superform q_I satisfying $q_I(h_{+i}, h_{-i}) = 1$ and $q_I(h_{\pm i}, h_{\pm j}) = 0$ for $i \neq j$. One can make $H(I, A)$ into a Jordan supertriple with quadratic maps given by $P_{\bar{0}}(m_{\bar{0}})n = q_I(m_{\bar{0}}, n)m_{\bar{0}} - q_I(m_{\bar{0}})n$ and $\{m\, n\, p\} = q_I(m, n)p + m q_I(n, p) - (-1)^{|n||p|} q_I(m, p)n$. The corresponding quadratic form superpair $\mathbb{EQ}_I(A, q_I) = (H(I, A), H(I, A))$ is covered by an even quadratic form grid $\mathcal{Q}_e(I)$. Conversely, any Jordan superpair covered by an even quadratic form grid $\mathcal{Q}_e(I)$, $|I| \geq 3$, is isomorphic to some $\mathbb{EQ}_I(A, q_I)$ ([**16**, 5.14]). Here $\mathcal{C} = A$.

(f) We let again A be a superextension of k and suppose that X is an A-module with an A-quadratic form q_X with a base point $e \in X_{\bar 0}$ satisfying $q_X(e) = 1$. For $I \neq \emptyset$ we put $M = H(I, A) \oplus X$, $q = q_I \oplus q_X$. The corresponding quadratic form superpair $(M, M) = \mathbb{OQ}_I(A, q_X)$ is covered by an odd quadratic form grid $\mathcal{Q}_o(I)$. Conversely, any Jordan superpair covered by an odd quadratic form grid $\mathcal{Q}_o(I)$, $|I| \geq 2$, is isomorphic to some $\mathbb{OQ}_I(A, q_X)$ ([**16**, 5.16]). In this case we put $\mathcal{C} = (A, X, q_X)$.

(g) For a superextension A of k we denote by $\mathrm{Alt}(I, A)$ the A-module of all alternating matrices $x \in \mathrm{Mat}(I, I; A)$, i.e., $x^{\mathrm{T}} = -x$ and all diagonal entries $x_{ii} = 0$. The alternating matrix superpair $\mathbb{A}_I(A) = (\mathrm{Alt}(I, A), \mathrm{Alt}(I, A))$ is a subpair of $\mathbb{M}_{II}(A)$; it is covered by an alternating grid $\mathcal{A}(I)$. Conversely, any Jordan superpair covered by an alternating grid $\mathcal{A}(I)$, $|I| \geq 4$, is isomorphic to some $\mathbb{A}_I(A)$ ([**16**, 5.18]). We put $\mathcal{C} = A$.

(h) The examples (e) and (g) are *superextensions of a Jordan pair* U, i.e., have the form $A \otimes U$ where A is a superextension of k and U is a Jordan pair, cf. 1.1. Moreover, U is *split of type* $\mathcal{G}$, i.e., $U^\sigma = \bigoplus_{g \in \mathcal{G}} k \cdot g^\sigma$. This is also so for the remaining two standard examples. A Jordan superpair over k is covered by a Bi-Cayley grid $\mathcal{B}$ if and only if it is isomorphic to the Bi-Cayley superpair $\mathbb{B}(A) = A \otimes_k \mathbb{M}_{12}(\mathbb{O}_k)$, the A-extension of the rectangular matrix superpair $\mathbb{B}(k) = \mathbb{M}_{12}(\mathbb{O}_k)$ for $\mathbb{O}_k$ the split Cayley algebra over k ([**16**, 5.20]). Here $\mathcal{C} = A$.

(i) A Jordan superpair V over k is covered by an Albert grid $\mathcal{A}$ if and only if there exists a superextension A of k such that V is isomorphic to the Albert superpair $\mathbb{AB}(A) = A \otimes_k \mathbb{AB}(k)$, the A-extension of the split Jordan pair $\mathbb{AB}(k) = \mathbb{H}_3(\mathbb{O}_k, k \cdot 1, \pi)$ where $\mathbb{O}_k$ is the split Cayley algebra over k with canonical involution π ([**16**, 5.22]). Here again $\mathcal{C} = A$.

1.8. 3-graded Lie superalgebras. There is an important connection between Jordan superpairs and so-called Jordan 3-graded Lie superalgebras. This sometimes allows one to transfer results from the category of Jordan superpairs to Lie superalgebras. We will review the basic constructions.

A *3-grading* of a Lie superalgebra L over k is a decomposition $L = L_1 \oplus L_0 \oplus L_{-1}$ where each L_i is a k-superspace, hence $L_i = L_{i\bar 0} \oplus L_{i\bar 1}$ for $i = 0, \pm 1$ satisfies $[L_i, L_j] \subset L_{i+j}$ with the understanding that $L_{i+j} = 0$ if $i + j \neq 0, \pm 1$. In other words, $L = L_1 \oplus L_0 \oplus L_{-1}$ is a $\mathbb{Z}$-grading with at most three non-zero homogeneous spaces. Because of this, 3-gradings are sometimes also called *short* $\mathbb{Z}$*-gradings*, e.g. in [**21**]. A Lie superalgebra is called *3-graded* if it has a 3-grading. If L is a 3-graded Lie superalgebra, its Grassmann envelope is a 3-graded Lie algebra in the sense of [**19**, 1.5].

A 3-graded Lie superalgebra $L = L_1 \oplus L_0 \oplus L_{-1}$ will be called *Jordan* 3*-graded* if

(i) $[L_1, L_{-1}] = L_0$, and

(ii) there exists a Jordan superpair structure on (L_1, L_{-1}) whose Jordan triple product is related to the Lie product by

$$\{x\,y\,z\} = [[xy]z] \text{ for all } x, z \in L_{\sigma 1}, y \in L_{-\sigma 1}, \sigma = \pm. \tag{1}$$

In this case, $V = (L_1, L_{-1})$ will be called the *associated Jordan superpair*. If char $k \neq 2$ the associated Jordan superpair is unique: its product is given by (1) and by $Q_{\bar{0}}(x_{\bar{0}})y = \frac{1}{2}[[x_{\bar{0}}, y]x_{\bar{0}}]$. Conversely, these two formulas define a pair structure on (L_1, L_{-1}) which will be a Jordan superpair in any situation where Jordan superpairs are defined by linear identities. For example, by [**16**, (3.2.1)], a 3-graded Lie superalgebra L over k with $[L_1, L_{-1}] = L_0$ is Jordan 3-graded as soon as char $k \neq 2, 3$.

So far we have associated a Jordan superpair to any Jordan 3-graded Lie superalgebra. Even more important is the fact that every Jordan superpair V arises in this way. Without going into details let us just recall that one can define a Lie superalgebra product on $\mathfrak{K}(V) = V^+ \oplus \mathrm{IDer}V \oplus V^-$ where $\mathrm{IDer}V$ denotes the Lie superalgebra of inner derivations of V. This so-called *Tits-Kantor-Koecher superalgebra* $\mathfrak{K}(V)$ is a 3-graded Lie algebra with $\mathfrak{K}(V)_{\pm 1} = V^{\pm}$ and $\mathfrak{K}(V)_{\bar{0}} = \mathrm{IDer}V$. It is obviously Jordan 3-graded. For more details, see [**7**].

