Extended affine Lie algebras and other generalizations of affine Lie algebras – a survey

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Summary. This is a survey on extended affine Lie algebras and related types of Lie algebras, which generalize affine Lie algebras.
Keywords: affine Lie algebras, extended affine Lie algebra, Lie torus
MSC 2000 Primary 17B65, 17B67. Secondary 16W50, 17B70

1 Introduction

Motivation. The theory of affine (Kac-Moody) Lie algebras has been a tremendous success story. Not only has one been able to generalize essentially all of the well-developed theory of finite-dimensional simple Lie algebras and their associated groups to the setting of affine Lie algebras, but these algebras have found many striking applications in other parts of mathematics. It is natural to ask – why? What is so special about affine Lie algebras? What really makes them "tick"? One way to understand this, is to generalize affine Lie algebras and see where things go wrong or don't. After all, it is conceivable that there is a whole theory of Lie algebras, ready to be discovered, for which affine Lie algebras are just one example.

Of course, one immediately thinks about Kac-Moody Lie algebras, for which finite-dimensional simple and affine Lie algebras are the basic examples. But is it the right generalization? This is not such a sacrilegious question as it seems. After all, one the founders and main contributor to the theory does not seem to be convinced himself ([Kac3]). So where to go? There are several directions that have been pursued: toroidal algebras, Borcherds algebras, GIM-algebras, root-graded algebras, to name just a few.

The first generalization. The generalizations that are of interest for this paper arose in the work of Saito and Slodowy on elliptic singularities and in the paper by the physicists Høegh-Krohn and Torrésani [HT] on Lie algebras of interest to quantum gauge field theory. The latter paper, written by

* Support of the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

physicists, did however not stand up to mathematical scrutiny. Fortunately, a mathematical sound foundation of the theory was later provided by Allison, Azam, Berman, Gao and Pianzola in their AMS memoirs [AABGP]. They also describe the type of root systems appearing in these algebras and give many examples. Last not least they coined the name extended affine Lie algebras for this new class of algebras.

Before we give the definition, it is convenient to introduce the basic concept of a toral pair over $\mathbb{K}$. Throughout this paper $\mathbb{K}$ denotes a field of characteristic 0. A *toral pair* (E,T) over $\mathbb{K}$ consists of a Lie algebra E over $\mathbb{K}$ and a subspace $T \subset E$ acting ad-diagonalizably on E, i.e., putting

$$\begin{aligned} E_\alpha &= \{e \in E : [t,e] = \alpha(t)e \text{ for all } t \in T\} \quad \text{for } \alpha \in T^* \text{ and} \\ R &= \{\alpha \in T^* : E_\alpha \neq 0\}, \end{aligned}$$

we have $E = \bigoplus_{\alpha \in R} E_\alpha$. It turns out that T is abelian, so $T \subset E_0$ and hence $0 \in R$ (unless $T = \{0\} = E$). In general T is a proper subspace of E_0. If, however, $T = E_0$ we will call T a *splitting Cartan subalgebra* and, following common usage, will use the letter H instead of T. The following definition is essentially the one given in [AABGP]:

An *extended affine Lie algebra*, or EALA for short, is a toral pair (E,H) over $\mathbb{K} = \mathbb{C}$ satisfying the conditions (E1)–(E6) below:

(E1) E has an invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$.

(E2) H is a non-trivial, finite-dimensional splitting Cartan subalgebra.

Since $H = E_0$, it is immediate that the restriction of $(\cdot|\cdot)$ to H is nondegenerate. We can therefore represent any $\alpha \in H^*$ by a unique vector $t_\alpha \in H$ via $\alpha(h) = (t_\alpha|h)$. Hence

$$R = R^0 \cup R^{\mathrm{an}} \tag{1}$$

for $R^0 = \{\alpha \in R : (t_\alpha|t_\alpha) = 0\}$ (*null roots*) and $R^{\mathrm{an}} = \{\alpha \in R : (t_\alpha|t_\alpha) \neq 0\}$ (*anisotropic roots*). We can now state the remaining axioms (E3)–(E6):

(E3) For any $\alpha \in R^{\mathrm{an}}$ and $x \in E_\alpha$, the adjoint map $\operatorname{ad} x$ is locally nilpotent on E.

(E4) R^{an} is connected in the sense that any partition $R^{\mathrm{an}} = R_1 \cup R_2$ with $(R_1|R_2) = 0$ either has $R_1 = \emptyset$ or $R_2 = \emptyset$.

(E5) $R^0 \subset R^{\mathrm{an}} + R^{\mathrm{an}}$.

(E6) R is a discrete subset of H^*, equipped with the natural topology.

Let (E,H) be an EALA. By (E6), the $\mathbb{Z}$-span of R^0 in H^* is free of finite rank, called the *nullity* of (E,H) (it turns out that this is independent of $(\cdot|\cdot)$). The *core of* (E,H) is the ideal E_c of E generated by all subspaces E_α, $\alpha \in R^{\mathrm{an}}$. We then have a Lie algebra homomorphism $E \to \operatorname{Der} E_c$, given by $e \mapsto \operatorname{ad} e \mid E_c$, and one says that (E,H) is *tame* if the kernel of this map lies in E_c. The idea of tameness is that one can recover E from E_c and its derivations. Although there are examples of non-tame EALAs, it is natural to assume tameness for the purpose of classification.

We can now convince the reader that extended affine Lie algebras indeed do what they are supposed to do: A tame EALA of nullity 0 is the same as a finite-dimensional simple Lie algebra (exercise), and a tame EALA of nullity 1 is the same as an affine Lie algebra, proven by Allison-Berman-Gao-Pianzola in [ABGP]. Tame EALAs of nullity 2 are closely related to the Lie algebras studied by Saito and Slodowy, and toriodal Lie algebras provide examples of tame EALAs of arbitrarily high nullity. What makes EALAs an interesting class of Lie algebras is that there are other, non-toroidal examples requiring noncommutative coordinates.

The revisions. The system of axioms (E1)–(E6) has been very successful. It describes an important and accessible class of Lie algebras, about which much is known today. But there are drawbacks. Foremost among them is the restriction to the ground field $\mathbb{C}$, which is necessary for the axiom (E6). Looking at the basic examples of EALAs given in [AABGP], one sees that they make sense over arbitrary fields $\mathbb{K}$. Hence, in [Neh7] the author proposed a new definition of EALAs, valid for Lie algebras over $\mathbb{K}$. It used the same set of axioms, but replaced (E6) with its essence:

(E6′) the $\mathbb{Z}$-span of R is free of finite rank.

(To be more precise, only tame EALAs were considered in [Neh7].) It turned out that one can develop a satisfactory theory for EALAs over $\mathbb{K}$, partially described in [Neh7] and [Neh6].

Some drawbacks however still remained: The theory only covers the "split case" because of the condition in (E2) that H be a splitting Cartan subalgebra, and it does not include finite-dimensional semisimple Lie algebras because of (E4), nor infinite rank affine Lie algebras because of the requirement in (E2) that H be finite-dimensional. Some of these issues were addressed in papers by Azam [Az4], Azam-Khalili-Yousofzadeh [AKY] and Morita-Yoshii [MY] (see 6.17–6.19).

A radically new look at the system of axioms seems to be called for: Delete the less important axioms (E4)–(E6) and modify (E2) so that all the aforementioned examples can be included. The purpose of this survey and research announcement is to propose such an approach by defining a new class of Lie algebras and to explain its main features.

The new definition. An *invariant affine reflection algebra*, an IARA for short, is a toral pair (E,T) over $\mathbb{K}$ satisfying the axioms (IA1)–(IA3) below (recall $E=\bigoplus_{\alpha\in R}E_\alpha$):

(IA1) E has an invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$ whose restriction to T is nondegenerate and which represents every root $\alpha\in R$: There exists $t_\alpha\in T$ such that $\alpha(t)=(t_\alpha|t)$ for all $t\in T$.

(IA2) For every $\alpha \in R$, $\alpha \neq 0$, there are $e_{\pm\alpha} \in E_{\pm\alpha}$ such that $0 \neq [e_\alpha,\, e_{-\alpha}] \in T$.

(IA3) = (E3) For every $\alpha \in R$ with $(t_\alpha | t_\alpha) \neq 0$ and for all $x_\alpha \in E_\alpha$ the adjoint map $\operatorname{ad} x_\alpha$ is locally nilpotent on E.

The following are examples of invariant affine reflection algebras: finite-dimensional reductive Lie algebras, infinite-rank affine Lie algebras, Lie algebras satisfying (E1)–(E3), in particular EALAs, and all the generalizations of EALAs that had been considered before.

To describe the structure of IARAs, we will use the inclusions of categories

$$\text{affine Lie algebras} \quad \subset \quad \text{EALAs} \quad \subset \quad \text{IARAs} \tag{2}$$

as a leitmotif. We will first discuss the combinatorics, viz. the roots appearing in IARAs, and then describe a procedure to determine their algebra structure.

The roots. The roots for the algebras in (2) form, respectively, an

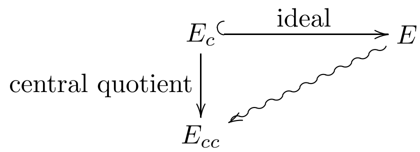
$$\text{affine root sytem} \quad \subset \quad \text{EARS} \quad \subset \quad \text{ARS}$$

where the affine root systems are the roots appearing in affine Lie algebras, EARS is the acronym for an extended affine root system, the roots of EALAs, and ARS stands for affine reflection system (3.6), of which the roots of an IARA are an example. To explain the general structure of an ARS, recall (2.12) that to any affine root system R one can associate a finite irreducible, but not necessarily reduced root system S, a torsion-free abelian group Λ and a family $(\Lambda_\alpha : \alpha \in S)$ of subsets $\Lambda_\alpha \subset \Lambda$ such that

$$R = \bigcup_{\xi \in S} (\xi \oplus \Lambda_\xi) \quad \subset \quad \operatorname{span}_{\mathbb{K}}(R) = \operatorname{span}_{\mathbb{K}}(S) \oplus \operatorname{span}_{\mathbb{K}}(\Lambda). \tag{3}$$

Indeed, $\Lambda = \mathbb{Z}\delta$ where δ is the basic null root. Now (3) also holds for EARS if one takes Λ to be free of finite rank. Since the toral subalgebra of an IARA may be infinite-dimensional, it is not surprising that root systems of infinite rank will occur. Indeed, if one allows S to be a locally finite root system (3.6) and Λ to be any torsion-free abelian group, then (3) also holds for ARSs. Moreover, the basic relations between the subsets Λ_ξ, $\xi \in S$, developed in [AABGP, Ch.II] for EARS, actually also hold in the more general setting of affine reflection systems. Hence, their theory is about as easy or complicated as the theory of EARS.

The core and centreless core. Recall that the core of an EALA (E, H) is the ideal generated by all E_α, $\alpha \in R^{\mathrm{an}}$. It is a perfect Lie algebra. Hence $E_{cc} = E_c / \operatorname{Z}(E_c)$ has centre $\{0\}$ and is justifiably called the *centreless core*. By (IA1) the decomposition $R = R^0 \cup R^{\mathrm{an}}$ is also available for IARAs, and we can therefore define E_c and E_{cc} as in the EALA-case. Hence, from any IARA E we can go down to its centreless core E_{cc}:



This diagram describes the fundamental approach to the study of the Lie algebras (E,T) in (2):

(A) Describe the class $\mathcal{C}$ of Lie algebras that arise as centreless cores E_{cc} in a way which is independent of (E,T),

(B) for every Lie algebra L in $\mathcal{C}$ provide a construction which yields all the Lie algebras E with centreless core L, and

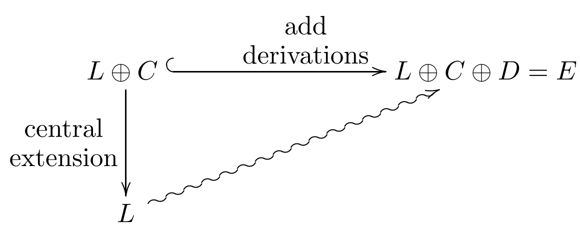
(C) classify the Lie algebras in $\mathcal{C}$.

That this approach has good chances of success, can already be witnessed by looking at affine Lie algebras. For them, the core and the centreless core are the basic ingredients of the structure theory: E_c is the derived algebra of the affine Lie algebra E and E_{cc} is the (twisted or untwisted) loop algebra that one uses to construct E from E_{cc}. The centreless cores of EALAs were described by Allison-Gao [AG] and then put into an axiomatic framework as centreless Lie $\mathbb{Z}^n$-tori by Yoshii [Yo5] and the author [Neh6]. One can show that the centreless core of an IARA is a certain type of root-graded Lie algebra, a so-called invariant predivision-(S,Λ)-graded Lie algebra for (S,Λ) as above: S is a locally finite root system and Λ is a torsion-free abelian group. Conversely, every such Lie algebra arises as the centreless core of some IARA (5.1, 6.8, 6.10). Thus, for the Lie algebras in (2) the classes $\mathcal{C}$ are

loop algebras $\subset$ centreless Lie $\mathbb{Z}^n$-tori $\subset$
invariant predivision-root-graded Lie algebras, Λ torsion-free.

This solves problem (A).

As for (B), we describe in 6.9 a construction, which starts with any invariant predivision-root-graded Lie algebra L with a torsion-free Λ and produces an invariant affine reflection algebra (E,T) with centreless core L. It maintains the main features of the construction of an affine Lie algebra from a loop algebra, i.e. take a central extension and then add on some derivations:



If L is a Lie torus this construction yields all EALAs with centreless core L, and hence settles problem (B) for them.

Finally, the efforts of many people have now lead to a complete solution of problem (C) for Lie tori, see the survey by Allison-Faulkner in [AF, §12]. The classification of predivision-root-graded Lie algebras is not known in general, but in several important cases, see the review in 5.10. What certainly is fascinating about this class and root-graded Lie algebras in general, is that there are many examples requiring noncommutative coordinates. For example, for $S = \mathrm{A}_l$, $l \geq 3$, one has to consider central quotients of $\mathfrak{sl}_{l+1}(A)$ where A is an associative algebra, a crossed product algebra to be more precise. Moreover, all the important classes of non-associative algebras appear as coordinates: alternative algebras for $S = \mathrm{A}_2$, Jordan algebras for A_1, structurable algebras for BC_1 and BC_2.

In the end, what have we gained? What is emerging, is the picture that EALAs and IARAs share many structural features with affine Lie algebras, for example they all have concrete realizations (unlike arbitrary Kac-Moody algebras). But they allow many more possibilities and hence possible applications, than affine Lie algebras. We are just at the beginning of this new theory and many problems have not been solved (representation theory, quantization, to name just two).

Contents. The theory of affine reflection systems is described in §3 after we have have laid the background for more general reflection systems in §2. Both sections are based on the author's joint paper with Ottmar Loos [LN2]. We then move to graded algebras. In §4 we introduce what is needed from the theory of arbitrary graded algebras, e.g. the centroid and centroidal derivations. Following this, we describe in §5 the necessary background from the theory of root-graded Lie algebras and the subclasses of predivision-root-graded algebras and Lie tori. These two sections contain few new results. The new theory of invariant affine reflection algebras and even more general algebras is presented in §6. Finally, in §7 we consider the example $\mathfrak{sl}_n(A)$ in detail. In particular, we describe central extensions and derivations, the important ingredients of the EALA- and IARA-construction.

Acknowledgements: The author thanks Bruce Allison for stimulating discussions on the topics of this paper and the referee for helpful comments on a preliminary version of this article.

2 Root systems and other types of reflection systems

In this section and the following section §3 we will describe the combinatorial background for the sets of roots that appear in the theory of extended affine Lie algebras, Lie tori and their generalizations. See 2.14 for references.

Throughout we work in vector spaces over a field $\mathbb{K}$ of characteristic 0. For a vector space X we denote by $\mathrm{GL}(X)$ the general linear group of X, by Id_X the identity map on X, by $X^* = \mathrm{Hom}(X, \mathbb{K})$ the dual space of X and by $\langle \cdot, \cdot \rangle \colon X \times X^* \to \mathbb{K}$ the canonical pairing given by $\langle x, \lambda \rangle = \lambda(x)$.

2.1 Pre-reflection and reflection systems. A *pre-reflection system* is a triple (R, X, s) where R is a subset of a vector space X and $s\colon R \to \mathrm{GL}(X)$, $\alpha \mapsto s_\alpha$ is a map satisfying the following axioms:

(ReS0) $0 \in R$, R spans X and for all $\alpha \in R$ we have $\mathrm{codim}\{x \in X : s_\alpha x = x\} \le 1$ and $s_\alpha^2 = \mathrm{Id}_X$. Thus, either $s_\alpha = \mathrm{Id}_X$ or s_α is a reflection with fixed point space of codimension 1, leading to a partition $R = R^{\mathrm{re}} \cup R^{\mathrm{im}}$ where

$$\begin{aligned} R^{\mathrm{re}} &= \{\alpha \in R : s_\alpha \ne \mathrm{Id}_X\} \quad (\textit{reflective} \text{ or } \textit{real roots}) \\ R^{\mathrm{im}} &= \{\alpha \in R : s_\alpha = \mathrm{Id}_X\} \quad (\textit{imaginary roots}). \end{aligned}$$

(ReS1) If $\alpha \in R^{\mathrm{re}}$ then $s_\alpha(\alpha) = -\alpha \ne \alpha$.

(ReS2) For all $\alpha \in R$ we have $s_\alpha(R^{\mathrm{re}}) = R^{\mathrm{re}}$ and $s_\alpha(R^{\mathrm{im}}) = R^{\mathrm{im}}$.

A *reflection system* is a pre-reflection system (R, X, s) for which, in addition to (ReS0)–(ReS2), we also have

(ReS3) $s_{c\alpha} = s_\alpha$ whenever $0 \ne c \in \mathbb{K}$ and both α and $c\alpha$ belong to R^{re}.

(ReS4) $s_\alpha s_\beta s_\alpha = s_{s_\alpha \beta}$ for all α, $\beta \in R$.

Some explanations are in order. First, following recent practice (e.g. [AABGP] and [LN1]) we assume that 0 is a root, in fact $0 \in R^{\mathrm{im}}$ because of (ReS1). While this is not really important, it will turn out to be convenient later on. The terminology of "real" and "imaginary" roots is of course borrowed from the theory of Kac-Moody algebras, whose roots provide an example of a reflection system. It is immediate from (ReS1) and (ReS2) that

$$R^{\mathrm{re}} = -R^{\mathrm{re}} \quad \text{and} \quad s_\alpha(R) = R \quad \text{for all } \alpha \in R.$$

But we do not assume $R = -R$ (this would exclude some examples in Lie superalgebras, see 2.11).

2.2 Some basic concepts. Let (R, X, s) be a pre-reflection system, often just denoted (R, X) or even simply R. For all pre-reflection systems we will use the same letter s to denote the reflections.

Let $\alpha \in R^{\mathrm{re}}$. Because of (ReS1) there exists a unique $\alpha^\vee \in X^*$ such that s_α is given by the usual formula

$$s_\alpha(x) = x - \langle x, \alpha^\vee\rangle \alpha \quad \text{for } x \in X. \tag{1}$$

Indeed, $\alpha^\vee$ is determined by $\alpha^\vee(\alpha) = 2$ and $\mathrm{Ker}\,\alpha^\vee = \{x \in X : s_\alpha x = x\}$. So that (1) holds for all $\alpha \in R$, we put $\alpha^\vee = 0$ for $\alpha \in R^{\mathrm{im}}$. We thus get a map ${}^\vee\colon R \to X^*$ which uniquely determines the map $s\colon R \to \mathrm{GL}(X)$.

One obtains a category of pre-reflection systems, and a fortiori a full subcategory of reflection systems, by defining a *morphism* $f\colon (R, X, s) \to (S, Y, s)$ of pre-reflection systems as a $\mathbb{K}$-linear map $f\colon X \to Y$ such that $f(R) \subset S$ and

$f(s_\alpha\beta) = s_{f(\alpha)} f(\beta)$ hold for all $\alpha, \beta \in R$. It is immediate that any morphism f satisfies

$$f(R^{\rm im}) \subset S^{\rm im} \quad \text{and} \quad f(R^{\rm re}) \subset S^{\rm re} \cup \{0\}. \tag{2}$$

Note that the axiom (ReS4) of reflection systems is equivalent to the condition that s_α be an automorphism for every $\alpha \in R$.

The subgroup of $\mathrm{GL}(X)$ generated by all reflections s_α is called the *Weyl group* of R and denoted $W(R)$.

A *subsystem* of R is a subset $S \subset R$ with $0 \in S$ and $s_\gamma\delta \in S$ for all $\gamma, \delta \in S$. It follows that $(S, \mathrm{span}(S))$ together with the obvious restriction of s is a pre-reflection system such that the inclusion $S \hookrightarrow R$ is a morphism. For any subsystem S we have $S^{\rm im} = S \cap R^{\rm im}$ and $S^{\rm re} = S \cap R^{\rm re}$. An example of a subsystem is

$$\mathrm{Re}(R) = R^{\rm re} \cup \{0\},$$

called the *real part of* R. Also, for any subspace Y of X the intersection $R \cap Y$ is a subsystem of R.

A pre-reflection system R is called

(i) *reduced* if $\alpha \in R^{\rm re}$, $0 \neq c \in \mathbb{K}$ and $c\alpha \in R^{\rm re}$ imply $c = \pm 1$ (note that we do not require this condition for roots in $R^{\rm im}$);

(ii) *integral* if $\langle R, R^\vee \rangle \subset \mathbb{Z}$;

(iii) *nondegenerate* if $\bigcap_{\alpha \in R} \mathrm{Ker}(\alpha^\vee) = \{0\}$;

(iv) *symmetric* if $R = -R$, equivalently, $R^{\rm im} = -R^{\rm im}$;

(v) *coherent* if for $\alpha, \beta \in R^{\rm re}$ we have $\langle \alpha, \beta^\vee \rangle = 0 \iff \langle \beta, \alpha^\vee \rangle = 0$,

(vi) *tame* if $R^{\rm im} \subset R^{\rm re} + R^{\rm re}$.

2.3 Direct sums, connectedness. Let $(R_i, X_i)_{i \in I}$ be a family of pre-reflection systems. Put $R = \bigcup_{i \in I} R_i$, $X = \bigoplus_{i \in I} X_i$ and define $s\colon R \to \mathrm{GL}(X)$ by extending each s_{α_i}, $\alpha_i \in R_i$ to a reflection of X by $s_{\alpha_i}|X_j = \mathrm{Id}$ for $i \neq j$. Then (R, X, s) is a pre-reflection system, called the *direct sum of the pre-reflection systems* (R_i, X_i). Obviously, $R^{\rm re} = \bigcup_i R_i^{\rm re}$ and $R^{\rm im} = \bigcup_i R_i^{\rm im}$. We will say that a pre-reflection system with a non-empty set of real roots is *indecomposable* if it is not isomorphic to a direct sum of two pre-reflection systems, each of which has a non-empty set of real roots.

Suppose R is coherent. Two roots $\alpha, \beta \in R^{\rm re}$ are called *connected* if there exist finitely many roots $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ such that $\langle \alpha_i, \alpha_{i-1}^\vee \rangle \neq 0$ for $i = 1, \ldots, n$, in particular all $\alpha_i \in R^{\rm re}$. It is easily seen that connectedness is an equivalence relation on $R^{\rm re}$. A *connected component of* $\mathrm{Re}(R)$ is by definition the union of $\{0\}$ and an equivalence class of $R^{\rm re}$, and $\mathrm{Re}(R)$ is called *connected* if $\mathrm{Re}(R)$ has only one connected component. Equivalently, $\mathrm{Re}(R)$ is connected if and only if for any decomposition $R^{\rm re} = R_1 \cup R_2$ with $\langle R_1, R_2^\vee \rangle = 0$ necessarily $R_1 = \emptyset$ or $R_2 = \emptyset$. Each connected component of $\mathrm{Re}(R)$ is a subsystem of R.

2.4 Lemma. *If* (R, X, s) *is a nondegenerate coherent pre-reflection system, the subsystem* $\operatorname{Re}(R)$ *is the direct sum of its connected components. In particular,* $\operatorname{Re}(R)$ *is connected if and only if it is indecomposable.*

2.5 Invariant bilinear forms. Let (R, X) be a pre-reflection system. A symmetric bilinear form $b\colon X \times X \to \mathbb{K}$ is called *invariant* if $b(wx, wy) = b(x, y)$ holds for all $w \in W(R)$ and $x, y \in X$. It is of course equivalent to require $b(s_\alpha x, s_\alpha y) = b(x, y)$ for all $x, y \in X$ and $\alpha \in R^{\mathrm{re}}$. Moreover, invariance of b is also equivalent to

$$2\, b(x, \alpha) = \langle x, \alpha^\vee \rangle\, b(\alpha, \alpha) \quad \text{for all } x \in X \text{ and } \alpha \in R^{\mathrm{re}}. \tag{1}$$

We will call an invariant bilinear form *strictly invariant* if (1) not only holds for $\alpha \in R^{\mathrm{re}}$ but for all $\alpha \in R$. It easily follows that an invariant symmetric bilinear form b is strictly invariant if and only if $R^{\mathrm{im}} \subset \{x \in X : b(x, X) = 0\} =: \operatorname{Rad} b$, the *radical of* b. For such a bilinear form b we have

$$R \cap \operatorname{Rad} b = \{\alpha \in R : b(\alpha, \alpha) = 0\} \quad \text{and} \quad \bigcap_{\alpha \in R} \operatorname{Ker} \alpha^\vee \subset \operatorname{Rad} b. \tag{2}$$

We note that we need not distinguish between invariant and strictly invariant forms if $R = \operatorname{Re}(R)$, for example if R is a locally finite root system as defined in 2.7.

2.6 Root strings. Let R be a pre-reflection system, let $\alpha \in R^{\mathrm{re}}$, $\beta \in R$, and assume that $a := -\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$. The *$\alpha$-string through β* is defined by

$$\mathbb{S}(\beta, \alpha) := (\beta + \mathbb{Z}\alpha) \cap R.$$

The reflection s_α leaves $\mathbb{S}(\beta, \alpha)$ invariant and corresponds to the reflection $i \mapsto a - i$ of

$$\mathbb{Z}(\beta, \alpha) := \{i \in \mathbb{Z} : \beta + i\alpha \in R\}$$

about the point $a/2$. The set $\mathbb{Z}(\beta, \alpha)$ is bounded if and only if it is bounded on one side. In this case, we put $-q = \min \mathbb{Z}(\beta, \alpha)$ and $p = \max \mathbb{Z}(\beta, \alpha)$ and then have $p, q \in \mathbb{N}$ and $p - q = a = -\langle \beta, \alpha^\vee \rangle$. Moreover, the following conditions are equivalent:

(i) the *root string* $\mathbb{S}(\beta, \alpha)$ *is unbroken*, i.e., $\mathbb{Z}(\beta, \alpha)$ is a finite interval in $\mathbb{Z}$ or equals $\mathbb{Z}$,
(ii) for all $\gamma \in \mathbb{S}(\beta, \alpha)$ and all integers i between 0 and $\langle \gamma, \alpha^\vee \rangle$ we have $\gamma - i\alpha \in \mathbb{S}(\beta, \alpha)$,
(iii) for all $\gamma \in \mathbb{S}(\beta, \alpha)$, $\langle \gamma, \alpha^\vee \rangle > 0$ implies $\gamma - \alpha \in R$ and $\langle \gamma, \alpha^\vee \rangle < 0$ implies $\gamma + \alpha \in R$.

We will now present some examples of (pre-)reflection systems.

2.7 Examples: Finite and locally finite root systems. A root system à la Bourbaki ([Bo2, VI, §1.1]) with 0 added will here be called a *finite root system*. Using the terminology introduced above, a finite root system is the same as a pre-reflection system (R, X) which is integral, coincides with its real part $\mathrm{Re}(R)$, and moreover satisfies the finiteness condition that R be a finite set.

Replacing the finiteness condition by the local-finiteness condition, defines a locally finite root system. Thus, a *locally finite root system* is a pre-reflection system (R, X) which is integral, coincides with its real part $\mathrm{Re}(R)$, and is locally-finite in the sense that $|R \cap Y| < \infty$ for any finite-dimensional subspace Y of X.

We mention some special features of locally finite root systems, most of them well-known from the theory of finite root systems. By definition, a locally finite root system R is integral and symmetric. One can show that it is also a reflection system whose root strings are unbroken and which is nondegenerate and coherent, but not necessarily reduced. Moreover, R is the direct sum of its connected components, see 2.4. In particular, R is connected if and only if R is indecomposable, in which case R is traditionally called *irreducible*.

A root $\alpha \in R$ is called *divisible* or *indivisible* according to whether $\alpha/2$ is a root or not. We put

$$R_{\mathrm{ind}} = \{0\} \cup \{\alpha \in R : \alpha \text{ indivisible}\} \quad \text{and} \quad R_{\mathrm{div}} = \{\alpha \in R : \alpha \text{ divisible}\}. \tag{1}$$

Both R_{ind} and R_{div} are subsystems of R.

A *root basis* of a locally finite root system R is a linearly independent set $B \subset R$ such that every $\alpha \in R$ is a $\mathbb{Z}$-linear combination of B with coefficients all of the same sign. A root system R has a root basis if and only if all irreducible components of R are countable ([LN1, 6.7, 6.9]).

The set $R^\vee = \{\alpha^\vee \in X^* : \alpha \in R\}$ is a locally finite root system in $X^\vee = \mathrm{span}_{\mathbb{K}}(R^\vee)$, called the *coroot system of* R. The root systems R and $R^{\vee\vee}$ are canonically isomorphic ([LN1, Th. 4.9]).

2.8 Classification of locally finite root systems. ([LN1, Th. 8.4]) Examples of possibly infinite locally finite root systems are the so-called *classical root systems* $\dot{\mathrm{A}}_I$ – BC_I, defined as follows. Let I be an arbitrary set and let $X_I = \bigoplus_{i\in I} \mathbb{K}\epsilon_i$ be the vector space with basis $\{\epsilon_i : i \in I\}$. Then

$$\begin{aligned}
\dot{\mathrm{A}}_I &= \{\epsilon_i - \epsilon_j : i, j \in I\},\\
\mathrm{B}_I &= \{0\} \cup \{\pm\epsilon_i : i \in I\} \cup \{\pm\epsilon_i \pm \epsilon_j : i, j \in I, i \neq j\} = \{\pm\epsilon_i : i \in I\} \cup \mathrm{D}_I,\\
\mathrm{C}_I &= \{\pm\epsilon_i \pm \epsilon_j : i, j \in I\} = \{\pm 2\epsilon_i : i \in I\} \cup \mathrm{D}_I,\\
\mathrm{D}_I &= \{0\} \cup \{\pm\epsilon_i \pm \epsilon_j : i, j \in I, i \neq j\}, \text{ and}\\
\mathrm{BC}_I &= \{\pm\epsilon_i \pm \epsilon_j : i, j \in I\} \cup \{\pm\epsilon_i : i \in I\} = \mathrm{B}_I \cup \mathrm{C}_I
\end{aligned}$$

are locally finite root systems which span X_I, except for $\dot{\mathrm{A}}_I$ which only spans a subspace $\dot{X}_I$ of codimension 1 in X_I. The notation $\dot{\mathrm{A}}$ is supposed to indicate this fact. For finite I, say $|I| = n \in \mathbb{N}$, we will use the usual notation $\mathrm{A}_n = \dot{\mathrm{A}}_{\{0,\dots,n\}}$, but B_n, C_n, D_n, BC_n for $|I| = n$. The root systems $\dot{\mathrm{A}}_I, \dots, \mathrm{D}_I$ are reduced, but $R = \mathrm{BC}_I$ is not.

Any locally finite root system R is the direct limit of its finite subsystems, which are finite root systems, and if R is irreducible it is a direct limit of irreducible finite subsystems. It is then not too surprising that an irreducible locally finite root system is isomorphic to exactly one of

(i) the finite exceptional root systems $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$ or
(ii) one of the root systems $\dot{\mathrm{A}}_I$ ($|I| \geq 1$), B_I ($|I| \geq 2$), C_I ($|I| \geq 3$), D_I ($|I| \geq 4$) or BC_I ($|I| \geq 1$).

