THE CENTROID OF EXTENDED AFFINE AND ROOT GRADED LIE ALGEBRAS

GEORGIA BENKART[A,1] AND ERHARD NEHER[B,2,*]

[A] *Department of Mathematics, University of Wisconsin, Madison, WI 53706-1388 USA*

[B] *Department of Mathematics and Statistics, University of Ottawa, Ottawa, Ontario, K1N 6N5, Canada*

To Professor Bruce Allison with our best wishes on his sixtieth birthday

Abstract. We develop general results on centroids of Lie algebras and apply them to determine the centroid of extended affine Lie algebras, loop-like and Kac-Moody Lie algebras, and Lie algebras graded by finite root systems.

1. Introduction

Our main focus will be on centroids of Lie algebras. When $\mathcal{L}$ is a Lie algebra, the centroid $\mathrm{Cent}(\mathcal{L})$ is just the space of $\mathcal{L}$-module homomorphisms χ on $\mathcal{L}$: $\chi([x,y]) = [x, \chi(y)]$ for all $x, y \in \mathcal{L}$, (viewing $\mathcal{L}$ as an $\mathcal{L}$-module under the adjoint action). Our interest in the centroid stems from investigations of extended affine Lie algebras (see [AABGP]). These Lie algebras are natural generalizations of the affine and toroidal Lie algebras, which have played such a critical role in many different areas of mathematics and physics. Their root systems (the so-called extended affine root systems) feature prominently in the work of Saito ([S1], [S2]) and Slodowy [Sl] on singularities. In the classification of the extended affine Lie algebras, elements of the centroid are essential in constructing the portion of the algebra

Date: May 28, 2005.

[1] Supported in part by NSF Grant #DMS–0245082 and NSA Grant MDA904-03-1-0068.

[2] Supported in part by Natural Sciences and Engineering Research Council of Canada Discovery Grant #8836–2001.

[*] Corresponding author

E-mail addresses: benkart@math.wisc.edu (G. Benkart), neher@uottawa.ca (E. Neher)

2000 Mathematical Subject Classification: Primary 17B40; Secondary 17A36.

that lies outside of the core (see for example, [N2]). This is the part of the extended affine Lie algebra $\mathcal{E}$ that is nondegenerately paired with the centre under the invariant bilinear form on $\mathcal{E}$. Thus, results on the centroid are a key ingredient in the classification of extended affine Lie algebras.

The centroid also plays an important role in understanding forms of an algebra: All scalar extensions of a simple algebra remain simple if and only if its centroid just consists of the scalars in the base field. In particular, for finite-dimensional simple associative algebras, the centroid is critical in investigating Brauer groups and division algebras. Another area where the centroid occurs naturally is in the study of derivations of an algebra. If $\chi \in \mathrm{Cent}(\mathcal{A})$ and ∂ is a derivation of $\mathcal{A}$, then $\chi\partial$ is also a derivation of $\mathcal{A}$, so centroidal transformations can be used to construct derivations of an algebra.

We will develop general results, extending some earlier work of other authors, and then apply them to determine the centroid of several families of (mostly infinite-dimensional) Lie algebras: tensor product and in particular loop-like algebras (Prop. 2.19, Rem. 2.24), centreless Lie tori (Prop. 3.13), extended affine Lie algebras (Cor. 4.13), Lie algebras graded by finite root systems (Thm. 5.15 and Thm. 5.18), and Kac-Moody Lie algebras (Cor. 3.5), which are distinguished because of their substantial applications in a diverse array of subjects.

We take this opportunity to thank Alberto Elduque and Arturo Pianzola for their helpful comments on an earlier version of this paper.

2. Centroids of algebras

2.1. **Some general results.** We begin with a little background on centroids for arbitrary (not necessarily Lie, associative, etc.) algebras $\mathcal{A}$. Proofs of the results quoted here can be found for example in [J, X.1] for finite-dimensional algebras and in [Mc, II, 1.6, 1.7] in general.

It is natural and important for our approach to centroids to consider algebras over a unital commutative associative ring, (for example, a perfect algebra $\mathcal{A}$ over its centroid, which may not be a field unless $\mathcal{A}$ is simple). Thus, let $\mathcal{A}$ be an arbitrary algebra over a unital commutative associative ring $\mathbb{K}$. The *centroid* of $\mathcal{A}$ is the space of $\mathbb{K}$-linear transformations on $\mathcal{A}$ given by $\mathrm{Cent}(\mathcal{A}) = \{\chi \in \mathrm{End}_{\mathbb{K}}(\mathcal{A}) \mid \chi(ab) = a\chi(b) = \chi(a)b \text{ for all } a, b \in \mathcal{A}\}$.

We will write $\mathrm{Cent}_{\mathbb{K}}(\mathcal{A})$ for $\mathrm{Cent}(\mathcal{A})$ if it is important to emphasize the dependence on $\mathbb{K}$. Clearly, $\mathrm{Cent}(\mathcal{A})$ is simply the centralizer algebra of the *multiplication algebra* $\mathrm{Mult}(\mathcal{A}) = \mathrm{Mult}_{\mathbb{K}}(\mathcal{A})$, the unital subalgebra of $\mathrm{End}_{\mathbb{K}}(\mathcal{A})$ generated by the left and multiplication operators of $\mathcal{A}$. The centroid is always a subalgebra of the associative algebra $\mathrm{End}_{\mathbb{K}}(\mathcal{A})$.

For $a, b, c \in \mathcal{A}$, their *associator* is defined as $(a, b, c) = (ab)c - a(bc)$. The *centre* of $\mathcal{A}$ consists of all $z \in \mathcal{A}$ satisfying $za = az$ and $(a, b, z) = (a, z, b) = (z, a, b) = 0$ for all $a, b \in \mathcal{A}$. The centre is always a commutative associative subalgebra of $\mathcal{A}$. Moreover, if $\mathcal{A}$ has an identity element 1, then $\mathrm{Cent}(\mathcal{A}) \to \mathcal{A}$, $\chi \mapsto \chi(1)$, is an algebra isomorphism between the centroid and the centre of $\mathcal{A}$. We denote by $\mathcal{A}^{(1)}$ the $\mathbb{K}$-span of all products ab for $a, b \in \mathcal{A}$. If $\mathcal{A}$ is *perfect*, (i.e., $\mathcal{A}$ equals $\mathcal{A}^{(1)}$), then the centroid is necessarily commutative, as $\chi\psi(ab) = \chi(\psi(a)b) = \psi(a)\chi(b) = \psi(a\chi(b)) = \psi\chi(ab)$ holds for all $a, b \in \mathcal{A}$, $\chi, \psi \in \mathrm{Cent}(\mathcal{A})$. If $\mathrm{Cent}(A)$ is commutative, we may regard $\mathcal{A}$ as an algebra over its centroid, $\chi a = a\chi$ and $\chi(ab) = (\chi a)b = a(\chi b)$ for all $a, b \in \mathcal{A}$, $\chi \in \mathrm{Cent}(\mathcal{A})$. If $\mathcal{A}$ is *prime* in the sense that $\mathcal{A}$ has no nonzero ideals I, J with $IJ = 0$, then $\mathrm{Cent}(\mathcal{A})$ is an integral domain and $\mathcal{A}$ is a torsion-free $\mathrm{Cent}(A)$-module. If $\mathcal{A}$ is *simple*, i.e., $\mathcal{A}^{(1)} \neq 0$ and the only ideals of $\mathcal{A}$ are 0 and $\mathcal{A}$, then $\mathrm{Cent}(\mathcal{A})$ must be a field by Schur's Lemma. When the centroid of an algebra coincides with the base ring $\mathbb{K}$ (more precisely, it equals $\mathbb{K}\,\mathrm{id}$), the algebra is said to be *central*, and in the special case of a simple algebra, it is said to be *central simple*. Every simple algebra is central simple over its centroid.