1.9. Root graded Lie superalgebras. Jordan 3-graded Lie superalgebras whose associated Jordan superpairs are covered by a grid are precisely the Lie superalgebras graded by a 3-graded root system ([**7**]). For the convenience of the reader we review here the basic definitions.

Let R be a reduced (possibly infinite) root system in the sense of [**18**] (so $0 \notin R$), and let $\mathcal{Q}(R) = \mathbb{Z}[R]$ be the $\mathbb{Z}$-span of R (the *root lattice*). Let L be a Lie superalgebra over k. We say L is *R-graded* if there exists a decomposition $L = \bigoplus_{\alpha \in R \cup \{0\}} L_\alpha$ into subspaces $L_\alpha = L_{\alpha\bar{0}} \oplus L_{\alpha\bar{1}}$ and subalgebras $\mathfrak{h} \subset \mathfrak{g} \subset L_{\bar{0}}$ such that the following conditions are satisfied:

- (i) the decomposition $L = \bigoplus_{\alpha \in R_1} L_\alpha$ is a $\mathcal{Q}(R)$-grading;
- (ii) $L_0 = \sum_{\alpha \in R} [L_\alpha, L_{-\alpha}]$;
- (iii) there exists a family $(x_\alpha : \alpha \in R)$ of non-zero elements $x_\alpha \in L_{\alpha\bar{0}}$ such that, putting $h_\alpha = -[x_\alpha, x_{-\alpha}]$, we have

$$\mathfrak{h} = \textstyle\sum_{\alpha \in R} k \cdot h_\alpha\,, \quad \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} k \cdot x_\alpha \quad \text{and}$$
$$[h_\alpha, y_\beta] = \langle \beta, \alpha^\vee \rangle y_\beta \text{ for all } \alpha \in R \text{ and } y_\beta \in L_\beta, \beta \in R \cup \{0\}.$$

This definition is a straightforward generalization of the notion of a root-graded Lie algebra studied in [**19**]. In case L is a Lie algebra, k is a field of characteristic 0 and R is finite, it is equivalent to the one considered by Berman-Moody [**2**] and Benkart-Zelmanov [**1**]. In this case R can be identified with a set of linear forms on $\mathfrak{h}$, the superspaces L_α are then given by $L_\alpha = \{x \in L : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$, $\{h_\alpha : \alpha \in R\}$ is isomorphic to the dual root system of R and $\mathfrak{h}$ is a splitting Cartan subalgebra of the finite-dimensional semisimple Lie algebra $\mathfrak{g}$.

2. Gelfand-Kirillov dimension.

2.1. Gelfand-Kirillov dimension of Jordan superstructures. Let V be a Jordan superpair over a field k. For any subspace $U = U_{\bar{0}} \oplus U_{\bar{1}}$ of V we define $U^{(n)} = (U^{+(n)}, U^{-(n)})$ and $U^{[n]} = (U^{+[n]}, U^{-[n]})$ for odd $n \in \mathbb{N}$ inductively by

$$\begin{aligned} U^{\sigma(1)} &= U^{\sigma}, \\ U^{\sigma(n)} &= \sum_{l+k+m=n} \{U^{\sigma(l)}, U^{-\sigma(k)}, U^{\sigma(m)}\} + \sum_{2l+k=n} Q_{\bar{0}}(U_{\bar{0}}^{\sigma(l)})U^{-\sigma(k)}, \qquad (1) \\ U^{\sigma[n]} &= \sum_{1 \le i \le n} U^{\sigma(i)}. \end{aligned}$$

The *Gelfand-Kirillov dimension of a Jordan superpair* V, called the *GK-dimension* for short, is defined as

$$\operatorname{GKdim} V = \sup_{U} \left(\limsup_{\text{odd } n} \frac{\ln(\dim U^{+[n]} + \dim U^{-[n]})}{\ln n} \right), \qquad (2)$$

where the supremum is taken over all finite dimensional subspaces U of V. Here and in the following we write dim for $\dim_k$. Obviously, the GK-dimension of any subpair of V is less than or equal to the GK-dimension of V.

As in the proof of [**10**, 1.1], it can be shown that for a finitely generated superpair V the GK-dimension of V is independent of the particular choice of the generating subspace U. Thus in this case

$$\operatorname{GKdim} V = \limsup_{\text{odd } n} \frac{\ln(\dim U^{+[n]} + \dim U^{-[n]})}{\ln n}, \qquad (3)$$

where U is any finite dimensional generating subspace of V. In the general situation it is of course not necessary to take the supremum over *all* subspaces. Rather, it is sufficient to consider a class of "special" subspaces, adapted to the Jordan superpair under investigation, with the property that every finite dimensional subspace is contained in a special one. Moreover, we have the following obvious reduction principle. Suppose $V = \bigcup_i V^{(i)}$ is the union of subpairs such that

(a) every finite dimensional subspace of V lies in some $V^{(i)}$,
(b) $\operatorname{GKdim} V^{(i)} = c$ is constant.

Then

$$\operatorname{GKdim} V = c. \qquad (4)$$

2.2. Remarks. (a) It will follow from our results in 3.9 and 3.13 that the GK-dimension of a Jordan superpair over a field of characteristic $\neq 2$ is either 0 or ≥ 1.

(b) If V is the direct sum of ideals U_i, $i \in I$, the GK-dimension of V equals the supremum of the GK-dimensions of the ideals U_i:

$$\operatorname{GKdim}\Big(\bigoplus_{i\in I} U_i\Big) = \sup_{i\in I}\Big(\operatorname{GKdim} U_i\Big).$$

Indeed, since $\operatorname{GKdim} U_i \leq \operatorname{GKdim} V$ for any $i \in I$, we have $\sup_{i\in I}(\operatorname{GKdim} U_i) \leq \operatorname{GKdim} V$. Conversely, if B is a fixed finite dimensional subspace of V, then B lies in an ideal $\oplus_{j\in J} U_j$ of U, where now J is a finite subset of I. Arguing as in [**10**, 3.2], we have that $\operatorname{GKdim}(\oplus_{j\in J} U_j) = \max_{j\in J}(\operatorname{GKdim} U_j)$, hence

$$\limsup_{\text{odd } n} \frac{\ln(\dim B^{+[n]} + \dim B^{-[n]})}{\ln n} \leq \max_{j\in J}(\operatorname{GKdim} U_j) \leq \sup_{i\in I}(\operatorname{GKdim} U_i),$$

so $\operatorname{GKdim} V \leq \sup_{i\in I}(\operatorname{GKdim} U_i)$.