Conversely, all the root systems listed in (i) and (ii) are irreducible.

The classification of locally finite root systems is independent of the base field $\mathbb{K}$, which could therefore be taken to be $\mathbb{Q}$ or $\mathbb{R}$. But since the root systems which we will later encounter naturally "live" in $\mathbb{K}$-vector spaces, fixing the base field is not convenient.

Locally finite root systems can also be defined via invariant nondegenerate symmetric bilinear forms:

2.9 Proposition ([LN2, 2.10]) *For an integral pre-reflection system (R, X) the following conditions are equivalent:*

(i) *R is a locally finite root system;*
(ii) *there exists a nondegenerate strictly invariant form on (R, X), and for every $\alpha \in R$ the set $\langle R, \alpha^\vee \rangle$ is bounded as a subset of $\mathbb{Z}$.*

In this case, (R, X) has a unique invariant bilinear form $(\cdot|\cdot)$ which is normalized *in the sense that $2 \in \{(\alpha|\alpha) : 0 \neq \alpha \in C\} \subset \{2, 4, 6, 8\}$ for every connected component C of R. The normalized form is nondegenerate in general and positive definite for $\mathbb{K} = \mathbb{R}$.*

2.10 Example: Reflection systems associated to bilinear forms. Let R be a spanning set of a vector space X containing 0, and let $(\cdot|\cdot)$ be a symmetric bilinear form on X. For $\alpha \in X$ we denote the linear form $x \mapsto (\alpha|x)$ by $\alpha^\flat$. Let $\varPhi \subset \{\alpha \in R : (\alpha|\alpha) \neq 0\}$, define $^\vee : R \to X^*$ by

$$\alpha^\vee = \begin{cases} \frac{2\alpha^\flat}{(\alpha|\alpha)} & \text{if } \alpha \in \varPhi, \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and then define s_α by 2.2. Thus s_α is the orthogonal reflection in the hyperplane $\alpha^\perp$ if $\alpha \in \varPhi$, and the identity otherwise. If $s_\alpha(R) \subset R$ and $s_\alpha(\varPhi) \subset \varPhi$ for all $\alpha \in \varPhi$, then (R, X, s) is a coherent reflection system with the given subset $\varPhi$ as set of reflective roots. The bilinear form $(\cdot|\cdot)$ is invariant, and it is strictly invariant if and only if $R^{\mathrm{im}} = R \cap \operatorname{Rad}(\cdot|\cdot)$, i.e., $(\cdot|\cdot)$ is affine in the sense of 3.8.

By 2.9 every locally finite root system is of this type. But also the not necessarily crystallographic finite root systems, see for example [Hu, 1.2], arise in this way. The latter are not necessarily integral reflection systems.

2.11 Example: Roots in Lie algebras with toral subalgebras. Let L be a Lie algebra over a ring k containing $\frac{1}{2}$ (this generality will be used later). Following [Bo1, Ch.VIII, §11] we will call a non-zero triple (e, h, f) of elements of L an $\mathfrak{sl}_2$*-triple* if

$$[e, f] = -h, \quad [h, e] = 2e \quad \text{and} \quad [h, f] = -2f.$$

For example, in

$$\mathfrak{sl}_2(k) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in k \right\}$$

the elements

$$e_{\mathfrak{sl}_2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h_{\mathfrak{sl}_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f_{\mathfrak{sl}_2} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{1}$$

form an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}_2(k)$. In general, for any $\mathfrak{sl}_2$-triple (e, h, f) in a Lie algebra L there exists a unique Lie algebra homomorphism $\varphi\colon \mathfrak{sl}_2(k) \to L$ mapping the matrices in (1) onto the corresponding elements (e, h, f) in L.

Let now $k = \mathbb{K}$ be a field of characteristic 0, and let L be a Lie algebra over $\mathbb{K}$ and $T \subset L$ a subspace. We call $T \subset L$ a *toral subalgebra*, sometimes also called an ad-diagonalizable subalgebra, if

$$L = \textstyle\bigoplus_{\alpha \in T^*} L_\alpha(T) \quad \text{where} \quad L_\alpha(T) = \{x \in L : [t, x] = \alpha(t)x \text{ for all } t \in T\} \tag{2}$$

for any $\alpha \in T^*$. In this case, (2) is referred to as the *root space decomposition* of (L, T) and $R = \{\alpha \in T^* : L_\alpha(T) \neq 0\}$ as the *set of roots of* (L, T). We will usually abbreviate $L_\alpha = L_\alpha(T)$. Any toral subalgebra T is abelian, thus $T \subset L_0$ and $0 \in R$ unless $T = \{0\} = E$ (which is allowed but not interesting). We will say that a toral T is a *splitting Cartan subalgebra* if $T = L_0$.

We denote by R^{int} the subset of *integrable roots* of (L, T), i.e., those $\alpha \in R$ for which there exists an $\mathfrak{sl}_2$-triple $(e_\alpha, h_\alpha, f_\alpha) \in L_\alpha \times T \times L_{-\alpha}$ such that the adjoint maps $\operatorname{ad} e_\alpha$ and $\operatorname{ad} f_\alpha$ are locally nilpotent.

Let $X = \operatorname{span}_{\mathbb{K}}(R) \subset T^*$ and for $\alpha \in R$ define $s_\alpha \in \mathrm{GL}(X)$ by

$$s_\alpha(x) = \begin{cases} x - x(h_\alpha)\alpha, & \alpha \in R^{\text{int}}, \\ x & \text{otherwise.} \end{cases}$$

It is a straightforward exercise in $\mathfrak{sl}_2$-representation theory to verify that with respect to the reflections s_α defined above, (R, X) *is an integral pre-reflection system, which is reduced in case* T *is a splitting Cartan subalgebra.* A priori, s_α will depend on h_α, but this is not so for the Lie algebras which are of main interest here, the invariant affine reflection algebras of §6.

We note that several important classes of pre-reflection systems arise in this way from Lie algebras, e.g., locally finite root systems [Neh3], Kac-Moody root systems [Kac2], [MP] and extended affine root systems [AABGP], see 3.11. An example of the latter is described in detail in 2.12. We point out that in general R is not symmetric. For example, this is so for the set of roots of classical Lie superalgebras [Kac1] (they can be viewed in the setting above by forgetting the multiplication of the odd part).

2.12 Example: Affine root systems. Let S be an irreducible locally finite root system. By 2.9, S has a unique normalized invariant form, with respect to which we can introduce *short roots* $S_{\rm sh} = \{\alpha \in S : (\alpha|\alpha) = 2\}$ and *long roots* $S_{\rm lg} = \{\alpha \in S : (\alpha|\alpha) = 4 \text{ or } 6\} = S \setminus (S_{\rm sh} \cup S_{\rm div})$, where $S_{\rm div}$ are the divisible roots of 2.7. Both $S_{\rm sh} \cup \{0\}$ and $S_{\rm lg} \cup \{0\}$ are subsystems of S and hence locally finite root systems. Note that $S = \{0\} \cup S_{\rm sh}$ if S is simply-laced, $(S_{\rm div} \setminus \{0\}) \neq \emptyset \Leftrightarrow S = {\rm BC}_I$, and if $S_{\rm lg} \neq \emptyset$ the set of long roots is given by $S_{\rm lg} = \{\alpha \in S : (\alpha|\alpha) = k(S)\}$ for

$$k(S) = \begin{cases} 2, & S = {\rm F}_4 \text{ or of type } {\rm B}_I, {\rm C}_I, {\rm BC}_I, |I| \geq 2, \\ 3 & S = {\rm G}_2. \end{cases} \tag{1}$$

Let $X = {\rm span}(S) \oplus \mathbb{K}\delta$ and let $(\cdot|\cdot)$ be the symmetric bilinear form which restricted to ${\rm span}(S)$ is the normalized invariant form of S and which satisfies $(X \,|\, \mathbb{K}\delta) = 0$. Furthermore, choose a *tier number* $t(S) = 1$ or $t(S) = k(S)$, with $t(S) = 1$ in case $S = {\rm BC}_I$, $|I| \geq 2$. Finally, put

$$\Phi = \Big(\bigcup_{\alpha \in S_{\rm sh}} (\alpha \oplus \mathbb{Z}\delta)\Big) \cup \Big(\bigcup_{\alpha \in S_{\rm lg}} \big(\alpha \oplus t(S)\, \mathbb{Z}\delta\big)\Big) \cup \Big(\bigcup_{0 \neq \alpha \in S_{\rm div}} \big(\alpha + (1 + 2\mathbb{Z})\delta\big)\Big),$$
$$R = \mathbb{Z}\delta \cup \Phi.$$

Of course, if $S_{\rm lg} = \emptyset$ or $S_{\rm div} = \{0\}$ the corresponding union above is to be interpreted as the empty set. Note $(\beta|\beta) \neq 0$ for any $\beta \in \Phi$. One can easily verify that every reflection s_β defined in 2.10 leaves Φ and $\mathbb{Z}\delta$ invariant, hence $R = R(S, t(S))$ is a symmetric reflection system, called the *affine root system associated to S and $t(S)$*.

The following table lists all possibilities for $(S, t(S))$ where S is locally finite. For finite S of type ${\rm X}_I$, say $|I| = l$, we also include the corresponding labels of the affine root system $R(S, t(S))$ as defined [MP] (third column) and [Kac2] (fourth column).

S	$t(S)$	affine label [MP]	affine label [Kac2]
reduced	1	$S^{(1)}$	$S^{(1)}$
B_I ($\lvert I\rvert \geq 2$)	2	$\mathrm{B}_l^{(2)}$	$\mathrm{D}_{l+1}^{(2)}$
C_I ($\lvert I\rvert \geq 3$)	2	$\mathrm{C}_l^{(2)}$	$\mathrm{A}_{2l-1}^{(2)}$
F_4	2	$\mathrm{F}_4^{(2)}$	$\mathrm{E}_6^{(2)}$
G_2	3	$\mathrm{G}_2^{(3)}$	$\mathrm{D}_4^{(3)}$
BC_1	–	$\mathrm{BC}_1^{(2)}$	$\mathrm{A}_2^{(2)}$
BC_I ($\lvert I\rvert \geq 2$)	1	$\mathrm{BC}_l^{(2)}$	$\mathrm{A}_{2l}^{(2)}$

If S is reduced then $R = S \oplus \mathbb{Z}\delta$, i.e., $t(S) = 1$ in the definition of Φ, is a so-called *untwisted affine root system.*

2.13 Affine Lie algebras. Every affine root system appears as the set of roots of a Lie algebra E with a splitting Cartan subalgebra H in the sense of 2.11 (the letters (E, H) are chosen in anticipation of the definition of an extended affine Lie algebra 6.11). This is easily verified in the untwisted case.

Indeed, let $\mathfrak{g}$ be a finite-dimensional split simple Lie $\mathbb{K}$-algebra, e.g., a finite-dimensional simple Lie algebra over an algebraically closed field $\mathbb{K}$, or let $\mathfrak{g}$ be a centreless infinite rank affine algebra [Kac2, 7.11]. Equivalently, $\mathfrak{g}$ is a locally finite split simple Lie algebra, classified in [NS] and [Stu], or $\mathfrak{g}$ is the Tits-Kantor-Koecher algebra of a Jordan pair spanned by a connected grid, whose classification is immediate from [Neh3]. In any case, $\mathfrak{g}$ contains a splitting Cartan subalgebra $\mathfrak{h}$ such that the set of roots of $(\mathfrak{g}, \mathfrak{h})$ is an irreducible locally finite root system S, which is finite if and only if $\mathfrak{g}$ is finite-dimensional. Moreover, $\mathfrak{g}$ carries an invariant nondegenerate symmetric bilinear form κ, unique to a scalar (in the finite-dimensional case κ can be taken to be the Killing form). The Lie algebra E will be constructed in three steps:

(I) $L = \mathfrak{g} \otimes \mathbb{K}[t^{\pm 1}]$ is the so-called *untwisted loop algebra of* $\mathfrak{g}$. Its Lie algebra product $[\cdot,\cdot]_L$ is given by

$$[x \otimes p,\, y \otimes q]_L = [x, y]_{\mathfrak{g}} \otimes pq$$

for $x, y \in \mathfrak{g}$ and $p, q \in \mathbb{K}[t^{\pm 1}]$. Although L could be viewed as a Lie algebra over $\mathbb{K}[t^{\pm 1}]$, we will view L as a Lie algebra over $\mathbb{K}$.

(II) $K = L \oplus \mathbb{K}c$ with Lie algebra product $[.,.]_K$ given by

$$[l_1 \oplus s_1 c,\, l_2 \oplus s_2 c]_K = [l_1,\, l_2]_L \oplus \sigma(l_1, l_2)c$$

for $l_i \in L$, $s_i \in \mathbb{K}$ and $\sigma\colon L \times L \to \mathbb{K}$ the $\mathbb{K}$-bilinear map determined by $\sigma(x_1 \otimes t^m,\, x_2 \otimes t^n) = \delta_{m+n,0}\, m\, \kappa(x_1, x_2)$.

(III) $E = K \oplus \mathbb{K}d$. Here d is the derivation of K given by $d(x \otimes t^m) = mx \otimes t^m$, $d(c) = 0$, and E is the semidirect product of K and $\mathbb{K}d$, i.e.,

$$[k_1 \oplus s_1 d,\, k_2 \oplus s_2 d]_E = [k_1,\, k_2]_K + s_2 d(k_2) - s_1 d(k_1).$$

One calls E the *untwisted affine Lie algebra associated to* $\mathfrak{g}$.

Denote the root spaces of $(\mathfrak{g}, \mathfrak{h})$ by $\mathfrak{g}_\alpha$, so that $\mathfrak{h} = \mathfrak{g}_0$. Then

$$H = (\mathfrak{h} \otimes \mathbb{K}1) \oplus \mathbb{K}c \oplus \mathbb{K}d$$

is a toral subalgebra of E whose set of roots in E is the untwisted affine root system associated to the root system S of $(\mathfrak{g}, \mathfrak{h})$: $E_0 = H$, $E_{m\delta} = \mathfrak{h} \otimes \mathbb{K}t^m$ for $0 \neq m \in \mathbb{Z}$ and $E_{\alpha \oplus m\delta} = \mathfrak{g}_\alpha \otimes t^m$ for $0 \neq \alpha \in S$ and $m \in \mathbb{Z}$.

One can also realize the *twisted affine root systems*, i.e. those with $t(S) > 1$, by replacing the untwisted loop algebra L in (I) by a twisted version. To define it in the case of a finite-dimensional $\mathfrak{g}$, let σ be a non-trivial diagram automorphism of $\mathfrak{g}$ of order r. Hence $\mathfrak{g}$ is of type A_l ($l \geq 2, r = 2$), D_l ($l \geq 4, r = 2$), D_4 ($r = 3$) or E_6 ($r = 2$). For an infinite-dimensional $\mathfrak{g}$ (where diagrams need not exist) we take as σ the obvious infinite-dimensional analogue of the corresponding matrix realization of σ for a finite-dimensional $\mathfrak{g}$, e.g. $x \mapsto -x^t$ for type A. The *twisted loop algebra* L is the special case $n = 1$ of the multi-loop algebra $\mathcal{M}_r(\mathfrak{g}, \sigma)$ defined in 4.7. The construction of K and E are the same as in the untwisted case. Replacing the untwisted $\mathfrak{h}$ in the definition of H above by the subspace $\mathfrak{h}^\sigma$ of fixed points under σ defines a splitting Cartan subalgebra of E in the twisted case, whose set of roots is a twisted affine root systems and all twisted affine root systems arise in this way, see [Kac2, 8.3] for details.

In the later sections we will generalize affine root systems and affine Lie algebras. Affine root systems will turn out to be examples of extended affine root systems and locally extended affine root systems (3.11), namely those of nullity 1. Extended and locally extended affine root systems are in turn special types of affine reflection systems, which we will describe in the next section. In the same vein, untwisted affine Lie algebras are examples of (locally) extended affine Lie algebras, which in turn are examples of affine reflection Lie algebras which we will study in §6. From the point of view of extended affine Lie algebras, 6.11, the Lie algebras K and L are the core and centreless core of the extended affine Lie algebra (E, H). As Lie algebras, they are Lie tori, defined in 5.1. The construction from L to E described above is a special case of the construction 6.13, see the example in 6.14.

2.14 Notes. With the exceptions mentioned in the text and below, all results in this section are proven in [LN1] and [LN2]. Many of the concepts introduced here are well-known from the theory of finite root systems. The notion of tameness (2.2) goes back to [ABY]. It is equivalent to the requirement that R not have isolated roots, where $\beta \in R$ is called an isolated root (with respect to

R^{re}), if $\alpha+\beta \notin R$ for all $\alpha \in R^{\mathrm{re}}$. In the setting of the Lie algebras considered in §6, tameness of the root system is a consequence of tameness of the algebra, see 6.8.

Locally finite root systems are studied in [LN1] over $\mathbb{R}$. But as already mentioned in [LN1, 4.14] there is a canonical equivalence between the categories of locally finite root systems over $\mathbb{R}$ and over any field $\mathbb{K}$ of characteristic 0. In particular, the classification is the "same" over any $\mathbb{K}$. The classification itself is proven in [LN1, Th. 8.4]. A substitute for the non-existence of root bases are grid bases, which exist for all infinite reduced root systems ([Neh1]).

Our definition of an $\mathfrak{sl}_2$-triple follows Bourbaki ([Bo1, Ch.VII, §11]). It differs from the one used in other texts by a sign. Replacing f by $-f$ shows that the two notions are equivalent. Bourbaki's definition is more natural in the setting here and avoids some minus signs later. The concept of an integrable root (2.11) was introduced by Neeb in [Nee1].

For a finite S, affine root systems (2.12) are determined in [Kac2, Prop. 6.3]. The description given in 2.12 is different, but of course equivalent to the one in loc. cit. It is adapted to viewing affine root systems as extended affine root systems of nullity 1. In fact, the description of affine root systems in 2.12 is a special case of the Structure Theorem for extended affine root systems [AABGP, II, Th. 2.37], keeping in mind that a semilattice of nullity 1 is a lattice by [AABGP, II, Cor. 1.7]. The table in 2.12 reproduces [ABGP, Table 1.24].

Other examples of reflection systems are the set of roots associated to a "root basis" in the sense of Hée [He]. Or, let R be the root string closure of the real roots associated to root data à la Moody-Pianzola [MP, Ch. 5]. Then R is a symmetric, reduced and integral reflection system.

The Weyl group of a pre-reflection system is in general not a Coxeter group. For example, this already happens for locally finite root systems, see [LN1, 9.9]. As a substitute, one can give a so-called presentation by conjugation, essentially the relation (ReS4), see [LN1, 5.12] for locally finite root systems and [Ho] for Weyl groups of extended affine root systems (note that the Weyl groups in [Ho] and also in [AABGP] are defined on a bigger space than the span of the roots and hence our Weyl groups are homomorphic images of the Weyl groups in loc. cit.).

3 Affine reflection systems

In this section we describe extensions of pre-reflection systems and use them to define affine reflection systems, which are a generalization of extended affine root systems. The roots of the Lie algebras, which we will be studying in §6, will turn out to be examples of affine reflection systems. Hints to references are given in 3.12.

3.1 Partial sections. Let $f\colon (R,X) \to (S,Y)$ be a morphism of pre-reflection systems with $f(R) = S$, and let $S' \subset S$ be a subsystem spanning Y. A *partial section of f over S'* is a morphism $g\colon (S',Y) \to (R,X)$ of pre-reflection systems such that $f \circ g = \mathrm{Id}_Y$. The name is (partially) justified by the fact that a partial section leads to a partial section of the canonical epimorphism $W(R) \to W(S)$, namely a section defined over $W(S')$.

Let $f\colon (R,X) \to (S,Y)$ be a morphism of pre-reflection systems satisfying $f(R) = S$. In general, a partial section of f over all of S need not exist. However, partial sections always exist in the category of reflection systems.

3.2 Extensions. Recall 2.2: $f(R^{\mathrm{im}}) \subset S^{\mathrm{im}}$ and $f(R^{\mathrm{re}}) \subset \{0\} \cup S^{\mathrm{re}}$ for any morphism $f\colon (R,X) \to (S,Y)$ of pre-reflection systems. We call f an *extension* if $f(R^{\mathrm{im}}) = S^{\mathrm{im}}$ and $f(R^{\mathrm{re}}) = S^{\mathrm{re}}$. We will say that R *is an extension of* S if there exists an extension $f\colon R \to S$.

We mention some properties of an extension $f\colon R \to S$. By definition, f is surjective. Also, R is coherent if and only if S is so, and in this case f induces a bijection $C \mapsto f(C)$ between the set of connected components of $\mathrm{Re}(R)$ and of $\mathrm{Re}(S)$. In particular,

$$\mathrm{Re}(R) \text{ is connected} \iff \mathrm{Re}(S) \text{ is connected} \iff \mathrm{Re}(R) \text{ is indecomposable.}$$

Moreover, R is integral if and only if S is so. Finally, f maps a root string $\mathbb{S}(\beta,\alpha)$, $\beta \in R$, $\alpha \in R^{\mathrm{re}}$, injectively to $\mathbb{S}\big(f(\beta), f(\alpha)\big)$.

If R is an extension of a nondegenerate S, e.g. a locally finite root system, then S is unique up to isomorphism and will be called the *quotient pre-reflection system of R*. On the other hand, if R is nondegenerate every extension $f\colon R \to S$ is injective, hence an isomorphism. In particular, a locally finite root system R does not arise as a non-trivial extension of a pre-reflection system S. But as we will see below, a locally finite root system does have many interesting extensions.

A first example of an extension is the canonical projection $R \to S$ for R an affine root system associated to a locally finite irreducible root system S, 2.12. In this case, S is the quotient pre-reflection system = quotient root system of R.

3.3 Extension data. Let (S,Y) be a pre-reflection system, let S' be a subsystem of S with $\mathrm{span}(S') = Y$ and let Z be a $\mathbb{K}$-vector space. A family $\mathfrak{L} = (\Lambda_\xi)_{\xi\in S}$ of nonempty subsets of Z is called an *extension datum of type* (S, S', Z) if

(ED1) for all $\xi, \eta \in S$ and all $\lambda \in \Lambda_\xi$, $\mu \in \Lambda_\eta$ we have $\mu - \langle \eta, \xi^\vee\rangle \lambda \in \Lambda_{s_\xi(\eta)}$,
(ED2) $0 \in \Lambda_{\xi'}$ for all $\xi' \in S'$, and
(ED3) Z is spanned by the union of all Λ_ξ, $\xi \in S$.

The only condition on Λ_0 is (ED2): $0 \in \Lambda_0$. Moreover, Λ_0 is related to the other Λ_ξ, $0 \neq \xi \in S$, only via the axiom (ED3). As (ED3) only serves to determine Z and can always be achieved by replacing Z by the span of the

Λ_ξ, $\xi \in S$, it follows that one can always modify a given extension datum by replacing Λ_0 by some other set containing 0. However, as 3.6 shows, this may change the properties of the associated pre-reflection system.

Any extension datum $\mathfrak{L}$ has the following properties, where $W_{S'} \subset W(S)$ is the subgroup generated by all $s_{\xi'}$, $\xi' \in S'$.

$$\begin{aligned} \Lambda_{-\xi} &= -\Lambda_\xi \quad \text{for all } \xi \in S^{\mathrm{re}}, \\ 2\Lambda_\xi - \Lambda_\xi &\subset \Lambda_\xi \quad \text{for all } \xi \in S^{\mathrm{re}}, \\ \Lambda_\eta &= \Lambda_{w'(\eta)} \quad \text{for all } \eta \in S \text{ and } w' \in W_{S'}\,, \\ \Lambda_\eta - \langle \eta, \xi'^\vee \rangle \Lambda_{\xi'} &\subset \Lambda_\eta \quad \text{for } \xi' \in S',\ \eta \in S \quad \text{and} \\ \Lambda_{\xi'} &= \Lambda_{-\xi'} \quad \text{for } \xi' \in S'. \end{aligned}$$

In particular, Λ_ξ is constant on the $W_{S'}$-orbits of S. However, in general, the Λ_ξ are not constant on all of S, see the examples in 3.7.

One might wonder if the conditions (ED1)–(ED3) are strong enough to force the Λ_ξ to be subgroups of $(Z,+)$. But it turns out that this is not the case. For example, this already happens for the extension data of locally finite root systems, 3.7. On the other hand, if R is an integral pre-reflection system and Λ a subgroup of $(Z,+)$ which spans Z, then $\Lambda_\xi \equiv \Lambda$ is an example of an extension datum, the so-called *untwisted* case, cf. the definition of an untwisted affine root system in 2.12.

The conditions above have appeared in the context of reflection spaces (not to be confused with a reflection system). Recall ([Loo]) that a *reflection space* is a set S with a map $S \times S \to S : (s,t) \mapsto s \cdot t$ satisfying $s \cdot s = s$, $s \cdot (s \cdot t) = t$ and $s \cdot (t \cdot u) = (s \cdot t) \cdot (s \cdot u)$ for all $s,t,u \in S$. A *reflection subspace* of a reflection space $(S,\cdot)$ is a subset $T \subset S$ such that $t_1 \cdot t_2 \in T$ for all $t_1, t_2 \in T$. In this case, T is a reflection space with the induced operation. Any abelian group $(Z,+)$ is a reflection space with respect to the operation $x \cdot y = 2x - y$ for $x, y \in Z$. Correspondingly, a reflection subspace of the reflection space $(Z,\cdot)$ is a subset $A \subset Z$ satisfying $2a - b \in A$ for all $a, b \in A$, symbolically $2A - A \subset A$. Hence all Λ_ξ, $\xi \in S^{\mathrm{re}}$, are reflection subspaces of $(Z,\cdot)$. Moreover, for any subset A of Z it is easily seen that

$$A - 2A \subset A \;\Leftrightarrow\; A = -A \text{ and } 2A + A \subset A \;\Leftrightarrow\; A = -A \text{ and } 2A - A \subset A. \quad (1)$$

A subset A satisfying (1) will be called a *symmetric reflection subspace*. We will consider 0 as the base point of the reflection space Z. Also, we denote by $\mathbb{Z}[A]$ the subgroup of $(Z,+)$ generated by $A \subset Z$. Then the following are equivalent ([NY, 2.1]):

(i) $0 \in A$ and $A - 2A \subset A$,

(ii) $0 \in A$ and $2A - A \subset A$,

(iii) $2\mathbb{Z}[A] \subset A$ and $2\mathbb{Z}[A] - A \subset A$,

(iv) A is a union of cosets modulo $2\mathbb{Z}[A]$, including the trivial coset $2\mathbb{Z}[A]$.

In this case, A will be called a *pointed reflection subspace*. It is immediate from the above that every $\Lambda_{\xi'}$, $\xi' \in S'^{\mathrm{re}}$, is a pointed reflection subspace. We note that pointed reflection subspaces are necessarily symmetric. It is obvious from (iv) above that a pointed reflection subspace is in general not a subgroup of $(Z, +)$.

The following theorem characterizes extensions in terms of extension data.

3.4 Theorem *Let (S, Y) be a pre-reflection system.*

(a) *Let $\mathfrak{L} = (\Lambda_\xi)_{\xi \in S}$ be an extension datum of type (S, S', Z). Put $X := Y \oplus Z$, denote by $\pi\colon X \to Y$ the projection with kernel Z, and define*

$$R := \bigcup_{\xi \in S} \xi \oplus \Lambda_\xi \subset X \quad \textit{and} \quad s_\alpha(x) := s_\xi(y) \oplus \big(z - \langle y, \xi^\vee \rangle \lambda\big), \tag{1}$$

for all $\alpha = \xi \oplus \lambda \in \xi \oplus \Lambda_\xi \subset R$ and all $x = y \oplus z \in X$. Then R is a pre-reflection system in X, denoted $\mathcal{E} = \mathcal{E}(S, S', \mathfrak{L})$. Moreover, $\pi\colon (R, X) \to (S, Y)$ is an extension of pre-reflection systems, and the canonical injection $\iota\colon Y \to X$ is a partial section of π over S'.

(b) *Conversely, let $f\colon (R, X) \to (S, Y)$ be an extension and let $g\colon S' \to R$ be a partial section of f, cf.* 3.1*. For every $\xi \in S$ define $R_\xi \subset R$ and $\Lambda_\xi \subset Z := \mathrm{Ker}(f)$ by*

$$R_\xi = R \cap f^{-1}(\xi) = g(\xi) \oplus \Lambda_\xi. \tag{2}$$

Then $\mathfrak{L} = (\Lambda_\xi)_{\xi \in S}$ is an extension datum of type (S, S', Z), and the vector space isomorphism $\varphi\colon Y \oplus Z \cong X$ sending $y \oplus z$ to $g(y) \oplus z$ is an isomorphism $\mathcal{E}(S, S', \mathfrak{L}) \cong R$ of pre-reflection systems making the following diagram commutative:

$$\begin{array}{ccccc} & & S' & & \\ & {\scriptstyle\iota}\swarrow & & \searrow{\scriptstyle g} & \\ \mathcal{E} & & \xrightarrow{\quad\varphi\quad} & & R\,. \\ & {\scriptstyle\pi}\searrow & & \swarrow{\scriptstyle f} & \\ & & S & & \end{array}$$

(c) *In the setting of* (b)*, the following are equivalent for $g' \in \mathrm{Hom}_{\mathbb{K}}(Y, Z)$:*

(i) *$g'\colon S' \to R$ is another partial section of f,*
(ii) *there exists $\varphi \in \mathrm{Hom}_{\mathbb{K}}(Y, Z)$ such that $g' = g + \varphi$ and $\varphi(\xi') \in \Lambda_{\xi'}$ for all $\xi' \in S'$.*

In this case, the extension datum $\mathfrak{L}' = (\Lambda'_\xi)_{\xi \in S}$ defined by (2) *with respect to g' is related to the extension datum $\mathfrak{L}$ by*

$$\Lambda'_\xi = \Lambda_\xi - \varphi(\xi) \quad \textit{for } \xi \in S. \tag{3}$$

In the context of root-graded Lie algebras, the partial sections g and g' in (c) lead to isotopic Lie algebras, see 5.2 and 6.4.

3.5 Corollary. *Let* (S, Y) *be a pre-reflection system and let* $\mathfrak{L} = (\Lambda_\xi)_{\xi \in S}$ *be an extension datum of type* (S, S', Z). *Let* $R = \mathcal{E}(S, S', Z)$ *be the pre-reflection system defined in* 3.4(a).

(a) *R is reduced if and only if, for all* $0 \neq \xi \in S$,

$$\xi,\ c\xi \in S \text{ for } c \in \mathbb{K} \setminus \{0, \pm 1\} \quad \Longrightarrow \quad \Lambda_{c\xi} \cap c\Lambda_\xi = \emptyset.$$

(b) *R is symmetric if and only if S is symmetric and* $\Lambda_{-\xi} = -\Lambda_\xi$ *for all* $\xi \in S^{\mathrm{im}}$.

(c) *R is a reflection system if and only if S is a reflection system.*

3.6 Affine reflection systems. A pre-reflection system is called an *affine reflection system* if it is an extension of a locally finite root system. A *morphism* between affine reflection systems is a morphism of the underlying pre-reflection systems.

The name affine reflection system, as opposed to affine pre-reflection system, is justified in view of 3.5(c), since a locally finite root system is a reflection system and hence so is any extension of it.

A morphism between affine reflection systems is a morphism of the underlying reflection systems (2.2). Isomorphisms have the following characterization.