For any subset $\mathcal{B}$ of $\mathcal{A}$, the *annihilator of $\mathcal{B}$ in $\mathcal{A}$* is $\mathrm{Ann}_{\mathcal{A}}(\mathcal{B}) = \{z \in \mathcal{A} \mid z\mathcal{B} = 0 = \mathcal{B}z\}$. Any $\mathbb{K}$-submodule of $\mathcal{A}$ containing $\mathcal{A}^{(1)}$ or contained in $\mathrm{Ann}_{\mathcal{A}}(\mathcal{A})$ is an ideal of $\mathcal{A}$. If $\mathcal{A}$ is a Lie algebra, we follow the usual convention of denoting the product of $a, b \in \mathcal{A}$ by $[a, b]$. In this case, $\mathrm{Ann}_{\mathcal{A}}(\mathcal{B})$ is simply the *centralizer* $\mathcal{C}_{\mathcal{A}}(\mathcal{B}) = \{z \in \mathcal{A} \mid [z, \mathcal{B}] = 0\}$ of $\mathcal{B}$ in $\mathcal{A}$. In particular, $\mathrm{Ann}_{\mathcal{A}}(\mathcal{A}) = Z(\mathcal{A})$, the usual centre of $\mathcal{A}$, which coincides with the general definition of the centre as given above if $\frac{1}{2} \in \mathbb{K}$. Let $\mathrm{Der}(\mathcal{A})$ denote the algebra $\mathrm{Der}_{\mathbb{K}}(\mathcal{A})$ of $\mathbb{K}$-linear derivations of an algebra $\mathcal{A}$. Then we have the following basic facts:

Lemma 2.1. *Let $\mathcal{A}$ be an algebra over a unital commutative associative ring $\mathbb{K}$ and let $\mathcal{B}$ be a subset of $\mathcal{A}$.*

(a) $\mathrm{Ann}_{\mathcal{A}}(\mathcal{B})$ *is invariant under* $\mathrm{Cent}(\mathcal{A})$*, as is any perfect ideal of $\mathcal{A}$.*

(b) *For any* $\mathrm{Cent}(\mathcal{A})$*-invariant ideal* $\mathcal{B}$ *of* $\mathcal{A}$*, the* vanishing ideal

$$\mathcal{V}(B) := \{\chi \in \mathrm{Cent}(\mathcal{A}) \mid \chi(\mathcal{B}) = 0\}$$

is isomorphic to $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(\mathcal{A}/\mathcal{B}, \mathrm{Ann}_{\mathcal{A}}(\mathcal{B}))$*, which is the set of* $\mathbb{K}$*-linear maps* $f : \mathcal{A}/\mathcal{B} \to \mathrm{Ann}_{\mathcal{A}}(\mathcal{B})$ *satisfying* $f(xy) = f(x)y = xf(y)$ *for all* $x, y \in \mathcal{A}/\mathcal{B}$*, where* $\mathcal{A}/\mathcal{B} \times \mathrm{Ann}_{\mathcal{A}}(\mathcal{B}) \to \mathcal{A}$ *is defined by* $(a + \mathcal{B})z = az$ *and similarly for* $\mathrm{Ann}_{\mathcal{A}}(\mathcal{B}) \times \mathcal{A}/\mathcal{B} \to \mathcal{A}$*. In particular, if* $\mathrm{Cent}(\mathcal{B}) = \mathbb{K}\,\mathrm{id}$*, then*

$$\mathrm{Cent}(\mathcal{A}) = \mathbb{K}\,\mathrm{id} \oplus \mathcal{V}(\mathcal{B}). \tag{2.2}$$

(c)

$$\begin{aligned}
\mathrm{Cent}(\mathcal{A}) \cap \mathrm{Der}(\mathcal{A}) &= \{\psi \in \mathrm{End}_{\mathbb{K}}(\mathcal{A}) \mid \mathcal{A}^{(1)} \subseteq \ker\psi,\ \mathrm{im}\ \psi \subseteq \mathrm{Ann}_{\mathcal{A}}(\mathcal{A})\} \\
&= \{\psi \in \mathrm{Cent}(\mathcal{A}) \mid \mathcal{A}^{(1)} \subseteq \ker\psi\} = \mathcal{V}(\mathcal{A}^{(1)}) \\
&= \{\psi \in \mathrm{Der}(\mathcal{A}) \mid \mathrm{im}\ \psi \subseteq \mathrm{Ann}_{\mathcal{A}}(\mathcal{A})\}, \\
&\cong \mathrm{Hom}_{\mathcal{A}/\mathcal{A}^{(1)}}(\mathcal{A}/\mathcal{A}^{(1)}, \mathrm{Ann}_{\mathcal{A}}(\mathcal{A})) \\
&\cong \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}/\mathcal{A}^{(1)}, \mathrm{Ann}_{\mathcal{A}}(\mathcal{A})).
\end{aligned}$$

(d) $\mathcal{A}$ *is indecomposable (cannot be written as the direct sum of two non-trivial ideals, or equivalently, is an indecomposable* $\mathrm{Mult}(\mathcal{A})$*-module) if and only if* $\mathrm{Cent}(\mathcal{A})$ *does not contain idempotents* $\neq 0, \mathrm{id}$*.*

(e) *Suppose* $\mathcal{A}$ *is an indecomposable* $\mathrm{Mult}(\mathcal{A})$*-module of finite length* n*, and denote by* $\mathrm{rad}\,\mathrm{Cent}(\mathcal{A})$ *the Jacobson radical of* $\mathrm{Cent}(\mathcal{A})$*. Then* $\mathrm{Cent}(\mathcal{A})$ *is a local ring, i.e.,* $\mathrm{Cent}(\mathcal{A})/\mathrm{rad}\,\mathrm{Cent}(\mathcal{A})$ *is a division ring (see for example,* [L, Sec. 19]*), and* $\big(\mathrm{rad}\,\mathrm{Cent}(\mathcal{A})\big)^n = 0$*. Thus,* $\mathrm{rad}\,\mathrm{Cent}(\mathcal{A})$ *is nilpotent and coincides with the set of nilpotent transformations in* $\mathrm{Cent}(\mathcal{A})$*. In particular, if* $\mathcal{A}$ *is a finite-dimensional indecomposable algebra over a perfect field* $\mathbb{F}$ *then there exists a division algebra* $\mathfrak{D}$ *over* $\mathbb{F}$ *such that* $\mathrm{Cent}(\mathcal{A}) = \mathfrak{D}\,\mathrm{id} \oplus \mathrm{rad}\,\mathrm{Cent}(\mathcal{A})$*.*

(f) *If* $\mathcal{A}$ *is perfect, every* $\chi \in \mathrm{Cent}(\mathcal{A})$ *is symmetric with respect to any invariant form on* $\mathcal{A}$*.*

Proof. (a) – (d) are straightforward. Part (d) can be found in [Me, Sec. 1] for Lie algebras or in [Po, Lem. 1] for more general algebras, where a description of $\mathrm{Cent}(\mathcal{A})$ for decomposable $\mathcal{A}$ is also given.

(e) Since $\mathrm{Cent}(\mathcal{A})$ consists of the $\mathrm{Mult}(\mathcal{A})$-module endomorphisms of $\mathcal{A}$, the first part of (e) follows from [L, Thm. 19.17]. Under the assumptions of the second part we know that $\mathrm{Cent}(\mathcal{A})$ is a local $\mathbb{F}$-algebra. The claim then follows from Wedderburn's Principal Theorem.

(f) Let $(\,|\,)$ be an *invariant* $\mathbb{K}$-bilinear form on $\mathcal{A}$, so that $(ab\,|\,c) = (a\,|\,bc)$ for all $a, b, c \in \mathcal{A}$, and let $\chi \in \mathrm{Cent}(\mathcal{A})$. Then $(\chi(ab)\,|\,c) = (a\chi(b)\,|\,c) = (a\,|\,\chi(b)c) = (a\,|\,b\chi(c)) = (ab\,|\,\chi(c))$ for all $a, b, c \in \mathcal{A}$. $\square$

Remark 2.3. Let $\mathcal{L}$ be a Lie algebra over a field $\mathbb{F}$. A derivation from $\mathcal{L}$ to an $\mathcal{L}$-module $\mathfrak{M}$ is an $\mathbb{F}$-linear map $\delta : \mathcal{L} \to \mathfrak{M}$ such that $\delta\big([x, y]\big) = x.\delta(y) - y.\delta(x)$ for all $x, y \in \mathcal{L}$. The space $\mathrm{Der}(\mathcal{L}, \mathfrak{M})$ of such derivations contains the inner derivations $\mathrm{IDer}(\mathcal{L}, \mathfrak{M}) = \{\delta_m \mid m \in \mathfrak{M}\}$, where $\delta_m(x) = x.m$ for all $x \in \mathcal{L}$. Then the *first cohomology group of* $\mathcal{L}$ *with values in* $\mathfrak{M}$ is the quotient $\mathtt{H}^1(\mathcal{L}, \mathfrak{M}) := \mathrm{Der}(\mathcal{L}, \mathfrak{M})/\mathrm{IDer}(\mathcal{L}, \mathfrak{M})$ (see for example, [Bo2, Ch. I, Sec. 3, Ex. 12] or [J, Ch. V.6]). Now for any Lie algebra $\mathcal{L}$, an ideal $\mathfrak{M}$ of $\mathcal{L}$ is an $\mathcal{L}$-module under the adjoint action. Examples of $\mathrm{Cent}(\mathcal{L})$-invariant ideals are the centre $Z(\mathcal{L})$ and all the ideals in the derived series, lower (descending) central series, and ascending central series of $\mathcal{L}$. In particular if $\mathfrak{M} = Z(\mathcal{L})$, we have $\mathrm{IDer}(\mathcal{L}, Z(\mathcal{L})) = 0$. Thus $\mathtt{H}^1(\mathcal{L}, Z(\mathcal{L})) = \mathrm{Der}(\mathcal{L}, Z(\mathcal{L}))$, and by Lemma 2.1, we have a canonical identification

$$\begin{aligned} \mathtt{H}^1(\mathcal{L}, Z(\mathcal{L})) &= \Big\{\psi \in \mathrm{End}_{\mathbb{F}}(\mathcal{L}) \,\Big|\, [\psi(\mathcal{L}), \mathcal{L}] = 0 = \psi(\mathcal{L}^{(1)})\Big\} = \mathcal{V}(\mathcal{L}^{(1)}) \quad (2.4) \\ &\cong \mathrm{Hom}_{\mathbb{F}}\big(\mathcal{L}/\mathcal{L}^{(1)}, Z(\mathcal{L})\big) \end{aligned}$$

as $\mathrm{Cent}(\mathcal{L})$-modules.