2.3. GK-dimension of Jordan supertriples and superalgebras. Now let T be a Jordan supertriple. Any subspace U of T gives rise to a subspace $\mathcal{U} = (U, U)$ of the associated Jordan superpair $V = (T, T)$, and we define $U^{(n)}$ and $U^{[n]}$ for odd n by

$$\mathcal{U}^{(n)} = (U^{(n)}, U^{(n)}) \quad \text{and} \quad \mathcal{U}^{[n]} = (U^{[n]}, U^{[n]}). \tag{1}$$

The *Gelfand-Kirillov dimension* of T is then defined in analogy to 2.1.2 as

$$\operatorname{GKdim} T = \sup_U \left(\limsup_{\text{odd } n} \frac{\ln(\dim U^{[n]})}{\ln n}\right)$$

where the supremum is taken over all finite dimensional subspaces of T. For a (quadratic) unital Jordan superalgebra J we put

$$\operatorname{GKdim} J = \operatorname{GKdim} J^T$$

where J^T is the underlying Jordan supertriple, see [**16**, 3.11].

2.4. Lemma. (a) *Let $V = (T, T)$ be the Jordan superpair associated to a Jordan supertriple T. Then $\operatorname{GKdim} V = \operatorname{GKdim} T$.*

(b) *Let $V = (V^+, V^-)$ be a Jordan superpair, and let $T(V) = V^+ \oplus V^-$ be the associated polarized Jordan supertriple with quadratic map P given by*

$$P_{\bar 0}(x^+_{\bar 0} \oplus x^-_{\bar 0})(y^+ \oplus y^-) = Q_{\bar 0}(x^+_{\bar 0})y^- \oplus Q_{\bar 0}(x^-_{\bar 0})y^+ \quad \textit{and}$$
$$\{x^+ \oplus x^-, y^+ \oplus y^-, z^+ \oplus z^-\} = \{x^+, y^-, z^+\} \oplus \{x^-, y^+, z^-\}.$$

Then $\operatorname{GKdim} V = \operatorname{GKdim} T(V)$.

Proof. (a) Let U be a finite dimensional subspace of T. Then $\mathcal{U} = (U, U)$ is a finite dimensional subspace of V with $\mathcal{U}^{[n]} = (U^{[n]}, U^{[n]})$, so $\operatorname{GKdim} V \geq \operatorname{GKdim} T$ follows from

$$\begin{aligned}\operatorname{GKdim} V &\geq \limsup_{\text{odd } n} \frac{\ln(\dim U^{[n]} + \dim U^{[n]})}{\ln n} = \limsup_{\text{odd } n} \frac{\ln(2 \dim U^{[n]})}{\ln n} \\ &= \limsup_{\text{odd } n} \frac{\ln(\dim U^{[n]})}{\ln n}.\end{aligned}$$

On the other hand, if (U^+, U^-) is a finite dimensional subspace of V then $U^+ \oplus U^-$ is a finite dimensional subspace of T with $(U^+ \oplus U^-)^{[n]} = U^{+[n]} \oplus U^{-[n]}$. Then $\operatorname{GKdim} T \geq \operatorname{GKdim} V$ in view of

$$\begin{aligned}\operatorname{GKdim} T &\geq \limsup_{\text{odd } n} \frac{\ln(\dim (U^+ \oplus U^-)^{[n]})}{\ln n} \\ &= \limsup_{\text{odd } n} \frac{\ln(\dim U^{+[n]} + \dim U^{-[n]})}{\ln n}.\end{aligned}$$

(b) Since any finite dimensional subspace of $T(V)$ imbeds in a finite dimensional subspace of the form $U^+ \oplus U^-$ for $U = (U^+, U^-) \subset V$, the assertion is immediate from the definitions.

2.5. Gelfand-Kirillov dimension of nonassociative superalgebras. Let A be a nonassociative superalgebra over k. For any subspace B of A and any $n \in \mathbb{N}$ we put $B^{(1)} = B$, $B^{(n)} = \sum_{i+j=n} B^{(i)} B^{(j)}$, $B^{[n]} = \sum_{1 \leq i \leq n} B^{(i)}$. Then the *Gelfand-Kirillov dimension* of A is defined as

$$\operatorname{GKdim} A = \sup_B \left(\limsup_n \frac{\ln(\dim B^{[n]})}{\ln n} \right), \tag{1}$$

where the supremum is taken over all finite dimensional subspaces B of A. It is well-known that in case of a finitely generated superalgebra A, the GK-dimension of A is independent of the particular choice of the generating subspace B, thus the analogous formula to 2.1.3 holds.

For Jordan superalgebras over fields of characteristic $\neq 2$ we now have two definitions for the Gelfand-Kirillov dimension. That they in fact coincide can be proven in the same way as the corresponding result in the non-supercase [**13**, Thm. 3.1]:

2.6. Lemma. *Let J be a Jordan superalgebra over k with $\frac{1}{2} \in k$, and denote by J_{lin} the underlying linear Jordan superalgebra structure. Then*

$$\operatorname{GKdim} J = \operatorname{GKdim} J_{\text{lin}}$$

where $\operatorname{GKdim} J$ *is defined in* 2.1 *while* $\operatorname{GKdim} J_{\text{lin}}$ *is given in* 2.5.

We will determine the GK-dimension of Jordan superpairs covered by a grid. For doing so, the following general result will be useful.

2.7. Lemma. *Let A be a superextension of k and let X be a finite dimensional Jordan pair. The GK-dimension of the Jordan superpair $A \otimes_k X = (A \otimes X^+, A \otimes X^-)$, the A-extension of X, then satisfies the inequality* $\mathrm{GKdim}(A \otimes X) \leq \mathrm{GKdim}\, A$ *and this is an equality if X is linearly perfect in the sense that $X = \{X\,,\,X\,,\,X\}$:*

$$\mathrm{GKdim}(A \otimes_k X) = \mathrm{GKdim}\, A.$$

Proof. We will first establish the inequality $\mathrm{GKdim}(A \otimes X) \leq \mathrm{GKdim}\, A$. Any finite-dimensional subspace of $A \otimes X$ is contained in one of the form $B \otimes X$ where B is a finite dimensional subspace of A containing 1. Hence for the calculation of $\mathrm{GKdim}(A \otimes X)$ it is sufficient to consider these special subspaces $B \otimes X$. We will prove by induction

$$(B \otimes X)^{\sigma(n)} \subset B^{(n)} \otimes X^{\sigma}. \tag{1}$$

Indeed, for odd $j, l, m \in \mathbb{N}$ with $j + l + m = n$ we have, using the definition of the product in $A \otimes X$ and associativity of A,

$$\begin{aligned}
&\{(B \otimes X)^{\sigma(j)}\,,\,(B \otimes X)^{-\sigma(l)}\,,\,(B \otimes X)^{\sigma(m)}\} \\
&\quad \subset \{B^{(j)} \otimes X^{\sigma}\,,\,B^{(l)} \otimes X^{-\sigma}\,,\,B^{(m)} \otimes X^{\sigma}\} \\
&\quad = B^{(j)} B^{(l)} B^{(m)} \otimes \{X^{\sigma}\,,\,X^{-\sigma}\,,\,X^{\sigma}\} \subset B^{(n)} \otimes X^{\sigma}.
\end{aligned}$$