> *A vector space isomorphism* $f\colon X \to X'$ *is an isomorphism of the reflection systems* (R, X) *and* (R', X') *if and only if* $f(R^{\mathrm{im}}) = R'^{\mathrm{im}}$ *and* $f(R^{\mathrm{re}}) = R'^{\mathrm{re}}$. (1)

By definition, the *nullity* of an affine reflection system R is the rank of the torsion-free abelian group $\langle R^{\mathrm{im}} \rangle$ generated by R^{im}, i.e., nullity $= \dim_{\mathbb{Q}}(\langle R^{\mathrm{im}} \rangle \otimes_{\mathbb{Z}} \mathbb{Q})$. It is clear from the definitions that a locally finite root system is an affine reflection system of nullity 0 and, conversely, any affine reflection system of nullity 0 is a locally finite root system. The affine root systems $R(S, t(S))$ of 2.12 are affine reflection systems of nullity 1, see also 3.11(a).

In the following let (R, X) be an affine reflection system and let $f\colon (R, X) \to (S, Y)$ be an extension where (S, Y) is a locally finite root system. As mentioned in 3.2, R is then coherent and integral. Also, non-degeneracy of S implies that S is unique, up to a unique isomorphism. We will call S the *quotient root system of R* in this context and refer to f as the *canonical projection*.

Let $Z = \mathrm{Ker}(f)$. One can show that the extension f has a partial section g over S_{ind}. Let $\mathfrak{L} = (\Lambda_\xi)_{\xi \in S}$ be the extension datum of type (S, S_{ind}, Z) associated to f and g in Th. 3.4(b). Thus, *up to an isomorphism we may assume that R is given by* (3.4.1). The extension datum $\mathfrak{L}$ appearing there has some special properties besides the ones mentioned in 3.5. Namely, for $0 \neq \xi, \eta \in S$ and $w \in W(S)$ we have

$$\begin{aligned} &\Lambda_\xi = \Lambda_{-\xi} = -\Lambda_\xi = \Lambda_{w(\xi)} \quad \text{for all } w \in W(S), \\ &\Lambda_\xi \supset \Lambda_\xi - \langle \xi, \eta^\vee \rangle \Lambda_\eta, \quad \text{and} \\ &\Lambda_\xi \supset \Lambda_{2\xi} \quad \text{whenever } 2\xi \in S. \end{aligned}$$

Moreover, define $\Lambda_{\text{diff}} := \bigcup_{0 \neq \xi \in S} (\Lambda_\xi - \Lambda_\xi)$. Then $\mathbb{Z}\Lambda_{\text{diff}} = \Lambda_{\text{diff}}$ and we have:

(a) R is symmetric if and only if $\Lambda_0 = -\Lambda_0$.

(b) All root strings $\mathbb{S}(\beta, \alpha)$, $\beta \in R$, $\alpha \in R^{\text{re}}$, are unbroken if and only if $\Lambda_{\text{diff}} \subset \Lambda_0$.

(c) R is tame (cf. 2.2) if and only if $\Lambda_0 \subset \Lambda_{\text{diff}}$.

(d) $|\mathbb{S}(\beta, \alpha)| \le 5$ for all $\beta \in R$ and $\alpha \in R^{\text{re}}$.

We now describe extension data in the irreducible case in more detail.

3.7 Extension data for irreducible locally finite root systems. Let S be an irreducible locally finite root system. Recall the decomposition $S = S_{\text{sh}} \cup S_{\text{lg}} \cup S_{\text{div}}$ of 2.12. Let $\mathfrak{L} = (\Lambda_\xi)_{\xi \in S}$ be an extension datum of type (S, S_{ind}, Z). One knows that $W(S)$ operates transitively on the roots of the same length ([LN1, 5.6]). By 3.6 we can therefore define Λ_{sh}, Λ_{lg} and Λ_{div} by

$$\Lambda_\alpha = \begin{cases} \Lambda_{\text{sh}} & \text{for } \alpha \in S_{\text{sh}}, \\ \Lambda_{\text{lg}} & \text{for } \alpha \in S_{\text{lg}}, \\ \Lambda_{\text{div}} & \text{for } 0 \neq \alpha \in S_{\text{div}}. \end{cases} \tag{1}$$

By convention, Λ_{lg} and Λ_{div} are only defined if the corresponding set of roots S_{lg} and $S_{\text{div}} \setminus \{0\}$ are not empty (we always have $S_{\text{sh}} \neq \emptyset$). To streamline the presentation, this will not be specified in the following. It will (hopefully) always be clear from the context what is meant.

(a) We have already seen in 3.3 that in general Λ_{sh} and Λ_{lg} are pointed reflection subspaces and that Λ_{div} is a symmetric reflection subspace of Λ. In the setting of this subsection we have in addition the following:
(i) Λ_{sh} is a subgroup of Z, if $S_{\text{sh}} \cup \{0\}$ contains a subsystem of type A_2,
(ii) Λ_{lg} is a subgroup of Z, if $S_{\text{lg}} \cup \{0\}$ contains a subsystem of type A_2, and
(iii) the following relations hold for S as indicated and $k = k(S)$ defined in 2.12

$$\begin{array}{lll} \Lambda_{\text{sh}} + \Lambda_{\text{lg}} \subset \Lambda_{\text{sh}}, & \Lambda_{\text{lg}} + k\Lambda_{\text{sh}} \subset \Lambda_{\text{lg}}, & (S_{\text{lg}} \neq \emptyset) \\ \Lambda_{\text{sh}} + \Lambda_{\text{div}} \subset \Lambda_{\text{sh}}, & \Lambda_{\text{div}} + 4\Lambda_{\text{sh}} \subset \Lambda_{\text{div}}, & (S = \text{BC}_1) \\ \Lambda_{\text{lg}} + \Lambda_{\text{div}} \subset \Lambda_{\text{lg}}, & \Lambda_{\text{div}} + 2\Lambda_{\text{lg}} \subset \Lambda_{\text{div}}, & (S = \text{BC}_I,\ |I| \ge 2). \end{array}$$

(b) Conversely, given a subset Λ_0 with $0 \in \Lambda_0$, pointed reflection subspaces Λ_{sh} and Λ_{lg} (if $S_{\text{lg}} \neq \emptyset$) and a symmetric reflection subspace $\Lambda_{\text{div}} \subset \Lambda$ (if $\{0\} \subsetneq S_{\text{div}}$) satisfying (i) – (iii) of (a), then (1) defines an extension datum of type (S, S_{ind}, Z). Moreover, the following hold where, we recall k is defined in (2.12.1):
(i)′ Λ_{sh} is a subgroup of Z if $S \neq \text{A}_1$, B_I or BC_I for any I,

(ii)' $\Lambda_{\rm lg}$ is a subgroup of Z if $S = {\rm B}_I$ or ${\rm BC}_I$ and $|I| \ge 3$, or if $S = {\rm F}_4$ or ${\rm G}_2$,
(iv) $k\Lambda_{\rm sh} \subset \Lambda_{\rm lg} \subset \Lambda_{\rm sh}$, $4\Lambda_{\rm sh} \subset \Lambda_{\rm div} \subset \Lambda_{\rm sh}$ and $\Lambda_{\rm div} \subset \Lambda_{\rm lg} \subset \Lambda_{\rm sh}$.
(v) The inclusions $\Lambda_{\rm sh} + \Lambda_{\rm div} \subset \Lambda_{\rm sh}$ and $\Lambda_{\rm div} + 4\Lambda_{\rm sh} \subset \Lambda_{\rm div}$ hold for all $S = {\rm BC}_I$.

One therefore obtains the following description of $\mathfrak{L}^\times := (\Lambda_{\rm sh}, \Lambda_{\rm lg}, \Lambda_{\rm div})$ for the various types of irreducible root systems. Note that the only condition on Λ_0 is $0 \in \Lambda_0$ because of (ED2).

(I) S is simply laced, i.e., $S = \dot{\rm A}_I$, ${\rm D}_I$ ($|I| \ge 4$), ${\rm E}_6, {\rm E}_7$ or ${\rm E}_8$: $\mathfrak{L}^\times = (\Lambda_{\rm sh})$, where $\Lambda_{\rm sh}$ is a pointed reflection subspace for $S = {\rm A}_1$ and a subgroup of Z otherwise.

(II) $S = {\rm B}_I(|I| \ge 2), {\rm C}_I(|I| \ge 3)$ or ${\rm F}_4$: $\mathfrak{L}^\times = (\Lambda_{\rm sh}, \Lambda_{\rm lg})$, where $\Lambda_{\rm sh}$ and $\Lambda_{\rm lg}$ are pointed reflection subspaces satisfying $\Lambda_{\rm sh} + \Lambda_{\rm lg} \subset \Lambda_{\rm sh}$ and $\Lambda_{\rm lg} + 2\Lambda_{\rm sh} \subset \Lambda_{\rm lg}$. Moreover, $\Lambda_{\rm sh}$ is a subgroup of Z if $S = {\rm C}_I$ or ${\rm F}_4$, while $\Lambda_{\rm lg}$ is a subgroup if $S = {\rm B}_I$, $|I| \ge 3$ or ${\rm F}_4$.

(III) $S = {\rm G}_2$: $\mathfrak{L}^\times = (\Lambda_{\rm sh}, \Lambda_{\rm lg})$, where $\Lambda_{\rm sh}$ and $\Lambda_{\rm lg}$ are subgroups of Z satisfying $\Lambda_{\rm sh} + \Lambda_{\rm lg} \subset \Lambda_{\rm sh}$ and $\Lambda_{\rm lg} + 3\Lambda_{\rm sh} \subset \Lambda_{\rm lg}$.

(IV) $S = {\rm BC}_1$: $\mathfrak{L}^\times = (\Lambda_{\rm sh}, \Lambda_{\rm div})$, where $\Lambda_{\rm sh}$ is a pointed reflection subspace, $\Lambda_{\rm div}$ is a symmetric reflection subspace and $\Lambda_{\rm sh} + \Lambda_{\rm div} \subset \Lambda_{\rm sh}$ and $\Lambda_{\rm div} + 4\Lambda_{\rm sh} \subset \Lambda_{\rm div}$.

(V) $S = {\rm BC}_I, |I| \ge 2$: $\mathfrak{L}^\times = (\Lambda_{\rm sh}, \Lambda_{\rm lg}, \Lambda_{\rm div})$ where $\Lambda_{\rm sh}$ and $\Lambda_{\rm lg}$ are pointed reflection subspaces, $\Lambda_{\rm div}$ is a symmetric reflection subspace and $\Lambda_{\rm sh} + \Lambda_{\rm lg} \subset \Lambda_{\rm sh}$, $\Lambda_{\rm lg} + 2\Lambda_{\rm sh} \subset \Lambda_{\rm lg}$, $\Lambda_{\rm lg} + \Lambda_{\rm div} \subset \Lambda_{\rm lg}$, $\Lambda_{\rm div} + 2\Lambda_{\rm lg} \subset \Lambda_{\rm div}$. Moreover, if $|I| \ge 3$ we require that $\Lambda_{\rm lg}$ is a subgroup of Z.

3.8 Affine forms. Our definition of affine reflection systems follows the approach of [Bo2] where root systems are defined without reference to a bilinear form. In the literature, it is customary to define affine root systems and their generalizations, the extended affine root systems (EARS), in real vector spaces using positive semidefinite forms. We will therefore give a characterization of affine reflections systems in terms of affine invariant forms where, by definition, an *affine form* of a pre-reflection system (R, X) over $\mathbb{K}$ is an invariant form b satisfying $R^{\rm im} = R \cap \operatorname{Rad} b$, cf. (2.5.2). In particular, affine forms are strictly invariant in the sense of 2.5.

As an example, the forms used in the theory of EARS are affine forms in our sense. In particular, the form $(\cdot|\cdot)$ in 2.12 is an affine form of the affine root system R.

3.9 Theorem. *Let (R, X) be a pre-reflection system. Then (R, X) is an affine reflection system if and and only if it satisfies the following conditions:*

(i) (R, X) *is integral,*
(ii) (R, X) *has an affine form, and*
(iii) $\langle R, \alpha^\vee \rangle$ *is bounded for every* $\alpha \in R^{\rm re}$.

In this case:

(a) *Let b be an affine form for (R,X) and let $f\colon X \to X/\operatorname{Rad} b$ be the canonical map. Then $(S,Y) = (f(R), X/\operatorname{Rad} b)$ is the quotient root system of R and f its canonical projection. Moreover,* $\operatorname{Re}(R)$ *is connected if and only if S is irreducible.*

(b) *There exists a unique affine form $(\cdot|\cdot)_a$ on (R,X) that is* normalized *in the sense of* 2.9*, i.e., for every connected component C of* $\operatorname{Re}(R)$ *we have*

$$2 \in \{(\alpha|\alpha)_a : 0 \neq \alpha \in C\} \subset \{2,3,4,6,8\}.$$

The form $(\cdot|\cdot)_a$ satisfies

$$\{(\alpha|\alpha)_a : 0 \neq \alpha \in C\} \in \big\{\{2\},\{2,4\},\{2,6\},\{2,8\},\{2,4,8\}\big\}.$$

Its radical is $\operatorname{Rad}(\cdot|\cdot)_a = \operatorname{Ker} f$. *If $\mathbb{K} = \mathbb{R}$ then $(\cdot|\cdot)_a$ is positive semidefinite.*

3.10 Corollary. *A pre-reflection system over the reals is an affine reflection system if and only if it is integral and has a positive semidefinite affine form.*

3.11 Special types of affine reflection systems. As usual, the *rank* of a reflection system (R,X) is defined as $\operatorname{rank}(R,X) = \dim X$. If R has finite rank and $\mathbb{K} = \mathbb{R}$, we will say that R is *discrete* if R is a discrete subset of X.

(a) Let (R,X) be an affine reflection system over $\mathbb{K} = \mathbb{R}$ with the following properties: R has finite rank, $\operatorname{Re}(R)$ is connected, $R = -R$ and R is discrete. Then R is called

(i) an EARS, an abbreviation for *extended affine root system*, if R is reduced, tame (see (3.6.c)), and all root strings are unbroken;

(ii) a SEARS, an abbreviation of *Saito's extended affine root system*, if $R = \operatorname{Re}(R)$.

It was shown in [Az3, Th. 18] that every reduced SEARS can be uniquely extended to an EARS and, conversely, the reflective roots of an EARS are the non-zero roots of a SEARS. This is now immediate. Indeed, by the results mentioned in 3.6, an affine reflection system is tame and has unbroken root strings if and only $R^{\mathrm{im}} = \Lambda_{\mathrm{diff}}$. The affine root systems $R(S, t(S))$ of 2.12 with S finite are precisely the EARSs of nullity 1 ([ABGP]).

(b) In [MY], Morita and Yoshii define a LEARS, an abbreviation for *locally extended affine root system*. In our terminology, this is a symmetric affine reflection system R over $\mathbb{K} = \mathbb{R}$ such that $R = \operatorname{Re}(R)$ is connected. The equivalence of this definition with the one in [MY] follows from 3.10.

(c) In [Az4], Azam defines a GRRS, an abbreviation of a *generalized reductive root system*. In our terminology, this is a symmetric real reduced, discrete affine reflection system R which has finite rank and unbroken root strings.

3.12 Notes. With the exceptions mentioned in the text and below, all results in this section are proven in [LN2].

If S is an integral reflection system, our definition of an extension datum in 3.3 makes sense for any abelian group Z instead of a $\mathbb{K}$-vector space. This generality is however not needed for extension data arising from Lie algebras with toral subalgebras, 2.11, like the generalizations of affine Lie algebras to be discussed later. But we note that the results on extension data stated in 3.6 and 3.7 are true for an abelian group Z. In this generality, but for S a finite irreducible root system, extension data without Λ_0 were defined by Yoshii in [Yo4] as "root systems of type S extended by Z". The results stated in 3.7 extend [Yo4, Th. 2.4] to the setting of locally finite root systems.

We point out that our definition of an EARS is equivalent to the one given by Azam, Allison, Berman, Gao and Pianzola in [AABGP, II, Def. 2.1]. That our definition of a SEARS is the same as Saito's definition of an "extended affine root system" in [Sa], follows from 3.10.

Our characterization of extensions in terms of extension data 3.4 generalizes the Structure Theorem for extended affine root systems [AABGP, II, Th. 2.37] as well as [MY, Prop. 4.2] and – modulo the limitations mentioned above – the description of root systems extended by an abelian group in [Yo4, Th. 3.4]. The isomorphism criterion (3.6.1) is proven in [Yo6,] for the case of LEARSs (3.11(c)). A proof of the general case will be contained in [Neh8].

The extension data arising in the theory of extended affine root systems are studied in detail in [AABGP, Ch. II] and [Az1]. These references are only a small portion of what is presently known on extended affine root systems and their Weyl groups. Many of these results likely have generalizations to the setting of affine reflection systems.

4 Graded algebras

In the previous sections §2 and §3 we reviewed the "combinatorics" needed to describe extended affine Lie algebras and their generalizations in §6. In this and the following section we will introduce the necessary algebraic concepts. All of them have to do with algebras graded by an arbitrary abelian group, usually denoted Λ and written additively. (The reader will notice that at least in the first few subsections Λ need not be abelian.)

4.1 Algebras per se. Throughout, we will consider algebras A over a unital commutative associative ring of scalars k. Unless explicitly stated otherwise, we do not assume that A belongs to some special variety of algebras, like associative algebras or Lie algebras. Therefore, an *algebra* or a *k-algebra* if we want to be more precise, is simply a k-module A together with a k-bilinear map $A \times A \to A$, $(a, b) \mapsto ab$, the *product of* A. As is customary, if A happens to be a Lie algebra, its product will be denoted $[a, b]$ or sometimes just $[a\, b]$ instead of ab.

Let A be an algebra. The k-module of all its k-linear derivations is always a Lie algebra, denoted $\mathrm{Der}_k A$. A symmetric bilinear form $(\cdot|\cdot) : A \times A \to k$ is called *invariant* if $(ab|c) = (a|bc)$ holds for all $a, b, c \in A$; it is *nondegenerate* if $(a|A) = 0$ implies $a = 0$. We put $AA = \mathrm{span}_k\{ab : a, b \in A\}$. The algebra A is *perfect* if $A = AA$, and is *simple* if $AA \neq 0$ and if $\{0\}$ and A are the only ideals of A. Any unital algebra is perfect, and any simple algebra is perfect since AA is an ideal of A.

4.2 The centroid of an algebra. The *centroid* of an algebra A is the subalgebra $\mathrm{Cent}_k(A)$ of the endomorphism algebra $\mathrm{End}_k(A)$ consisting of the k-linear endomorphisms of A, which commute with the left and right multiplications of A. One always has $k\,\mathrm{Id}_A \subset \mathrm{Cent}_k(A)$, but in general this is a proper inclusion. If the canonical map $k \to \mathrm{Cent}_k(A)$ is an isomorphism, A is called *central*, and one says that A is *central-simple* if A is central and simple, see 4.4 for a characterization of central-simple algebras.

The centroid is always a unital associative algebra, but not necessarily commutative in general. For example, the centroid of the null algebra (all products are 0) is the full endomorphism algebra. On the other hand, the centroid of a perfect algebra is always commutative. For example, let A be a unital associative algebra and denote by $\mathrm{Z}(A) = \{z \in A : za = az$ for all $a \in A\}$ its *centre*, then

$$\mathrm{Z}(A) \to \mathrm{Cent}_k(A), \quad z \mapsto L_z \qquad (A \text{ associative}) \tag{1}$$

is an isomorphism of algebras, where L_z denotes the left multiplication by z. However, even if the names are quite similar, one should not confuse the centroid with the centre of A! A non-zero Lie algebra L may well be *centreless* in the sense that $\mathrm{Z}(L) = \{0\}$, but always has $0 \neq k\ \mathrm{Id}_L \in \mathrm{Cent}(L)$, see 4.12(a) for an example.

Since $\mathrm{Cent}_k(A)$ is a subalgebra of $\mathrm{End}_k(A)$ one can consider A as a left module over $\mathrm{Cent}_k(A)$. This change of perspective is particularly useful if $\mathrm{Cent}_k(A)$ is commutative since then A is an algebra over $\mathrm{Cent}_k(A)$. In general, an algebra A is called *fgc* (for f̲initely g̲enerated over its c̲entroid) if A is a finitely generated $\mathrm{Cent}_k(A)$-module. Fgc algebras are more tractable than general algebras, and we will characterize various classes of fgc algebras throughout this paper, see 4.6, 4.7, 4.13 and 5.8.

4.3 Graded algebras. Recall that Λ is an abelian group. A *Λ-graded algebra* is an algebra A together with a family $(A^\lambda : \lambda \in \Lambda)$ of submodules A^λ of A satisfying $A = \bigoplus_{\lambda\in\Lambda} A^\lambda$ and $A^\lambda A^\mu \subset A^{\lambda+\mu}$ for all $\lambda, \mu \in \Lambda$. We will usually indicate this by saying "Let $A = \bigoplus_\lambda A^\lambda$ be a Λ-graded algebra ..." or simply "Let A be graded algebra...".

We will encounter many example of graded algebras. But an immediate and important example is $k[\Lambda]$, the *group algebra of Λ* over k. By definition, $k[\Lambda]$ is a free k-module with a k-basis in bijection with Λ, say by $\lambda \mapsto z^\lambda$, and product determined by $z^\lambda z^\mu = z^{\lambda+\mu}$ for $\lambda, \mu \in \Lambda$. It is graded by $(k[\Lambda])^\lambda = k\,z^\lambda$ and is a unital associative algebra.

We will need some more terminology for a graded algebra A. The submodules A^λ of A are referred to as *homogeneous spaces*. We will occasionally use subscripts to describe them, in particular whenever we consider an algebra with two gradings, say $A = \bigoplus_{\lambda \in \Lambda} A^\lambda$ and $A = \bigoplus_{q \in \mathcal{Q}} A_q$. The two gradings are called *compatible* if, putting $A_q^\lambda := A^\lambda \cap A_q$, we have $A^\lambda = \bigoplus_{q \in \mathcal{Q}} A_q^\lambda$ for all $\lambda \in \Lambda$. We will indicate compatible gradings by saying that A *is* $(\mathcal{Q}, \Lambda)$*-graded.*

The *support* of a Λ-graded algebra A is the set $\operatorname{supp}_\Lambda A = \{\lambda \in \Lambda : A^\lambda \neq 0\}$. In general $\operatorname{supp}_\Lambda A$ is a proper subset of Λ. The subgroup of Λ generated by $\operatorname{supp}_\Lambda A$ will be denoted $\langle \operatorname{supp}_\Lambda A \rangle$. One says that A *has full support* if $\langle \operatorname{supp}_\Lambda A \rangle = \Lambda$. Since the grading of A only depends on $\langle \operatorname{supp}_\Lambda A \rangle$, it is of course always possible, and sometimes even useful, to assume that a graded algebra has full support.

Two graded algebras A and A' with grading groups Λ and Λ' respectively are called *isograded-isomorphic* if there exists an isomorphism $f\colon A \to A'$ of the underlying k-algebras and a group isomorphism $\varphi\colon \langle \operatorname{supp}_\Lambda A \rangle \to \langle \operatorname{supp}_{\Lambda'} A' \rangle$ satisfying $f(A^\lambda) = A'^{\varphi(\lambda)}$ for all $\lambda \in \operatorname{supp}_\Lambda A$. Note that φ is uniquely determined by f. A more restrictive concept is that of a *graded-isomorphism* between two Λ-graded algebras A and A', which by definition is an isomorphism $f\colon A \to A'$ of the ungraded algebras satisfying $f(A^\lambda) = A'^\lambda$ for all $\lambda \in \Lambda$.

A Λ-graded algebra is called *graded-simple* if $AA \neq 0$ and $\{0\}$ and A are the only Λ-graded ideals of A. For example, $k[\Lambda]$ is graded-simple if and only if k is a field. But of course in this case $k[\Lambda]$ may have many non-trivial ungraded ideals.

A bilinear form $(\cdot|\cdot)\colon A \times A \to k$ on A is Λ*-graded* if $(A^\lambda | A^\mu) = 0$ for $\lambda + \mu \neq 0$. For example, the Killing form of a finite-dimensional Λ-graded Lie algebra is Λ-graded. The radical of an invariant Λ-graded symmetric bilinear form $(\cdot|\cdot)$ is a graded ideal of A, namely $\operatorname{Rad}(\cdot|\cdot) = \{a \in A : (a|A) = 0\}$. A Λ-graded bilinear form $(\cdot|\cdot)$ is nondegenerate, i.e. $\operatorname{Rad}(\cdot|\cdot) = \{0\}$, if and only if for all $\lambda \in \Lambda$ the restriction of $(\cdot|\cdot)$ to $A^\lambda \times A^{-\lambda}$ is a nondegenerate pairing.

We denote by $\operatorname{GIF}(A, \Lambda)$ the k-module of all Λ-graded invariant symmetric bilinear forms on A. If L is a perfect Λ-graded Lie algebra, then $\operatorname{Rad}(\cdot|\cdot) \supset \operatorname{Z}(L) = \{z \in L : [z, L] = 0\}$, the *centre* of L. Since $\operatorname{Z}(L)$ is also Λ-graded, the quotient algebra inherits a canonical Λ-grading, and every $(\cdot|\cdot) \in \operatorname{GIF}(L, \Lambda)$ induces an invariant symmetric bilinear form on $L/\operatorname{Z}(L)$ by $(\bar{x}|\bar{y}) = (x|y)$, where $x \mapsto \bar{x}$ denotes the canonical map. This gives rise to an isomorphism

$$\operatorname{GIF}(L, \Lambda) \cong \operatorname{GIF}(L/\operatorname{Z}(L), \Lambda) \qquad \text{(Lie algebras).} \tag{1}$$

If A is a unital algebra, say with identity element 1, then necessarily $1 \in A^0$. If in addition A is associative, then

$$\operatorname{GIF}(A, \Lambda) \cong (A^0/[A, A]^0)^* \qquad \text{(associative algebras)} \tag{2}$$

by assigning to any linear form φ of $A^0/[A, A]^0$ the bilinear form $(a \mid b)_\varphi = \varphi((ab)^0 + [A, A]^0)$. Here $[A, A] = \operatorname{span}_k\{ab - ba : a, b \in A\}$, and $(ab)^0$ and

$[A,A]^0$ denote the 0-component of ab and $[A,A]$. For example, the group algebra $k[\Lambda]$ has up to scalars just one invariant symmetric bilinear form, which assigns to (a,b) the z^0-coefficient of ab with respect to the canonical k-basis.

An endomorphism f of the underlying k-module of A is said to have *degree* λ if $f(A^\mu) \subset A^{\lambda+\mu}$ for all $\mu \in \Lambda$. The submodules $(\mathrm{End}_k A)^\lambda$, consisting of all endomorphisms of degree λ, are the homogeneous spaces of the Λ-graded subalgebra

$$\mathrm{grEnd}_k(A) = \bigoplus_{\lambda\in\Lambda} (\mathrm{End}_k A)^\lambda$$

of the associative algebra $\mathrm{End}_k(A)$. We put

$$\mathrm{grDer}_k(A) = \mathrm{grEnd}_k(A) \cap \mathrm{Der}_k(A) = \bigoplus_{\lambda\in\Lambda} (\mathrm{Der}_k A)^\lambda, \quad \text{and}$$
$$\mathrm{grCent}_k(A) = \mathrm{grEnd}_k(A) \cap \mathrm{Cent}_k(A) = \bigoplus_{\lambda\in\Lambda} (\mathrm{Cent}_k A)^\lambda.$$

Then $(\mathrm{Der}_k A)^\lambda$ and $(\mathrm{Cent}_k A)^\lambda$ are, respectively, the derivations and centroidal transformations of A of degree λ. In general, $\mathrm{grDer}_k(A)$ or $\mathrm{grCent}_k(A)$ are proper subalgebras of $\mathrm{Der}_k(A)$ and $\mathrm{Cent}_k(A)$. However, $\mathrm{grDer}_k(A) = \mathrm{Der}_k(A)$ if A is a finitely generated algebra ([Fa, Prop. 1]). Similarly, $\mathrm{grCent}_k(A) = \mathrm{Cent}_k(A)$ if A is finitely generated as ideal. For example, any unital algebra is finitely generated as ideal (namely by $1 \in A$). We will discuss some examples of (gr) $\mathrm{Der}_k A$ and (gr) $\mathrm{Cent}_k A$ later, see 4.12, 4.13, 7.9 and 7.12.

We call A *graded-central* if the canonical map $k \to (\mathrm{Cent}_k A)^0$ is an isomorphism, and *graded-central-simple* if A is graded-central and graded-simple. It is well-known that an (ungraded) algebra A over a field k is central-simple if and only if $A \otimes_k K$ is simple for every extension field K of k. The following theorem extends this to the case of graded algebras.

4.4 Theorem. *Let A be a graded-simple k-algebra, where k is a field. Then the following are equivalent:*

(i) *A is graded-central-simple,*

(ii) *for any extension field K of k the base field extension $A \otimes_k K$ is a graded-central-simple K-algebra.*

(iii) *for any extension field K of k the algebra $A \otimes_k K$ is a graded-simple K-algebra.*

(iii) *$A \otimes_k (\mathrm{Cent}_k A)^0$ is graded-simple.*

In this theorem, "base field extension" means that we consider $A \otimes_k K$ as the K-algebra with product $(a_1 \otimes x_1)\,(a_2 \otimes x_2) = (a_1 a_2) \otimes (x_1 x_1)$ for $a_i \in A$ and $x_i \in K$ and the Λ-grading determined by $(A \otimes_k K)^\lambda = A^\lambda \otimes_k K$.

We will describe the centroids of graded-simple algebras in 4.6, after we have introduced special classes of associative graded algebras.

4.5 Associative (pre)division-graded algebras and tori. A Λ-graded unital associative algebra $A = \bigoplus_{\lambda\in\Lambda} A^\lambda$ is called

- *predivision-graded*, if every non-zero homogeneous space contains an invertible element;
- *division-graded* if every non-zero homogenous element is invertible;
- an *associative torus* if A is predivision-graded, k is a field and $\dim_k A^\lambda \le 1$.

For example, any group algebra $k[\Lambda]$ is predivision-graded. Moreover, $k[\Lambda]$ is division-graded if and only if k is a field, and in this case $k[\Lambda]$ is a torus.

Group algebras cover only a small part of all possible predivision-graded algebras. If A is such an algebra, it is easily seen that the support of A is a subgroup of the grading group Λ. To simplify the notation,

we assume $\operatorname{supp}_\Lambda A = \Lambda$ *in this subsection.*

Let $(u_\lambda : \lambda \in \Lambda)$ be a family of invertible elements $u_\lambda \in A^\lambda$ and put $B = A^0$. Then (u_λ) is a B-basis of the left B-module A, and the product of A is uniquely determined by the two rules

$$u_\lambda\, u_\mu = \tau(\lambda,\mu)\, u_{\lambda+\mu} \quad \text{and} \quad u_\lambda\, b = b^{\sigma(\lambda)}\, u_\lambda \quad (b \in B) \tag{1}$$

where $\tau\colon \Lambda \times \Lambda \to B^\times$ and $\sigma\colon \Lambda \to \operatorname{Aut}_k(B)$ are functions. Associativity of A leads to the two identities

$$\begin{aligned} \tau(\lambda,\mu)^{\sigma(\nu)}\, \tau(\nu, \lambda+\mu) &= \tau(\nu,\lambda)\, \tau(\nu\lambda, \mu), \ \text{and} \\ b^{\sigma(\lambda)})^{\sigma(\nu)}\, \tau(\nu,\lambda) \quad &= \ \tau(\nu,\lambda)\, b^{\sigma(\nu+\lambda)} \end{aligned} \tag{2}$$

for $\nu, \lambda, \mu \in \Lambda$ and $b \in B$. Conversely, given any unital associative k-algebra B and functions τ, σ as above satisfying (2), one can define a Λ-graded k-algebra by (1). It turns out to be an associative predivision-graded algebra. Algebras arising in this way are called *crossed product algebras* and denoted $(B, (u_\lambda), \tau, \sigma)$. To summarize,

any associative predivision-graded algebra A with support Λ is graded-isomorphic to a crossed product algebra $(B, (u_\lambda), \tau, \sigma)$. A is division-graded iff B is a division algebra, and A is a torus iff $B = k$ is a field. (3)

The description of predivision-graded algebras as crossed product algebras may not be very illuminating, except that it (hopefully) demonstrates how general this class is. The reader may be more comfortable with the subclass of *twisted group algebras* defined by the condition that σ is trivial, i.e., $\sigma(\lambda) = \mathrm{Id}_B$ for all $\lambda \in \Lambda$, and denoted $B^t[\Lambda]$. For example, any commutative crossed product algebra is a twisted group algebra. Or, up to graded isomorphism, a Λ-torus with full support is the same as a twisted group algebra $k^t[\Lambda]$. In the torus case, (2) says that τ is a 2-*cocycle* of the group Λ with values in $k^\times$. We will describe the special case of associative $\mathbb{Z}^n$-tori in 4.13.