Example 2.5. For any Lie algebra $\mathcal{L}$ over a field $\mathbb{F}$,

$$\mathcal{V}(\mathcal{L}^{(1)}) = \Big\{\psi \in \mathrm{End}_{\mathbb{F}}(\mathcal{L}) \,\Big|\, [\psi(\mathcal{L}), \mathcal{L}] = 0 = \psi(\mathcal{L}^{(1)})\Big\},$$

as in (2.4). Thus, if $Z(\mathcal{L}) \neq 0$ and $\mathcal{L} \neq \mathcal{L}^{(1)}$, we have

$$\mathbb{F}\,\mathrm{id} \subsetneq \mathbb{F}\,\mathrm{id} \oplus \mathcal{V}(\mathcal{L}^{(1)}) \subseteq \mathrm{Cent}_{\mathbb{F}}(\mathcal{L}).$$

So in order for $\mathcal{L}$ to be central, a necessary condition is that $Z(\mathcal{L}) = 0$ or $\mathcal{L}$ is perfect. Later results (Corollaries 3.4 and 4.13) will treat various classes of Lie algebras for which $\mathbb{F}\,\mathrm{id} \subsetneq \mathbb{F}\,\mathrm{id} \oplus \mathcal{V}(\mathcal{L}^{(1)}) = \mathrm{Cent}(\mathcal{L})$. *Heisenberg Lie algebras* provide easy examples of this phenomenon. A Heisenberg Lie algebra $\mathcal{H}$ has a basis $\{a_i, b_i \mid i \in \mathcal{I}\} \cup \{c\}$, such that $[a_i, b_j] = \delta_{i,j}c$, $[a_i, a_j] = 0 = [b_i, b_j]$, and $[\mathcal{H}, c] = 0$, where $\delta_{i,j}$ is the Kronecker delta. By (2.2) applied to $\mathcal{B} = \mathcal{H}^{(1)} = Z(\mathcal{H})$,

$$\mathrm{Cent}(\mathcal{H}) = \mathbb{F}\,\mathrm{id} \oplus \mathcal{V}(\mathcal{H}^{(1)}).$$

For nilpotent Lie algebras (in particular, for $\mathcal{L} = \mathcal{H}$), $\mathcal{V}(\mathcal{L}^{(1)}) \neq 0$, but it may well be that $\mathbb{F}\,\mathrm{id} \oplus \mathcal{V}(\mathcal{L}^{(1)}) \subsetneq \mathrm{Cent}(\mathcal{L})$ for an arbitrary nilpotent Lie algebra (see [Me, Prop. 2.7]).

Remark 2.6. Indecomposable Lie algebras $\mathcal{L}$ having a small centroid, i.e. those for which $\mathrm{Cent}(\mathcal{L}) = \mathbb{F}\,\mathrm{id} \oplus \mathcal{V}(\mathcal{L}^{(1)})$, have been investigated by Melville [Me] and Ponomarëv [Po] under certain assumptions (e.g. in [Me], when $\mathcal{L}$ is finite-dimensional).

Lemma 2.7. *Let $\pi : \mathcal{A} \to \mathcal{B}$ be an epimorphism of $\mathbb{K}$-algebras. For every $f \in \mathrm{End}_{\mathbb{K}}(\mathcal{A}; \ker\pi) := \{g \in \mathrm{End}_{\mathbb{K}}(\mathcal{A}) \mid g(\ker\pi) \subseteq \ker\pi\}$, there exists a unique $\bar{f} \in \mathrm{End}_{\mathbb{K}}(\mathcal{B})$ satisfying $\pi \circ f = \bar{f} \circ \pi$. Moreover the following hold:*

(a) *The map*

$$\pi_{\mathrm{End}} : \mathrm{End}_{\mathbb{K}}(\mathcal{A}; \ker\pi) \to \mathrm{End}_{\mathbb{K}}(\mathcal{B}), \quad f \mapsto \bar{f}$$

is a unital algebra homomorphism with the following properties:

$$\pi_{\mathrm{End}}(\mathrm{Mult}(\mathcal{A})) = \mathrm{Mult}(\mathcal{B}), \tag{2.8}$$

$$\pi_{\mathrm{End}}\big(\mathrm{Cent}(\mathcal{A}) \cap \mathrm{End}_{\mathbb{K}}(\mathcal{A}; \ker\pi)\big) \subseteq \mathrm{Cent}(\mathcal{B}). \tag{2.9}$$

By restriction, there is an algebra homomorphism

$$\pi_{\mathrm{Cent}} : \big(\mathrm{Cent}(\mathcal{A}) \cap \mathrm{End}_{\mathbb{K}}(\mathcal{A}; \ker\pi)\big) \to \mathrm{Cent}(\mathcal{B}), \quad f \mapsto \bar{f} \tag{2.10}$$

If $\ker\pi = \mathrm{Ann}_{\mathcal{A}}(\mathcal{A})$, every $\chi \in \mathrm{Cent}(\mathcal{A})$ leaves $\ker\pi$ invariant, and hence π_{Cent} is defined on all of $\mathrm{Cent}(\mathcal{A})$.

(b) *Suppose $\mathcal{A}$ is perfect and $\ker\pi \subseteq \mathrm{Ann}_{\mathcal{A}}(\mathcal{A})$. Then*

$$\pi_{\mathrm{Cent}} : \big(\mathrm{Cent}(\mathcal{A}) \cap \mathrm{End}_{\mathbb{K}}(\mathcal{A}; \ker\pi)\big) \to \mathrm{Cent}(\mathcal{B}), \quad f \mapsto \bar{f} \tag{2.11}$$

is injective.

(c) *If $\mathcal{A}$ is perfect, $\mathrm{Ann}_{\mathcal{B}}(\mathcal{B}) = 0$ and $\ker\pi \subseteq \mathrm{Ann}_{\mathcal{A}}(\mathcal{A})$, then $\pi_{\mathrm{Cent}} : \mathrm{Cent}(\mathcal{A}) \to \mathrm{Cent}(\mathcal{B})$ is an algebra monomorphism.*

The main application of this lemma will be to Lie algebras. In this case, an epimorphism $\pi : \mathcal{A} \to \mathcal{B}$ with $\ker\pi \subseteq Z(\mathcal{A})$ is nothing but a central extension, see Section 4 for more on central extensions and centroids.

Proof. (a) That π_{End} is an algebra homomorphism is well-known and easily seen. Since $\ker\pi$ is an ideal, all left and right multiplication operators of $\mathcal{A}$ leave $\ker\pi$ invariant, whence $\mathrm{Mult}(\mathcal{A}) \subseteq \mathrm{End}_{\mathbb{K}}(\mathcal{A};\ker\pi)$. Also, for the left multiplication operator L_x on $\mathcal{A}$ we have $\pi\circ L_x = L_{\pi(x)}\circ\pi$ which shows $\pi_{\mathrm{End}}(L_x) = L_{\pi(x)}$. Since the analogous formula holds for the right multiplication operators and since π_{End} is an algebra epimorphism, we have (2.8). Let $\chi \in \mathrm{Cent}(\mathcal{A}) \cap \mathrm{End}_{\mathbb{K}}(\mathcal{A};\ker\pi)$. Then for $x, y \in \mathcal{A}$, we have $\bar{\chi}\big(\pi(x)\pi(y)\big) = \bar{\chi}\pi(xy) = \pi\chi(xy) = \pi\big(x\chi(y)\big) = \pi(x)(\bar{\chi}\pi(y)) = \pi(\chi(x)y) = \bar{\chi}(\pi(x))\pi(y)$, which proves $\bar{\chi} \in \mathrm{Cent}(\mathcal{B})$.