Moreover, arguing in a similar way, for $2l + k = n$ we have $Q_{\bar{0}}((B_{\bar{0}} \otimes X)^{\sigma(l)})(B \otimes X)^{-\sigma(k)}) \subset B^{(n)} \otimes X^{\sigma}$. Hence we have proven the inclusion (1), and this easily implies $(B \otimes X)^{\sigma[n]} \subset B^{[n]} \otimes X^{\sigma}$. For the special subspace $U = B \otimes X$ we then obtain, using $\dim X = \dim X^+ + \dim X^-$,

$$\begin{aligned}
&\limsup_{\text{odd}\, n} \frac{\ln(\dim U^{+[n]} + \dim U^{-[n]})}{\ln n} \leq \limsup_{\text{odd}\, n} \frac{\ln(\dim B^{[n]} \dim X)}{\ln n} \\
&= \limsup_{\text{odd}\, n} \frac{\ln(\dim B^{[n]})}{\ln n} \leq \limsup_{n} \frac{\ln(\dim B^{[n]})}{\ln n} \leq \mathrm{GKdim}\, A,
\end{aligned}$$

which implies $\mathrm{GKdim}\, A \otimes X \leq \mathrm{GKdim}\, A$.

Now suppose that X is linearly perfect. For the other inequality, $\mathrm{GKdim}\, A \leq \mathrm{GKdim}\, A \otimes X$, we take B again to be a finite-dimensional subspace of A containing 1. Because of this and associativity, we have $B^{[n]} = B^{(n)} = B^{(i)} B^{(n-i)}$ for $1 \leq i \leq n$. Using perfectness of X it then follows that $B^{[2]} \otimes X^{\sigma} = \{B \otimes X^{\sigma}\,,\,1 \otimes X^{-\sigma}\,,\,B \otimes X^{\sigma}\} \subset (B \otimes X)^{\sigma[3]}$, and by induction $B^{[n]} \otimes X^{\sigma} \subset (B \otimes X)^{\sigma[2n-1]}$. But then

$$\begin{aligned}
\limsup_{n} \frac{\ln \dim B^{[n]}}{\ln n} &= \limsup_{n} \frac{\ln \dim (B^{[n]} \otimes X^{\sigma})}{\ln n} \\
&\leq \limsup_{n} \frac{\ln \dim (B \otimes X)^{\sigma[2n-1]}}{\ln n} \leq \mathrm{GKdim}\, A \otimes X
\end{aligned}$$

hence $\mathrm{GKdim}\, A \leq \mathrm{GKdim}\, A \otimes X$.

2.8. GK-dimension of Jordan superpairs covered by grids. Let V be a Jordan superpair covered by a grid with associated root system (R, R_1). We know from 1.5.2 that V is a direct sum of ideals $V = \bigoplus_{i\in I} V^{(i)}$, each covered by a connected grid, and hence by Remark 2.2(b)

$$\operatorname{GKdim} V = \operatorname{GKdim}(\oplus_{i\in I} V^{(i)}) \;=\; \sup_{i\in I}(\operatorname{GKdim} V^{(i)}). \tag{1}$$

In view of the formula above we will from now on consider Jordan superpairs covered by a connected grid.

2.9. Theorem. *Let V be a Jordan superpair covered by a connected grid $\mathcal{G}$. If $\mathcal{G}$ contains a pair of collinear idempotents let A be the associated McCrimmon-Meyberg algebra. Otherwise let $A = J$ where $V \cong \mathbb{J} = (J, J)$. Then*

$$\operatorname{GKdim} V = \operatorname{GKdim} A. \tag{1}$$

Proof. If V is covered by a connected grid $\mathcal{G}$ which does not contain collinear idempotents then either $\mathcal{G}$ is a single idempotent or it is associated to a triangle of idempotents. In both cases $\mathbb{V} \cong (J, J)$ for a unital Jordan superalgebra J and so $\operatorname{GKdim} V = \operatorname{GKdim} J$ by 2.4 and the definition of $\operatorname{GKdim} J$.

We can now assume that $\mathcal{G}$ contains a pair of collinear idempotents. We will first show that it is enough to consider Jordan superpairs covered by a finite grid $\mathcal{G}$. Indeed, if $\mathcal{G}$ or, equivalently, its associated irreducible 3-graded root system (R, R_1) is infinite, it is obvious from the classification of 3-graded root systems that R is a union of finite subsystems $R^{(i)}$ of the same type containing a given collinear pair. (In fact, this is part of the classification proof as given in [**12**].) Correspondingly, we have $V = \bigcup_i V^{(i)}$ where $V^{(i)} = \bigoplus_{\alpha\in R^{(i)}} V_\alpha$ is covered by the grid $\{g_\alpha : \alpha \in R^{(i)}\}$. Because of 2.1.4 it then suffices to prove (1) for a finite $\mathcal{G}$. Our next aim is to show

$$\operatorname{GKdim} V \le \operatorname{GKdim} A. \tag{2}$$

For the calculation of $\operatorname{GKdim} V$ it is sufficient to consider a class of special subspaces with the property that any finite dimensional subspace of V is contained in one of them. These special subspaces U will be defined below. They all have the following two properties. Firstly, U is split with respect to the root grading $V = \bigoplus_{\alpha\in R_1} V_\alpha$, i.e., $U = \bigoplus_{\alpha\in R_1} U_\alpha$, $U_\alpha = U \cap V_\alpha$. Clearly all $U^{(n)}$ and hence also all $U^{[n]}$ are then split too. Secondly, if we define $B = U^+ \cap A$ (keeping in mind that A is defined on some V_β^+), then there exists a constant c_U depending on U such that for all odd n and all $\alpha \in R_1$

$$\dim U_\alpha^{\sigma(n)} \le c_U \dim B^{(n)}. \tag{3}$$

We claim that this is sufficient to establish (2). Indeed, for a special U we have

$$\limsup_{\text{odd } n} \frac{\ln(\dim U^{+[n]} + \dim U^{-[n]})}{\ln n} \le \limsup_{\text{odd } n} \frac{\ln(2c_U|\mathcal{G}|\dim B^{[n]})}{\ln n}$$
$$= \limsup_{\text{odd } n} \frac{\ln(\dim B^{[n]})}{\ln n} \le \limsup_{n} \frac{\ln(\dim B^{[n]})}{\ln n} \le \operatorname{GKdim} A,$$

which implies (2). The class of special subspaces satisfying (3) will be defined by making use of the coordinatization theorems. Since by Lemma 2.7 the inequality (2) holds for superextensions of finite-dimensional Jordan pairs and since we assumed the covering grid to contain a pair of collinear idempotents, we only have to consider the cases (b), (d) and (f) of 1.7.