Twisted group algebras arise naturally as centroids of graded-simple algebras.

4.6 Proposition. *Let* $A = \bigoplus_{\lambda \in \Lambda} A^\lambda$ *be a graded-simple* k*-algebra.*

(a) *Then* $\mathrm{Cent}_k(A)$ *is a commutative associative division-graded* k*-algebra,*

$$\mathrm{Cent}_k(A) = \mathrm{grCent}_k(A) = \bigoplus\nolimits_{\gamma \in \Gamma} (\mathrm{Cent}_k A)^\gamma \quad \textit{where} \tag{1}$$

$$\Gamma = \mathrm{supp}_\Lambda \mathrm{Cent}_k A$$

is a subgroup of Λ*, called the* central grading group of A. *Hence* $\mathrm{Cent}_k(A)$ *is a twisted group algebra over the field* $K = (\mathrm{Cent}_k A)^0$.

(b) A *is a graded-central-simple* K*-algebra.*

(c) A *is a free* $\mathrm{Cent}_k(A)$*-module with* $\Gamma + \mathrm{supp}_\Lambda A \subset \mathrm{supp}_\Lambda A$. *Moreover,* A *has finite rank as* $\mathrm{Cent}_k(A)$*-module, i.e.* A *is fgc, if and only if*

$$[\mathrm{supp}_\Lambda A : \Gamma] < \infty \quad \textit{and} \quad \dim_K(A^\lambda) < \infty \textit{ for all } \lambda \in \Lambda. \tag{2}$$

The interest in algebras which are graded-central-simple and also fgc comes from the following important realization theorem.

4.7 Theorem. ([ABFP1, Cor. 8.3.5]) *Let* k *be an algebraically closed field of characteristic* 0 *and let* Λ *be free abelian of finite rank. Then the following are equivalent for a* Λ*-graded* k*-algebra* B*:*

(i) B *is graded-central-simple, fgc and* $[\Lambda : \Gamma] < \infty$ *where* Γ *is the central grading group of* B*;*

(ii) B *is isograded-isomorphic to a multiloop algebra* $\mathcal{M}_{\mathbf{m}}(A, \boldsymbol{\sigma})$ *for some finite-dimensional simple* k*-algebra* A*, described below.*

The multiloop algebras referred to in (ii) provide an interesting class of examples of graded algebras. They are defined as follows (A can be arbitrary, but we assume that k is a field with enough roots of unity). We let $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}_+^n$ be an n-tuple of positive integers and let $\pi : \mathbb{Z}^n \to \mathbb{Z}/(m_1) \oplus \cdots \oplus \mathbb{Z}/(m_n) =: \Xi$ be the canonical map. Furthermore, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ is a family of n pairwise commuting automorphisms of the (ungraded) algebra A such that $\sigma_i^{m_i} = \mathrm{Id}$ for $1 \le i \le n$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ we define

$$A^{\pi(\lambda)} = \{a \in A : \sigma_i(a) = \zeta_{m_i}^{\lambda_i} a \text{ for } 1 \le i \le n\}.$$

where for each $l \in \{m_1, \ldots, m_n\}$ we have chosen a primitive l^{th}-root of unity $\zeta_l \in k$. Then $A = \bigoplus_{\xi \in \Xi} A^\xi$ is a Ξ-grading of A. Denoting by $k[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ the Laurent polynomial algebra over k in the variables $z_1, \ldots, z_n$,

$$\mathcal{M}_{\mathbf{m}}(A, \boldsymbol{\sigma}) = \bigoplus\nolimits_{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n} A^{\pi(\lambda_1, \ldots, \lambda_n)} \otimes k\, z_1^{\lambda_1} \cdots z_n^{\lambda_n}$$

is a $\mathbb{Z}^n$-graded algebra, called the *multiloop algebra* of $\boldsymbol{\sigma}$ based on A and relative to $\mathbf{m}$.

Thus, the theorem in particular says that $\mathcal{M}_{\mathbf{m}}(A, \boldsymbol{\sigma})$ is graded-central-simple. In fact, the centroid can be determined precisely.

4.8 Proposition. ([ABP3, Cor. 6.6]) *The centroid of a multiloop algebra* $\mathcal{M}_{\mathbf{m}}(A, \boldsymbol{\sigma})$ *based on a finite-dimensional central-simple algebra A is graded-isomorphic to the Laurent polynomial ring* $k[z_1^{\pm m_1}, \dots, z_n^{\pm m_n}]$.

4.9 Centroidal derivations. There is a close connection between the centroid and the derivations of an algebra A: $[d, \chi] \in \mathrm{Cent}_k(A)$ for $d \in \mathrm{Der}_k(A)$ and $\chi \in \mathrm{Cent}_k(A)$. In other words, $\mathrm{Cent}_k(A)$ is a submodule of the $\mathrm{Der}_k(A)$-module $\mathrm{End}_k(A)$. In the graded case one can define special derivations using the centroid.

Let $A = \bigoplus_{\lambda \in \Lambda} A^\lambda$ be a Λ-graded algebra. We abbreviate $\mathrm{C} = \mathrm{Cent}_k(A)$, $\mathrm{grC} = \mathrm{grCent}_k(A)$ and $\mathrm{C}^\lambda = (\mathrm{Cent}_k\, A)^\lambda$, thus $\mathrm{grC} = \bigoplus_\lambda C^\lambda$. We denote by $\mathrm{Hom}(\Lambda, \mathrm{C}) \cong \mathrm{Hom}_{\mathrm{C}}(\Lambda \otimes_{\mathbb{Z}} \mathrm{C}, \mathrm{C})$ the (left) C-module of all abelian group homomorphisms $\theta\colon \Lambda \to (\mathrm{C}, +)$. Every $\theta \in \mathrm{Hom}(\Lambda, \mathrm{C})$ gives rise to a so-called *centroidal derivation*, defined by

$$\partial_\theta(a^\lambda) = \theta(\lambda)\,(a^\lambda) \qquad \text{for } a^\lambda \in A^\lambda. \tag{1}$$

We put

$$\mathrm{CDer}(A, \Lambda) = \{\partial_\theta : \theta \in \mathrm{Hom}(\Lambda, \mathrm{C})\},$$

sometimes also denoted $\mathrm{CDer}(A)$ if Λ is clear from the context. We emphasize that $\mathrm{CDer}(A, \Lambda)$ not only depends on C but also on Λ. For example, any algebra A can be graded by $\Lambda = \{0\}$ in which case $\mathrm{CDer}(A, \Lambda) = \{0\}$.

We will also consider the Λ-graded submodule $\mathrm{grCDer}(A, \Lambda)$ of $\mathrm{CDer}\, A$,

$$\begin{aligned} \mathrm{grCDer}(A, \Lambda) &= \textstyle\bigoplus_{\lambda \in \Lambda} (\mathrm{CDer}\, A)^\lambda \qquad \text{where} \\ (\mathrm{CDer}\, A)^\lambda &= \mathrm{CDer}(A) \cap (\mathrm{End}_k\, A)^\lambda = \{\, \partial_\theta : \theta \in \mathrm{Hom}(\Lambda, \mathrm{C}^\lambda)\}. \end{aligned}$$

Suppose now that $\mathrm{Cent}_k(A)$ *is commutative*, e.g. that A is perfect. Then any ∂_θ is C^0-linear and $\mathrm{CDer}(A)$ is a C^0-subalgebra of the Lie algebra $\mathrm{Der}_{\mathrm{C}^0}(A)$ of all C^0-linear derivations of A. Indeed, $[\partial_\theta, \partial_\psi] = \partial_{\theta * \psi - \psi * \theta}$ for $\theta * \psi \in \mathrm{Hom}(\Lambda, \mathrm{C})$ defined by

$$(\theta * \psi)\,(\lambda) = \textstyle\sum_{\mu \in \Lambda} \theta(\mu)\, \psi(\lambda)^\mu \tag{2}$$

where $\psi(\lambda)^\mu$ denotes the μ-component of $\psi(\lambda) \in \mathrm{C}$. The sum in (2) converges in the finite topology. In particular, for $\theta \in \mathrm{Hom}(\Lambda, \mathrm{C}^\lambda)$ and $\psi \in \mathrm{Hom}(\Lambda, \mathrm{C}^\mu)$ one has

$$[\partial_\theta, \partial_\psi] = \theta(\mu)\partial_\psi - \psi(\lambda)\partial_\theta \in (\mathrm{CDer}\, A)^{\lambda+\mu}, \tag{3}$$

in particular $(\mathrm{CDer}\, A)^0$ is abelian.

Suppose A is perfect and $(\cdot|\cdot)$ *is a* Λ*-graded invariant nondegenerate symmetric bilinear form on A.* We denote by

$$\mathrm{grSCDer}(A, \Lambda) = \textstyle\bigoplus_{\lambda \in \Lambda} (\mathrm{SCDer}\, A)^\lambda$$

the graded skew centroidal derivations of A, i.e. the graded subalgebra of $\mathrm{grCDer}(A, \Lambda)$ consisting of those graded centroidal derivations of A which are skew-symmetric with respect to $(\cdot|\cdot)$. Since A is perfect, any centroidal transformation is symmetric with respect to $(\cdot|\cdot)$. This implies that

$$(\mathrm{SCDer}\, A)^\lambda = \{\partial_\theta \in (\mathrm{CDer}\, A)^\lambda : \theta(\lambda) = 0\},$$

in particular $(\mathrm{SCDer}\, A)^0 = (\mathrm{CDer}\, A)^0$. Because of (3) we have

$$[(\mathrm{SCDer}\, A)^\lambda,\, (\mathrm{SCDer}\, A)^{-\lambda}] = 0$$

for any $\lambda \in \Lambda$. Hence $\mathrm{grSCDer}(A, \Lambda)$ is the semidirect product of the ideal spanned by the non-zero homogeneous spaces and the abelian subalgebra $(\mathrm{CDer}\, A)^0$.

4.10 Degree derivations. Let again $A = \bigoplus_\lambda A^\lambda$ be a Λ-graded k-algebra. Recall that $k\,\mathrm{Id}_A \subset (\mathrm{Cent}_k\, A)^0$. We can therefore consider the submodule

$$\mathcal{D} = \mathcal{D}(A, \Lambda) = \{\partial_\theta : \theta \in \mathrm{Hom}(\Lambda,\, k\,\mathrm{Id}_A)\} \subset (\mathrm{CDer}\, A)^0$$

of so-called *degree derivations of* (A, Λ). For example, if $\Lambda = \mathbb{Z}^n$ we have $\mathcal{D} = \mathrm{span}_k\{\partial_i : 1 \le i \le n\}$ where $\partial_i(a^\lambda) = \lambda_i a^\lambda$ for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$.

Observe that $[\partial_\theta,\, \partial_\psi] = \theta(\mu)\, \partial_\psi$ for $\partial_\theta \in \mathcal{D}$ and $\psi \in \mathrm{Hom}(\Lambda, \mathrm{Cent}_k(A)^\mu)$. So the action of $\mathcal{D}$ on $\mathrm{grCDer}\, A$ is diagonalizable. In order to actually get a toral action,

we will assume in this subsection and in 4.11 that k is a field and that A has full support.

In this case, we can identify $\mathrm{Hom}(\Lambda, k\,\mathrm{Id}_A) \equiv \mathrm{Hom}_{\mathbb{Z}}(\Lambda, k)$. Moreover, the canonical map $\mathrm{Hom}(\Lambda, k) \to \mathcal{D} = \{\partial_\theta : \theta \in \mathrm{Hom}(\Lambda, k)\}$ is an isomorphism. We therefore have an evaluation map $\mathrm{ev}\colon\ \Lambda \to \mathcal{D}^*$, defined by

$$\mathrm{ev}_\lambda(\partial_\theta) = \theta(\lambda).$$

Observe that (4.9.1) says

$$A^\lambda \subset \{a \in A : d(a) = \mathrm{ev}_\lambda(d)\, a \text{ for all } d \in \mathcal{D}\}. \qquad (1)$$

The following lemma specifies a condition under which (1) becomes an equality.

4.11 Lemma. *As in* 4.10 *we suppose that k is a field and A is a Λ-graded k-algebra with full support. We let $T \subset \mathcal{D}$ be a subspace satisfying the condition that*

$$\text{the restricted evaluation map } \Lambda \to T^*,\ \lambda \mapsto \mathrm{ev}_\lambda|T, \text{ is injective.} \qquad (1)$$

Then $A^\lambda = \{a \in A : d(a) = \mathrm{ev}_\lambda(t)$ for all $t \in T\}$, in particular equality holds in (4.10.1). *Moreover, if $\mathrm{Cent}_k(A)$ is commutative, T is a toral subalgebra of* grCDer$_k(A)$*, and if* A *is perfect and has a* Λ*-graded invariant nondegenerate symmetric bilinear form, then* T *is also a toral subalgebra of* grSCDer$_k(A)$*.*

We mention that (1) always holds for $T = \mathcal{D}$, Λ torsion-free and k a field of characteristic 0.

4.12 Examples. (a) Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra over a field $\mathbb{K}$ of characteristic 0, e.g. a finite-dimensional simple Lie algebra over an algebraically closed $\mathbb{K}$. Hence $\mathfrak{g}$ has a root space decomposition $\mathfrak{g} = \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ with respect to some splitting Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$ of $\mathfrak{g}$, where $R \subset \mathfrak{h}^*$ is the root system of $(\mathfrak{g}, \mathfrak{h})$. We consider this decomposition as a $\mathcal{Q}(R)$-grading. It is well-known that $\mathfrak{g}$ is central (hence central-simple): Any centroidal transformation χ leaves the root space decomposition of $\mathfrak{g}$ invariant and, by simplicity, is uniquely determined by $\chi|\mathfrak{g}_\alpha$ for some $0 \neq \alpha \in R$.

For any $\theta \in \mathrm{Hom}(\mathcal{Q}(R), \mathbb{K}) \cong \mathrm{Hom}_{\mathbb{K}}(\mathfrak{h}^*, \mathbb{K})$ there exists a unique $h_\theta \in \mathfrak{h}$ such that $\alpha(h_\theta) = \theta(\alpha)$. Therefore $\partial_\theta = \mathrm{ad}\, h_\theta$ and

$$\mathrm{CDer}(\mathfrak{g}, \mathcal{Q}(R)) = \mathrm{grCDer}(\mathfrak{g}, \mathcal{Q}(R)) = \mathrm{grSCDer}(\mathfrak{g}, \mathcal{Q}(R)) = \mathrm{ad}\, \mathfrak{h}.$$

(b) Let $\mathfrak{g}$ be as in (a) and let C be a unital commutative associative Λ-graded $\mathbb{K}$-algebra. We view $L = \mathfrak{g} \otimes C$ as a Λ-graded Lie algebra over $\mathbb{K}$ with product $[x \otimes c, x' \otimes c'] = [x, x'] \otimes cc'$ for $x, x' \in \mathfrak{g}$ and $c, c' \in C$ and with homogeneous spaces $L^\lambda = \mathfrak{g} \otimes C^\lambda$ (here and in the following all tensor products are over $\mathbb{K}$). The centroid of L is determined in [ABP3, Lemma 2.3] or [BN, Cor. 2.23]:

$$\mathrm{Cent}_{\mathbb{K}}(L) = \mathbb{K}\, \mathrm{Id}_{\mathfrak{g}} \otimes C. \tag{1}$$

(We note in passing that (2) also holds for the infinite-dimensional versions of $\mathfrak{g}$ defined in 2.13, as well as for the version of $\mathfrak{g}$ over rings which we will introduce in 5.5.)

By [Bl2, Th. 7.1] or [BM, Th. 1.1], the derivation algebra of L is

$$\mathrm{Der}_{\mathbb{K}}(L) = \big(\mathrm{Der}(\mathfrak{g}) \otimes C\big) \oplus \big(\mathbb{K}\, \mathrm{Id}_{\mathfrak{g}} \otimes \mathrm{Der}_{\mathbb{K}}(C)\big) = \mathrm{IDer}(L) \oplus \big(\mathbb{K}\, \mathrm{Id}_{\mathfrak{g}} \otimes \mathrm{Der}_{\mathbb{K}}(C)\big) \tag{2}$$

where $\mathrm{IDer}\, L$ is the ideal of inner derivations of L. It is then immediate that $\mathrm{CDer}(L)$ and $\mathrm{grCDer}(L)$ are given by the formulas

$$\mathrm{(gr)CDer}(L, \Lambda) = \mathbb{K}\, \mathrm{Id}_{\mathfrak{g}} \otimes \mathrm{(gr)CDer}(C, \Lambda).$$

where here and in the following (gr) indicates that the formula is true for the graded as well as the ungraded case.

Let κ be the Killing form of $\mathfrak{g}$ and suppose that ϵ is a Λ-graded invariant nondegenerate symmetric bilinear form on C. The bilinear form $\kappa \otimes \epsilon$ of L, defined by $(\kappa \otimes \epsilon)(x \otimes c, x' \otimes c') = \kappa(x, x')\, \epsilon(c, c')$, is Λ-graded, invariant, nondegenerate and symmetric (if $\mathfrak{g}$ is simple, any such form arises in this way by [Be, Th. 4.2]). With respect to $\kappa \otimes \epsilon$, the skew-symmetric derivations and skew-symmetric centroidal derivations are

$$\begin{aligned}(\mathrm{gr})\mathrm{SDer}_{\mathbb{K}}(L) &= \mathrm{IDer}\, L \oplus \big(\mathbb{K}\ \mathrm{Id}_{\mathfrak{g}} \otimes (\mathrm{gr})\mathrm{SDer}_{\mathbb{K}}(C)\big),\\ (\mathrm{gr})\mathrm{SCDer}_{\mathbb{K}}(L) &= \qquad\qquad \mathbb{K}\ \mathrm{Id}_{\mathfrak{g}} \otimes (\mathrm{gr})\mathrm{SCDer}_{\mathbb{K}}(C).\end{aligned}$$

In particular, let $C = \mathbb{K}[t^{\pm 1}]$ be the $\mathbb{K}$-algebra of Laurent polynomials over $\mathbb{K}$, which we view as $\mathbb{Z}$-graded algebra with $C^i = \mathbb{K}t^i$. Its derivation algebra is (isomorphic to) the Witt algebra,

$$\mathrm{Der}_{\mathbb{K}}\big(\mathbb{K}[t^{\pm 1}]\big) = \textstyle\bigoplus_{i\in\mathbb{Z}} \mathbb{K}d^{(i)},$$

where $d^{(i)}$ acts on $\mathbb{K}[t^{\pm 1}]$ by $d^{(i)}(t^j) = jt^{i+j}$. Hence $d^{(i)}$ is the centroidal derivation given by the homomorphism $\theta^{(i)}\colon \mathbb{Z} \to C^i$, $\theta^{(i)}(j) = jt^i$. It follows that

$$\mathrm{Der}\big(\mathbb{K}[t^{\pm 1}]\big) = \mathrm{grDer}\big(\mathbb{K}[t^{\pm 1}]\big) = \mathrm{CDer}\big(\mathbb{K}[t^{\pm 1}]\big) = \mathrm{grCDer}\big(\mathbb{K}[t^{\pm 1}]\big).$$

Up to scalars, the algebra $\mathbb{K}[t^{\pm 1}]$ has a unique $\mathbb{Z}$-graded invariant nondegenerate symmetric bilinear form ϵ given by $\epsilon(t^i, t^j) = \delta_{i+j,0}$. With respect to this form $\mathrm{SCDer}\big(\mathbb{K}[t^{\pm 1}]\big) = \mathbb{K}d^{(0)}$ and hence the skew derivations of the untwisted loop algebra $L = \mathfrak{g} \otimes \mathbb{K}[t^{\pm 1}]$ are

$$\mathrm{SCDer}(\mathfrak{g} \otimes \mathbb{K}[t^{\pm 1}]) = \mathbb{K}d^{(0)}$$

It is important for later that $\mathrm{SCDer}(L)$ are precisely the derivations needed to construct the untwisted affine Lie algebra in terms of L.

(c) Let E be an affine Lie algebra (2.13). Then (b) applies to

$$L = [E, E] \big/ \mathrm{Z}([E, E]) \cong \mathfrak{g} \otimes \mathbb{K}[t^{\pm 1}]$$

(the centreless core when E is considered as an extended affine Lie algebra, see 6.11). In particular, L has an infinite-dimensional centroid. But the centroid of the core $K := [E, E]$ is 1-dimensional: $\mathrm{Cent}(K) = \mathbb{K}\ \mathrm{Id}_K$, while the centroid of E is 2-dimensional, namely isomorphic to $\mathbb{K}\ \mathrm{Id}_E \oplus \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}c, \mathbb{K}d)$ where $\mathbb{K}c$ is the centre of the derived algebra K and $E = K \rtimes \mathbb{K}d$ ([BN, Cor. 3.5]).

4.13 Quantum tori. In short, a *quantum torus* is an associative $\mathbb{Z}^n$-torus. As in 4.5 we may assume that a quantum torus has full support. In this case it can be described as follows.

Let $k = F$ be a field and let $q = (q_{ij}) \in \mathrm{Mat}_n(F)$ be a *quantum matrix*, i.e., $q_{ij}\, q_{ji} = 1 = q_{ii}$ for all $1 \le i, j \le n$. The *quantum torus associated to* q is the associative F-algebra F_q presented by generators $t_i, {t_i}^{-1}$ and relations

$$t_i{t_i}^{-1} = 1 = {t_i}^{-1}t_i \quad \text{and} \quad t_it_j = q_{ij}t_jt_i \quad \text{for } 1 \le i, j \le n.$$

Observe that if all $q_{ij} = 1$ then $F_q = F[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$ is the Laurent polynomial ring in n variables. Hence, for a general q, a quantum torus is a non-commutative version (a quantization) of the Laurent polynomial algebra $F[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$, which is the coordinate ring of an n-dimensional algebraic torus. This explains the name "quantum torus" for F_q.

Let $A = F_q$ be a quantum torus. It is immediate that

$$F_q = \bigoplus_{\lambda \in \mathbb{Z}^n} F t^\lambda, \qquad \text{for} \quad t^\lambda = t_1^{\lambda_1} \cdots t_n^{\lambda_n}.$$

The multiplication of F_q is determined by the 2-cocycle $\tau \colon \mathbb{Z}^n \times \mathbb{Z}^n \to F^\times$ where

$$t^\lambda t^\mu = \tau(\lambda, \mu)\, t^{\lambda+\mu}, \qquad \tau(\lambda, \mu) = \prod_{1 \le j < i \le n} q_{ij}^{\lambda_i \mu_j}. \tag{1}$$

It then clear that F_q is an associative $\mathbb{Z}^n$-torus with $(F_q)^\lambda = F t^\lambda$. Conversely, every associative $\mathbb{Z}^n$-torus is graded-isomorphic to some quantum torus.

The structure of quantum tori is well understood. We mention some facts which are relevant for this paper. Let $A = F_q$, $q \in \mathrm{Mat}_n(F)$, be a quantum torus.

(a) ([OP, Lemma 3.1], [Nee3, Th. 2.4]) Let $\tilde q \in \mathrm{Mat}_n(F)$ be another quantum matrix and denote by $\tilde\tau$ the 2-cocycle associated in (1) to the quantum torus $F_{\tilde q} = \bigoplus_{\lambda \in \mathbb{Z}^n} F \tilde t^\lambda$. Then the following are equivalent:

(i) F_q and $F_{\tilde q}$ are isomorphic as algebras;
(ii) F_q and $F_{\tilde q}$ are isograded-isomorphic as $\mathbb{Z}^n$-graded algebras;
(iii) there exists $M \in \mathrm{GL}_n(\mathbb{Z})$ such that the cohomology classes of the 2-cocycles τ and $\tilde\tau \circ (M \times M)$ agree, i.e., $\tau(\lambda, \mu)/\tilde\tau(M\lambda, M\mu) = v_\lambda\, v_\mu\, v_{\lambda+\mu}^{-1}$ for some function $v \colon \mathbb{Z}^n \to k^\times$.

In this case, the map $f \colon F_q \to F_{\tilde q}$, given by $f(t^\lambda) \mapsto v_\lambda\, \tilde t^{\,M\lambda}$, is an isograded-isomorphism, and every isomorphism arises in this way for suitable (v_λ) and M.

(b) ([OP, Lemma 1.1], [BGK, Prop. 2.44]) The centre of F_q has the description $\mathrm{Z}(A) = \bigoplus_{\gamma \in \Gamma} F t^\gamma$ where

$$\begin{aligned} \Gamma &= \{\gamma \in \Lambda : \tau(\gamma, \mu) = \tau(\mu, \gamma) \text{ for all } \mu \in \mathbb{Z}^n\} \\ &= \{\gamma \in \Lambda : \textstyle\prod_{1 \le j \le n} q_{ij}^{\gamma_j} = 1 \text{ for } 1 \le i \le n\} \end{aligned}$$

is a subgroup of $\mathbb{Z}^n$. The centre is isomorphic to $F[\Gamma]$, hence to a Laurent polynomial ring in m variables, $0 \le m \le n$. Recall from (4.2.1) that $\mathrm{Z}(A) \cong \mathrm{Cent}_F(A)$, hence Γ is the centroidal grading group in the sense of 4.6. It is also useful to know that $F_q = \mathrm{Z}(A) \oplus [F_q, F_q]$ where

$$[F_q, F_q] = \mathrm{span}_F\{ab - ba : a, b \in F_q\}.$$

(c) The following are equivalent, cf. 4.6(c):
(i) all q_{ij} are roots of unity;
(ii) $[\mathbb{Z}^n : \Gamma] < \infty$;
(iii) F_q is fgc (4.2).

(d) ([BGK, Remark 2.45]) Up to scalars, F_q has a unique $\mathbb{Z}^n$-graded invariant symmetric bilinear form $(\cdot|\cdot)$, given by $(t^\lambda|t^\mu) = \pi(t^\lambda t^\mu)$ where $\pi\colon F_q \to Ft^0$ is the canonical projection. The form $(\cdot|\cdot)$ is nondegenerate.

(e) ([OP, Cor. 2.3], [BGK, 2.48–2.56]) The derivation algebra of F_q is the semidirect product of the ideal $\mathrm{IDer}(F_q) = \{\mathrm{ad}\, a : a \in F_q\}$ of inner derivations, where $(\mathrm{ad}\, a)(b) = ab - ba$, and the subalgebra $\mathrm{CDer}(F_q)$ of centroidal derivations:

$$\mathrm{Der}_F(F_q) = \mathrm{IDer}(F_q) \rtimes \mathrm{CDer}(F_q).$$

Both $\mathrm{IDer}(F_q)$ and $\mathrm{CDer}(F_q)$ are graded subalgebras of $\mathrm{Der}_F(F_q)$:

$$\begin{aligned} \mathrm{IDer}(F_q) &= \textstyle\bigoplus_{\lambda\notin\Gamma} (\mathrm{Der}_F(F_q))^\lambda \cong [F_q, F_q] = \bigoplus_{\lambda\notin\Gamma}(Ft^\lambda), \\ \mathrm{CDer}(F_q) &= \textstyle\bigoplus_{\lambda\in\Gamma} (\mathrm{Der}(F_q))^\lambda = \mathrm{Z}(F_q)\cdot \mathcal{D}, \quad \text{where} \\ \mathcal{D} &= \mathrm{span}_F\{\partial_1, \dots, \partial_n\} \end{aligned}$$

are the degree derivations of the $\mathbb{Z}^n$-graded algebra F_q with ∂_i operating as $\partial_i(t^\lambda) = \lambda_i t^\lambda$.

4.14 Notes. The notion of the centroid of an algebra arose in the theory of forms of algebras, see e.g. [Ja, Ch.X, §1] or [Mc, Ch.II, §1.6]. The centroids of not necessarily split finite-dimensional simple Lie algebras over a field of characteristic 0 were determined in [Se, V.1]. Some recent references regarding the centroids of Lie algebras are the papers [ABP3], [ABFP1] and [BN].

Th. 4.4 is a standard result in the ungraded case, see for example [Ja, Ch.X, §1] or [Mc, Ch.II, §1.6, Th. 1.7.1]. The graded case can be proven by a graded version of the Density Theorem, [Neh8]. The implication (i) $\Rightarrow$ (ii) is mentioned in [ABFP1, Remark 4.3.2(ii)].

For more information about crossed product algebras see for example the book [Pa]. Prop. 4.6(a) and (b) are proven in [BN, Prop. 2.16]. Part (c) is shown in [ABFP1, Prop. 4.4.5]; it was independently discovered by the author and used in the proof of [Neh6, Th. 7(b)].

Theorem 4.7 is only one of several realization theorems proven in [ABFP1]. The special case of Lie-$\mathbb{Z}^n$-tori is developed in detail in [ABFP2]. Multiloop algebras are studied in detail in [ABP1, ABP2, ABP3]. Their derivations are determined in [Az5]. A cohomological approach to the classification of multiloop algebras is developed in [GiP1] and [GiP2].

Quantum tori are the algebraic counterparts of the so-called rotation algebras, a special class of C^*-algebras which has a very well-developed theory. The terminology "quantum torus" seems to go back to Manin [Ma, Ch. 4, §6]. Much more is known about quantum tori than what is mentioned in 4.13. For example, the structure of quantum tori F_q for which all q_{ij} roots of unity is elucidated in [ABFP1, Prop. 9.3.1], [Ha, Th. 4.8] and [Nee3, Th. III.4]. The involutions of quantum tori are determined in [Yo3].

5 Lie algebras graded by root systems

While the previous section §4 provided some background material for arbitrary graded algebras, in this section we consider Lie algebras with a special type of grading by the root lattice of a root system, so-called root-graded Lie algebras. We will only describe that part of the theory which is of importance for the following sections.

Throughout, and unless specified otherwise, all algebras are defined over a unital commutative associative base ring k in which 2 and 3 are invertible. As in §4, Λ denotes an abelian group, written additively.

5.1 Root-graded Lie algebras. Let R be a locally finite root system. It is convenient to put

$$R^\times = R \setminus \{0\} \quad \text{and} \quad R^\times_{\text{ind}} = R^\times \cap R_{\text{ind}},$$

see 2.7. We denote the $\mathbb{Z}$-span of R by $\mathcal{Q}(R)$.

Let L be a $(\mathcal{Q}(R), \Lambda)$-graded Lie algebra. We will always use subscripts to indicate the $\mathcal{Q}(R)$-grading and superscripts for the Λ-grading: $L = \bigoplus_{q \in \mathcal{Q}(R)} L_q$ and $L = \bigoplus_{\lambda \in \Lambda} L^\lambda$. By compatibility we therefore have

$$L = \textstyle\bigoplus_{q \in \mathcal{Q}(R),\, \lambda \in \Lambda} L_q^\lambda \qquad \text{for } L_q^\lambda = L_q \cap L^\lambda.$$

To stress the analogy between the definitions in 4.5 and the ones below, we will first define invertible elements of L. A non-zero element $e \in L_\alpha^\lambda$ for $\alpha \in R^\times$ is called *invertible*, more precisely, an *invertible element of the* $(\mathcal{Q}(R), \Lambda)$-*graded Lie algebra* L, if there exists $f \in L_{-\alpha}^{-\lambda}$ such that $h = [f,\, e]$ acts on L_q, $q \in \mathcal{Q}(R)$, by

$$[h,\, x_q] = \langle q, \alpha^\vee \rangle\, x_q \qquad (x_q \in L_q). \tag{1}$$

In this case, (e, h, f) is an $\mathfrak{sl}_2$-triple. If $L_{3\alpha} = \{0\}$ (a condition which will always be fulfilled later), then $f \in L_{-\alpha}^{-\lambda}$ is uniquely determined by (1). We will therefore refer to f as the *inverse* of e.