(b) If $\bar{\chi} = 0$ for $\chi \in \mathrm{Cent}(\mathcal{A}) \cap \mathrm{End}_{\mathbb{K}}(\mathcal{A};\ker\pi))$, then $\chi(\mathcal{A}) \subseteq \ker\pi \subseteq \mathrm{Ann}_{\mathcal{A}}(\mathcal{A})$. So $\chi(xy) = \chi(x)y = 0$ for all $x, y \in \mathcal{A}$, and because $\mathcal{A}$ is perfect, it must be that $\chi = 0$.

(c) It follows readily from $\pi\big(\mathrm{Ann}_{\mathcal{A}}(\mathcal{A})\big) \subseteq \mathrm{Ann}_{\mathcal{B}}(\mathcal{B}) = 0$ that $\ker\pi = \mathrm{Ann}_{\mathcal{A}}(\mathcal{A})$. By (a), $\pi_{\mathrm{Cent}} : \mathrm{Cent}(\mathcal{A}) \to \mathrm{Cent}(\mathcal{B})$ is then a well-defined algebra homomorphism, which is injective by (b). $\square$

2.2. **Centroids of graded algebras.** We recall some concepts and results from the theory of graded algebras and graded modules ([Bo1, Sec. 11]).

Definition 2.12. (1) Let Λ be an abelian group written additively. An algebra $\mathcal{A}$ over some base ring $\mathbb{K}$ is said to be Λ-*graded* if $\mathcal{A} = \bigoplus_{\lambda\in\Lambda} \mathcal{A}^\lambda$ is a direct sum of $\mathbb{K}$-submodules $\mathcal{A}^\lambda$ satisfying $\mathcal{A}^\lambda\mathcal{A}^\mu \subseteq \mathcal{A}^{\lambda+\mu}$ for all $\lambda, \mu \in \Lambda$. In this case, $\mathrm{supp}_{\mathcal{A}} = \{\lambda \in \Lambda \mid \mathcal{A}^\lambda \neq 0\}$ is called the *support* of $\mathcal{A}$, and the elements of $\mathcal{A}^\lambda$ are said to be *homogeneous of degree* λ. A subalgebra (or ideal) $\mathcal{B}$ of $\mathcal{A}$ is *graded* if $\mathcal{B} = \bigoplus_{\lambda\in\Lambda} (\mathcal{B} \cap \mathcal{A}^\lambda)$. Then $\mathcal{A}$ is *graded-simple* if $\mathcal{A}^{(1)} \neq 0$, and every graded ideal $\mathcal{B}$ of $\mathcal{A}$ is *trivial*, i.e., $\mathcal{B} = 0$ or $\mathcal{B} = \mathcal{A}$.

(2) A Λ-graded unital associative algebra $\mathcal{A}$ is said to be a *division-graded* algebra if every nonzero homogeneous element of $\mathcal{A}$ is invertible.

When $\mathcal{A}$ is a division-graded associative algebra, $\mathrm{supp}_{\mathcal{A}}$ is a subgroup of Λ; $\mathcal{A}^0$ is division algebra; and $\mathcal{A}$ is a crossed product algebra $\mathcal{A} = \mathcal{A}^0 * \mathrm{supp}_{\mathcal{A}}$. Conversely, every crossed product algebra over a division algebra is a division-graded associative algebra. In particular, a commutative associative division-graded algebra $\mathcal{A}$ is the same as a twisted group ring $\mathbb{E}^t[\mathrm{supp}_{\mathcal{A}}]$ for $\mathbb{E} = \mathcal{A}^0$ (see for example, [P, Sec. 1]).

Now let $\mathcal{B}$ be a Λ-graded unital associative $\mathbb{K}$-algebra. A left $\mathcal{B}$-module $\mathcal{M}$ is Λ-*graded* if $\mathcal{M}$ is a direct sum of $\mathbb{K}$-submodules, $\mathcal{M} = \bigoplus_{\lambda\in\Lambda} \mathcal{M}^\lambda$,

such that $\mathcal{B}^\lambda \mathcal{M}^\mu \subseteq \mathcal{M}^{\lambda+\mu}$ for all $\lambda, \mu \in \Lambda$. In this case, we denote by $\mathrm{End}_{\mathcal{B}}(\mathcal{M})^\lambda$ the $\mathbb{K}$-submodule of all $f \in \mathrm{End}_{\mathcal{B}}(\mathcal{M})$ satisfying $f\mathcal{M}^\mu \subseteq \mathcal{M}^{\lambda+\mu}$ for all $\mu \in \Lambda$. The (internal) sum of the subspaces $\mathrm{grEnd}_{\mathcal{B}}(\mathcal{M})^\lambda$ is direct, and we set

$$\mathrm{grEnd}_{\mathcal{B}}(\mathcal{M}) = \bigoplus_{\lambda \in \Lambda} \mathrm{End}_{\mathcal{B}}(\mathcal{M})^\lambda.$$

This is a Λ-graded associative subalgebra of $\mathrm{End}_{\mathcal{B}}(\mathcal{M})$ such that $\mathcal{M}$ is canonically a Λ-graded left module over $\mathrm{grEnd}_{\mathcal{B}}(\mathcal{M})$. In general $\mathrm{grEnd}_{\mathcal{B}}(\mathcal{M})$ is a proper subalgebra of $\mathrm{End}_{\mathcal{B}}(\mathcal{M})$. However, the following is proven in [Bo1, Sec. 11.6, Rem.]:

Lemma 2.13. *If $\mathcal{M}$ is a finitely generated graded $\mathcal{B}$-module, then* $\mathrm{grEnd}_{\mathcal{B}}(\mathcal{M}) = \mathrm{End}_{\mathcal{B}}(\mathcal{M})$.

A Λ-graded $\mathcal{B}$-module $\mathcal{M}$ is *graded-irreducible* if the only graded $\mathcal{B}$-submodules $\mathcal{N} = \bigoplus_{\lambda \in \Lambda} (\mathcal{N} \cap \mathcal{M}^\lambda)$ are the trivial submodules $\mathcal{N} = 0$ and $\mathcal{N} = \mathcal{M}$. A straightforward adaptation of the usual proof of Schur's Lemma gives the graded version below, in which the equality $\mathrm{End}_{\mathcal{B}}(\mathcal{M}) = \mathrm{grEnd}_{\mathcal{B}}(\mathcal{M})$ follows from Lemma 2.13.

Lemma 2.14. *Let $\mathcal{B}$ be a Λ-graded associative algebra and let $\mathcal{M}$ be a Λ-graded $\mathcal{B}$-module which is graded-irreducible. Then* $\mathrm{End}_{\mathcal{B}}(\mathcal{M}) = \mathrm{grEnd}_{\mathcal{B}}(\mathcal{M})$ *is a division-graded algebra.*

We now apply the above to a Λ-graded algebra $\mathcal{A}$ over $\mathbb{K}$. The multiplication algebra $\mathrm{Mult}(\mathcal{A})$ is a graded subalgebra of the Λ-graded algebra $\mathrm{grEnd}_{\mathbb{K}}\mathcal{A}$, and $\mathcal{A}$ is a Λ-graded $\mathrm{Mult}(\mathcal{A})$-module. Then since $\mathrm{Cent}(\mathcal{A}) = \mathrm{End}_{\mathrm{Mult}(\mathcal{A})}\mathcal{A}$, we have

$$\begin{aligned} \mathrm{grCent}(\mathcal{A}) &= \mathrm{grEnd}_{\mathrm{Mult}(\mathcal{A})}\mathcal{A} = \bigoplus_{\lambda \in \Lambda} \mathrm{Cent}(\mathcal{A})^\lambda, \quad \text{where} \\ \mathrm{Cent}(\mathcal{A})^\lambda &= \mathrm{Cent}(\mathcal{A}) \cap \mathrm{End}_{\mathbb{K}}(\mathcal{A})^\lambda. \end{aligned}$$

By Lemma 2.13 we see that

$$\mathrm{grCent}(\mathcal{A}) = \mathrm{Cent}(\mathcal{A}) \ \textit{ if } \mathcal{A} \textit{ is a finitely generated } \mathrm{Mult}(\mathcal{A})\textit{-module.} \tag{2.15}$$

The graded version of Schur's Lemma now yields the following result.