Case (b): R is of type A, so $V \cong \mathbb{M}_{JK}(A)$ for $|J| + |K| \ge 3$. The special subspaces are $\mathbb{M}_{JK}(B)$ where B is a subspace of A. It follows from the multiplication rules in $\mathbb{M}_{JK}(A)$, [**16**, 5.4] for A alternative, $|J| + |K| = 3$, or [**16**, 5.6] for A associative and $|J| + |K| \ge 3$, that $U^{(n)} \subset \mathbb{M}_{JK}(B^{(n)})$ which proves (3) with $c_U = 1$. More precisely, if A is associative then, using $B^{(n)} = BB^{(n-1)}$, we even have

$$\mathbb{M}_{JK}(B)^{(n)} = \mathbb{M}_{JK}(B^{(n)}) \quad (A \text{ associative}) \tag{4}$$

Case (d): V is a hermitian matrix superpair $\mathbb{H}_I(A, A_0, \pi)$, $|I| \ge 3$. Thus R is of type C_I. Here the special subspaces are $U = \mathbb{H}_I(B, B \cap A_0, \pi)$ where B is a π-invariant subspace of A. It follows from the multiplication rules in [**16**, 5.11] that $U^{(n)} \subset \mathbb{H}_I(B^{(n)}, B^{(n)} \cap A_0, \pi)$ whence (3) holds (with $c_U = 1$).

Case (f): $V = \mathbb{OQ}_I(A, q_X)$ is an odd quadratic form superpair with $|I| \ge 2$, thus R is of type $\mathrm{B}_{1+|I|}$. In this case, using the notation of [**16**, 5.13 and 5.15], the special subspaces are $U = (Y, Y) \oplus \mathbb{EQ}_I(B)$ where $Y \subset X$ and $B \subset A$ are finite dimensional ($\mathbb{Z}_2$-graded) subspaces satisfying $h_0 \in Y$ and $k \cdot 1 + b_X(Y, Y) + q_X(Y_{\bar{0}}) \subset B$ for b_X the polar of q_X. The condition $1 \in B$ implies that $B^{[n]} = B^{(n)}$ for all n and $U^{(n)} \subset (B^{(n)}Y, B^{(n)}Y) \oplus \mathbb{EQ}_I(B^{(n)})$ for odd n, whence also $U^{[n]} \subset (B^{(n)}Y, B^{(n)}Y) \oplus \mathbb{EQ}_I(B^{(n)})$. (In fact, it can be proven by induction that $U^{(n)} = (B^{(n-1)}Y, B^{(n-1)}Y) \oplus \mathbb{EQ}_I(B^{(n)})$ but we will not need this.) It follows that (3) holds with $c_U = \dim Y$.

We have now established (3) in all cases and hence (2) holds. For the proof of the other inequality, $\operatorname{GKdim} A \le \operatorname{GKdim} V$, we observe that $\mathcal{G}$ contains a pair of collinear idempotents, say g_α, g_β, and it is further no harm to assume that A is the McCrimmon-Meyberg superalgebra of g_α, g_β. We claim that $U = V_\alpha \oplus V_\beta$ is a subpair. Indeed, this follows from the following facts: V_α and V_β are subpairs, $\{V_\alpha\, V_\alpha, V_\beta\} \subset V_\beta$ (since $\alpha - \alpha + \beta = \beta$), $\{V_\alpha V_\beta V_\alpha\} = 0$ (since for collinear α, $2\alpha - \beta$ is not a root), and the analogous formulas for α and β exchanged. The subpair U is covered by the grid $\{g_\alpha, g_\beta\}$. The rectangular Coordinatization Theorem [**16**, 5.5] then implies that $V_\alpha \oplus V_\beta \cong \mathbb{M}_{12}(A)$, hence V contains a subpair $U \cong \mathbb{M}_{12}(A)$. Because $\operatorname{GKdim} U \le \operatorname{GKdim} V$ it is then sufficient to prove $\operatorname{GKdim} A \le$

$\operatorname{GKdim} \mathbb{M}_{12}(A)$. To this end, let B be a finite dimensional subspace of A. It is no harm to assume that B contains the identity element of A. It then follows by induction, using the product formula of the McCrimmon-Meyberg algebra (see 1.6), that $\mathbb{M}_{12}(B^{(n)}) \subset \mathbb{M}_{12}(B)^{(2n-1)}$. Therefore, $\dim B^{(n)} = \frac{1}{2} \dim \mathbb{M}_{12}(B^{(n)})^{\sigma} \leq \frac{1}{2} \dim \mathbb{M}_{12}(B)^{\sigma(2n-1)}$, and

$$\begin{aligned}
\limsup_n \frac{\ln \dim B^{(n)}}{\ln n} &\leq \limsup_n \frac{\ln \dim \mathbb{M}_{12}(B)^{\sigma(2n-1)}}{\ln n} \\
&= \limsup_{\text{odd } n} \frac{\ln \dim \mathbb{M}_{12}(B)^{\sigma(n)}}{\ln n} \leq \operatorname{GKdim} \mathbb{M}_{12}(A),
\end{aligned}$$

hence $\operatorname{GKdim} A \leq \operatorname{GKdim} V$.

2.10. Proposition. *The Gelfand-Kirillov dimension of a Jordan 3-graded Lie superalgebra $L = L_1 \oplus [L_1, L_{-1}] \oplus L_{-1}$ over a field of characteristic different from 2 coincides with the Gelfand-Kirillov dimension of its associated Jordan superpair $V = (L_1, L_{-1})$.*

The special case of a finitely generated Jordan pair V, which by [**7**, 2.4(b)] is equivalent to L being finitely generated, has been proven in [**13**, Thm. 3.2]. Our proof is more elaborate since we do not assume finite generation.

Proof. Let U be a finite dimensional subspace of V and put $W = U^+ \oplus U^- \subset L$. Since $\frac{1}{2} \in k$, we have $U^{\sigma[n]} = \sum_{l+k+m \leq n} \{U^{\sigma(l)}, U^{-\sigma(k)}, U^{\sigma(m)}\}$ for odd l, k, m and n, so $U^{+[n]} \oplus U^{-[n]} \subset W^{[n]}$ for all odd n. Therefore

$$\begin{aligned}
\limsup_{\text{odd } n} \frac{\ln(\dim U^{+[n]} + \dim U^{-[n]})}{\ln n} &\leq \limsup_{\text{odd } n} \frac{\ln(\dim W^{[n]})}{\ln n} \\
&\leq \limsup_n \frac{\ln(\dim W^{[n]})}{\ln n} \leq \operatorname{GKdim} L,
\end{aligned}$$

whence $\operatorname{GKdim} V \leq \operatorname{GKdim} L$. Conversely, if B is a finite dimensional subspace of L, then there exists a finite dimensional subspace $W = U^+ \oplus U^-$ such that $B \subset W + [W, W] = U^+ \oplus [U^+, U^-] \oplus U^- = W^{[2]}$. Then, for all $n \in \mathbb{N}$, $B^{[n]} \subset (W^{[2]})^{[n]} \subset W^{[2n]}$.