A $(\mathcal{Q}(R), \Lambda)$-graded Lie algebra $L = \bigoplus_{q \in \mathcal{Q}(R),\, \lambda \in \Lambda} L_q^\lambda$ is called (R, Λ)-*graded* if it satisfies the axioms (RG1)–(RG3) below:

(RG1) $\operatorname{supp}_{\mathcal{Q}(R)} L \subset R$, whence $L = \bigoplus_{\alpha \in R} L_\alpha^\lambda$,

(RG2) every L_α^0, $\alpha \in R^\times_{\text{ind}}$, contains an invertible element of L, and

(RG3) $L_0 = \sum_{0 \neq \alpha \in R} [L_\alpha,\, L_{-\alpha}]$.

An (R, Λ)-graded Lie algebra L is called

- *predivision*-(R, Λ)-*graded* if every non-zero L_α^λ, $\alpha \in R^\times$, contains an invertible element;
- *division*-(R, Λ)-*graded* if every non-zero element of L_α^λ, $\alpha \in R^\times$, is invertible;

- *a Lie-(R, Λ)-torus* if L is predivision-graded, defined over a field k and $\dim_k L_\alpha^\lambda \le 1$ for all $\alpha \in R^\times$;
- *invariant* if L has a Λ-graded invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$ such that for some choice of invertible elements $e_\alpha \in L_\alpha^0$, $\alpha \in R_{\mathrm{ind}}^\times$, with inverses f_α the restriction of $(\cdot|\cdot)$ to $\mathrm{span}_k\{[e_\alpha, f_\alpha] : 0 \ne \alpha \in R_{\mathrm{ind}}\}$ is nondegenerate too.

Let L be (R, Λ)-graded. By (RG1) and (RG2) we have

$$R_{\mathrm{ind}} \subset \mathrm{supp}_{\mathcal{Q}(R)} L = \{\alpha \in R : L_\alpha \ne 0\} \subset R,$$

whence L has full $\mathcal{Q}(R)$-support: $\langle \mathrm{supp}_{\mathcal{Q}(R)} L \rangle = \mathcal{Q}(R)$. One can show that $\mathrm{supp}_{\mathcal{Q}(R)} L$ is a subsystem of R. We can therefore always assume that $\mathrm{supp}_{\mathcal{Q}(R)} L = R$, if we so wish. Analogously, we can always assume that L has full Λ-support in the sense that $\Lambda = \langle \mathrm{supp}_\Lambda L \rangle$.

If in the definitions above $\Lambda = \{0\}$, we will speak of an *R-graded* or a *(pre)division-R-graded* Lie algebra. Lie algebras that are R-graded for some locally finite root system R are simply called *root-graded*. Similarly, a *predivision-root-graded Lie algebra* is a Lie algebra which is predivision-(S, Λ)-graded for some (S, Λ), and a *Lie torus* is a Lie-(R, Λ)-torus for a suitable pair (R, Λ).

We will present examples of root-graded Lie algebras in 5.5 and in §7. But we mention here right away the following examples: the derived algebra of an affine Kac-Moody algebra (see 2.13), toroidal Lie algebras, Slodowy's intersection matrix algebras and finite-dimensional simple Lie algebras over fields of characteristic 0 which contain an $\mathfrak{sl}_2$-subalgebra. Moreover, predivision-graded Lie algebras will turn out to be the basic building blocks of the generalizations of affine Lie algebras that we will be describing in section §6.

5.2 Isomorphisms and isotopy. An (R, Λ)-graded Lie algebra L is called *isograded-isomorphic* to an (R', Λ')-graded Lie algebra L' if there exists an isomorphism $\varphi_a\colon L \to L'$ of Lie k-algebras, an isomorphism $\varphi_r\colon R \to R'$ of locally finite root systems and an isomorphism $\varphi_e\colon \Lambda \to \Lambda'$ of abelian groups satisfying

$$\varphi_a(L_\alpha^\lambda) = L_{\varphi_r(\alpha)}^{\varphi_e(\lambda)}$$

for all $\alpha \in R$ and $\lambda \in \Lambda$. We say L and L' are *graded-isomorphic* if there exists an isograded isomorphism with $\varphi_r = \mathrm{Id}$ and $\varphi_e = \mathrm{Id}$.

Let $\iota\colon \mathcal{Q}(R) \to \Lambda$ be a group homomorphism. We can then define a new $(\mathcal{Q}(R), \Lambda)$-graded Lie algebra $L^{(\iota)}$ by

$$\left(L^{(\iota)}\right)_\alpha^\lambda = L_\alpha^{\lambda + \iota(\alpha)}.$$

The following are equivalent:

(i) $L^{(\iota)}$ is (R, Λ)-graded,

(ii) $L_\alpha^{\iota(\alpha)}$ contains an invertible element for all $\alpha \in R_{\rm ind}^\times$.

If R has a root basis and, in addition to $\frac12, \frac13$ also $\frac15 \in k$, then (i) and (ii) are equivalent to

(iii) $L_\alpha^{\iota(\alpha)}$ contains an invertible element for all $\alpha \in B$.

In this case, we say that $L^{(\iota)}$ is an *isotope* of L. An (R', Λ')-graded Lie algebra L' is called *isotopic* to L if some isotope of L is isograded-isomorphic to L'. Being isotopic is an equivalence relation on the class of all root-graded Lie algebras.

Isotopy preserves the classes of (pre)division-graded Lie algebras and Lie tori. For these cases, condition (ii) can be replaced by $L^{\iota(\alpha)} \neq 0$ for all $\alpha \in R_{\rm ind}^\times$, and similarly for (iii).

5.3 Coverings and central quotients. Let Ξ be an abelian group and let L be a Ξ-graded Lie algebra. By definition, a *Ξ-graded central extension* of L is a Ξ-graded Lie algebra K together with a graded epimorphism $f\colon K \to L$ with $\operatorname{Ker} f$ contained in the centre of K. A Ξ-graded central extension $f : K \to L$ is called a *Ξ-covering* if K is perfect. We will use the terms *central extension* and *covering* in case $\Xi = \{0\}$.

A central extension $\mathfrak{u}\colon \mathfrak{L} \to L$ is called a *universal central extension* if for any central extension $f\colon K \to L$ there exists a unique homomorphism $\mathfrak{f}\colon \mathfrak{L} \to \mathfrak{k}$ such that $\mathfrak{u} = f \circ \mathfrak{f}$:

$$\begin{array}{ccc} \mathfrak{L} & \overset{\mathfrak{f}!}{\dashrightarrow} & K \\ & {\scriptstyle\mathfrak{u}}\searrow \quad \swarrow{\scriptstyle f} & \\ & L & \end{array}$$

It is obvious from this universal property that *two universal central extensions of L are isomorphic* in the sense that one has a commutative triangle as above with the horizontal map being an isomorphism. A Lie algebra L has a universal central extension, say $\mathfrak{u}\colon \mathfrak{uce}(L) \to L$, if and only if L is perfect. In this case, $\mathfrak{uce}(L)$ is perfect and is a universal central extension of all covering algebras of L as well as all central quotients. In particular, it is a universal central extension of the centreless algebra $L/\operatorname{Z}(L)$. If L is Ξ-graded, then $\mathfrak{u}\colon \mathfrak{uce}(L) \to L$ is a Ξ-covering. One calls L *simply connected* if L is a universal central extension of itself. For example, a universal central extension is always simply connected.

As an example we mention that a universal central extension of the untwisted and twisted loop algebra L of 2.13 is the Lie algebra K defined there, [Ga, Wi].

Let now L be an (R, Λ)-graded Lie algebra. We will apply the concepts above to the Ξ-grading of L where $\Xi = \mathcal{Q}(R) \oplus \Lambda$. We observe first that the centre of L is Λ-graded and contained in L_0,

$$\mathrm{Z}(L) = \bigoplus_{\lambda\in\Lambda} (\mathrm{Z}(L) \cap L_0^\lambda). \tag{1}$$

whence $\mathrm{Z}(L)$ is also Ξ-graded.

5.4 Proposition. *Let $f\colon K \to L$ be a Λ-covering of the Λ-graded Lie algebra L. Then K is (R,Λ)-graded if and only if L is (R,Λ)-graded. In this case $\operatorname{supp}_{\mathcal{Q}(R)} K = \operatorname{supp}_{\mathcal{Q}(R)} L$ and $f|K_\alpha$ is bijective for all $\alpha \in R^\times$. Moreover, K is predivision- or division-graded or a Lie torus if and only if L has the corresponding property.*

As a consequence of this proposition, the classification of root-graded Lie algebras and their subclasses introduced above can be done in three steps:

1. Describe all centreless root-graded Lie algebras,
2. determine their universal central extensions and
3. characterize those that are (pre)division-graded or Lie tori.

Note that any root-graded Lie algebra is perfect and therefore has a universal central extension. Examples of this approach are given in 5.5-5.7 and §7, see also 5.10 for a short survey.

5.5 Example. Let $\mathfrak{g}_\mathbb{Z}$ be a Chevalley order of a finite-dimensional semisimple complex Lie algebra $\mathfrak{g}_\mathbb{C}$ of type R, see [Bo1, Ch. VIII, §12]. Thus, we have a Chevalley system $(e_\alpha : 0 \neq \alpha \in R)$ such that, putting $h_\alpha = [e_{-\alpha}, e_\alpha]$,

$$\mathfrak{g}_\mathbb{Z} = \big(\bigoplus_{0\neq\alpha\in R} \mathbb{Z}e_\alpha\big) \oplus \big(\sum_{0\neq\alpha\in R} \mathbb{Z}h_\alpha\big)$$

is a Lie algebra over $\mathbb{Z}$ which is a $\mathbb{Z}$-form of $\mathfrak{g}_\mathbb{C}$, i.e. $\mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C} \cong \mathfrak{g}_\mathbb{C}$. Then $\mathfrak{g}_k = \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} k$ is a $\mathcal{Q}(R)$-graded Lie k-algebra with homogeneous spaces $\mathfrak{g}_{k,\,\alpha} = \mathfrak{g}_{\mathbb{Z},\alpha} \otimes_\mathbb{Z} k$. For every unital commutative associative Λ-graded k-algebra $C = \bigoplus_{\lambda\in\Lambda} C^\lambda$ the Lie k-algebra

$$L = \mathfrak{g}_k \otimes_k C \cong \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} C \tag{1}$$

is $(\mathcal{Q}(R),\Lambda)$-graded with homogeneous spaces $L_\alpha^\lambda = \mathfrak{g}_{k,\alpha} \otimes_k C^\lambda$. An element $e_\alpha \otimes c$ is invertible in L if and only if $c \in C^\times$. Hence, L is (R,Λ)-graded. It is (pre)division-(R,Λ)-graded if and only if C is (pre)division-graded. If k is a field (recall of characteristic $\neq 2,3$), then L is a Lie torus if and only if C is an associative torus.

Let κ be the Killing form of $\mathfrak{g}_\mathbb{C}$ and let $h^\vee$ be the dual Coxeter number. Then $\frac{1}{2h^\vee}\kappa$ is an invariant nondegenerate symmetric bilinear form on $\mathfrak{g}_\mathbb{Z}$ ([SS, I.4.8], [GrN, §5]). The canonical extension $(\cdot|\cdot)$ of this form to $\mathfrak{g}_k$ is then an invariant symmetric bilinear form on $\mathfrak{g}_k$. It is nondegenerate if R is irreducible and not of type A. In this case L is invariant with respect to $(\cdot|\cdot) \otimes \epsilon$, where ϵ is any Λ-graded invariant nondegenerate symmetric bilinear form on C, see (4.3.2).

If R is irreducible, the Lie algebra $\mathfrak{g}_k$ is simply connected ([Ste, 6.1], [vdK, Cor. 3.14], [Kas]). The universal extension $\mathfrak{u}\colon \mathfrak{uce}(L) \to L$ of an arbitrary L as in (1) is determined in [Kas, 3.8]. It turns out that the kernel of $\mathfrak{u}$ is $\Omega_{C/k}\big/dC$ where $\Omega_{C/k}$ is the module of Kähler differentials of C over k and $d\colon\ C \to \Omega_{C/k}$ the universal k-linear derivation. In view of (7.8.1) it is appropriate to point out that $\Omega_{C/k}\big/dC = \mathrm{HC}_1(C)$, the first cyclic homology group of C.

For root systems of type D or E the algebras of this subsections are in fact all examples of R-graded Lie algebras.

5.6 Theorem. ([BeMo], [Neh3]) *Suppose R is a finite root system of type D_l, $l \ge 4$ or E_l, $l = 6, 7$ or 8, and that k is a field of characteristic 0 in case of E_8. Then any (R, Λ)-graded Lie algebra is graded-isomorphic to a Λ-covering of a Lie algebra L as in* (5.5.1) *for a suitable C.*

If in case D one replaces $\mathfrak{g}$ by the corresponding infinite rank Lie algebra (2.13), the result remains true for (D_I, Λ)-graded Lie algebras, $|I| = \infty$. But it is no longer true for the other types of root systems, see §7, in particular 7.6 and 7.7, for R of type $\dot{\mathrm{A}}_I$ and 5.10 for a short summary of the classification results. The models used to describe R-graded Lie algebras are different for every type of R and involve all important classes of nonassociative algebras (Jordan algebras, alternative algebras, structurable algebras). On the other side, one has the following type-free approach for most root systems, not only finite ones.

5.7 Theorem. ([Neh3]) *Let R be a locally finite root system without an irreducible component of type E_8, F_4 or G_2. Then a Lie algebra L is R-graded if and only if L is a covering of the Tits-Kantor-Koecher algebra of a Jordan pair covered by a grid whose associated root system is R.*

5.8 Centroids of root-graded Lie algebras. Let $L = \bigoplus_{\alpha\in R,\, \lambda\in\Lambda} L_\alpha^\lambda$ be an (R, Λ)-root-graded Lie algebra over k.

(a) Since L is perfect, the centroid of L is a commutative (and of course also unital and associative) subalgebra of $\mathrm{End}_k(L)$. One can show that a centroidal transformation of L leaves every L_α, $\alpha \in R$, invariant. If R has only finitely many irreducible components, $\mathrm{Cent}_k(L)$ is Λ-graded. As an example, we mention that the centroid of the Lie algebra L of (5.5.1) is isomorphic to C.

(b) Suppose now that the Λ-graded algebra L is graded-simple. Then R is irreducible and L is also graded-simple with respect to the $\mathcal{Q}(R)$-grading of L. Hence, the centroid of L is a twisted group algebra $K^t[\Gamma]$ over the field $K = (\mathrm{Cent}\, L)^0$, as described in 4.6. The condition (4.6.2) characterizing fgc algebras now simplifies to the following, where $\Lambda_\alpha = \{\lambda \in \Lambda : L_\alpha^\lambda \ne 0\}$: L is fgc if and only if

(i) R is finite,
(ii) $[\Lambda_\alpha : \Gamma] < \infty$ for all short roots $\alpha \in R$, and
(iii) $\dim_K L_\alpha^\lambda < \infty$ for all short roots α and all $\lambda \in \Lambda_\alpha$.

If also $\frac{1}{5} \in k$, then for every $w \in W(R)$ there exists a Cent(L)-linear automorphism of L which for all $\alpha \in R$ and $\lambda \in \Lambda$ maps L_α^λ to $L_{w(\alpha)}^\lambda$. Therefore, in this case it is sufficient to require (ii) and (iii) for one short root α.

If L is in addition predivision-graded and $\Lambda = \langle \operatorname{supp}_\Lambda L \rangle$ is finitely generated, then the condition (ii) can be replaced by

(ii′) $[\Lambda : \Gamma] < \infty$.

(c) Let L be division-graded with $\mathrm{Z}(L) = \{0\}$ and suppose that R is irreducible. Then L is Λ-graded-simple. So its centroid is described in (b). Moreover, if Λ is torsion-free, L is a prime algebra, i.e., $[I, J] = 0$ for ideals I, J of L implies $I = 0$ or $J = 0$. In this case, $\mathrm{Cent}_k(L)$ is an integral domain, and L embeds into the $\tilde{C}$-algebra $\tilde{L} = L \otimes \tilde{C}$, where $\tilde{C}$ is the quotient field of $\mathrm{Cent}_k(L)$. The algebra $\tilde{L}$ is central and prime ([EMO]). If the Lie algebra L is fgc, then $\tilde{L}$ is finite-dimensional. Hence, if C has characteristic 0, then $\tilde{L}$ is a finite-dimensional central-simple $\tilde{C}$-algebra.

(d) Let L be a centreless Lie-(R, Λ)-torus and suppose that R is irreducible. Then L is graded-central-simple by (c) and (b). In particular, if $\Lambda = \langle \operatorname{supp}_\Lambda L \rangle$ is finitely generated, then L is fgc if and only if R is finite and $[\Lambda : \Gamma] < \infty$. See also 6.12.

We will now consider root-graded Lie algebras in characteristic 0. It will turn out that the $\mathcal{Q}(R)$-grading of a root-graded Lie algebra is in fact the root space decomposition with respect to a toral subalgebra, as we will explain now.

5.9 Proposition. *Let L be an (R, Λ)-graded Lie algebra, defined over a field $\mathbb{K}$ of characteristic 0. We suppose $\operatorname{supp}_{\mathcal{Q}(R)} L = R$. For $\alpha \in R_{\mathrm{ind}}^\times$ let $e_\alpha \in L_\alpha^0$ be an invertible element with inverse f_α. We put $h_\alpha = [f_\alpha, e_\alpha]$, $h_0 = 0$, $h_{2\alpha} = \frac{1}{2} h_\alpha$ in case $2\alpha \in R$, and*

$$\mathfrak{h} = \operatorname{span}_{\mathbb{K}}\{h_\alpha : \alpha \in R\}.$$

(a) *For every $\alpha \in R$ there exists a unique $\tilde{\alpha} \in \mathfrak{h}^*$ defined by $\tilde{\alpha}(h_\beta) = \langle \alpha, \beta^\vee \rangle$ for $\beta \in R$. The set $\tilde{R} = \{\tilde{\alpha} \in \mathfrak{h}^* : \alpha \in R\}$ is a locally finite root system, isomorphic to R via the map $\alpha \mapsto \tilde{\alpha}$.*

(b) *The set $\{h_\alpha : \alpha \in R\}$ is a locally finite root system in $\mathfrak{h}$, canonically isomorphic to the coroot system $R^\vee$ via the map $\alpha^\vee \mapsto h_\alpha$.*

(c) *The subspace $\mathfrak{h}$ is a toral subalgebra of L whose root spaces are the homogeneous subspaces L_α with corresponding linear form $\tilde{\alpha}$:*

$$L_\alpha = \{l \in L : [h, l] = \tilde{\alpha}(h) l \text{ for all } h \in \mathfrak{h}\}.$$

(d) *Let R be finite and let B be a root basis of R. Then $\{e_\beta, f_\beta : \beta \in B\}$ generates a finite-dimensional split semisimple subalgebra $\mathfrak{g}$ of L with splitting Cartan subalgebra $\mathfrak{h}$. The root system of $(\mathfrak{g}, \mathfrak{h})$ is $\tilde{R}_{\mathrm{ind}}$.*

It follows immediately from this result that for $\mathbb{K}$ a field of characteristic 0, an (R,Λ)-graded Lie $\mathbb{K}$-algebra for R an irreducible root system and Λ an abelian group can be equivalently defined as follows: $L = \bigoplus_{\lambda\in\Lambda} L^\lambda$ is a Λ-graded Lie algebra over $\mathbb{K}$ satisfying (i)–(iv) below:

(i) L^0 contains as a subalgebra a finite-dimensional split simple Lie algebra $\mathfrak{g}$, called the *grading subalgebra*, with splitting Cartan subalgebra $\mathfrak{h}$,

(ii) either R is reduced and equals the root system $R_\mathfrak{g} \subset \mathfrak{h}^*$ of $(\mathfrak{g},\mathfrak{h})$, or $R = \mathrm{BC}_I$ and $R_\mathfrak{g} = \mathrm{A}_1$ in case $|I| = 1$ or $R_\mathfrak{g} = \mathrm{B}_I$ for $|I| \geq 2$.

(iii) $\mathfrak{h}$ is a toral subalgebra of L, and the weights of L relative $\mathfrak{h}$ are in R, whence $L = \bigoplus_{\alpha\in R} L_\alpha$,

(iv) $L_0 = \bigoplus_{0\neq\alpha\in R}[L_\alpha,\, L_{-\alpha}]$;

This is the original definition of a root-graded Lie algebra, see 5.10.

5.10 Notes. Unless stated otherwise, the results mentioned in 5.1–5.9 are proven in [Neh8]. Some of them have also been proven by others, as indicated below.

The definition of a root-graded Lie algebra in 5.1 is in the spirit of the definition in [Neh3], which introduces root-graded Lie algebras over rings for reduced root systems. The definition given at the end of 5.9 is the original one of [BeMo], where R-graded Lie algebras were introduced R simply laced. It is mentioned in [Neh3, Remark 2 of 2.1] that in characteristic 0 the two definitions are equivalent. A_1-graded Lie algebras had been studied much earlier in [Ti] and then in [Kan1, Kan2, Kan3] and [Ko1, Ko2].

The paper [BeMo] contains a classification of centreless R-graded Lie algebras for R simply laced. The classification for reduced, non-simply laced R is given in [BZ]. The theory for a non-reduced R, i.e. $R = \mathrm{BC}_I$, is developed in [ABG2] for $R = \mathrm{BC}_l$, $2 \leq l < \infty$) and [BS] for $R = \mathrm{BC}_1$. In this case, one can be more general by allowing the grading subalgebra $\mathfrak{g}$ to have type C or D. This too can be done in the setting of 5.1, but we will not need this in the following and have therefore refrained from describing the more general setting. The papers [BeMo], [BZ], [ABG2] and [BS] work with finite root systems and Lie algebras over fields of characteristic 0. The extensions to locally finite root systems and Lie algebras over rings is given in [Neh3] for R reduced and 3-graded. As already mentioned in [GaN, 2.10], the classification results in [Neh3], viz. 5.7, can easily be extended to the (R,Λ)-graded and (R,Λ)-(pre)division-graded case. Finite-dimensional simple Lie algebras and affine Lie algebras graded by finite respectively affine root systems are described in [Ner1] and [Ner2].

At present, predivision-(R,Λ)-graded Lie algebras have been classified in the centreless case only for certain root systems and base rings k: R finite of type A_l, ($l \geq 3$), D or E, $\Lambda = \mathbb{Z}^n$, $k = \mathbb{K}$ a field of characteristic 0 in [Yo1]; $R = \mathrm{B}_r$, $3 \leq r < \infty$, L division-graded, $k = \mathbb{K}$ a field of characteristic

0 in [Yo4]; $R = \mathrm{C}_r$, $2 \le r < \infty$, L division-graded over $k = \mathbb{K}$ a field of characteristic 0 in [BY]; $R = \mathrm{B}_2 = \mathrm{C}_2$, Λ torsion-free, L graded-simple (e.g. division-graded), k arbitrary in [NT]. Regarding the classification of Lie tori, the situation the reader is referred to the survey in [AF, §12].

The concept of isotopy was introduced in [AF] for Lie tori over fields of characteristic 0 and related to isotopy of their coordinate algebras. In the context of Lie tori in characteristic 0, the equivalence of (i)–(iii) in 5.2 is proven in [ABFP2, Prop. 2.2.3].

Universal central extensions of Lie algebras over rings were described in [vdK, §1] (this reference also describes the universal central extensions of the Lie algebras $\mathfrak{g}_k$ of 5.5, where k not necessarily contains $\frac{1}{2}$ and $\frac{1}{3}$. Later references on universal central extensions are [Ga, §1], [MP, 1.9], [We, 7.9] or [Neh4,] (the list is incomplete). In characteristic 0, the universal central extensions of root-graded Lie algebras are determined in [ABG1], [ABG2] and [BS].

Prop. 5.4 is standard, proven in [ABG2, Th. 5.36], [BS, §5], [BZ, Prop. 1.5–1.8] and [BeMo, Prop. 1.29] for the root-graded Lie algebras studied in these papers, viz. Lie algebras over fields of characteristic 0, where one can use that the grading subalgebra $\mathfrak{g}$ is simply connected. The proof of 5.4 in [Neh8] is for arbitrary k and uses the model of the universal central extension from [vdK].

5.5: We give some more information on this example in case $k = F$ is a field (recall of characteristic $\neq 2, 3$). It follows as in 5.5 that $\mathfrak{g}_F$ is central. Moreover, the description of $\mathrm{Der}_F\, L$ given there remains true, except that $\mathrm{Der}_F(\mathfrak{g}_F)$ need not be equal to $\mathrm{IDer}\, \mathfrak{g}_F$ and hence $\mathrm{IDer}\, L \neq \mathrm{Der}_F(\mathfrak{g}_F) \otimes C$ in general. If $R \neq \dot{\mathrm{A}}$, L is centreless, see 7.4 for the case $R = \dot{\mathrm{A}}$.

Th. 5.6 is proven in [BeMo] for k a field of characteristic 0 and in [Neh3, 5.4, 7.2, 7.3] for arbitrary k (of course containing $\frac{1}{2}$ and $\frac{1}{3}$) and $R \neq \mathrm{E}_8$. The methods of [Neh3] do not allow to treat the case E_8, but the result is probably also true in this case for any k containing $\frac{1}{30}$. Similarly, Th. 5.7 can be generalized to cover all locally finite root systems by using Kantor pairs instead of Jordan pairs. With this method one can likely extend the classification results to cover all root-graded Lie algebras (R not 3-graded and Lie algebras over rings).

5.8: That a centreless division-(R, Λ)-graded Lie algebra with R irreducible is graded-simple is proven in [Yo4, Lemma 4.4] for finite R (the proof easily generalizes). The characterization of fgc Lie tori in 5.8(d) is proven in [ABFP2, Prop. 1.4.2]. For R a finite root system, $R \neq \mathrm{BC}_1$ or BC_2, the centroid of an R-graded Lie algebra over a field of characteristic 0 is described in [BN, §5] in terms of the centre of the coordinate system associated to an R-graded Lie algebra.

For 3-graded R, Prop. 5.9(a) was shown in [Neh2, 3.2]. It is also proven in [ABFP2, Prop. 1.2.2] for Lie tori.

Invariant forms and derivations of root-graded Lie algebras over fields of characteristic 0 are described in [Be], [ABG2] and [BS].

6 Extended affine Lie algebras and generalizations

In this section we will introduce Lie algebras whose set of roots are affine reflection systems (6.2). We will then discuss special types, most importantly invariant affine reflection algebras (6.7) and extended affine Lie algebras (6.11).

Unless specified otherwise, in this section all algebras are defined over a field $\mathbb{K}$ of characteristic 0. Throughout, (E,T) will be a *toral pair*, i.e., a nonzero Lie algebra E with a toral subalgebra T. We will denote the set of roots of (E,T) by R and put $X = \operatorname{span}_{\mathbb{K}}(R) \subset T^*$, cf. 2.11.

6.1 Core and tameness. Let (E,T) be a toral pair. Recall that $0 \in R$ if $T \neq 0$ and that E has a root space decomposition

$$E = \textstyle\bigoplus_{\alpha\in R} E_\alpha. \tag{1}$$

If (R,X) is a pre-reflection system with respect to some set of real roots $R^{\rm re} \subset R$, we define the *core of* (E,T) as the subalgebra E_c of E generated by $\{E_\alpha : \alpha \in R^{\rm re}\}$. It is a graded subalgebra with respect to the root space decomposition (1),

$$E_c = \big(\textstyle\bigoplus_{\alpha\in R^{\rm re}} E_\alpha\big) \oplus \big(\textstyle\bigoplus_{\beta\in R^{\rm im}} (E_c \cap E_\beta)\big), \tag{2}$$

and also with respect to any Λ-grading of E which is compatible with the root space decomposition (1). If E_c is perfect (which will be the case from 6.2 on), the quotient algebra

$$E_{cc} = E_c / \operatorname{Z}(E_c) \tag{3}$$

is centreless and is called the *centreless core*. We call (E,T) *tame* if $\operatorname{C}_E(E_c) \subset E_c$. Tameness of (E,T) is not the same as tameness of R as defined in 2.2, cf. 6.5, 6.8 and 6.10.

6.2 Affine reflection Lie algebras. Let α be an integrable root of (E,T). Hence, by definition, there exists an integrable $\mathfrak{sl}_2$-triple $(e_\alpha, h_\alpha, f_\alpha) \subset E_\alpha \times T \times E_{-\alpha}$ of (E,T). It gives rise to a reflection s_α of (R,X), defined by

$$s_\alpha(x) = x - x(h_\alpha)\alpha, \quad \text{whence } \langle x, \alpha^\vee\rangle = x(h_\alpha). \tag{1}$$

We will call $(E,T,R^{\rm re})$ or simply (E,T) an *affine reflection Lie algebra* (abbreviated ARLA) if (R,X) is an affine reflection system with $R^{\rm re} \subset R^{\rm int}$ and the reflections s_α, $\alpha \in R^{\rm re}$, defined by (1). Thus $R^{\rm re} \subset R^{\rm int}$, but not necessarily $R^{\rm re} = R^{\rm int}$.

Because of (3.6.1), other choices of integrable $\mathfrak{sl}_2$-triples for $\alpha \in R^{\rm re}$ lead to the same reflection s_α. We will therefore not consider the family $(e_\alpha : \alpha \in R^{\rm re})$ as part of the structure of an affine reflection Lie algebra. Following this point of view, two affine reflection Lie algebras $(E,T,R^{\rm re})$ and $(E',T',R'^{\rm re})$ are called *isomorphic* if there exists a Lie algebra isomorphism $\varphi\colon E \to E'$ with $\varphi(T) = T'$ and $\varphi_r(R^{\rm re}) = R'^{\rm re}$ where $\varphi_r(\alpha) = \alpha \circ (\varphi|T)^{-1}$.

Recall from 3.2 and 3.6 that any affine reflection system is integral, coherent and has finite root strings. The affine reflection systems occurring as roots in an affine reflection Lie algebra have the following additional properties:

- all root strings $\mathbb{S}(\beta,\alpha)$ for $\beta\in R$, $\alpha\in R^{\rm re}$, are unbroken.
- R is reduced in case T is a splitting Cartan subalgebra.

By definition, the *nullity of* $(E,T,R^{\rm re})$ is the nullity of the affine reflection system R, i.e., the rank of the subgroup Λ of $(T^*,+)$ generated by $R^{\rm im}$, cf. 3.6. We always have

$$\text{nullity} \quad \geq \quad \dim_{\mathbb{K}}\big(\operatorname{span}_{\mathbb{K}}(R^{\rm im})\big). \tag{2}$$

In general this is a strict inequality, see 6.17.