Proposition 2.16. *Let $\mathcal{A}$ be a Λ-graded $\mathbb{K}$-algebra that is graded-simple. Then* $\mathrm{grCent}(\mathcal{A}) = \mathrm{Cent}(\mathcal{A})$ *is a division-graded commutative associative algebra, i.e., a twisted group ring $\mathbb{E}^t[\Gamma]$ over an extension field $\mathbb{E} = \mathrm{Cent}(\mathcal{A})^0$ of $\mathbb{K}$ where $\Gamma = \{\lambda \in \Lambda \mid \mathrm{Cent}(\mathcal{A})^\lambda \neq 0\}$ is a subgroup of Λ. Moreover, for every nonzero homogeneous $a \in \mathcal{A}$, the evaluation map*

$$\mathtt{ev}_a : \mathrm{Cent}(\mathcal{A}) \to \mathcal{A}, \quad \chi \mapsto \chi(a)$$

is injective and has degree equal to the degree of a.

Proof. By assumption, $\mathcal{A}$ is a graded-irreducible $\mathrm{Mult}(\mathcal{A})$-module. From Lemma 2.14, we know that $\mathrm{grCent}(\mathcal{A}) = \mathrm{Cent}(\mathcal{A})$ is a division-graded associative algebra. It is commutative, since $\mathcal{A}^{(1)}$ is a graded ideal and hence $\mathcal{A}$ is perfect. The map $\mathtt{ev}_a$ is injective since the $\mathrm{Mult}(\mathcal{A})$-submodule generated by a is all of $\mathcal{A}$, i.e., $\mathrm{Mult}(\mathcal{A}).a = \mathcal{A}$. □

2.3. **Centroids of tensor products and loop algebras.** In the following all tensor products will be over a unital commutative associative ring $\mathbb{K}$. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbb{K}$-algebras. There exists a unique $\mathbb{K}$-algebra structure on $\mathcal{A} \otimes \mathcal{B}$ satisfying $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1a_2) \otimes (b_1b_2)$ for $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$. Also, for $f \in \mathrm{End}_{\mathbb{K}}(\mathcal{A})$ and $g \in \mathrm{End}_{\mathbb{K}}(\mathcal{B})$ there exists a unique map $f\widetilde{\otimes}g \in \mathrm{End}_{\mathbb{K}}(\mathcal{A} \otimes \mathcal{B})$ such that $(f\widetilde{\otimes}g)(a \otimes b) = f(a) \otimes g(b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. The map $f\widetilde{\otimes}g$ should not be confused with the element $f \otimes g$ of the tensor product $\mathrm{End}_{\mathbb{K}}(\mathcal{A}) \otimes \mathrm{End}_{\mathbb{K}}(\mathcal{B})$. Of course, one has a canonical map

$$\omega : \mathrm{End}_{\mathbb{K}}(\mathcal{A}) \otimes \mathrm{End}_{\mathbb{K}}(\mathcal{B}) \to \mathrm{End}_{\mathbb{K}}(\mathcal{A} \otimes \mathcal{B}) : \ f \otimes g \mapsto f\widetilde{\otimes}g, \tag{2.17}$$

It is straightforward to see that if $\chi_{\mathcal{A}} \in \mathrm{Cent}(\mathcal{A})$ and $\chi_{\mathcal{B}} \in \mathrm{Cent}(\mathcal{B})$ then $\chi_{\mathcal{A}}\widetilde{\otimes}\chi_{\mathcal{B}} \in \mathrm{Cent}(\mathcal{A} \otimes \mathcal{B})$, and so $\mathrm{Cent}(\mathcal{A})\widetilde{\otimes}\mathrm{Cent}(\mathcal{B}) \subseteq \mathrm{Cent}(\mathcal{A} \otimes \mathcal{B})$ where $\mathrm{Cent}(\mathcal{A})\widetilde{\otimes}\mathrm{Cent}(\mathcal{B})$ is the $\mathbb{K}$-span of all endomorphisms $\chi_{\mathcal{A}}\widetilde{\otimes}\chi_{\mathcal{B}}$.

We will say that $\chi \in \mathrm{Cent}(\mathcal{A} \otimes \mathcal{B})$ *has finite $\mathcal{A}$-image* if for every $b \in \mathcal{B}$ there exist finitely many $b_1, \dots, b_n \in \mathcal{B}$ such that $\chi(\mathcal{A} \otimes \mathbb{K}b) \subseteq \mathcal{A} \otimes \mathbb{K}b_1 + \cdots + \mathcal{A} \otimes \mathbb{K}b_n$. It is easily seen that

$$\mathrm{Cent}(\mathcal{A})\widetilde{\otimes}\mathrm{Cent}(\mathcal{B}) \subseteq \{\chi \in \mathrm{Cent}(\mathcal{A} \otimes \mathcal{B}) \mid \chi \text{ has finite } \mathcal{A}\text{-image}\}. \tag{2.18}$$

Proposition 2.19. *Let $\mathcal{A}$ be a perfect $\mathbb{K}$-algebra and let $\mathcal{B}$ be a unital $\mathbb{K}$-algebra. Then*

(a) *$\mathcal{A} \otimes \mathcal{B}$ is perfect, and $\chi \in \mathrm{Cent}(\mathcal{A} \otimes \mathcal{B})$ has finite $\mathcal{A}$-image as soon as $\chi(\mathcal{A} \otimes 1) \subseteq \mathcal{A} \otimes \mathbb{K}b_1 + \cdots + \mathcal{A} \otimes \mathbb{K}b_n$ for suitable $b_i \in \mathcal{B}$.*

(b) *Every* $\chi \in \mathrm{Cent}(\mathcal{A}\otimes\mathcal{B})$ *has finite* $\mathcal{A}$*-image if* $\mathcal{B}$ *is free as a* $\mathbb{K}$*-module and either of the following conditions holds:*

(b.1) $\mathcal{A}$ *is finitely generated as a* $\mathrm{Mult}(\mathcal{A})$*- or as a* $\mathrm{Cent}(\mathcal{A})$*-module, or*

(b.2) $\mathbb{K}$ *is a domain, and* $\mathcal{A}$ *is central and a torsion-free* $\mathbb{K}$*-module.*

(c) *If* $\mathrm{Cent}(\mathcal{A})$ *and* $\mathcal{B}$ *are a free* $\mathbb{K}$*-modules and the map* ω *of (2.17) is injective, then*

$$\mathrm{Cent}(\mathcal{A})\widetilde{\otimes}\mathrm{Cent}(\mathcal{B}) = \{\chi \in \mathrm{Cent}(\mathcal{A}\otimes\mathcal{B}) \mid \chi \textit{ has finite } \mathcal{A}\textit{-image}\}.$$

Proof. (a) Since $\mathcal{A}$ is perfect, any $a\otimes b \in \mathcal{A}\otimes\mathcal{B}$ can be written as a finite sum $a\otimes b = \sum_i (a_i' a_i'')\otimes b = \sum_i (a_i'\otimes b)(a_i''\otimes 1)$, where 1 denotes the identity element of $\mathcal{B}$. Hence $\mathcal{A}$ is perfect, and for χ as in the statement of (a) we have $\chi(a\otimes b) = \sum_i (a_i'\otimes b)\chi(a_i''\otimes 1) \in \sum_j \mathcal{A}\otimes\mathbb{K}bb_j$.

(b.1) Set $\mathcal{M} = \mathrm{Mult}(\mathcal{A})$, and observe that $\mathcal{M}\otimes\mathrm{id} \subseteq \mathrm{Mult}(\mathcal{A}\otimes\mathcal{B})$ since $\mathcal{B}$ is unital. Suppose $\mathcal{A} = \mathcal{M}a_1 + \cdots + \mathcal{M}a_n$ for $a_1,\ldots,a_n \in \mathcal{A}$ and fix $\chi \in \mathrm{Cent}(\mathcal{A}\otimes\mathcal{B})$ and $b \in \mathcal{B}$. There exist finite families $(a_{ij}) \subseteq \mathcal{A}$ and $(b_{ij}) \subseteq \mathcal{B}$ such that $\chi(a_i\otimes b) = \sum_j a_{ij}\otimes b_{ij}$ for $1 \le i \le n$, and hence $\chi(\mathcal{A}\otimes b) = \sum_{i,j} \chi\big((\mathcal{M}\widetilde{\otimes}\,\mathrm{id})(a_i\otimes b)\big) = \sum_{i,j}(\mathcal{M}\widetilde{\otimes}\,\mathrm{id})(a_{ij}\otimes b_{ij}) \subseteq \sum_{i,j} \mathcal{A}\otimes b_{ij}$.

By (a) the centroid $\mathrm{Cent}(\mathcal{A}\otimes\mathcal{B})$ is commutative. The same argument as above with $\mathcal{M}$ replaced by $\mathrm{Cent}(\mathcal{A})$ then shows that every $\chi \in \mathrm{Cent}(\mathcal{A}\otimes\mathcal{B})$ has finite $\mathcal{A}$-image if $\mathcal{A}$ is a finitely generated $\mathrm{Cent}(\mathcal{A})$-module.