By the Jacobi identity we have $W^{(n)} = [W, W^{(n-1)}]$ for all $n \geq 2$, whence $W^{[n]} = W + [W, W^{[n-1]}]$. Using this, one shows by induction that

$$\begin{aligned}
W^{[2n]} &\subset U^{+[2n-1]} \oplus ([U^+, U^{-[2n-1]}] + [U^{+[2n-1]}, U^-]) \oplus U^{-[2n-1]}, \\
W^{[2n+1]} &\subset U^{+[2n+1]} \oplus ([U^+, U^{-[2n-1]}] + [U^{+[2n-1]}, U^-]) \oplus U^{-[2n+1]}.
\end{aligned}$$

In particular, $\dim W^{[2n]} \leq (1 + \dim U^+ + \dim U^-)(\dim U^{+[2n-1]} + \dim U^{-[2n-1]})$, and hence

$$
\begin{aligned}
\limsup_n \frac{\ln \dim B^{[n]}}{\ln n} &\le \limsup_n \frac{\ln \dim W^{[2n]}}{\ln n} \\
&\le \limsup_n \frac{\ln((1+\dim U^+ + \dim U^-)(\dim U^{+[2n-1]} + \dim U^{-[2n-1]}))}{\ln n} \\
&= \limsup_n \frac{\ln(\dim U^{+[2n-1]} + \dim U^{-[2n-1]})}{\ln n} \\
&= \limsup_n \frac{\ln(\dim U^{+[2n-1]} + \dim U^{-[2n-1]})}{\ln(2n-1)} \\
&= \limsup_{\mathrm{odd}\, n} \frac{\ln(\dim U^{+[n]} + \dim U^{-[n]})}{\ln n}.
\end{aligned}
$$

So we have the other inequality $\mathrm{GKdim}\, L \le \mathrm{GKdim}\, V$.

Coming back to the general case, we have shown in [**7**, 2,9] that a Lie superalgebra graded by a 3-graded root system R is a central extension of the Tits-Kantor-Koecher superalgebra of a Jordan superpair V covered by a grid with associated root system R. In particular, assuming that R is irreducible we can associate to L the coordinate superalgebra A of V as in 2.9. Using 2.10 we thus arrive at the following.

2.11. Corollary. *Suppose k has characteristic $\neq 2, 3$, and let L be a Lie superalgebra over k which is graded by an irreducible 3-graded root system R. Then* $\mathrm{GKdim}\, L = \mathrm{GKdim}\, A$ *where A is the associated coordinate superalgebra.*

3. Local finiteness.

3.1. Definition. A nonassociative superalgebra is called *locally finite* if every finitely generated subalgebra is finite dimensional. The concept of a subalgebra of a "linear" superalgebra, given by a bilinear product, is of course obvious. For a unital quadratic Jordan superalgebra J, a subalgebra of J is defined as a subspace invariant under $U = (U_{\bar 0}, U(.,.))$ and the *squaring operation* $x_{\bar 0}^2 = U_{\bar 0}(x_{\bar 0})1$. Similarly, a Jordan superpair or Jordan supertriple is called *locally finite* if every finitely generated subpair, respectively subsystem, is finite dimensional.

The following lemmata 3.2 – 3.4 give some preliminary results on locally finite superalgebras and Jordan superpairs. Some of them may be known, but we could not find a suitable reference. Most of the proofs are straightforward and will therefore be left to the reader.

3.2. Lemma. (a) *A subalgebra of a locally finite superalgebra is locally finite.*

(b) *Assume the superalgebra $A = \bigoplus_{i\in I} A^{(i)}$ is a direct sum of ideals $A^{(i)}$. Then A is locally finite if and only if every ideal $A^{(i)}$ is so. The analogous result holds for Jordan superpairs.*

(c) *Let S be a unital k-superalgebra (e.g. a superextension of k) and let B be a finite dimensional algebra. Then the superalgebra $A = S \otimes B$ with product* 1.1.3 *is locally finite if and only if S is so.*

Proof. For the proof of (c), let U be a finitely generated subalgebra of A. Since any element of $A = S \otimes B$ is a finite sum of pure tensors $s \otimes b \in S \otimes B$, there exist finitely many homogeneous $s_1, \dots, s_n \in S$ and $b_1, \dots, b_n \in B$ such that U is a subalgebra of the subalgebra $T \otimes C$ of A where $T \subset S$ is the subalgebra generated by the s_i, $1 \le i \le n$, while the subalgebra $C \subset B$ is generated by the b_i. Hence, if S is locally finite then so is A. Conversely, let A be locally finite and let $T \subset S$ be a subalgebra generated by finitely many $s_1, \dots, s_n \in S$. The subalgebra of A generated by $\{s_i \otimes B : 1 \le i \le n\}$ is finite dimensional and equals $T \otimes B$, whence T is finite dimensional too.

3.3. Lemma. (a) *Let $V = (T, T)$ be the Jordan superpair associated to a Jordan supertriple T. Then V is locally finite if and only if T is so.*

(b) *Let $V = (V^+, V^-)$ be a Jordan superpair and denote by $T(V)$ the polarized supertriple with product defined in* 2.4.b. *Then V is locally finite if and only if $T(V)$ is so.*

(c) *Let J be a unital Jordan superalgebra and denote by J^T the underlying Jordan supertriple. Then J is locally finite if and only if J^T is so.*

3.4. Lemma. *Let A be an alternative unital superalgebra. If A is locally finite then so is the Jordan superalgebra $A^{(+)}$.*

3.5. Local finiteness of Jordan superpairs covered by a grid. Recall from 1.5.2 that a Jordan superpair V covered by a grid is a direct sum of ideals, each covered by a connected grid. Because of Lemma 3.2(b), it is therefore enough to study local finiteness in the case of connected grids. For our characterization of local finiteness for these Jordan superpairs in 3.7 below, the following is a useful preliminary result.

3.6. Lemma. *Let J and K be finite index sets with $2 \le |K|$ and let A be a unital alternative superalgebra which we assume to be associative if $|J| + |K| \ge 4$. Then the rectangular matrix superpair $\mathbb{M}_{JK}(A)$ is locally finite if and only if A is so.*

Proof. Let X be a subspace of A and let B be the subalgebra of A generated by X. We then claim that the subpair U of $\mathbb{M}_{JK}(A)$ generated by the subspace $\mathbb{M}_{JK}(X)$ is $U = \mathbb{M}_{JK}(B)$. Indeed, since the multiplication rules of $\mathbb{M}_{JK}(A)$ are expressed in terms of the multiplication in A, it follows that $\mathbb{M}_{JK}(B)$ is a subpair containing the generators of U, hence U is contained in $\mathbb{M}_{JK}(B)$. To prove the converse, note that the product of A can be expressed as a product in $\mathbb{M}_{JK}(A)$. Namely, denoting by E_{jk}, $j \in J, k \in K$ the canonical matrix units we have for $a, a' \in A$ the formula $aa'E_{jk} = \{aE_{jk'}\,, E_{k'j}\,, a'E_{jk}\}$ where k, k' are two distinct elements of K. As a consequence, any bE_{jk} for $b \in B$ of the form $b = x_1 \cdots x_n$

with $x_i \in X$ is then a product with factors in $\mathbb{M}_{JK}(X)$, hence lies in U, which implies $\mathbb{M}_{JK}(B) \subset U$.