6.3 The core of an affine reflection Lie algebra. To study the core of an affine reflection Lie algebra $(E,T,R^{\rm re})$ we need some more notation:

- (S,Y) is the quotient root system of (R,X) (3.6);
- $f\colon (R,X)\to(S,Y)$ is the corresponding extension (3.2);
- $Z=\operatorname{Ker}(f)=\operatorname{span}_{\mathbb{K}}(R^{\rm im})$;
- $g\colon Y\to X$, $y\mapsto g(y)=:\dot y$, is a partial section over $S_{\rm ind}$ (3.1);
- $\mathfrak{L}=(\Lambda_\xi)_{\xi\in S}$ is the extension datum associated to f and g, see 3.3–3.7; thus $f^{-1}(\xi)\cap R=g(\xi)+\Lambda_\xi$ and the set R can be decomposed as

$$R=\textstyle\bigcup_{\xi\in S}\big(\dot\xi\oplus\Lambda_\xi\big)\ \subset\ X=g(Y)\oplus Z \quad\text{with}\quad R^{\rm im}=\dot 0\oplus\Lambda_0=\Lambda_0. \tag{1}$$

- Λ is the subgroup of $(Z,+)$ generated by $\bigcup_{\xi\in S}\Lambda_\xi$,
- $K=E_c$ is the core of $(E,T,R^{\rm re})$, and $K_\xi^\lambda\subset E_{g(\xi)\oplus\lambda}$, K_ξ and K^λ are defined for $\xi\in S$ and $\lambda\in\Lambda$ by

$$\begin{aligned}
K_\xi^\lambda &= E_{\dot\xi\oplus\lambda}\,, && (\xi\neq 0)\\
K_0^\lambda &= \textstyle\sum_{\tau\in S^\times,\,\mu\in\Lambda}\,[E_{\dot\tau\oplus\mu},E_{-\dot\tau\oplus(\lambda-\mu)}]\,, &&\\
K_\xi &= \textstyle\bigoplus_{\lambda\in\Lambda} K_\xi^\lambda, && (\xi\in S),\\
K^\lambda &= \textstyle\bigoplus_{\xi\in S} K_\xi^\lambda, && (\lambda\in\Lambda).
\end{aligned}$$

- To distinguish the Λ-support sets of K and the extension datum $(\Lambda_\xi)_{\xi\in S}$ we put

$$\Lambda_\xi^K=\{\lambda\in\Lambda : K_\xi^\lambda\neq 0\}\quad\text{and}\quad \Lambda^K=\operatorname{span}_{\mathbb{Z}}\{\Lambda_\xi^K:\xi\in S\}.$$

The data S, f, Z and the Lie algebra K are invariants of $(E,T,R^{\rm re})$. Moreover, since $K_\xi=\bigoplus_{f(\alpha)=\xi}E_\alpha$ for $\xi\in S^\times$, also the subspaces K_ξ, $\xi\in S$, are uniquely determined by $(E,T,R^{\rm re})$. But this is not so for the partial section g and hence for the family $(\Lambda_\xi)_{\xi\in S}$. Part (d) of the following Theorem 6.4 says that another choice of g leads to an isotope of K as defined in 5.2.

6.4 Proposition. *We use the notation of* 6.3. *The core K of an affine reflection Lie algebra (E, T, R^{re}) is a perfect ideal of E and a predivision-(S, Λ^K)-graded Lie algebra with respect to the decomposition*

$$K = \bigoplus_{\xi \in S,\, \lambda \in \Lambda} K_\xi^\lambda.$$

Moreover, K has the following properties.

(a) *The root system S embeds into $\mathfrak{h}_K^*$ where $\mathfrak{h}_K = \operatorname{span}_{\mathbb{K}}\{h_{g(\xi)} : \xi \in S_{\text{ind}}^\times\}$, such that $K = \bigoplus_{\xi \in S} K_\xi$ is the root space decomposition of K with respect to the toral subalgebra $\mathfrak{h}_K$ of K. The set of roots of $(K, \mathfrak{h}_K)$ is S, unless $K = \{0\}$ and thus $R = R^{\text{im}}$ and $S = \{0\}$.*

(b) *The family $\mathfrak{L}^K = (\Lambda_\xi^K : \xi \in S)$ of Λ-supports of K is an extension datum of type $(S, S_{\text{ind}}, \operatorname{span}_{\mathbb{K}}(\Lambda^K))$ with $\Lambda_\xi = \Lambda_\xi^K$ for $0 \neq \xi \in S$ and*

$$\Lambda_0 \supset \Lambda_0^K = \operatorname{supp}_\Lambda K = \bigcup_{\xi \in S^\times} (\Lambda_\xi + \Lambda_\xi),$$

hence $\langle \Lambda_0 \rangle = \Lambda \supset \Lambda^K = \langle \Lambda_0^K \rangle$.

(c) *The following are equivalent:*

(i) *K is a Lie torus,*

(ii) $\dim_{\mathbb{K}} E_\alpha \le 1$ *for all $\alpha \in R^{\text{re}}$,*

(iii) $[E_\alpha, E_{-\alpha}] \subset T$ *for all $\alpha \in R^{\text{re}}$.*

In particular, if T is a splitting Cartan subalgebra, then K and the centreless core E_{cc} are Lie tori.

(d) *Suppose (E, T) is isomorphic to the affine reflection Lie algebra (E', T'). Denote by R', S', Λ' and K' the data for (E', T') corresponding to the ones introduced in 6.3 for (E, T). Then R and R' are isomorphic as reflection systems, and the (S, Λ)-graded Lie algebra K is isotopic to the (S', Λ')-graded Lie algebra K'.*

We have excluded $K = \{0\}$ in (a) because, by definition in 2.11, the set of roots of $(K, \mathfrak{h}_K)$ consists of those $\alpha \in \mathfrak{h}^*$ for which the corresponding root space $K_\alpha(\mathfrak{h}) \neq 0$. Of course, $K = \{0\}$ can still be viewed as a root space decomposition.

6.5 A construction of an affine reflection Lie algebra. We have seen in 6.4 that the core of an affine reflection Lie algebra (E, T) is a predivision-(S, Λ^K)-graded Lie algebra, where S is the quotient root system of the affine reflection system of (E, T, R^{re}) and where Λ^K is a torsion-free abelian group. Changing notation (replacing Λ^K by Λ) we will now show that, conversely,

> *every predivision-(S, Λ)-graded Lie algebra, where S is any locally finite root system and where Λ is any torsion-free abelian group, is the core of an affine reflection Lie algebra.* (1)

Our construction uses K and D, where K is a predivision-(S, Λ)-graded Lie algebra over $\mathbb{K}$ with Λ a torsion-free abelian group and where D is described below. Since $\operatorname{supp}_{\mathcal{Q}(S)} K$ is a subsystem of S, we can assume $\operatorname{supp}_{\mathcal{Q}(S)} K = S$. Also, the family $\mathfrak{L} = (\Lambda_\xi : \xi \in S)$, $\Lambda_\xi = \{\lambda \in \Lambda : K_\xi^\lambda \neq 0\}$, is an extension datum of type S satisfying $\operatorname{supp}_\Lambda K = \Lambda_0$. Again it does no harm to assume that Λ is spanned by $\operatorname{supp}_\Lambda K$.

Let $(e_\xi^\lambda \in K_\xi^\lambda : 0 \neq \xi \in S, \lambda \in \Lambda_\xi)$ be a family of invertible elements of the $(\mathcal{Q}(S), \Lambda)$-graded Lie algebra K. We denote by $f_\xi^\lambda \in L_{-\xi}^{-\lambda}$ the inverse of e_ξ^λ, put $h_\xi^\lambda = [e_\xi^\lambda, f_\xi^\lambda] \in K_0^0$ and define

$$\mathfrak{h} = \operatorname{span}_{\mathbb{K}}\{h_\xi^0 : \xi \in S_{\text{ind}}^\times\} \ \subset \ T_K = \operatorname{span}_{\mathbb{K}}\{h_\xi^\lambda : \xi \in S^\times, \lambda \in \Lambda_\xi\}.$$

Then $\mathfrak{h}$ is a toral subalgebra of K and S canonically embeds into the dual space $\mathfrak{h}^*$ such that $\langle \xi, \tau^\vee \rangle = \xi(h_\tau^0)$ for $\sigma, \tau \in S_{\text{ind}}$, $\tau \neq 0$, and the root spaces of $(K, \mathfrak{h})$ are the subspaces K_ξ, $\xi \in S$. Since $T_K \subset \mathfrak{h} + \mathrm{Z}(K)$, T_K is also a toral subalgebra of K with the same root spaces as $\mathfrak{h}$. We view S (non-canonically) embedded into T_K^*. Let $\mathcal{D} = \{\partial_\theta : \theta \in \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{K})\}$ be the space of degree derivations of the Λ-graded Lie algebra K, 4.10. Since Λ is torsion-free, there exists an embedding of the abelian group Λ into $\mathcal{D}^*$ mapping λ to the linear form $\operatorname{ev}_\lambda$ given by $\operatorname{ev}_\lambda(\partial_\theta) = \theta(\lambda)$, 4.11.

Let $D \subset \mathcal{D}$ be a subspace such that $\lambda \mapsto \operatorname{ev}_\lambda |D$ is injective, e.g. $D = \mathcal{D}$. For such a D, the subspaces $K^\lambda \subset K$ are determined by D, namely $K^\lambda = \{x \in K : [d, x] = \operatorname{ev}_\lambda(d) \text{ for all } d \in D\}$.

It is now clear that $T = T_K \oplus D$ is a toral subalgebra of the semidirect product $E = K \rtimes D$ whose root spaces are

$$E_0 = K_0^0 \oplus D \quad \text{and} \quad E_{\xi \oplus \lambda} = K_\xi^\lambda \text{ for } (\xi, \lambda) \neq (0,0).$$

where for $\xi \in S$ and $\lambda \in \Lambda$ the linear form $\xi \oplus \lambda \in T^*$ is given by

$$(\xi \oplus \lambda)\,(t_K \oplus d) = \xi(t_K) + \operatorname{ev}_\lambda(d) \quad \text{for } t_K \in T_K \text{ and } d \in D.$$

Thus the set of roots of (E, T) is $R = \bigcup_{\xi \in S} (\xi \oplus \Lambda_\xi) \subset T^*$.

We let $R^{\text{re}} = \{\xi \oplus \lambda : \xi \in S^\times, \lambda \in \Lambda_\xi\}$ and observe that $R^{\text{re}} \subset R^{\text{int}}$. The corresponding reflection s_α, $\alpha = \xi \oplus \lambda$, maps $\beta = \tau \oplus \mu \in R$, $\mu \in \Lambda_\tau$, to

$$s_\alpha(\beta) = \beta - \beta(h_\xi^\lambda)\alpha = \beta - \tau(h_\xi^\lambda)\alpha = \beta - \langle \tau, \xi^\vee \rangle \alpha.$$

Hence R with the reflections s_α, $\alpha \in R^{\text{re}}$, is the affine reflection system associated to S and the extension datum $(\Lambda_\xi : \xi \in S)$ in 3.4. Thus, (E, T) is an affine reflection Lie algebra. Moreover, since K is R-graded, it equals the core of (E, T). This proves (1).

Note that $\Lambda_0 = \Lambda_0^K$ in the notation of 6.2. This implies that the affine reflection system R of (E, T, R^{re}) is tame. However, (E, T) is tame if and only if $\operatorname{IDer} K \cap D = \{0\}$.

We will now give a sufficient criterion for a toral pair (E, T) to be an affine reflection Lie algebra.

6.6 Theorem. *Suppose (E,T) satisfies the conditions* (AR1)–(AR3) *below.*

(AR1) *There exists a symmetric bilinear form $(\cdot|\cdot)$ on $X = \mathrm{span}_{\mathbb{K}}(R)$,*

(AR2) *for every $\alpha \in R^{\mathrm{an}} = \{\alpha \in R : (\alpha|\alpha) \neq 0\}$ there exists an integrable $\mathfrak{sl}_2$-triple $(e_\alpha, h_\alpha, f_\alpha)$ of (E,T) such that*

$$(\beta|\alpha) = \tfrac{(\alpha|\alpha)}{2}\, \beta(h_\alpha)$$

holds for all $\beta \in R$, and

(AR3) *for every $\alpha \in R^0 = \{\alpha \in R : (\alpha|\alpha) = 0\}$ there exists a triple $(e_\alpha, t_\alpha, f_\alpha) \in E_\alpha \times T \times E_{-\alpha}$ such that*

$$[e_\alpha,\, f_\alpha] = t_\alpha \quad \textit{and} \quad (\beta|\alpha) = \beta(t_\alpha)$$

holds for all $\beta \in R$.

Then, with respect to the orthogonal reflections s_α, $\alpha \in R^{\mathrm{an}}$, of 2.10*, (R,X) is an affine reflection system with $R^{\mathrm{re}} = R^{\mathrm{an}} \subset R^{\mathrm{int}}$, $R^{\mathrm{im}} = R^0$ and the bilinear form $(\cdot|\cdot)$ of* (AR1) *as affine form. In particular, (E,T,R^{an}) is an affine reflection Lie algebra.*

It follows that there exists an affine form which is positive-semidefinite on the rational span of R and hence on the real span of R if $\mathbb{R} \subset \mathbb{K}$ (Kac's conjecture). We also note that the triple $(e_\alpha, t_\alpha, f_\alpha)$ in (AR3) spans a 2-dimensional abelian or a 3-dimensional Heisenberg algebra, depending on $t_\alpha = 0$ or $t_\alpha \neq 0$.

6.7 Invariant affine reflection algebras. Recall that any bilinear form $(\cdot|\cdot)$ on a vector space V gives rise to a linear map $\flat \colon V \to V^* : v \mapsto v^\flat$, where $v^\flat$ is defined by $v^\flat(u) = (v|u)$ for $v, u \in V$. The map $\flat$ is injective if and only if $(\cdot|\cdot)$ is nondegenerate. In this case, $\flat$ is an isomorphism if and only if V is finite-dimensional.

We will call (E,T) an *invariant affine reflection algebra*, or IARA for short, if the axioms (IA1)–(IA3) below hold.

(IA1) E has an invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$ such that the restriction $(\cdot|\cdot)_T$ of the form to $T \times T$ is nondegenerate.

(IA2) For every $0 \neq \alpha \in R$ there exists $e_{\pm\alpha} \in E_{\pm\alpha}$ such that $0 \neq [e_\alpha,\, e_{-\alpha}] \in T$.

Suppose (E,T) satisfies (IA1) and (IA2). Then for every $\alpha \in R$ there exists a unique $t_\alpha \in T$ such that $(t_\alpha)^\flat = \alpha$, i.e., $(t_\alpha \mid t)_T = \alpha(t)$ for all $\alpha \in R$ and $t \in T$. We use $\flat \colon T \to T^*$ to transport the bilinear form $(\cdot|\cdot)_T$ to a symmetric bilinear form $(\cdot|\cdot)_X$ on $X = \mathrm{span}_{\mathbb{K}}(R)$. Thus, by definition $(\alpha \mid \beta)_X = (t_\alpha \mid t_\beta)_T = \alpha(t_\beta)$ for $\alpha, \beta \in R$. We define the *null* and *anisotropic* roots as

$$R^0 = \{\alpha \in R : (\alpha|\alpha)_X = 0\} \quad \text{and} \quad R^{\mathrm{an}} = \{\alpha \in R : (\alpha|\alpha)_X \neq 0\}. \qquad (1)$$

We can now introduce the remaining axiom (IA3):

(IA3) If $\alpha \in R^{\rm an}$ there exist $e_{\pm\alpha} \in E_{\pm\alpha}$ such that $\operatorname{ad} e_{\pm\alpha}$ is locally nilpotent and $0 \neq [e_\alpha, e_{-\alpha}] \in T$.

Remarks. (a) Of particular interest are invariant affine reflection algebras, whose toral subalgebras are finite-dimensional splitting Cartan subalgebras. They have the following characterization: A toral pair (E, T) is an invariant affine reflection algebra with T a finite-dimensional splitting Cartan subalgebra if and only if

- E has an invariant nondegenerate symmetric bilinear form,
- $T = E_0$ is finite-dimensional, and
- (IA3) holds.

(b) We point out that an IARA may well have an empty set of anisotropic roots. For example, the extended Heisenberg algebra, constructed in [Kac2, 2.9] and denoted $\mathfrak{g}(0)$ there, is an IARA with respect to the subalgebra $\mathfrak{h}$ of loc. cit. with $R^{\rm an} = \emptyset$.

6.8 Theorem. *Suppose* (E, T) *is an invariant affine reflection algebra. Then the conditions* (AR1)–(AR3) *of* 6.6 *are satisfied for the form* $(\cdot|\cdot)_X$ *defined in* 6.7*, and* (R, X) *is a symmetric affine reflection system with* $R^{\rm re} = R^{\rm an} = R^{\rm int}$ *and* $R^0 = R^{\rm im}$*. Moreover, the following hold.*

(a) *If* (E, T) *is tame, then* R *is tame.*

(b) *The centreless core* $L = E_{cc} = K/\operatorname{Z}(K)$ *is a predivision-*(S, Λ)*-graded Lie algebra, which is invariant with respect to the form induced from* $(\cdot|\cdot) \mid K \times K$ *by the canonical epimorphism* $K \to L$*.*

This result and the examples below indicate that the bilinear form $(\cdot|\cdot)$ on an invariant affine reflection algebra E plays only a secondary role. We therefore define an isomorphism of IARAs as an isomorphism of the underlying affine reflection algebras, i.e., two IARAs (E, T) and (E', T') are *isomorphic* if there exists a Lie algebra isomorphism $\varphi\colon\ E \to E'$ such that $\varphi(T) = T'$. It then follows that $\varphi_r(R^{\rm an}) = R'^{\rm an}$.

6.9 A construction of invariant affine reflection algebras. We have seen in 6.8 that the centreless core of an invariant affine reflection algebra is an invariant predivision-(S, Λ)-graded Lie algebra with Λ a torsion-free abelian group. We will now describe a construction of invariant affine reflection algebras which starts from any invariant predivision-root-graded Lie algebra. In particular, it will follow from 6.10 that

every invariant predivision-(S, Λ)*-graded Lie algebra* L *with* S *a locally finite root system and* Λ *a torsion-free abelian group arises as the centreless core of an invariant affine reflection algebra.* (1)

Our construction uses data $(L, D, T_D, C, T_C, \tau)$ as described below in (IAa) – (IAf).

(IAa) $L = \bigoplus_{\lambda\in\Lambda,\,\xi\in S} L_\xi^\lambda$ is an invariant predivision-(S,Λ)-graded Lie algebra with Λ torsion-free.

As explained in 6.5, it is no harm to assume that $\operatorname{supp}_{\mathcal{Q}(S)} L = S$ and that Λ is spanned by $\operatorname{supp}_\Lambda L = \bigcup_{\xi\in S}\Lambda_\xi$ where $\Lambda_\xi = \{\lambda\in\Lambda : L_\xi^\lambda \neq 0\}$. Since L is predivision-graded, we can choose families $(e_\xi^\lambda,\, h_\xi^\lambda,\, f_\xi^\lambda) \in L_\xi^\lambda \times L_0^0 \times L_{-\xi}^{-\lambda}$, $\xi\in S$ and $\lambda\in\Lambda_\xi$, such that e_ξ^λ for $\xi\neq 0$ is invertible with inverse f_ξ^λ, in particular $(e_\xi^\lambda, h_\xi^\lambda, f_\xi^\lambda)$ is an integrable $\mathfrak{sl}_2$-triple, and such that $(e_0^\lambda|f_0^\lambda)_L \neq 0$ but $[f_0^\lambda, e_0^\lambda] = 0 = h_0^\lambda$. We put

$$T_L = \mathfrak{h} = \operatorname{span}_{\mathbb{K}}\{h_\xi^0 : \xi\in S_{\rm ind}^\times\}.$$

We denote by $\operatorname{grSCDer} L = \bigoplus_{\lambda\in\Lambda}(\operatorname{SCDer} L)^\lambda$ the Λ-graded subalgebra of centroidal derivations which are skew-symmetric with respect to $(\cdot|\cdot)_L$, see 4.9. One knows that $\mathcal{D} = \{\partial_\theta : \theta\in\operatorname{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{K})\}$, the space of Λ-degree derivations of L (4.10), is contained in $(\operatorname{SCDer} L)^0$. As in 6.5 every $\lambda\in\Lambda$ gives rise to a well-defined linear form $\operatorname{ev}_\lambda \in \mathcal{D}^*$. We can now describe the remaining data (D, T_D, C, T_C, τ).

(IAb) $D = \bigoplus_{\lambda\in\Lambda} D^\lambda$ is a Λ-graded subalgebra of $\operatorname{grSCDer} L$.

(IAc) $T_D \subset D^0 \cap \mathcal{D}$ is a subspace such that the restricted evaluation map $\Lambda \to T_D^*$, given by $\lambda \mapsto \operatorname{ev}_\lambda |T_D$, is injective.

(IAd) $C = \bigoplus_{\lambda\in\Lambda} C^\lambda$ is a Λ-graded subspace of $D^{\rm gr*}$, which is invariant under the contragredient action of D on $D^{\rm gr*}$ and contains $\operatorname{span}_{\mathbb{K}}\{\sigma_D(l_1,l_2) : l_i \in L\}$, where

$$\sigma_D\colon\ L\times L \to D^{\rm gr*}$$

is the central 2-cocycle defined by $\sigma_D(l_1,l_2)(d) = \big(d(l_1)\mid l_2\big)_L$.

(IAe) $T_C \subset C^0$ is a subspace such that the restriction map $T_C \to T_D^*$, $t_C \mapsto t_C|T_D$, is injective and such that $\sigma_D(e_\xi^\lambda, f_\xi^\lambda) \in T_C$ for all $\xi\in S$ and $\lambda\in\Lambda_\xi$.

(IAf) $\tau\colon\ D\times D\to C$ is an *invariant toral 2-cocycle*, i.e., τ is a bilinear map $\tau\colon\ D\times D\to C$ such that for $d, d_1, d_2, d_3 \in D$

$$\begin{aligned} \tau(d,d) &= 0,\\ \textstyle\sum_{\circlearrowright} \tau([d_1\, d_2], d_3) &= \textstyle\sum_{\circlearrowright} d_1\cdot\tau(d_2,d_3),\\ \tau(d_1,d_2)(d_3) &= \tau(d_2,d_3)(d_1) \quad \text{and}\\ \tau(T_D, D) &= 0. \end{aligned}$$

Here $\sum_{\circlearrowright}$ indicates the sum over all cyclic permutations of $(1,2,3)$.

Assume now that $(L, T_L, D, T_D, C, T_C, \tau)$ satisfy (IAa) – (IAf). Then $E = C\oplus L\oplus D$ becomes a Lie algebra with respect to the product

$$[c_1 \oplus l_1 \oplus d_1\,,\, c_2 \oplus l_2 \oplus d_2] = \big(\sigma_D(l_1, l_2) + d_1 \cdot c_2 - d_2 \cdot c_1 + \tau(d_1, d_2)\big) \oplus \big([l_1,\, l_2]_L + d_1 \cdot l_2 - d_2 \cdot l_1\big) \oplus [d_1,\, d_2]_D$$

where $c_i \in C$, $l_i \in L$, $d_i \in D$, and $[.,.]_L$ and $[.,.]_D$ denote the Lie algebra product of L and D respectively. Moreover, $T = T_C \oplus \mathfrak{h} \oplus T_D$ is a toral subalgebra of E such that (IA1.i) holds for (E,T) with respect to the bilinear form $(\cdot|\cdot)_E$ given by

$$(\, c_1 \oplus l_1 \oplus d_1 \mid c_2 \oplus l_2 \oplus d_2 \,)_E = c_1(d_2) + c_2(d_1) + (l_1 \mid l_2)_L. \qquad (2)$$

To describe the roots of (E,T), recall that S uniquely embeds into $\mathfrak{h}^*$. For $\xi \in S$ and $\lambda \in \Lambda_\xi$ we define a linear form $\xi \oplus \lambda \in T^*$ by

$$(\xi \oplus \lambda)\,(t_C \oplus h \oplus t_D) = \xi(h) + \mathrm{ev}_\lambda(t_D)$$

($t_C \in T_C$, $h \in \mathfrak{h}$ and $t_D \in T_D$). The root spaces of (E,T) then are

$$E_{\xi\oplus\lambda} = \begin{cases} C^\lambda \oplus L_0^\lambda \oplus D^\lambda, & \xi = 0, \lambda \in \Lambda_0, \\ L_\xi^\lambda, & \xi \neq 0, \lambda \in \Lambda_\xi. \end{cases}$$

Hence the set of roots of (E,T) is $R = \{\xi \oplus \lambda \in T^* : \xi \in S, \lambda \in \Lambda_\xi\}$.

Examples. Suppose L satisfies (IAa). Of course, $D = \mathrm{grSCDer}\, L$ or $D = \mathcal{D}$ are always possible choices for D. By 4.9, $\mathrm{grSCDer}\, L$ is the semidirect product of the abelian subalgebra $(\mathrm{CDer}\, L)^0$ and the ideal $\bigoplus_{\lambda\neq 0}(\mathrm{SCDer}\, L)^\lambda$, which indicates that there are lots of possibilities for D. An example for C is $C_{\min} = \mathrm{span}_{\mathbb{K}}\{\sigma_D(l_1, l_2) : l_i \in L\}$. In general, any D-submodule between $C_{\min}$ and $C_{\max} = (\mathrm{grSCDer}\, L)^{\mathrm{gr}*}$ is a possible choice for C.

The conditions on T_D and T_C are interrelated. Since $\mathbb{K}$ has characteristic 0 and Λ is torsion-free, $T_D = \mathcal{D}$ fulfills the conditions in (IAc), see 4.11. Then $T_{C,\min} = \mathrm{span}_{\mathbb{K}}\{\sigma_D(e_\xi^\lambda, f_\xi^\lambda) : \xi \in S, \lambda \in \Lambda_\xi\}$ satisfies (IAe), and hence any bigger space does too.

For τ, we can of course always take $\tau = 0$. But there are in general many more interesting and non-trivial cocycles. For example, this already happens in the case where $D = \mathrm{grSCDer}\, L$ is a generalized Witt algebra and $C = D^{\mathrm{gr}*}$, see [ERM].

In particular, the choices $(D, T_D, C, T_C) = (\mathcal{D}, \mathcal{D}, C_{\min}, C_{\min})$ and $(D, T_D, C, T_C) = (\mathrm{grSCDer}\, L, \mathcal{D}, (\mathrm{grSCDer}\, L)^{\mathrm{gr}*}, \mathcal{D}^{\mathrm{gr}*})$ fulfill all conditions (IAb)–(IAe).

6.10 Theorem. *The pair (E,T) constructed in* 6.9 *is an invariant affine reflection algebra with respect to the form* (6.9.2)*. Its centreless core is L. Moreover:*

(a) *The core E_c of E is always contained in $C_{\min} \oplus L$, and (E,T) is tame if and only if $E_c = C \oplus L$, which is in turn equivalent to $C \oplus L$ being perfect.*

(b) *If* $D = T_D = \mathcal{D}$ *and* $C = T_C = C_{\min}$ *then* (E, T) *is tame.*

(c) *The root system* R *of* (E, T) *is always tame.*

(d) $E_0 = T$ *if and only if* L *is an invariant Lie torus,* $T_D = D^0$ *and* $T_C = C^0$.

6.11 Extended affine Lie algebras. These are special types of invariant affine reflection algebras and are defined in the introduction. However, since the goal here is a description of all extended affine Lie algebras in terms of their centreless cores (6.14), we will only consider tame extended affine Lie algebras in this section. To simplify the presentation, we will include tameness in the definition of an EALA. Hence, *an EALA here is the same as a tame EALA in the introduction.*

For the convenience of the reader, who is solely interested in extended affine Lie algebras, and also for easier comparison with [Neh7] and other papers on extended affine Lie algebras, we list the complete set of axioms, following [Neh7]. Moreover, following the tradition in EALA theory, we will denote the toral subalgebra of E by H (and not by T). This is "justified" since (1) shows that H is a splitting Cartan subalgebra of E.

And now, without any further ado, an *extended affine Lie algebra* or EALA for short, is a pair (E, H) consisting of a Lie algebra E over a field $\mathbb{K}$ of characteristic 0 and a toral subalgebra H satisfying the following axioms (EA1)–(EA6). We denote by $R \subset H^*$ the set of roots of (E, H), hence E has a root space decomposition $E = \bigoplus_{\alpha \in R} E_\alpha$ with respect to H.

(EA1) E has an invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$.

(EA2) H is nontrivial, finite-dimensional and self-centralizing subalgebra

$$E_0 = H. \tag{1}$$

Before we can state the other axioms, we need to draw some consequences of (EA1) and (EA2). The condition (1) implies that the restriction of the form $(\cdot|\cdot)$ to $H \times H$ is nondegenerate. Hence (IA1) is satisfied. As in 6.7 we can therefore define $t_\alpha \in H$ for all $\alpha \in H^*$, transport the restricted form $(\cdot|\cdot) \mid H \times H$ to a symmetric bilinear form $(\cdot|\cdot)_X$ on $X = H^*$ and define isotropic roots $R^0 = \{\alpha \in R : (\alpha|\alpha)_X = 0\}$ and anisotropic roots $R^{\mathrm{an}} = \{\alpha \in R : (\alpha|\alpha)_X \neq 0\}$. We let $K = E_c$ be the core of (E, H), i.e., the subalgebra of E generated by $\bigcup\{E_\alpha : \alpha \in R^{\mathrm{an}}\}$. We can now state the remaining axioms.

(EA3) For every $\alpha \in R^{\mathrm{an}}$ and $x_\alpha \in E_\alpha$, the operator $\operatorname{ad} x_\alpha$ is locally nilpotent on E.

(EA4) R^{an} is connected in the sense that for any decomposition $R^{\mathrm{an}} = R_1 \cup R_2$ with $(R_1 \mid R_2) = 0$ we have $R_1 = \emptyset$ or $R_2 = \emptyset$.

(EA5) (E, H) is tame.

(EA6) The subgroup $\Lambda = \langle R^0 \rangle \subset H^*$ generated by R^0 in $(H^*, +)$ is a free abelian group of finite rank.

Suppose the pair (E, H) satisfies (EA1)–(EA3). Then (IA1) holds, and because of (EA3) also (IA2) and (IA3) hold. It therefore follows that

> *an EALA is an IARA* (E, H) *with the following* 4 *additional properties:* H *is a finite-dimensional splitting Cartan subalgebra and* (EA4)–(EA6) *hold.*

In particular, we will define an *isomorphism* of EALAs as an isomorphism of their underlying IARAs.

Since any pair satisfying (EA1)–(EA3) is an IARA, Th. 6.8 implies that (R, X) is an affine reflection system. It therefore follows from 3.2 that (EA4) is equivalent to

(EA4′) The quotient root system (S, Y) of the affine reflection system (R, X) is an irreducible finite root system.

Without axiom (EA6), Λ is always a torsion-free abelian group. Because a torsion-free finitely generated abelian group is free of finite rank, the axiom (EA6) is equivalent to Λ being finitely generated. As we mentioned earlier, Λ is an invariant of an EALA whose rank is called the nullity of E.

Since an EALA (E, H) is in particular an affine reflection algebra, the root system of (E, H) and its core are described in 6.3 and 6.4. Because of (1), the core and the centreless core are invariant Lie-(S, Λ)-tori.

We will next describe some properties of this class of Lie tori, without assuming that they are the centreless core of an EALA. This is important since in 6.13 we will specialize the construction 6.9 to obtain EALAs starting with any invariant Lie-(S, Λ)-torus. Because of the special properties of Lie tori we will in fact obtain a greatly simplified version of 6.9.

6.12 Some properties of invariant Lie tori. Let L be an invariant Lie-(S, Λ)-torus where, as in 6.11, S is a finite irreducible root system and Λ is a free abelian group of finite rank. Thus L has a $(\mathcal{Q}(S), \Lambda)$-grading

$$L = \bigoplus\nolimits_{\xi \in S,\, \lambda \in \Lambda} L_\xi^\lambda. \tag{1}$$

(i) (5.9) The decomposition $L = \bigoplus_{\xi \in S} L_\xi$, where $L_\xi = \bigoplus_{\lambda \in \Lambda} L_\xi^\lambda$, is the root space decomposition with respect to the toral subalgebra $\mathfrak{h}_L = \operatorname{span}_{\mathbb{K}}\{h_\alpha : \alpha \in S_{\rm ind}^\times\}$. We have $L_0^0 = \mathfrak{h}_L$.

(ii) L is finitely generated as Lie algebra and has bounded dimension with respect to the $(\mathcal{Q}(S), \Lambda)$-grading (1), i.e., there exists a constant C such that $\dim_{\mathbb{K}} L_\xi^\lambda \le C$ for all $\xi \in S$ and $\lambda \in \Lambda$ (this is of course only of interest for $\xi = 0$).