(b.2) and (c): (This part of the proof is inspired by [BM, Lem. 1.2].) Let $\{b_r\}_{r\in\mathfrak{R}}$ be a basis of $\mathcal{B}$, and let $\chi \in \mathrm{Cent}(\mathcal{A}\otimes\mathcal{B})$. We define $\chi_r \in \mathrm{End}_{\mathbb{K}}(\mathcal{A})$ by

$$\chi(a\otimes 1) = \sum_{r\in\mathfrak{R}} \chi_r(a)\otimes b_r\,. \tag{2.20}$$

For $a_1, a_2 \in \mathcal{A}$, we then have

$$\begin{aligned}
\chi(a_1a_2\otimes 1) &= \sum_{r\in\mathfrak{R}} \chi_r(a_1a_2)\otimes b_r \\
&= \chi\big((a_1\otimes 1)(a_2\otimes 1)\big) = \big(\chi(a_1\otimes 1)\big)(a_2\otimes 1) \\
&= \sum_{r\in\mathfrak{R}} \chi_r(a_1)a_2\otimes b_r \\
&= (a_1\otimes 1)\chi(a_2\otimes 1) = \sum_{r\in\mathfrak{R}} a_1\chi_r(a_2)\otimes b_r,
\end{aligned}$$

whence $\chi_r(a_1a_2) = \chi_r(a_1)a_2 = a_1\chi_r(a_2)$ for all $a_i \in \mathcal{A}$, so all $\chi_r \in \mathrm{Cent}(\mathcal{A})$. We can now finish the proof of (b.2): By assumption there exist scalars $x_r \in \mathbb{K}$ such that $\chi_r = x_r\,\mathrm{id}$, hence $\chi(a\otimes 1) = \sum_{r\in\mathfrak{R}} x_r a\otimes b_r$. Fix a

non-zero $a \in \mathcal{A}$. Since almost all $x_r a = 0$, the torsion-freeness of $\mathcal{A}$ implies that almost all $x_r = 0$. Thus, there exists a finite subset $\mathfrak{F}$ of $\mathfrak{R}$ such that $\chi(a' \otimes 1) = \sum_{r\in\mathfrak{F}} x_r a' \otimes b_r$ holds for all $a' \in \mathcal{A}$. By (a), this shows that χ has finite $\mathcal{A}$-image.

We continue with the proof of (c). Because of (2.18), we only need to prove the inclusion from right to left. So suppose that $\chi \in \mathrm{Cent}(\mathcal{A} \otimes \mathcal{B})$ has finite $\mathcal{A}$-image. Then there exists a finite subset $\mathfrak{F} \subseteq \mathfrak{R}$ such that (2.20) becomes

$$\chi(a \otimes 1) = \sum_{r\in\mathfrak{F}} \chi_r(a) \otimes b_r \,.$$

For $a_1, a_2 \in \mathcal{A}$ and $b \in \mathcal{B}$, we then get $\chi(a_1 a_2 \otimes b) = \chi\big((a_1 \otimes 1)(a_2 \otimes b)\big) = \big(\chi(a_1 \otimes 1)\big)(a_2 \otimes b) = \sum_{r\in\mathfrak{F}} \chi_r(a_1)a_2 \otimes b_r b = \sum_{r\in\mathfrak{F}} \chi_r(a_1 a_2) \otimes b_r b$. Since $\mathcal{A}$ is perfect, this implies

$$\chi = \sum_{r\in\mathfrak{F}} \chi_r \widetilde{\otimes} \lambda_r \tag{2.21}$$

where λ_r is the left multiplication in $\mathcal{B}$ by b_r.

Let $\{\psi_s \mid s \in \mathfrak{S}\}$ be a $\mathbb{K}$-basis of $\mathrm{Cent}(\mathcal{A})$. Then there exists a finite subset $\mathfrak{T} \subseteq \mathfrak{S}$ and scalars $x_{rs} \in \mathbb{K}$ $(r \in \mathfrak{F}, s \in \mathfrak{T})$ such that $\chi_r = \sum_{s\in\mathfrak{T}} x_{rs}\psi_s$. We then get $\chi = \sum_{r\in\mathfrak{F}}\sum_{s\in\mathfrak{T}} x_{rs}\psi_s \widetilde{\otimes} \lambda_s = \sum_{s\in\mathfrak{T}} \psi_s \widetilde{\otimes} \phi_s$ for $\phi_s = \sum_{r\in\mathfrak{F}} x_{rs}\lambda_r$, and it remains to show that $\phi_s \in \mathrm{Cent}(\mathcal{B})$. For $a_i \in \mathcal{A}$, $b_i \in \mathcal{B} (i = 1, 2)$ we have

$$\begin{aligned}
\sum_{s\in\mathfrak{T}} \psi_s(a_1 a_2) \otimes \phi_s(b_1 b_2) &= \chi(a_1 a_2 \otimes b_1 b_2) = \big(\chi(a_1 \otimes b_1)\big)(a_2 \otimes b_2) \\
&= \sum_{s\in\mathfrak{T}} \psi_s(a_1)a_2 \otimes \phi_s(b_1)b_2 \\
&= \sum_{s\in\mathfrak{T}} \psi_s(a_1 a_2) \otimes \phi_s(b_1)b_2 \,.
\end{aligned}$$

Because $\mathcal{A}^{(1)} = \mathcal{A}$, this implies $\sum_{s\in\mathfrak{T}} \psi_s(a) \otimes (\phi_s(b_1 b) - \phi_s(b_1)b) = 0$ for all $a \in \mathcal{A}$ and $b, b_1 \in \mathcal{B}$, and then $\sum_{s\in\mathfrak{T}} \psi_s \widetilde{\otimes} \mu_s = 0$ where $\mu_s \in \mathrm{End}_{\mathbb{K}}(\mathcal{B})$ is defined as $\mu_s(b) = \phi_s(b_1 b) - \phi_s(b_1)b$. Since by assumption ω is injective, we also have $\sum_{s\in\mathfrak{T}} \psi_s \otimes \mu_s = 0$. So by the linear independence of the ψ_s, we see that $\mu_s = 0$. But then $\psi_s \in \mathrm{Cent}(\mathcal{B})$ follows, and hence $\chi \in \mathrm{Cent}(\mathcal{A}) \widetilde{\otimes} \mathrm{Cent}(\mathcal{B})$. $\square$

Remark 2.22. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbb{K}$-algebras such that $\mathcal{A}$ is a finitely generated $\mathrm{Mult}(\mathcal{A})$-module and $\mathcal{B}$ is unital commutative associative $\mathbb{K}$-algebra which

is free as a $\mathbb{K}$-module. It is shown in [ABP2, Lem. 2.3] that then

$$\mathrm{Cent}(\mathcal{A}) \otimes \mathrm{Cent}(\mathcal{B}) \xrightarrow{\cong} \mathrm{Cent}(\mathcal{A}) \widetilde{\otimes} \mathrm{Cent}(\mathcal{B}) = \mathrm{Cent}(\mathcal{A} \otimes \mathcal{B}).$$

Indeed, equation (2.20), (which in fact holds even if $\mathcal{A}$ is not perfect), together with (b.1) of Proposition 2.19 shows that $\mathrm{Cent}(\mathcal{A}) \widetilde{\otimes} \mathrm{Cent}(\mathcal{B}) = \mathrm{Cent}(\mathcal{A} \otimes \mathcal{B})$. The isomorphism $\mathrm{Cent}(\mathcal{A}) \otimes \mathrm{Cent}(\mathcal{B}) \cong \mathrm{Cent}(\mathcal{A}) \widetilde{\otimes} \mathrm{Cent}(\mathcal{B})$ follows from the linear independence of the set $(b_r)_{r \in \mathfrak{R}}$.

From now on, unless explicitly stated otherwise, all algebras will be over some field $\mathbb{F}$. For easier reference we state the following consequence of Proposition 2.19.

Corollary 2.23. *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras over a field $\mathbb{F}$ such that $\mathcal{A}$ is perfect and $\mathcal{B}$ is unital. Then*

$$\begin{aligned} \mathrm{Cent}(\mathcal{A}) \otimes \mathrm{Cent}(\mathcal{B}) &\cong \mathrm{Cent}(\mathcal{A}) \widetilde{\otimes} \mathrm{Cent}(\mathcal{B}) \\ &= \{\chi \in \mathrm{Cent}(\mathcal{A} \otimes \mathcal{B}) \mid \chi \textit{ has finite } \mathcal{A}\textit{-image}\}. \end{aligned}$$

Moreover, we have

$$\mathrm{Cent}(\mathcal{A}) \widetilde{\otimes} \mathrm{Cent}(\mathcal{B}) = \mathrm{Cent}(\mathcal{A} \otimes \mathcal{B})$$

in either one of the following cases:

(a) *$\mathcal{A}$ is finitely generated as* $\mathrm{Mult}(\mathcal{A})$*-module, e.g.,* $\dim_{\mathbb{F}} \mathcal{A} < \infty$*, or*
(b) $\mathrm{Cent}(\mathcal{A}) = \mathbb{F}\,\mathrm{id}$*, in which case* $\mathrm{Cent}(\mathcal{A} \otimes \mathcal{B}) = \mathrm{id} \otimes \mathrm{Cent}(\mathcal{B})$.