Now suppose that $\mathbb{M}_{JK}(A)$ is locally finite, and let X, B and U be as above. If X is finite-dimensional, the subspace $\mathbb{M}_{JK}(X)$ is finite-dimensional, hence U is finite-dimensional, hence B is finite-dimensional, proving that A is locally finite. Conversely, let A be locally finite and let $W \subset \mathbb{M}_{JK}(A)$ be a finite dimensional subspace. Then $W \subset \mathbb{M}_{JK}(X)$ for $X \subset A$ of finite dimension. The subalgebra B of A generated by X is then finite dimensional, hence so is $\mathbb{M}_{JK}(B)$. But by the above, $\mathbb{M}_{JK}(B)$ contains the subpair generated by W, proving that $\mathbb{M}_{JK}(A)$ is locally finite.

3.7. Theorem. *Let V be a Jordan superpair covered by a connected grid $\mathcal{G}$ and, as in* 2.9*, let A be the associated coordinate superalgebra. Then V is locally finite if and only if A is so.*

Proof. In case $\mathcal{G}$ does not contain a pair of collinear idempotents and hence $V \cong \mathbb{J}$, the claim follows from 3.3. Thus in the following we can assume that $\mathcal{G}$ does contain a pair of collinear idempotents, hence A is alternative.

That local finiteness of V implies local finiteness of A is easy: we have seen in the proof of Thm. 2.9 that V contains a subpair $U \cong \mathbb{M}_{12}(A)$ which is locally finite if V is so. But then A is locally finite by Lemma 3.6.

Let now A be locally finite, and let U be a finitely generated subpair of V. Decomposing each generator with respect to the Peirce decomposition 1.5.1, it is no harm to assume that U is generated by finitely many elements in joint Peirce spaces V_α, thus involving only a finite number of roots in R. It is obvious from the classification of 3-graded root systems in [**18**] that any finite number of roots in R lie in a finite subsystem of the same type (see [**12**] for a classification-free proof), and replacing $\mathcal{G}$ by the subfamily indexed by this subsystem shows that we can without loss of generality assume that $\mathcal{G}$ is finite.

We will now consider the different types arising in the coordinatization theorems 1.7 above. Because of our assumption that $\mathcal{G}$ contains a pair of collinear idempotents, these are the types (b) and (d) – (i) where, however, case (b) has already been dealt with in Lemma 3.6.

Case (d): $V \cong \mathbb{H}_I(A, A_0, \pi)$ for $3 \le |I| < \infty$. Then there exists a finitely generated subalgebra B of A such that $U \subset \mathbb{H}_I(B, B \cap A_0, \pi)$. We know that B has finite dimension since A is locally finite, whence $\dim U \le \dim \mathbb{H}_I(B, B \cap A_0, \pi) \le 2|I|^2 \dim B < \infty$.

Case (f): $V \cong \mathbb{OQ}_I(A, q_X)$ for $|I| \ge 2$. Here U is contained in a subpair of the form $(BY, BY) \oplus \mathbb{EQ}_I(B)$, where Y is a finite dimensional subspace of X and B is a finitely generated subalgebra of A. By local finiteness of A, $\dim B < \infty$, so $\dim U \le \dim((BY, BY) \oplus \mathbb{EQ}_I(B)) \le 2 \dim B \dim Y + \dim \mathbb{EQ}_I(B) < \infty$.

Cases (e), (g) – (i): Here V is the A-extension of a split Jordan pair $W = \oplus_{\alpha \in R_1} (kg_\alpha^+, kg_\alpha^-)$ of type $\mathcal{G}$ and one can argue as in the proof of Lemma 3.2(c).

Thus, in all cases we have proven that V is locally finite as soon as A is so.

3.8. Local finiteness and Gelfand-Kirillov dimension. It is immediate from the definition that *a locally finite superalgebra or a locally finite Jordan superpair has GK-dimension* 0. The goal of the remaining part of this section is to prove the converse for certain varieties of superalgebras and Jordan superstructures, see 3.9, 3.13 and 3.15. This will also provide an alternative (and quicker) proof of Theorem 3.7 and Corollary 3.16 below in case our base field has characteristic $\neq 2$.

It is important to note here that a nonassociative algebra of GK-dimension 0 need not be locally-finite as the following example, due to D. Finston [**3**], shows. Let A be the commutative algebra defined on the linear span of $y_i, i \in \mathbb{N}$ by the rule:

$$y_i y_j = \delta_{ij} y_{i+1}.$$

Thus A only has squares: $y_i^2 = y_{i+1}$ while $y_i y_j = 0$ for $i \neq j$. This algebra is not locally finite since it has infinite dimension yet it is "finitely generated" by y_1, but it can be shown ([**3**, p. 537]) that its GK-dimension is 0.

The following lemma provides a sufficient condition under which GK-dimension 0 does imply local finiteness.

3.9. Lemma. *Let A be a linear superalgebra such that there exists a $k \in \mathbb{N}$, $k \geq 1$, such that for all $n \geq k$ and for any subspace B of A we have*

$$B^{(n)} = BB^{(n-1)} + B^{(2)}B^{(n-2)} + \cdots + B^{(k)}B^{(n-k)}. \tag{1}$$

Then either A is locally finite and hence has GK-dimension 0*, or* $\operatorname{GKdim} A \geq 1$.

In particular, our assumption holds with $k = 1$ for an associative or Lie superalgebra A, which is therefore locally finite if and only if $\operatorname{GKdim} A = 0$.

That a Lie algebra is locally finite if and only if it has GK-dimension 0 is also proven in [**20**, Thm.1].

Proof. We first prove for a subspace B of A that

$$B^{[m]} = B^{[m+k]} \text{ for some } m \quad \Longrightarrow \quad B^{[m]} = B^{[m+l]} \text{ for all } l \in \mathbb{N}. \tag{2}$$

Indeed, since the $B^{[n]}$ form an ascending chain, our assumption implies $B^{[m]} = B^{[m+1]} = \cdots = B^{[m+k]}$, and hence $B^{(p)} \subset B^{[m]}$ for all $p \leq m + k$. For $l = k + 1$ we obtain

$$B^{[m+k+1]} = B^{[m]} + B^{(m+k+1)} = B^{[m]} + \sum_{i=1}^{k} B^{(i)} B^{(m+k+1-i)},$$

using (1). In the last sum each term satisfies $B^{(i)}B^{(m+k+1-i)} \subset B^{(i)}B^{[m]} \subset B^{[i+m]} = B^{[m]}$ since $1 \leq i \leq k$, whence $B^{[m]} = B^{[m+k+1]}$. An induction then proves (2). Observe that in this case $B^{[m]}$ is in fact a subalgebra of A.