Let t be the maximum cardinality of minimal generating sets of the finitely generated reflection spaces Λ_ξ, $\xi \in S_{\rm ind}$. Then L can be generated by $\le 2t \operatorname{rank} S$ elements and $\dim_{\mathbb{K}} L_0^\lambda \le 4(t \operatorname{rank} S)^2$.

(iii) Because of (ii), $\operatorname{Der}_{\mathbb{K}} L$ is $(\mathcal{Q}(S), \Lambda)$-graded:

$$\operatorname{Der}_{\mathbb{K}} L = \bigoplus_{\xi \in S,\, \lambda \in \Lambda} (\operatorname{Der} L)_\xi^\lambda,$$

where $(\operatorname{Der}_{\mathbb{K}} L)_\xi^\lambda$ consists of those derivations mapping L_τ^μ to $L_{\tau+\xi}^{\mu+\lambda}$. Moreover, the dimension of the homogeneous subspaces $(\operatorname{Der}_{\mathbb{K}} L)_\xi^\lambda$ is bounded. The analogous statements also hold for $\operatorname{SCDer} L$.

(iv) (5.8(c)) The Λ-graded algebra L is graded-simple and prime. The centroid of L is an integral domain, and L embeds into the $\tilde{C}$-algebra $L \otimes \tilde{C} = \tilde{L}$ (tensor product over $\operatorname{Cent}_{\mathbb{K}}(L)$), where $\tilde{C}$ be the quotient field of $\operatorname{Cent}_{\mathbb{K}}(L)$.

(v) (5.8(b)) $\Gamma = \operatorname{supp}_\Lambda \operatorname{Cent}_{\mathbb{K}}(L)$ is a subgroup of Λ, and $\operatorname{Cent}_{\mathbb{K}}(L)$ is isomorphic to the group algebra $\mathbb{K}[\Gamma]$ over $\mathbb{K}$, hence to a Laurent polynomial ring in m variables, $0 \le m \le \operatorname{rank} \Lambda$ (all possibilities occur). Because of $(\operatorname{Cent}_{\mathbb{K}} L)_0 \cong \mathbb{K} \cdot \operatorname{Id}$, it follows that $(\operatorname{SCDer} L)^0 = \mathcal{D}$.

(vi) (5.8(b)) L is free as $\operatorname{Cent}_{\mathbb{K}} L$-module; the following are equivalent:

(a) L is a finite rank $\operatorname{Cent}_k(L)$-module,
(b) L is fgc,
(c) the number $[\Lambda_\alpha : \Gamma]$ of cosets of Λ_α in Γ is finite for some (and hence for every) short root α,
(d) $[\langle \operatorname{supp}_\Lambda L \rangle : \Gamma] < \infty$.

In this case, the Lie algebra $\tilde{L}$ is finite-dimensional and central-simple over $\tilde{C}$.

(vii) L is fgc, if S is not of type A.

(viii) $\operatorname{Der} L$ is the semidirect product of the ideal $\operatorname{IDer} L$ of all inner derivations and the subalgebra $\operatorname{CDer} L$ of centroidal derivations: $\operatorname{Der} L = \operatorname{IDer} L \rtimes \operatorname{CDer} L$. Consequently, $\operatorname{SDer} L = \operatorname{IDer} L \rtimes \operatorname{SCDer} L$, where $\operatorname{SDer} L$ denotes the subalgebra of skew-symmetric derivations of L.

6.13 A construction of EALAs. We specialize the construction and the data (L, D, T_D, C, T_C) of 6.9 with the aim of obtaining an EALA (E, H). As explained in 6.10, L must therefore be an invariant Lie-(S, Λ)-torus where S is a finite irreducible root system and Λ is a free abelian group of finite rank. Also, since H is a splitting Cartan subalgebra, we must take $T_D = D^0$ and $T_C = C^0$, see 6.10. By 6.12, we have $\operatorname{grSCDer} L = \operatorname{SCDer} L$ with $(\operatorname{SCDer} L)^0 = \mathcal{D}$. Moreover $\operatorname{SCDer} L$ and hence any Λ-graded subalgebra have finite homogenous dimension. Since C and D are nondegenerately paired, it follows that $C = D^{\rm gr*}$. Therefore, specializing 6.9, we obtain the following construction, based on data (L, D, τ), where

(EAa) L is an invariant Lie-(S,Λ)-torus with (S,Λ) as above,

(EAb) D is a graded subalgebra of $\operatorname{SCDer} L$ such that the canonical map $\Lambda \to (D^0)^{\mathrm{gr}*}$, $\lambda \mapsto \mathrm{ev}_\lambda \,| D^0$, is injective, and

(EAc) $\tau\colon D \times D \to D^{\mathrm{gr}*}$ is an invariant toral 2-cocycle, as in (IAf).

We denote the resulting invariant affine reflection algebra by $E(L,D,\tau)$. By construction, $H = (D^0)^* \oplus \mathfrak{h} \oplus D^0$ is a splitting Cartan subalgebra and (EA4) and (EA6) hold. One can also show that $D^{\mathrm{gr}*} \oplus L$ is perfect, so that (E,H) is tame. Thus we have the first part of the following theorem, describing the structure of EALAs in terms of Lie tori.

6.14 Theorem. *The invariant affine reflection algebra $E(L,D,\tau)$ constructed in* 6.13 *is an EALA. Conversely, let E be an EALA and let L be its centreless core. Then there exist unique data (D,τ) as in* 6.13 *such that $E = E(L,D,\tau)$.*

Since $D = \mathcal{D}$ or $D = \operatorname{SCDer} L$ and $\tau = 0$ always satisfy the conditions (EAb) and (EAc), any invariant Lie torus arises as the centreless core of an EALA.

Example. If $L = \mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[t^{\pm 1}]$ is the untwisted loop algebra as in 2.13 and 4.12(b), then $\operatorname{grSCDer} L = \operatorname{SCDer} L = \mathbb{K}\, d^{(0)}$. Hence $D = \mathbb{K}\, d^{(0)}$ is the only possible choice for D and then necessarily $\tau = 0$. Thus, in this case $E(L,D,\tau)$ is the Lie algebra E constructed in 2.13(III), and this is the only extended affine Lie algebra based on $L = \mathfrak{g} \otimes \mathbb{K}[t^{\pm 1}]$.

In the remainder of this section we will discuss other classes of invariant affine reflection algebras.

6.15 Discrete extended affine Lie algebras. Let E be a Lie algebra over $\mathbb{K} = \mathbb{C}$ and let H be a toral subalgebra of E. We call (E,H) a *discrete extended affine Lie algebra* if (E,H) satisfies the axioms (EA1)–(EA5) above and in addition

(DE) R is a discrete subset of H^*.

In (DE) “discrete” refers to the natural topology of H^*. Suppose (E,H) satisfies (EA1)–(EA5). Then tameness of (E,H), which is axiom (EA5), implies tameness of R. Also, the quotient root system S is irreducible by (EA4). One can easily show that (DE) holds iff R^0 is discrete iff Λ is discrete. Since a discrete subgroup of H^* is always free of finite rank, (DE) implies (EA6). Hence, a discrete EALA is in particular an EALA, but not every EALA over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ is discrete, see 6.17.

In the construction 6.13, $E(L,D,\tau)$ is a discrete EALA if the evaluation map $\Lambda \to (D^0)^*$ has a discrete image. Conversely, any discrete EALA arises in this way. For example, the embedding $\Lambda \to \mathcal{D}^*$ always has a discrete image.

6.16 Reductive Lie algebras. Let E be a finite-dimensional reductive Lie algebra over $\mathbb{K}$, and let T be a maximal toral subalgebra of the semisimple subalgebra $[E, E]$. Then (E, T) is an invariant affine reflection algebra of nullity 0.

To discuss the core and tameness of (E, T) we first recall that a finite-dimensional simple Lie algebra $\mathfrak{s}$ is called *anisotropic* if $\mathfrak{s}$ does not contain a non-zero toral subalgebra. Otherwise, $\mathfrak{s}$ is said to be *isotropic*.

Coming back to the case of a reductive Lie algebra E with a maximal toral subalgebra T of $[E, E]$, we observe that the core of (E, T) is the ideal of $[E, E]$ spanned by all isotropic components of $[E, E]$. It equals the centreless core and is division-graded. Moreover, (E, T) is tame if and only if E is semisimple and every simple component of E is isotropic. In this case, E equals its core and centreless core.

One can characterize the class of semisimple Lie algebras with all simple components being isotropic as those tame IARAs of nullity 0 whose set of roots is finite and whose core is division-graded with finite-dimensional root spaces. In particular, (E, H) is an EALA of nullity 0 if and only if E is finite-dimensional split simple with splitting Cartan subalgebra H.

6.17 Locally extended affine Lie algebras. A toral pair (E, H) with set of roots R is called a *locally extended affine Lie algebra* or LEALA for short ([MY]), if (E, H) satisfies the following axioms:

(LEA1) $H = E_0$,

(LEA2) E has an invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$,

(LEA3) $\mathrm{ad}_E\, x$ is locally nilpotent for $x \in E_\alpha$ and

$$\alpha \in R^{\mathrm{an}} = \{\alpha \in R : (\alpha|\alpha) \neq 0\}.$$

(LEA4) R^{an} is connected.

Let (E, H) be such a Lie algebra. We point out that $R^{\mathrm{an}} = \emptyset$ is allowed. Since $(\cdot|\cdot) \mid H \times H$ is nondegenerate by invariance of $(\cdot|\cdot)$ and (LEA1), the axiom (IA1.i) is fulfilled. Observe (IA1.ii) = (LEA3) and (IA3) = (LEA4). Because also (IA2) holds, we have

> *a LEALA is the same structure as an invariant affine reflection algebra (E, H) with a splitting Cartan subalgebra H and a connected set of anisotropic roots.*

In this case, the core K of (E, H) and hence also the centreless core L are Lie tori, 6.4. Moreover, by 6.8, L is an invariant Lie torus.

We note that LEALAs generalize EALAs: A LEALA (E, H) is an EALA if and only if H is finite-dimensional, (E, H) is tame and $\langle R^0 \rangle$ is finitely generated.

Examples. (a) By 6.10(e), the invariant affine reflection algebras constructed in 6.9 are LEALAs if and only if $T_D = D^0$, $T_C = C^0$ and L is an invariant Lie torus, say of type (S, Λ) where S is an irreducible locally finite root system. For example, the IARAs $(L, \mathcal{D}, \mathcal{D}, C_{\min}(\mathcal{D}), C_{\min}(\mathcal{D}), \tau)$ are LEALAs.

(b) Let $\mathfrak{g}$ be a finite-dimensional split simple Lie $\mathbb{K}$-algebra or a centreless infinite rank affine algebra, see 2.12. Let Λ be a subgroup of $(\mathbb{K}, +)$ generated by a set B of $\mathbb{Q}$-linearly independent elements, and let $\mathbb{K}[\Lambda]$ be the group algebra of Λ. Then $L = \mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[\Lambda]$ is a Lie-(S, Λ)-torus which is invariant with respect to the bilinear form $\kappa \otimes \epsilon$, where κ is a invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$ and ϵ is the canonical trace form of $\mathbb{K}[\Lambda]$. Let $\theta \colon \Lambda \to \mathbb{K}$ be given by $\theta(\lambda) = \lambda$ and denote by ∂_θ the corresponding degree derivation. Then $D = \mathbb{K}\partial_\theta = T_D$ and $C = D^* = T_C$ satisfy (IAb)–(IAe), and hence the construction 6.9 produces a LEALA E, e.g. by taking $\tau = 0$. It is tame.

Note that for this example the affine root system is

$$R = \{\xi \oplus \lambda \in \mathfrak{h}^* \oplus (\mathbb{K}\partial_\theta)^* : \xi \in S, \lambda \in \Lambda\}$$

with isotropic roots $R^0 \cong \Lambda \subset (\mathbb{K}\partial_\theta)^*$. Hence, the nullity of (E, H) is the cardinality $|B|$, while $\dim_{\mathbb{K}} \operatorname{span}_{\mathbb{K}}(R^{\mathrm{im}}) = 1$. In particular, this shows that 6.2(2) is in general a strict inequality.

It also follows that this LEALA is an EALA if and only if S and B are finite. Assume this to be the case and let $\mathbb{K} = \mathbb{C}$. Then $\Lambda = R^0$ is a discrete subset of $D \cong \mathbb{K}$, i.e., E is a discrete EALA, if and only if B is linearly independent over $\mathbb{R}$. For example, the EALA corresponding to $B = \{1, \sqrt{2}\}$ is not discrete.

6.18 Generalized reductive Lie algebras. By definition [Az4], these are the invariant affine reflection algebras (E, H) over $\mathbb{K} = \mathbb{C}$ with the following properties:

(GRLA1) H is a non-trivial finite-dimensional splitting Cartan subalgebra,

(GRLA2) the set of roots R of (E, H) is a discrete subset of H^*.

We point out that R^{an} is not necessarily connected, i.e., the quotient root system of the corresponding affine reflection system is not necessarily irreducible. It follows from Th. 6.4(c), 6.8 and the axioms (GRLA1), (GRLA2) that the affine reflection system of such an algebra has finite rank, unbroken root strings and is symmetric, discrete and reduced. In other words, R is a generalized reductive root system in the sense of 3.11.

As in 6.17, the core and centreless core of such an algebra are invariant Lie tori, say of type (S, Λ). Here S is a finite, not necessarily irreducible root system and Λ is a free abelian group of finite rank.

The construction 6.9 provides examples of this type of IARAs, assuming in (IAa) that L is an invariant Lie-(S, Λ)-torus with S a finite root system and $\Lambda \cong \mathbb{Z}^n$ and in (IAc) and (IAe) that $T_D = D^0$ and $T_C = C^0$.

6.19 Toral type extended affine Lie algebras. By definition [AKY], these are invariant affine reflection algebras (E, T) which also satisfy the conditions (TEA1)–(TEA4):

(TEA1) T is non-trivial and finite-dimensional,

(TEA2) the affine root system R of (E, T) is tame and R^{an} is connected,

(TEA3) (E, T) has finite nullity, and

(TEA4) R is a discrete subset of $\mathrm{span}_{\mathbb{Q}}(R) \otimes \mathbb{R}$.

It follows from Th. 6.4(c), 6.8 and the axioms (TEA2)–(TEA4) that the affine reflection system of such an algebra has finite rank, a connected $\mathrm{Re}(R)$, unbroken root strings and is symmetric, discrete and tame. In other words, R is an extended affine root system as defined in 3.11 with the exception that R need not be reduced.

6.20 Kac-Moody algebras. Concerning Kac-Moody algebras we will follow the definitions and terminology of [Kac2]. We have the following characterization of invariant affine reflection algebras and extended affine Lie algebras among the Kac-Moody algebras:

Let $\mathfrak{g}(A)$ be a Kac-Moody algebra for some generalized Cartan matrix A and let $\mathfrak{h}$ be its standard Cartan subalgebra. Then the following are equivalent:

(i) $(\mathfrak{g}(A), \mathfrak{h})$ is an invariant affine reflection algebra,
(ii) A is symmetrizable and every imaginary root is a null root,
(iii) every connected component of the Dynkin diagram of A is either of finite or of affine type.

In this case, the anisotropic roots of the associated affine reflection system equals the set of real roots in the terminology of [Kac2], and the core and the centreless core of $\mathfrak{g}(A)$ are Lie tori. In particular, $(\mathfrak{g}(A), \mathfrak{h})$ is an EALA if and only if A is of finite or affine type, in which case the nullity of $\mathfrak{g}(A)$ is, respectively, 0 or 1, and $\mathfrak{g}(A)$ is discrete. In fact, one has a much stronger result:

6.21 Theorem. ([ABGP]) *A complex Lie algebra E is (isomorphic to) a discrete extended affine Lie algebra of nullity* 1 *if and only if E is (isomorphic to) an affine Kac-Moody algebra.*

6.22 Notes. The results in this section are due to the author, unless indicated otherwise in the text or below. Details will be contained in [Neh8]. Some results have been independently proven by others, as indicated below.

The importance of isotopy for the theory of extended affine Lie algebras is explained in [AF]. Assuming the Structure Theorem for extended affine Lie algebras, Th. 6.4(d) is in fact proven in [AF, Th. 6.1] for the case of extended affine Lie algebras.

The notion of “nullity” goes back to the early papers [BGK] and [BGKN] on EALAs, where it was implicitly assumed that 6.2(2) is an equality. This was later corrected in [Ga2]. Beware that in the paper [MY] the term “null rank” is used for the nullity as defined here, while nullity has another meaning.

Kac’s conjecture (6.6) for EALAs was proven in [AABGP, I, §2]. It was shown for LEALAs in [MY, Th. 3.10] and for GRLAs in [Az4, §1]. The proof of 6.6 generalizes techniques found in these papers. Th. 6.8(a) is proven in [ABP1, Lemma 3.62] for EALAs.

The definition of an extended affine Lie algebra in 6.11 is due to the author ([Neh7]). It extends the previous definition from [AABGP], which only made sense for Lie algebras over $\mathbb{C}$ or $\mathbb{R}$. Tame extended affine Lie algebras in the sense of [AABGP] are here called discrete extended affine Lie algebras (see 6.11). To describe the structure of the core and the centreless core of an EALA E, it would be enough to require that R is tame. We have included tameness of (E, H) in the definition of an EALA since there seems to be little hope to classify non-tame EALAs, as the examples in 6.10 hopefully demonstrate. The reader can find another example at the beginning of §3 of [BGK]. That a tame extended affine Lie algebra has a tame root system, see 6.8(b), is proven in [ABP1, 3.62]. The structure of the root system and the core of an EALA is determined in [AG, §1].

Some of the properties of Lie tori summarized in 6.12, the construction 6.13 and Th. 6.14 were announced in [Neh7]. The characterization of fgc Lie tori in 6.12(vi) is also proven in [ABFP2, Prop. 1.4.2]. The interest in fgc Lie tori comes from the multiloop realization described 4.7. This description is made more precise in [ABFP2]. In this context it is of importance to know 6.12(vii). Part (viii) is crucial for the Structure Theorem 6.14.

Yoshii has shown in [Yo5] that for S a finite root system every centreless complex Lie-$(S, \mathbb{Z}^n)$-torus automatically has a graded invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)_L$ with respect to which $(L, \mathfrak{h})$ is invariant. Such a form is unique, up to scalars. Hence, in this case one could leave out the quantifier “invariant”. Yoshii’s proof uses the classification of Lie tori of type $(\mathrm{A}_1, \mathbb{Z}^n)$ and hence ultimately Zelmanov’s Structure Theorem for strongly prime Jordan algebras ([Z], [MZ]).

The construction 6.13 with $D = \mathcal{D}$ and $\tau = 0$ appears in [AABGP, III] and a modified (twisted) version of it in [Az2]. For S of type $\dot{\mathrm{A}}_l$, $(l \geq 3)$, D or E, the construction with $D = \mathrm{SCDer}\, L$ and arbitrary τ can be found in [BGK]. The fact that the Moody-Rao cocycle provides a non-trivial example of an invariant toral 2-cocycle is also mentioned there. Finally, for $S = \mathrm{A}_2$ the general cosntruction is given in [BGKN], modulo the mistake mentioned above. The two papers [BGK] and [BGKN] rely heavily on a concrete realization of the corresponding Lie torus L, and do not present the construction in a way suitable for generalization to the case of an arbitrary S.

LEALAs (6.17) were introduced in [MY]. The Example(b) of 6.17 appears there. This paper also contains examples of LEALAs with $R^{\mathrm{an}} = \emptyset$ and a classification of LEALAs of nullity 0.

It is shown in [Az4] that the set of roots of a generalized reductive Lie algebra is a generalized root system and that its core and centreless core are Lie tori. A proof that the set of roots of a toral type extended affine Lie algebra is an extended affine root system is given in [AKY]. This paper also gives an example of a toral type EALA of type BC with a non-reduced set of roots.

7 Example: $\mathfrak{sl}_I(A)$ for A associative

We describe the various concepts introduced in the previous sections for the special linear Lie algebra $\mathfrak{sl}_I(A)$ and its central quotient $\mathfrak{psl}_I(A)$, where A is an associative algebra.

Unless specified otherwise, k is a commutative associative ring with unit element 1_k such that $2\cdot 1_k$ and $3\cdot 1_k$ are invertible in k. All algebraic structures are defined over k. Throughout, I is a not necessarily finite set of cardinality $|I| \ge 3$ and Λ denotes an abelian group, written additively.

7.1 Basic definitions. Let A be a unital associative k-algebra with identity element 1_A. We recall that $[a,b] := ab - ba$ for $a, b \in A$ denotes the *commutator* in A. We put $[A,A] = \mathrm{span}_k\{[a,b] : a,b \in A\}$ and denote by $\mathrm{Z}(A) = \{z \in A : [z,a] = 0\}$ the centre of A. The underlying k-module of A becomes a Lie algebra with respect to the commutator product $[.,.]$, denoted A^-.

We denote by $\mathrm{Mat}_I(A)$ the set of all finitary $I \times I$-matrices with entries from A. By definition, an element of $\mathrm{Mat}_I(A)$ is a square matrix $x = (x_{ij})_{i,j\in I}$ with all $x_{ij} \in A$ and $x_{ij} \neq 0$ for only finitely many indices $(ij) \in I \times I$. With respect to the usual addition and multiplication of matrices, $\mathrm{Mat}_I(A)$ is an associative k-algebra. For $i, j \in I$ we let $E_{ij} \in \mathrm{Mat}_I(A)$ be the matrix with entry 1_A at the position (ij) and 0 elsewhere. Then any $x \in \mathrm{Mat}_I(A)$ can be uniquely written as $x = \sum_{i,j} x_{ij}E_{ij}$, $x_{ij} \in A$, with almost all $x_{ij} = 0$.

We denote by $\mathfrak{gl}_I(A)$ the Lie algebra $\mathrm{Mat}_I(A)^-$ and by $\mathfrak{sl}_I(A) = [\mathfrak{gl}_I(A), \mathfrak{gl}_I(A)]$ its derived algebra. One easily verifies that $\mathfrak{sl}_I(A)$ is the subalgebra of $\mathfrak{gl}_I(A)$ generated by all off-diagonal matrices AE_{ij}, $i \neq j$ and that $\mathfrak{sl}_I(A)$ can also be described as

$$\mathfrak{sl}_I(A) = \{x \in \mathfrak{gl}_I(A) : \mathrm{tr}(x) \in [A,A]\} \tag{1}$$

where $\mathrm{tr}(x) = \sum_i x_{ii}$ is the *trace* of x. Since $\mathfrak{sl}_I(A) \cong \mathfrak{sl}_J(A)$ if $|I| = |J|$ we put

$$\mathfrak{sl}_n(A) = \mathfrak{sl}_I(A) \quad \text{if } |I| = n \in \mathbb{N}.$$

It is important that the algebra A can be recovered from the Lie algebra $\mathfrak{sl}_I(A)$ by the product formula:

$$ab\, E_{ij} = \big[\big[[aE_{ij},\, E_{jl}],\, E_{li}\big],\, bE_{ij}\big] \tag{2}$$

where $a, b \in A$ and $i, j, l \in I$ are distinct.

What makes $\mathfrak{sl}_I(A)$ somewhat delicate, are the diagonals of its elements. For a commutative A the description (1) of $\mathfrak{sl}_I(A)$ shows that $\mathfrak{sl}_I(A)$ is the "usual" Lie algebra:

$$\mathfrak{sl}_I(A) = \{x \in \mathfrak{gl}_I(A) : \operatorname{tr}(x) = 0\} \cong \mathfrak{sl}_I(k) \otimes_k A \qquad (A \text{ commutative}). \quad (3)$$

But (3) is no longer true for a non-commutative A. Rather, if $n \cdot 1 \in k^\times$ then any $x \in \mathfrak{sl}_n(A)$ can be uniquely written in the form $x = aE_n + x'$ with $a \in [A, A]$ and $x' \in \operatorname{Mat}_n(A)$ with $\operatorname{tr}(x') = 0$, namely

$$x = \frac{1}{n}\operatorname{tr}(x)E_n + (x - \frac{1}{n}\operatorname{tr}(x)E_n).$$

Hence, as k-module,

$$\mathfrak{sl}_n(A) = [A, A]E_n \oplus (\mathfrak{sl}_n(k) \otimes_k A) \qquad (n \cdot 1_k \in k^\times) \quad (4)$$

where E_n is the $n\times n$-identity matrix. One can describe the Lie algebra product with respect to this decomposition. To do so, let us abbreviate $a \circ b = ab + ba$ for a, b in an associative algebra B, like $B = A$ or $B = \operatorname{Mat}_I(A)$ (the experts will recognize $a \circ b$ as the "circle product" of the Jordan algebra associated to B). Then, for $x, y \in \mathfrak{sl}_n(k)$, $a, b \in A$ and $c, d \in [A, A]$,

$$\begin{aligned}
[x \otimes a,\, y \otimes b] &= (\tfrac{1}{n}\operatorname{tr}(xy)\,[a, b]\,E_n) \oplus \big([x, y] \otimes \tfrac{1}{2}(a \circ b) \\
&\qquad + (\tfrac{1}{2}(x \circ y) - \tfrac{1}{n}\operatorname{tr}(xy)E_n) \otimes [a, b]\big),\\
[cE_m,\, dE_n] &= [c, d]\,E_n,\\
[cE_n,\, x \otimes a] &= x \otimes [ca].
\end{aligned}$$

7.2 $\mathfrak{sl}_I(A)$ as $\dot{\mathrm{A}}_I$-graded Lie algebra. Recall the irreducible root system $R = \dot{\mathrm{A}}_I = \{\epsilon_i - \epsilon_j : i, j \in I\}$ defined in 2.8. For $\alpha = \epsilon_i - \epsilon_j \neq 0$ and $\beta = \epsilon_m - \epsilon_n \in R$ the integers $\langle \beta, \alpha^\vee \rangle$ are $\langle \epsilon_m - \epsilon_n, (\epsilon_i - \epsilon_j)^\vee \rangle = \delta_{mi} - \delta_{mj} - \delta_{ni} + \delta_{nj}$, where δ_{**} is the Kronecker delta.

The Lie algebra $\mathfrak{sl}_I(A)$ has a natural $\mathcal{Q}(\dot{\mathrm{A}}_I)$-grading with support $\dot{\mathrm{A}}_I$, whose homogeneous spaces are given by

$$\mathfrak{sl}_I(A)_\alpha = \begin{cases} AE_{ij}, & \alpha = \epsilon_i - \epsilon_j \neq 0, \\ \{x \in \mathfrak{sl}_I(A) : x \text{ diagonal}\}, & \alpha = 0. \end{cases} \quad (1)$$

Because of our assumption that 2 and 3 are invertible in k, the space $\mathfrak{sl}_I(A)_\alpha$, $\alpha = \epsilon_i - \epsilon_j \neq 0$, has the description

$$\mathfrak{sl}_I(A)_\alpha = \{x \in \mathfrak{sl}_I(A) : [E_{ii} - E_{jj},\, x] = 2x\} \quad (2)$$

An element $aE_{ij} \in \mathfrak{sl}_I(A)_\alpha$, $\alpha = \epsilon_i - \epsilon_j \neq 0$, is an invertible element of the $\mathcal{Q}(\dot{\mathrm{A}}_I)$-graded Lie algebra $\mathfrak{sl}_I(A)$, cf. 5.1, if and only if a is invertible in A. In this case, $-a^{-1}E_{ji}$ is the inverse of aE_{ij} and

$$(aE_{ij},\, E_{ii}-E_{jj},\, -a^{-1}E_{ji}) \textit{ is an } \mathfrak{sl}_2\textit{-triple.}$$

In particular, every matrix unit E_{ij}, $i \neq j$, is invertible in $\mathfrak{sl}_I(A)$. Since by construction

$$\mathfrak{sl}_I(A)_0 = \textstyle\sum_{0\neq\alpha\in R} [\mathfrak{sl}_I(A)_\alpha,\, \mathfrak{sl}_I(A)_{-\alpha}]$$

it follows that $\mathfrak{sl}_I(A)$ *is an* $\dot{\mathrm{A}}_I$*-graded Lie algebra* in the sense of 5.1. Hence, either by 5.4 or by an immediate verification,

$$\mathfrak{psl}_I(A) = \mathfrak{sl}_I(A)/\, \mathrm{Z}(\mathfrak{sl}_I(A))$$

and any covering of $\mathfrak{psl}_I(A)$ are $\dot{\mathrm{A}}_I$-graded Lie algebras. For rank $\dot{\mathrm{A}}_I \geq 3$ these are in fact all $\dot{\mathrm{A}}_I$-graded Lie algebras:

7.3 Theorem. *Let* $|I| \geq 4$. *Then a Lie algebra* L *is* $\dot{\mathrm{A}}_I$*-graded if and only if* L *is a covering of* $\mathfrak{psl}_I(A)$ *for a unital associative algebra* A.

In the light of this theorem and the definition of $\mathfrak{psl}_I(A)$ it is of interest to identify the centre of $\mathfrak{sl}_I(A)$, which will be done in 7.4, and the coverings of $\mathfrak{psl}_I(A)$. Since any covering of a perfect Lie algebra P is a quotient of its universal central extension $\mathfrak{uce}(P)$ by a central ideal, it is enough to describe the universal central extension $\mathfrak{uce}(\mathfrak{psl}_I(A)) \to \mathfrak{psl}_I(A)$. Because $\mathfrak{sl}_I(A)$ is a covering of $\mathfrak{psl}_I(A)$, the Lie algebras $\mathfrak{uce}(\mathfrak{psl}_I(A))$ and $\mathfrak{uce}(\mathfrak{sl}_I(A))$ are the same (in any case, in many interesting cases $\mathfrak{sl}_I(A) = \mathfrak{psl}_I(A)$, see 7.4 and 7.7). It is therefore sufficient to identify $\mathfrak{uce}(\mathfrak{sl}_I(A))$, which we will do in 7.8 after we introduced gradings in 7.5.

7.4 Centre of $\mathfrak{sl}_I(A)$. The centre of $\mathfrak{sl}_I(A)$ has the following description, cf. (5.3.1):

$$\mathrm{Z}\left(\mathfrak{sl}_I(A)\right) = \begin{cases} \{0\}, & \text{if } |I| = \infty, \\ \{zE_n \in \mathfrak{sl}_I(A) : z \in \mathrm{Z}(A)\}, & \text{if } |I| = n \in \mathbb{N}. \end{cases} \tag{1}$$

In particular $\mathfrak{sl}_I(A) = \mathfrak{psl}_I(A)$ for $|I| = \infty$, while

$$\mathrm{Z}\left(\mathfrak{sl}_n(A)\right) = \{zE_n : z \in \mathrm{Z}(A), nz \in [A, A]\} \quad \text{for } n \in \mathbb{N}, \tag{2}$$

and therefore also $\mathfrak{sl}_n(A) \cong \mathfrak{psl}_n(A)$ if, for example, k is a field of characteristic 0 and A is commutative.

Since $\mathrm{Z}(\mathfrak{sl}_I(A))$ is diagonal, the homogeneous spaces $\mathfrak{psl}_I(A)_\alpha$ are given by the obviously modified formula (7.2.1). In particular, we can identify $AE_{ij} = \mathfrak{psl}_I(A)_\alpha$ for $0 \neq \alpha = \epsilon_i - \epsilon_j$ and then also have (7.2.2) and the product formula (7.1.2) available in $\mathfrak{psl}_I(A)$.