Proof. Over fields the map ω is injective ([Bo1, Sec. 7.7, Prop. 16]). All assertions then follow from Proposition 2.19.

Remark 2.24. When $\mathfrak{g}$ is a simple Lie algebra with $\mathrm{Cent}(\mathfrak{g}) = \mathbb{F}\,\mathrm{id}$ (which is always true when $\mathbb{F}$ is algebraically closed and $\mathfrak{g}$ is finite-dimensional) Melville [Me, 3.2, 3.4, 3.5] has shown $\mathrm{Cent}(\mathfrak{g} \otimes \mathcal{B}) = \mathbb{F}\,\mathrm{id} \otimes \mathrm{Cent}(\mathcal{B})$ for $\mathcal{B} = \mathbb{F}[x_1, \ldots, x_n]$ a polynomial ring or an ideal $\mathcal{B} = t^m \mathbb{F}[t^n]$ of the polynomial ring $\mathbb{F}[t]$. Actually the proof given in [Me, 3.2] works for any unital commutative associative $\mathbb{F}$-algebra $\mathcal{B}$. Part (b) of Corollary 2.23 was proven by Allison-Berman-Pianzola ([ABP1, Lem. 4.2]) under the assumption that $\mathcal{B}$ is a unital commutative associative $\mathbb{F}$-algebra.

Example 2.25. Suppose $\mathcal{A}$ is the centreless Virasoro Lie algebra (often called the Witt algebra). Thus, $\mathcal{A}$ has a basis consisting of the elements $\{a_i \mid i \in \mathbb{Z}\}$ and multiplication given by $[a_i, a_j] = (j - i)a_{i+j}$. Suppose

$\chi \in \mathrm{Cent}(\mathcal{A})$. Then $j\chi(a_j) = \chi\big([a_0, a_j]\big) = [a_0, \chi(a_j)]$. Since $\{x \in \mathcal{A} \mid [a_0, x] = jx\} = \mathbb{F}a_j$, we see that $\chi(a_j) = \lambda_j a_j$ for some scalar λ_j. But then $\lambda_{i+j}(j-i)a_{i+j} = \chi\big([a_i, a_j]\big) = [a_i, \chi(a_j)] = (j-i)\lambda_j a_{i+j}$ (with $j = 0$) shows that $\lambda_i = \lambda_0$ for all i. Hence $\mathrm{Cent}(\mathcal{A}) = \mathbb{F}\,\mathrm{id}$ — this can also be derived easily from Proposition 3.1 below. Applying part (b) of the previous proposition, we obtain that $\mathrm{Cent}(\mathcal{A} \otimes \mathcal{B}) = \mathrm{id} \otimes \mathrm{Cent}(\mathcal{B})$ for any unital algebra $\mathcal{B}$.

Corollary 2.26. *Let $\mathcal{A}$ be a central, perfect $\mathbb{F}$-algebra, $\mathcal{B}$ be a unital commutative associative $\mathbb{F}$-algebra, and $\mathcal{C}$ be a unital subalgebra of $\mathcal{B}$ such that $\mathcal{B}$ is a free $\mathcal{C}$-module with a $\mathcal{C}$-basis containing the identity element 1 of $\mathcal{B}$. Suppose further that $\mathcal{L}$ is $\mathcal{B}/\mathcal{C}$-form of $\mathcal{A} \otimes_{\mathbb{F}} \mathcal{C}$, i.e., a $\mathcal{C}$-algebra $\mathcal{L}$ such that $\mathcal{L} \otimes_{\mathcal{C}} \mathcal{B} \cong (\mathcal{A} \otimes_{\mathbb{F}} \mathcal{C}) \otimes_{\mathcal{C}} \mathcal{B} \cong \mathcal{A} \otimes_{\mathbb{F}} \mathcal{B}$. Then* $\mathrm{Cent}_{\mathbb{F}}(\mathcal{L}) = \mathcal{C}\,\mathrm{id}$.

Proof. Since $\mathcal{B}$ is a free and hence faithfully flat $\mathcal{C}$-module, there exists a $\mathcal{C}$-subalgebra $\mathcal{L}'$ of $\mathcal{A} \otimes_{\mathbb{F}} \mathcal{B}$ which is isomorphic to $\mathcal{L}$ as a $\mathcal{C}$-algebra (see for example, [W, Ch. 17]). Therefore, we can assume that $\mathcal{L}$ is a subalgebra of the $\mathcal{C}$-algebra $\mathcal{A} \otimes_{\mathbb{F}} \mathcal{B}$. It follows from the assumptions that $\mathcal{C}\,\mathrm{id} \subseteq \mathrm{Cent}_{\mathbb{F}}(\mathcal{L})$, where $a \otimes b \mapsto a \otimes cb$ for all $c \in \mathcal{C}$. Conversely, let $\chi \in \mathrm{Cent}_{\mathbb{F}}(\mathcal{L})$ and extend χ to a $\mathbb{F}$-linear map $\widetilde{\chi}$ on $\mathcal{L} \otimes_{\mathcal{C}} \mathcal{B}$ by setting $\widetilde{\chi} = \chi \otimes \mathrm{id}$. Since $\mathcal{L} \otimes_{\mathcal{C}} \mathcal{B} \cong \mathcal{A} \otimes_{\mathbb{F}} \mathcal{B}$ and $\mathrm{Cent}_{\mathbb{F}}(\mathcal{A} \otimes_{\mathbb{F}} \mathcal{B}) \cong \mathrm{id} \otimes_{\mathbb{F}} \mathcal{B}$ by Corollary 2.23, it follows that there exists $c \in \mathcal{B}$ such that $\widetilde{\chi}$ is given by $x \otimes_{\mathcal{C}} b \mapsto x \otimes_{\mathcal{C}} cb$ for all $x \in \mathcal{L}$, $b \in \mathcal{B}$. Let $(b_i)_{i \in \mathcal{I}}$ be a $\mathcal{C}$-basis of $\mathcal{B}$ containing 1, say $b_0 = 1$. We can write c in the form $c = \sum_i c_i b_i$ for unique $c_i \in C$. Then $\widetilde{\chi}(x \otimes_{\mathcal{C}} 1) = x \otimes_{\mathcal{C}} c = \sum_i x \otimes_{\mathcal{C}} c_i b_i = \sum_i x c_i \otimes_{\mathcal{C}} b_i$. Since $\widetilde{\chi}(x \otimes_{\mathcal{C}} 1) = \chi(x) \otimes_{\mathcal{C}} 1 \in \mathcal{L} \otimes_{\mathcal{C}} 1$, it follows that $xc_i = 0$ for $i \neq 0$ and $\chi(x) = c_0 x$, i.e., $\chi = c_0\,\mathrm{id}$. Thus, $\mathrm{Cent}_{\mathbb{F}}(\mathcal{L}) \subseteq \mathcal{C}\,\mathrm{id}$, and we have the desired conclusion. $\square$

Remark 2.27. The assumptions on $\mathcal{B}$ and $\mathcal{C}$ are fulfilled for example when $\mathcal{B} = \mathbb{F}[t, t^{-1}]$, a Laurent polynomial ring, and $\mathcal{C} = \mathbb{F}[t^m, t^{-m}]$ for some positive $m \in \mathbb{N}$. Indeed, in this case $\{t^i \mid 0 \leq i < m\}$ is a $\mathcal{C}$-basis of $\mathcal{B}$ as required. Let $\mathcal{A}$ be a Lie algebra and σ be an automorphism of $\mathcal{A}$ of period m. Assume $\zeta \in \mathbb{F}$ is a primitive mth root of unity, and form the loop algebra $L(\mathcal{A}, \sigma) = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i \otimes_{\mathbb{F}} \mathbb{F}t^i$ where $\mathcal{A}_i$ is the ζ^i-eigenspace of σ. That $L(\mathcal{A}, \sigma)$ is indeed a $\mathcal{C}$-form of $\mathcal{A} \otimes_{\mathbb{F}} \mathcal{B}$ is shown by Allison-Berman-Pianzola [ABP1, Thm. 3.6]. In this particular situation, the result $\mathrm{Cent}(L(\mathcal{A}, \sigma)) = \mathcal{C}\,\mathrm{id}$ from Corollary 2.26 can be found in [ABP1, Lem. 4.3 (d)]. When $\mathcal{A}$ is taken to be a finite-dimensional split simple Lie algebra over a field $\mathbb{F}$ of characteristic 0, the loop algebra $L(\mathcal{A}, \sigma)$ is an example of a centreless core of an extended

affine Lie algebra. Later in Corollary 4.13, we will see that the cores of extended affine Lie algebras are always central.