There are now the following alternatives: either all finite dimensional subspaces satisfy (2) and hence A is locally finite, or there exists a subspace B such that $B^{[n]} \subset B^{[n+k]}$ is a proper inclusion for all $n \in \mathbb{N}$. For such a B we have $\dim B^{[n+k]} \geq 1 + \dim B^{[n]}$ which implies $\dim B^{[n]} \geq n/k$. But then

$$\limsup_n \frac{\ln(\dim B^{[n]})}{\ln n} \geq \limsup_n \frac{\ln(n/k)}{\ln n} = 1,$$

and therefore $\mathrm{GKdim}\, A \geq 1$.

We have $A^{(n)} = AA^{(n-1)}$ if A is associative, and this also holds in the Lie case by the Jacobi identity.

We will show below in Prop. 3.13 that for a Jordan superpair over a field of characteristic $\neq 2$ local finiteness is equivalent to GK-dimension zero. It is an open problem to extend this result to the case of characteristic 2. However, we can at least show this in the non-super setting. Our proof uses the following folklore lemma, proven in [**15**, Cor. 3 of Th. 1] for quadratic Jordan algebras.

3.10. Lemma. *Let J be a Jordan triple system generated by a subspace B. Then the multiplication algebra of J is generated by the identity and by operators of the form $P_a, L_{b,c}, P_{d,e}$ for $a, b, c, d, e \in B$.*

From this lemma one easily obtains the following.

3.11. Corollary. *Let J be a Jordan triple system and let B be any subspace of T. Then for any odd $n \in \mathbb{N}$ greater than 2 we have*

$$B^{[n]} = P_B B^{[n-2]} + \{B, B, B^{[n-2]}\} + \{B, B^{[n-2]}, B\}.$$

3.12. Proposition. *Let J denote a Jordan system (algebra, pair or triple system) over k. Then J is locally finite if and only if the GK-dimension of J is zero.*

Proof. If J is a Jordan triple system, the proof follows from 3.11 arguing as in 3.9. If J is a Jordan pair, it suffices to consider the associated polarized triple system $T(V) = V^+ \oplus V^-$ and to use 2.4.b and 3.3.b.

Finally, if J is a Jordan algebra, let $\hat{J} = J \oplus k \cdot 1$ be its unital hull. It is immediate to see that local finiteness is equivalent for J and for $\hat{J}$ and that $\mathrm{GKdim}\, J = \mathrm{GKdim}\, \hat{J}$. We may therefore assume that J is unital. By Lemma 3.3.c and the definition of $\mathrm{GKdim}\, J$, we then have J is locally finite $\Leftrightarrow$ J^T is locally finite $\Leftrightarrow$ $\mathrm{GKdim}\, J^T = \mathrm{GKdim}\, J = 0$.

3.13. Proposition. *For a Jordan 3-graded Lie superalgebra $L = L_1 \oplus [L_1, L_{-1}] \oplus L_{-1}$ over a field of characteristic different from 2 the following are equivalent:*

(i) *L is locally finite,*
(ii) $\operatorname{GKdim} L = 0$,
(iii) *the associated Jordan superpair $V = (L_1, L_{-1})$ is locally finite,*
(iv) $\operatorname{GKdim} V = 0$.

Proof. We know (i) $\Leftrightarrow$ (ii) from 3.9, (ii) $\Leftrightarrow$ (iv) from 2.10 and (iii) $\Rightarrow$ (iv) from 3.1. It therefore suffices to prove (i) $\Rightarrow$ (iii) which is immediate: if L is locally finite and $U \subset V$ is a finitely generated subpair of V then $U^+ \oplus U^- \subset U^+ \oplus [U^+, U^-] \oplus U^-$ which is a finitely generated, hence finite dimensional, subalgebra of L, whence U is finite dimensional.

3.14. Corollary. *A Jordan superpair over a field of characteristic $\neq 2$ is locally finite if and only if it has GK-dimension 0. The same holds for Jordan superalgebras.*

Proof. Since every Jordan superpair is the associated Jordan superpair of some Jordan 3-graded Lie superalgebra, e.g., the Tits-Kantor-Koecher superalgebra, the equivalence for Jordan superpairs is immediate from 3.13. For a Jordan superalgebra J we have the equivalences: J is locally finite if and only if $V = (J, J)$ is locally finite (by 3.3.a) if and only if $\operatorname{GKdim} V = 0$ if and only if $\operatorname{GKdim} J = 0$ (by 2.4.a).

3.15. Corollary. *Let A be a unital alternative superalgebra over a field of characteristic $\neq 2$. Then A is locally finite if and only if* $\operatorname{GKdim} A = 0$.

Proof. Indeed, we have the following equivalences: A is locally finite if and only if $\mathbb{M}_{12}(A)$ is locally finite (by 3.6) if and only if $\operatorname{GKdim} \mathbb{M}_{12}(A) = 0$ (by 3.13) if and only if $\operatorname{GKdim} A = 0$ (by 2.9).

We note that this result allows us to give a quicker proof of Theorem 3.7 in case the base field has characteristic $\neq 2$ and the coordinate superalgebra A is alternative: V is locally finite if and only if $\operatorname{GKdim} V = 0$ (by Prop. 3.13) if and only if $\operatorname{GKdim} A = 0$ (by Thm. 2.9) if and only if A is locally finite.

The following corollary can now be obtained by the same argument used in the proof of Cor. 2.11. Another proof can be given by combining 2.11 with 3.13, 3.14 and 3.15.

3.16. Corollary. *Let L be a Lie superalgebra over a field k of characteristic $\neq 2, 3$ which is graded by an irreducible 3-graded root system, and let A be the associated coordinate superalgebra. Then L is locally finite if and only if A is so.*

Remark. Even for Lie *algebras* this is a new result. With the appropriate concept of a coordinate algebra it is likely to be true in all cases. Indeed, if the irreducible root system R is not 3-graded we have $R = \mathrm{E}_8, \mathrm{F}_4$ or G_2. A Lie algebra L over a field of characteristic 0 graded by the root system E_8 has the form $L \cong A \otimes \mathfrak{g}$ where A is a unital associative commutative k-algebra and $\mathfrak{g}$ is the split simple Lie algebra of type E_8 over k ([**2**]). Hence Lemma 3.2(d) implies that

L is locally finite if and only if A is so. This leaves open the two cases $R = \mathrm{F}_4$ and $R = \mathrm{G}_2$ for which the corresponding R-graded Lie algebras were described in [**1**].

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