7.5 Graded coefficient algebras. Let $A = \bigoplus_{\lambda\in\Lambda} A^\lambda$ be a Λ-grading of A and define $\mathfrak{sl}_I(A)^\lambda$ as the submodule of matrices in $\mathfrak{sl}_I(A)$ whose entries all lie in A^λ. Then

$$\mathfrak{sl}_I(A) = \bigoplus_{\lambda\in\Lambda} \mathfrak{sl}_I(A)^\lambda \tag{1}$$

is a Λ-grading of $\mathfrak{sl}_I(A)$ which is compatible with the $\mathcal{Q}(\dot{\mathrm{A}}_I)$-grading (7.2.1). The homogeneous spaces

$$\mathfrak{sl}_I(A)^\lambda_\alpha = \mathfrak{sl}_I(A)_\alpha \cap \mathfrak{sl}_I(A)^\lambda$$

are given by $\mathfrak{sl}_I(A)^\lambda_\alpha = A^\lambda E_{ij}$, $\alpha = \epsilon_i - \epsilon_j \neq 0$, while $\mathfrak{sl}_I(A)^\lambda_0$ consists of the diagonal matrices whose entries all lie in A^λ. Since $1_A \in A^0$, it follows that $\mathfrak{sl}_I(A)^0_\alpha$, $\alpha \neq 0$, contains an invertible element. Thus

$$\mathfrak{sl}_I(A) = \bigoplus_{\alpha\in\dot{\mathrm{A}}_I,\,\lambda\in\Lambda} \mathfrak{sl}_I(A)^\lambda_\alpha \textit{ is an } (\dot{\mathrm{A}}_I, \Lambda)\textit{-graded Lie algebra.} \tag{2}$$

It is immediate from the product formula (7.1.2) that, conversely, any Λ-grading of $\mathfrak{sl}_I(A)$ which is compatible with the $\dot{\mathrm{A}}_I$-grading is obtained from a Λ-grading of A as described above.

Since the centre of A and $[A, A]$ are graded submodules, it follows from (7.1.2) that $\mathrm{Z}(\mathfrak{sl}_I(A))$ is Λ-graded. Hence $\mathfrak{psl}_I(A)$ is $(\mathcal{Q}(\dot{\mathrm{A}}_I), \Lambda)$-graded with homogenous spaces

$$\begin{aligned} &\mathfrak{psl}_I(A)^\lambda_\alpha \cong \mathfrak{sl}_I(A)^\lambda_\alpha \quad \text{for } 0 \neq \alpha \in \dot{\mathrm{A}}_I \text{ and} \\ &\mathfrak{psl}_I(A)^\lambda_0 = \mathfrak{sl}_I(A)^\lambda_0 / \mathrm{Z}(\mathfrak{sl}_I(A))^\lambda. \end{aligned}$$

This is of course a special case of 5.4. The graded version of 7.3 is the following Theorem.

7.6 Theorem. *Let $|I| \geq 4$. Then a Lie algebra L is $(\dot{\mathrm{A}}_I, \Lambda)$-graded if and only if L is a Λ-covering of $\mathfrak{psl}_I(A)$ for some unital associative Λ-graded algebra A.*

Combining this with the description of invertible elements in $\mathfrak{sl}_I(A)$ immediately shows that $\mathfrak{sl}_I(A)$ and $\mathfrak{psl}_I(A)$ are predivision- or division-$(\dot{\mathrm{A}}_I, \Lambda)$-graded Lie algebras if and only if A is, respectively, a predivision- or a division-Λ-graded algebra. Similarly, $\mathfrak{sl}_I(A)$ is a Lie torus of type $(\dot{\mathrm{A}}_I, \Lambda)$ if and only if A is an associative Λ-torus. Hence we get the following corollary (a special case of 5.4).

7.7 Corollary. *Let $|I| \geq 4$. Then L is a predivision- or a division-$(\dot{\mathrm{A}}_I, \Lambda)$-graded Lie algebra if and only if L is a Λ-covering of $\mathfrak{psl}_I(A)$ where A is, respectively, a predivision- or a division-Λ-graded algebra. If k is a field then L is a Lie torus of type $(\dot{\mathrm{A}}_I, \Lambda)$ if and only if L is a Λ-covering of $\mathfrak{psl}_I(A)$ where A is a Λ-torus.*

Refer to 4.5 for a discussion of (pre)division-graded associative algebras. Concerning the structure of Lie tori, we note that

$$A = \mathrm{Z}(A) \oplus [A,\, A] \tag{1}$$

for every associative Λ-torus A. Hence, by (7.4.1) and (7.4.2), we have $\mathfrak{sl}_I(A) = \mathfrak{psl}_I(A)$ for a Lie torus if $|I| = \infty$ or if A does not have $|I|$-torsion. In particular, the corollary says that for $\mathbb{K}$ a field of characteristic 0, a Lie algebra L over $\mathbb{K}$ is a Lie-$(\dot{\mathrm{A}}_I, \mathbb{Z}^n)$-torus if and only if L is a $\mathbb{Z}^n$-covering of $\mathfrak{sl}_I(\mathbb{K}_q)$ where $\mathbb{K}_q$ is a quantum torus (see 4.13).

7.8 The universal central extension of $\mathfrak{sl}_I(A)$ and $\mathfrak{psl}_I(A)$. Let $\langle A, A\rangle = (A \wedge A)/\mathcal{B}$ where $\mathcal{B} = \mathrm{span}_{\mathbb{K}}\{ab \otimes c + bc \otimes a + ca \otimes b : a, b, c \in A\}$. It follows that $\langle A, A\rangle$ is spanned by elements $\langle a, b\rangle = a \wedge b + \mathcal{B}$, $a, b \in A$, and that $\langle a, b\rangle \mapsto [a, b]$ is a well-defined map $\langle A, A\rangle \to [A, A]$. The kernel of this map is $\mathrm{HC}_1(A)$, the first cyclic homology group of A.

We also need to recall the *Steinberg Lie algebra* $\mathfrak{st}_I(A)$ associated to A. It is the Lie k-algebra presented by generators $X_{ij}(a)$, where $i, j \in I$, $i \neq j$, and $a \in A$, and relations (st1)–(st3):

(st1) The map $a \mapsto X_{ij}(a)$ is k-linear,
(st2) $[X_{ij}(a),\, X_{jl}(b)] = X_{il}(ab)$ for distinct $i, j, l \in I$, and
(st3) $[X_{ij}(a),\, X_{lm}(b)] = 0$ for $i \neq m$ and $j \neq l$.

Under our assumptions on k and I, *the universal central extension of* $\mathfrak{sl}_I(A)$ *is the map*

$$\mathfrak{u}\colon\ \mathfrak{st}_I(A) \to \mathfrak{sl}_I(A), \quad X_{ij}(a) \mapsto aE_{ij}. \quad \textit{Its kernel is } \mathrm{HC}_1(A), \tag{1}$$

cf. 5.5. The $\dot{\mathrm{A}}_I$-grading of $\mathfrak{st}_I(A)$ is determined by $\mathfrak{st}_I(A)_\alpha = X_{ij}(A)$ for $0 \neq \alpha = \epsilon_i - \epsilon_j$. Any invertible element $aE_{ij} \in \mathfrak{sl}_I(A)$ lifts to an invertible element $X_{ij}(a) \in \mathfrak{st}_I(A)$. It then follows that $\mathfrak{st}_I(A)$ is $\dot{\mathrm{A}}_I$-graded, as stated in Th. 7.3. Similarly, if A is Λ-graded, then $\mathfrak{st}_I(A)$ is $(\mathcal{Q}(\dot{\mathrm{A}}_I) \oplus \Lambda)$-graded with homogeneous spaces $\mathfrak{st}_I(A)^\lambda_\alpha = X_{ij}(A^\lambda)$ for $\alpha = \epsilon_i - \epsilon_j \neq 0$. This explains Th. 7.6 and Cor. 7.7, at least for $\mathfrak{st}_I(A)$, but then the general case easily follows.

If the extension $\mathfrak{u}$ is split on the level of k-modules, e.g. if $\mathfrak{sl}_I(A)$ is projective as k-module (for example k is a field), we can identify $\mathfrak{sl}_I(A)$ with a k-submodule of $\mathfrak{st}_I(A)$ such that

$$\mathfrak{st}_I(A) = \mathrm{HC}_1(A) \oplus \mathfrak{sl}_I(A)$$

as k-modules and the Lie algebra product $[.,.]_{\mathfrak{st}}$ of $\mathfrak{st}_I(A)$ is given in terms of the Lie algebra product $[.,.]$ of $\mathfrak{sl}_I(A)$ by $[\mathrm{HC}_1(A),\, \mathfrak{st}_I(A)]_{\mathfrak{s}}\mathfrak{t} = 0$ and

$$[x, y]_{\mathfrak{st}} = \textstyle\sum_{i,j} \langle x_{ij}, y_{ji}\rangle \oplus [x, y] \tag{2}$$

for $x = (x_{ij})$ and $y = (y_{ij}) \in \mathfrak{sl}_I(A)$.

In case $n \cdot 1_k \in k^\times$, one can give a description of $\mathfrak{uce}(\mathfrak{sl}_I(A))$ in the spirit of 7.1: We define a Lie algebra on

$$\mathfrak{uce}\big(\mathfrak{sl}_n(A)\big) = \langle A, A\rangle \oplus (\mathfrak{sl}_n(k) \otimes_k A) \tag{3}$$

by modifying the product formulas in 7.1 as follows:

$$\begin{aligned}[x\otimes a,\, y\otimes b] &= \left(\tfrac{1}{n}\operatorname{tr}(xy)\,\langle a,b\rangle\right)\oplus(\ \dots\ \text{as in 7.1}\ \dots)\\ [\langle c_1,c_2\rangle,\, \langle d_1,d_2\rangle] &= \langle\, [c_1,c_2],\, [d_1,d_2]\,\rangle,\\ [\langle c_1,c_2\rangle,\, x\otimes a] &= x\otimes\big[[c_1,c_2],\, a\big]\end{aligned}$$

for $x,y\in\mathfrak{sl}_n(k)$ and $a,b,c_i,d_i\in A$. The obvious map $\mathfrak{uce}\big(\mathfrak{sl}_I(A)\big)\to\mathfrak{sl}_I(A)$ is then a universal central extension.

For example, let $A=\mathbb{K}[t^{\pm 1}]$ be the Laurent polynomial ring in one variable t, where $\mathbb{K}$ is a field of characteristic 0. Since $\mathrm{HC}_1(A)$ is 1-dimensional, the Lie algebra $K=\mathfrak{sl}_I(\mathbb{K}[t^{\pm 1}])\oplus\mathbb{K}c$ constructed in 2.13 is a universal central extension of $\mathfrak{sl}_I(A)$.

7.9 The centroid of $\mathfrak{sl}_I(A)$. It is convenient here and in the description of derivations of $\mathfrak{sl}_I(A)$ to use the following notation. Any endomorphism $f\in\operatorname{End}_k(A)$, which leaves $[A,A]$ invariant, induces an endomorphism of $\mathfrak{sl}_I(A)$, denoted $\mathfrak{sl}_I(f)$, by applying f to every entry of a matrix in $\mathfrak{sl}_I(A)$:

$$\mathfrak{sl}_I(f)\,(x)=(f(x_{ij}))\quad\text{for } x=(x_{ij})\in\mathfrak{sl}_I(A). \tag{1}$$

The map

$$\mathfrak{sl}_I(.)\colon\ \{f\in\operatorname{End}_k(A): f([A,A])\subset[A,A]\}\to\operatorname{End}_k(\mathfrak{sl}_I(A))$$

is a monomorphism of k-algebras. If $f\in\operatorname{End}_k(A)$ also leaves the centre $\mathrm{Z}(A)$ of A invariant, the map $\mathfrak{sl}_I(f)$ descends to an endomorphism of $\mathfrak{psl}_I(A)$, naturally denoted $\mathfrak{psl}_I(f)$.

Recall (4.2) that $\mathrm{Z}(A)\to\mathrm{Cent}(A)$, $z\mapsto L_z$, is an isomorphism of k-algebras. For $z\in\mathrm{Z}(A)\cong\mathrm{Cent}_k(A)$ the endomorphism $\chi_z=\mathfrak{sl}_I(L_z)$ of $\mathfrak{sl}_I(A)$, mapping $x=(x_{ij})$ to $\chi_z(x)=(zx_{ij})$, belongs to the centroid of $\mathfrak{sl}_I(A)$. The map

$$\mathrm{Z(A)\to Cent\left(\mathfrak{sl}_I(A)\right),\quad z\mapsto\chi_z}$$

is an isomorphism of k-algebras. Moreover, $z\mapsto\mathfrak{psl}_I(L_z)$ is also an isomorphism between $\mathrm{Z}(A)$ and $\mathrm{Cent}(\mathfrak{psl}_I(A))$. If A is Λ-graded, then so is the centre $\mathrm{Z}(A)$ and hence also $\mathrm{Cent}(\mathfrak{sl}_I(A))$.

For the description of $\mathfrak{sl}_I(A)$ as a multiloop algebra ([ABFP1]) it is important to know when is $\mathfrak{sl}_I(A)$ finitely generated over its centroid. This is the case if and only if I is finite and A is finitely generated over $\mathrm{Z}(A)$. For example, if $A=k_q$, $q=(q_{ij})\in\mathrm{Mat}_n(k)$, is a quantum torus over a field k, A is a finitely generated module over its centre if and only all $q_{ij}\in k$ are roots of unity, cf. 4.13.

7.10 Invariant bilinear forms. For a Ξ-graded algebra X, Ξ an abelian group, we denote by $\mathrm{GIF}(X,\Xi)$, the k-module of Ξ-graded invariant symmetric bilinear forms on X, see 4.3. In this subsection we will determine

$\mathrm{GIF}(\mathfrak{sl}_I(A), \mathcal{Q}(\dot{\mathrm{A}}_I) \oplus \Lambda)$ for a Λ-graded algebra A. For simpler notation we put $\Xi = \mathcal{Q}(\dot{\mathrm{A}}_I) \oplus \Lambda$.

To $(\cdot|\cdot)_A \in \mathrm{GIF}(A, \Lambda)$ we associate the bilinear form $(\cdot|\cdot)_{\mathfrak{sl}}$ defined on $\mathfrak{sl}_I(A)$ by

$$\big(\sum_{i,j} x_{ij} E_{ij} \mid \sum_{p,q} y_{pq} E_{pq} \big)_{\mathfrak{sl}} = \sum_{i,j} (x_{ij} \mid y_{ji})_A. \tag{1}$$

It is an easy exercise to show that $(\cdot|\cdot)_{\mathfrak{sl}}$ is invariant. Since it is obviously also Ξ-graded, we have $(\cdot|\cdot)_{\mathfrak{sl}} \in \mathrm{GIF}(\mathfrak{sl}_I(A), \Xi)$. Observe

$$(aE_{ij} \mid bE_{ji})_{\mathfrak{sl}} = (a \mid b)_A \tag{2}$$

for $a, b \in A$ and distinct $i, j \in I$. Conversely, this formula assigns to every $(\cdot|\cdot)_{\mathfrak{sl}} \in \mathrm{GIF}(\mathfrak{sl}_I(A), \Xi)$ a graded bilinear form $(\cdot|\cdot)_A$ on A. It is immediate, using (7.1.2), that $(\cdot|\cdot)_A$ is invariant. Since every invariant form on $\mathfrak{sl}_I(A)$ is uniquely determined by its restriction to $AE_{ij} \times AE_{ji}$, $i \neq j$, we have an isomorphism of k-modules

$$\mathrm{GIF}(\mathfrak{sl}_I(A), \Xi) \xrightarrow{\cong} \mathrm{GIF}(A, \Lambda), \quad (\cdot|\cdot)_{\mathfrak{sl}} \mapsto (\cdot|\cdot)_A. \tag{3}$$

Moreover, by (4.3.2), $\mathrm{GIF}(A, \Lambda) \cong (A^0/[A,A]^0)^*$. Combining these two isomorphisms, it follows that *any Λ-graded invariant bilinear form b of $\mathfrak{sl}_I(A)$ can be uniquely written in the form*

$$b(x, y) = \varphi(\mathrm{tr}(xy) + [A, A]) \quad \textit{for } \varphi \in (A^0/[A,A]^0)^*. \tag{4}$$

Since $\mathfrak{sl}_I(A)$ is perfect, $\mathrm{GIF}(\mathfrak{sl}_I(A)) \cong \mathrm{GIF}(\mathfrak{psl}_I(A))$ by (4.3.1). Combining (3) with this isomorphism implies

$$\mathrm{GIF}(\mathfrak{psl}_I(A)) \cong \mathrm{GIF}(A, \Lambda) \cong (A^0/[A,A]^0)^*. \tag{5}$$

We now turn to the description of nondegenerate invariant forms. Since the centre of a perfect Lie algebra is contained in the radical of any invariant form, $\mathrm{Z}(\mathfrak{sl}_I(A)) = \{0\}$ is a necessary condition for the existence of such a form on $\mathfrak{sl}_I(A)$. But rather than assuming $\mathrm{Z}(\mathfrak{sl}_I(A)) = \{0\}$ we will work with the centreless Lie algebra $\mathfrak{psl}_I(A)$ and the isomorphism (5), written as $(\cdot|\cdot)_{\mathfrak{psl}} \mapsto (\cdot|\cdot)_A$. It is easily seen that

$$(\cdot|\cdot)_{\mathfrak{psl}} \textit{ is nondegenerate} \iff (\cdot|\cdot)_A \textit{ is nondegenerate.} \tag{6}$$

Example. Let A be a Λ-torus over a field F. Since $[A,A]^0 = [A,A] \cap A^0 = \{0\}$ by (7.7.1),

$$\mathrm{GIF}(\mathfrak{psl}_I(A)) \cong (A^0)^*$$

is a 1-dimensional space. Moreover, any non-zero graded invariant bilinear form on $\mathfrak{psl}_I(A)$ is nondegenerate. Indeed, any non-zero bilinear form $(\cdot|\cdot)$ on A is of type $(a_1 \mid a_2) = \psi(a_1 a_2)$ for a linear form $\psi \in A^*$ with $\mathrm{Ker}\,\psi = \bigoplus_{0 \neq \lambda} A^\lambda$. Since $\mathrm{Rad}(\cdot|\cdot)$ is graded and $1 \in zA$ for every non-zero homogeneous $z \in A$, it follows that $\mathrm{Rad}(\cdot|\cdot) = \{z \in A : zA \subset \mathrm{Ker}\,\psi\} = \{0\}$.

7.11 Derivations. We denote by $\widehat{\mathrm{Mat}}_I(A)$ the space of all row- and column-finite matrices with entries from A. This is an associative k-algebra with respect to the usual matrix multiplication. Hence it becomes a Lie algebra, denoted $\widehat{\mathfrak{gl}}_I(A)$, with respect to the commutator. It is immediate that $[\widehat{\mathfrak{gl}}_I(A), \mathfrak{sl}_I(A)] \subset \mathfrak{sl}_I(A)$. We obtain a Lie algebra homomorphism $\widehat{\mathfrak{gl}}_I(A) \to \mathrm{Der}_k \mathfrak{sl}_I(A)$, $\hat{x} \mapsto \mathrm{ad}\,\hat{x}|\mathfrak{sl}_I(A)$ with kernel $\{zE : z \in \mathrm{Z}(A)\}$ where E is the identity matrix of $\widehat{\mathfrak{gl}}_I(A)$. Its image will be denoted

$$\widehat{\mathrm{IDer}}\,\mathfrak{sl}_I(A) = \{\mathrm{ad}\,\hat{x}|\mathfrak{sl}_I(A) : \hat{x} \in \hat{\mathfrak{gl}}_I(A)\} \cong \widehat{\mathfrak{gl}}_I(A)/\{zE : z \in \mathrm{Z}(A)\}.$$

Of course, $\widehat{\mathrm{IDer}}\,\mathfrak{sl}_I(A) = \mathrm{IDer}\,\mathfrak{sl}_I(A)$ if I is finite. Since $[\hat{x}, \mathrm{Z}(\mathfrak{sl}_I(A))] = 0$ for $\hat{x} \in \mathfrak{gl}_I(A)$, $\mathrm{ad}\,\hat{x}$ descends to a derivation of $\mathfrak{psl}_I(A)$.

Another class of derivations of $\mathfrak{sl}_I(A)$ and $\mathfrak{psl}_I(A)$ are given by the maps $\mathfrak{sl}_I(d)$ and $\mathfrak{psl}_I(d)$ for $d \in \mathrm{Der}_k(A)$, cf. (7.9.1). The map $\mathfrak{sl}_I\colon\ \mathrm{Der}_k(A) \to \mathrm{Der}_k \mathfrak{sl}_I(A)$, $d \mapsto \mathfrak{sl}_I(d)$ is clearly a monomorphism of Lie algebras. We denote by $\mathfrak{sl}_I(\mathrm{Der}_k A)$ its image. One checks that

$$[\mathfrak{sl}_I(d), \mathrm{ad}\,\hat{x}] = \mathrm{ad}\,d(\hat{x}) \quad \text{for } \hat{x} \in \hat{\mathfrak{gl}}_I(A).$$

7.12 Theorem. (a) *The ungraded derivation algebras are*

$$\begin{aligned}
\mathrm{Der}_k\big(\mathfrak{sl}_I(A)\big) &= \widehat{\mathrm{IDer}}\,\mathfrak{sl}_I(A) + \mathfrak{sl}_I(\mathrm{Der}_k A) \quad \textit{with}\\
\mathfrak{sl}_I(\mathrm{IDer}\,A) &= \widehat{\mathrm{IDer}}\,\mathfrak{sl}_I(A) \cap \mathfrak{sl}_I(\mathrm{Der}_k A) \cong \mathrm{IDer}\,A,\\
\mathrm{CDer}\big(\mathfrak{sl}_I(A)\big) &= \mathfrak{sl}_I(\mathrm{CDer}\,A) \cong \mathrm{CDer}\,A,\\
\mathrm{SDer}\big(\mathfrak{sl}_I(A)\big) &= \widehat{\mathrm{IDer}}\,\mathfrak{sl}_I(A) + \mathfrak{sl}_I(\mathrm{SDer}A),
\end{aligned}$$

where in the last formula the derivations in SDer *are skew-symmetric with respect to a bilinear form* $(\cdot|\cdot)_{\mathfrak{sl}}$ *and the corresponding bilinear form* $(\cdot|\cdot)_A$*, cf.* (7.10.2)

(b) *Suppose A is Λ-graded. Then the homogeneous subspaces* $(\mathrm{Der}\ \mathfrak{sl}_I(A))^\lambda_\alpha$ *of derivations of degree* $\alpha \oplus \lambda$ *are given by*

$$\begin{aligned}
\big(\,\mathrm{Der}\ \mathfrak{sl}_I(A)\big)^\lambda_\alpha &= \mathrm{ad}\,\big(\mathfrak{sl}_I(A)^\lambda_\alpha\big) = \{\mathrm{ad}\,x : x \in A^\lambda E_{ij}\}, \qquad \alpha = \epsilon_i - \epsilon_j \neq 0,\\
\big(\,\mathrm{Der}\ \mathfrak{sl}_I(A)\big)^\lambda_0 &= \{\mathrm{ad}\,\hat{x} : \hat{x} \in \widehat{\mathfrak{gl}}_I(A) \textit{ diagonal, all } x_{ii} \in A^\lambda\} \oplus \mathfrak{sl}_I\big((\mathrm{Der}\ A)^\lambda\big)
\end{aligned}$$

In particular,

$$\mathrm{grSCDer}\,(\mathfrak{sl}_I(A)) = \mathfrak{sl}_I\big(\,\mathrm{grSCDer}(A)\big) \cong \mathrm{grSCDer}\,A.$$

(c) *All statements in* (a) *and* (b) *hold, mutatis mutandis, for* $\mathfrak{psl}_I(A)$.

7.13 The standard toral subalgebra. In this subsection we suppose that $k = \mathbb{K}$ is a field of characteristic 0. We put

$$\mathfrak{h} = \mathrm{span}_{\mathbb{K}}\{E_{ii} - E_{jj} : i, j \in I\}$$

which is the space of diagonal trace-0-matrices over $\mathbb{K}$. We write an element $h \in \mathfrak{h}$ as $h = \operatorname{diag}(h_i)_{i\in I}$ where $h_i \in \mathbb{K}$ is the entry of the matrix h at the position (ii). We embed the root system $\dot{\mathrm{A}}_I$ into $\mathfrak{h}^*$ by

$$(\epsilon_i - \epsilon_j)(h) = h_i - h_j \quad \text{for } h = \operatorname{diag}(h_i).$$

Then $\mathfrak{h}$ is a toral subalgebra of $\mathfrak{sl}_I(A)$ with set of roots equal to $\dot{\mathrm{A}}_I$.

Suppose $A = \bigoplus_{\lambda\in\Lambda} A^\lambda$ is Λ-graded and let $\psi \in (A^0)^*$ with $[A\,A]^0 \subset \operatorname{Ker}\psi$. Hence ψ gives rise to a Λ-graded invariant symmetric bilinear form $(\cdot|\cdot)_{A,\psi}$ on A, given by $(a_1|a_2)_{A,\psi} = \psi(a_1a_2)$. By (7.10.1) we can extend $(\cdot|\cdot)_{A,\psi}$ to an invariant $(\mathcal{Q}(\dot{\mathrm{A}}_I) \oplus \Lambda)$-graded symmetric bilinear form $(\cdot|\cdot)_{\mathfrak{sl},\psi}$ on $\mathfrak{sl}_I(A)$. Note that by (7.10.4) for $h = \operatorname{diag}(h_i)$ and $h' = \operatorname{diag}(h'_i)$ we get

$$(h|h')_{\mathfrak{sl},\psi} = \textstyle\sum_{i\in I} \psi(h_ih'_i) = \big(\sum_{i\in I} h_ih'_i\big)\,\psi(1).$$

This implies

$$(\cdot|\cdot)_{\mathfrak{sl},\psi} \mid \mathfrak{h}\times\mathfrak{h} \text{ is nondegenerate if and only if } \psi(1) \neq 0.$$

Indeed, $\psi(1) \neq 0$ is obviously necessary for nondegeneracy. Conversely, if $(h|\mathfrak{h})_{\mathfrak{sl},\psi} = 0$ for $h = \operatorname{diag}(h_i)$, then $h_i = h_j$ for all $i, j \in I$ whence $h = 0$ in case I is infinite, but this also holds in case of a finite I, say $|I| = n$, since then $h = h_1E_n$ has trace $nh_1 = 0$.

As we have seen in (7.10.6), the form $(\cdot|\cdot)_{\mathfrak{sl},\psi}$ descends to a $(\mathcal{Q}(\dot{\mathrm{A}}_I) \oplus \Lambda)$-graded invariant symmetric bilinear form $(\cdot|\cdot)_{\mathfrak{psl},\psi}$ on $\mathfrak{psl}_I(A)$. By (7.10.6),

> $\mathfrak{psl}_I(A)$ *is an invariant* $(\dot{\mathrm{A}}_I, \Lambda)$*-graded Lie algebra if and only if* A *has a* Λ*-graded invariant nondegenerate symmetric bilinear form, say* $(\cdot|\cdot)_{A,\psi}$*, and* $\psi(1) \neq 0$.

In particular, if A is a Λ-torus the Lie torus $\mathfrak{psl}_I(A)$ is invariant with respect to any form $(\cdot|\cdot)_{\mathfrak{psl},\psi}$ with $\psi \neq 0$.

7.14 Notes. The definition of $\mathfrak{sl}_I(A)$ is standard. Some authors call it the *elementary Lie algebra* and denote it by $\mathfrak{e}_I(A)$. One can of course also define $\mathfrak{sl}_I(A)$ for $|I| = 2$ (recall our convention $|I| \geq 3$, but then the claims in this section, e.g. (7.1.1), are no longer true. Some results on $\mathfrak{sl}_n(A)$, $n \in \mathbb{N}$, are proven in [BGK], e.g. the formula in 7.1 for the Lie algebra product of $\mathfrak{sl}_n(A)$ can be found in [BGK, (2.27)–(2.29)].

In the generality stated, Th. 7.3 is proven in [Neh3, (3.5)]. For k a field of characteristic 0 and I finite, i.e., $\dot{\mathrm{A}}_I$ finite, it was proven earlier by Berman-Moody [BeMo, 0.7]. Their proof shows at the same time that $\mathfrak{st}_n(A)$ is a universal central extension of any A_{n-1}-graded Lie algebra.

As stated, Th. 7.6 is an immediate consequence of [GaN, 2.11 and 3.4] and [Neh5, 4.7]. Cor. 7.7 is proven in [Yo1, Prop. 2.13] for I finite and k a field of characteristic 0. This paper also contains a construction of crossed products

$B * \mathbb{Z}^n$ ([Yo1, §3]). The description of Lie tori of type $(\mathrm{A}_l, \mathbb{Z}^n)$ is given in [BGK]. The formula (7.7.1) is proven in [NY, Prop. 3.3]. The case $\Lambda = \mathbb{Z}^n$ is already contained in [BGK, Prop. 2.44].

That (7.8.1) is a universal central extension is proven in [Bl1] for A commutative and $|I| \geq 5$, for A arbitrary and again $|I| \geq 5$ in [KL] and for $3 \leq |I| \leq 4$ in [Ga1, 2.63] (recall our assumption that $\frac{1}{6} \in k$). The universal central 2-cocycle (7.8.2) is also given in these papers. The results in 7.8 are no longer true if 2 or 3 are not invertible in k, see [GS] for the case of an algebra A that is free as k-module. If $\mathfrak{sl}_I(A)$ is a free k-module or a finitely generated and projective k-module, the kernel of the universal central extension is the second homology with trivial coefficients, i.e., $H_2\big(\mathfrak{sl}_I(A)\big) = \mathrm{HC}_1(A)$. That the Lie algebra (7.8.3) is the universal central extension of $\mathfrak{sl}_n(A)$ is shown in [BGK, page 359–360]. A lot is known about the cyclic homology groups $\mathrm{HC}_1(A)$, see [Lod]. For example, if A is commutative, then $\mathrm{HC}_1(A) \cong \Omega^1_{A|k}/dA$ [Lod, 2.1.14]. For a complex quantum torus $\mathrm{HC}_1(\mathbb{C}_q)$ is described in [BGK, Prop. 3.19] (the result is true for any field). In particular, it follows that $\dim \mathrm{HC}_1(\mathbb{K}[t^{\pm 1}]) = 1$. The formula (7.10.4) is proven in [BGK, Lemma 2.8].

The description of $\mathrm{Der}\,\mathfrak{sl}_I(A)$ in 7.12(a) is proven for finite I and k a field (but A arbitrary non-associative) in [BO, Th. 4.8 and Cor. 4.9]. The proof can easily be adapted to yield the result as stated. For $k = A$ a field of characteristic 0 one has $\mathrm{Der}_k(k) = \{0\}$, $\mathfrak{sl}_I(k) = \mathfrak{psl}_I(k)$ and hence $\mathrm{Der}_k\left(\mathfrak{sl}_I(k)\right) = \widehat{\mathrm{IDer}}\,\mathfrak{sl}_I(k)$, which is proven in [Nee2, Th. I.3] with a different method. The derivation algebra of $\mathbb{C}_q$ is determined in [BGK, Lemma 2.48].

The astute reader will have noticed that we have excluded the A_2-case in all statements referring to the description of $(\dot{\mathrm{A}}_I, \Lambda)$-graded Lie algebras (7.3, 7.6 and 7.7). The reason for this is that not all (A_2, Λ)-graded Lie algebra are Λ-coverings of $\mathfrak{psl}_3(A)$ for A a Λ-graded associative algebra. Rather, one has to allow alternative algebras A. Various models of A_2-graded Lie algebras have been given, see e.g. [BeMo, 3.3], [BGKN, §2], [Neh3, 3.3] or [AF, §9]. Once a replacement of $\mathfrak{psl}_3(A)$ has been defined, the Theorems 7.3, 7.6 and 7.7 hold, mutatis mutandis, by replacing the associative algebra A by an alternative algebra A. For example, for $\Lambda = \mathbb{Z}^n$ and Lie algebras over fields of characteristic 0 this has been worked out in [Yo2, §6], while the classification of the corresponding coordinate algebras, i.e., division-$\mathbb{Z}^n$-graded alternative algebras, is given in [Yo2, Th. 5.7] and is valid in characteristic $\neq 2$. For arbitrary Λ, special types of alternative division Λ-graded algebras, so-called quasi-algebras, are described in [AEP]. An alternative nonassociative $\mathbb{Z}^n$-torus over a field F of characteristic $\neq 2$ is graded-isomorphic to an octonion torus [BGKN, Th. 1.25], [Yo2, Cor. 5.13], see [AF, Example 9.2] for a presentation.

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