Corollary 2.28. *Let $\mathcal{A}$ be an algebra over a ring $\mathbb{K}$ and set $\mathcal{C} = \mathrm{Cent}_{\mathbb{K}}(\mathcal{A})$.*

(a) *Every $\mathbb{K}$-linear automorphism f of $\mathcal{A}$ determines a $\mathbb{K}$-linear automorphism $f_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}, \quad \chi \mapsto f_{\mathcal{C}}(\chi) = f \circ \chi \circ f^{-1}$ of the associative $\mathbb{K}$-algebra $\mathcal{C}$. The map*

$$\gamma : \mathrm{Aut}_{\mathbb{K}}(\mathcal{A}) \to \mathrm{Aut}_{\mathbb{K}}(\mathcal{C}), \quad f \mapsto f_{\mathcal{C}}$$

is a group homomorphism whose kernel is $\mathrm{Aut}_{\mathcal{C}}(\mathcal{A})$, the $\mathcal{C}$-linear automorphisms of $\mathcal{A}$.

(b) *Let $\mathcal{A}$ be a perfect, central algebra over a field $\mathbb{F}$ and $\mathcal{B}$ be a unital commutative associative $\mathbb{F}$-algebra. Then, after identifying $\mathrm{Aut}_{\mathbb{F}}(\mathcal{B})$ with a subgroup of $\mathrm{Aut}_{\mathbb{F}}(\mathcal{A} \otimes_{\mathbb{F}} \mathcal{B})$ via $g \mapsto \mathrm{id} \otimes g$, we have*

$$\mathrm{Aut}_{\mathbb{F}}(\mathcal{A} \otimes_{\mathbb{F}} \mathcal{B}) = \mathrm{Aut}_{\mathcal{B}}(\mathcal{A} \otimes_{\mathbb{F}} \mathcal{B}) \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathcal{B}) \quad (\textit{semidirect product}).$$

Proof. (a) is straightforward. For (b) we note that $\mathrm{Cent}(\mathcal{A} \otimes_{\mathbb{F}} \mathcal{B}) = \mathrm{id} \otimes \mathcal{B} \cong \mathcal{B}$ by Corollary 2.23. Every $g \in \mathrm{Aut}_{\mathbb{F}}(\mathcal{B})$ extends to an automorphism $\xi(g) = \mathrm{id} \otimes g$ of $\mathcal{A} \otimes_{\mathbb{F}} \mathcal{B}$. The map $\xi : \mathrm{Aut}_{\mathbb{F}}(\mathcal{B}) \to \mathrm{Aut}_{\mathbb{F}}(\mathcal{A} \otimes_{\mathbb{F}} \mathcal{B})$ is a group homomorphism which satisfies $(\gamma \circ \xi)(g) = g$, i.e., ξ is a section of γ. The claim then follows from standard facts in group theory. $\square$

Remarks 2.29. For special choices of $\mathcal{A}$ and $\mathcal{B}$, the automorphism group $\mathrm{Aut}_{\mathbb{F}}(\mathcal{A} \otimes_{\mathbb{F}} \mathcal{B})$ has been investigated by several authors; for example, by Benkart-Osborn [BO] when $\mathcal{A}$ is the algebra of $n \times n$-matrices over $\mathbb{F}$ and $\mathcal{B}$ is an arbitrary algebra with an Artinian nucleus, or by Pianzola [Pi] when $\mathcal{A}$ is a finite-dimensional split simple Lie algebra over a field $\mathbb{F}$ of characteristic 0 and $\mathcal{B}$ is an integral domain with trivial Picard group and with a maximal ideal $\mathfrak{m}$ which satisfies $\mathcal{B}/\mathfrak{m} \cong \mathbb{F}$.

An analogous result holds for derivations; we leave the proof as an exercise to the reader.

Corollary 2.30. *Let $\mathcal{A}$ be an algebra over a ring $\mathbb{K}$ and set $\mathcal{C} = \mathrm{Cent}_{\mathbb{K}}(\mathcal{A})$.*

(a) *Every $\mathbb{K}$-linear derivation d of $\mathcal{A}$ determines a $\mathbb{K}$-linear derivation $d_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}, \quad \chi \mapsto d_{\mathcal{C}}(\chi) := [d, \chi] = d \circ \chi - \chi \circ d$ of $\mathcal{C}$. The map*

$$\delta : \mathrm{Der}_{\mathbb{K}}(\mathcal{A}) \to \mathrm{Der}_{\mathbb{K}}(\mathcal{C}), \quad d \mapsto d_{\mathcal{C}}$$

is a $\mathbb{K}$-linear Lie algebra homomorphism whose kernel is $\mathrm{Der}_{\mathcal{C}}(\mathcal{A})$*, the $\mathcal{C}$-linear derivations of $\mathcal{A}$.*

(b) *Let $\mathcal{A}$ be a perfect, central algebra over a field $\mathbb{F}$ and $\mathcal{B}$ be a unital commutative associative $\mathbb{F}$-algebra. Then, after identifying* $\mathrm{Der}_{\mathbb{F}}(\mathcal{B})$ *with a subalgebra of* $\mathrm{Der}_{\mathbb{F}}(\mathcal{A}\otimes_{\mathbb{F}}\mathcal{B})$ *via $e\mapsto \mathrm{id}\otimes e$, we have*

$$\mathrm{Der}_{\mathbb{F}}(\mathcal{A}\otimes_{\mathbb{F}}\mathcal{B}) = \mathrm{Der}_{\mathcal{B}}(\mathcal{A}\otimes_{\mathbb{F}}\mathcal{B}) \rtimes \mathrm{Der}_{\mathbb{F}}(\mathcal{B}) \quad (\textit{semidirect product}).$$

Remarks 2.31. The result in Corollary 2.30(b) complements [BM, Thm. 1], which describes $\mathrm{Der}_{\mathbb{F}}(\mathcal{A}\otimes_{\mathbb{F}}\mathcal{B})$ when $\mathcal{A}$ is a finite-dimensional perfect $\mathbb{F}$-algebra and $\mathcal{B}$ is as above:

$$\mathrm{Der}_{\mathbb{F}}(\mathcal{A}\otimes_{\mathbb{F}}\mathcal{B}) = \big(\mathrm{Der}_{\mathbb{F}}(\mathcal{A})\otimes_{\mathbb{F}}\mathcal{B}\big) \ \oplus\ \big(\mathrm{Cent}_{\mathbb{F}}(\mathcal{A})\otimes_{\mathbb{F}}\mathrm{Der}_{\mathbb{F}}(\mathcal{B})\big). \tag{2.32}$$

We note that (2.32) is not a semidirect product in general.

3. Centroids of Lie algebras with toral subalgebras

3.1. **A general result.** In this section, $\mathcal{L}$ is a Lie algebra over some field $\mathbb{F}$, which will be assumed of characteristic 0 from Section 3.2 on. Recall that a subalgebra $\mathfrak{h}$ is a *toral subalgebra* of $\mathcal{L}$ if $\mathcal{L} = \bigoplus_{\alpha\in\mathfrak{h}^*}\mathcal{L}_\alpha$, where $\mathcal{L}_\alpha = \{x\in\mathcal{L} \mid [h,x]=\alpha(h)x \text{ for all } h\in\mathfrak{h}\}$. Necessarily $\mathfrak{h}$ is abelian, and $[\mathcal{L}_\alpha\,,\,\mathcal{L}_\beta]\subseteq\mathcal{L}_{\alpha+\beta}$ for $\alpha,\beta\in\mathfrak{h}^*$. If $\mathcal{L}_\alpha\neq 0$, then α is a *weight* (relative to $\mathfrak{h}$) and $\mathrm{supp}\,\mathcal{L} = \{\alpha\in\mathfrak{h}^* \mid \mathcal{L}_\alpha\neq 0\}$ is the set of weights.

Proposition 3.1. *Let $\mathcal{L}$ be a Lie algebra with a toral subalgebra $\mathfrak{h}$.*

(a) *If $\chi\in\mathrm{Cent}(\mathcal{L})$, then $\chi(\mathcal{L}_\alpha)\subseteq\mathcal{L}_\alpha$ for all $\alpha\in\mathfrak{h}^*$, $\chi(\mathfrak{h})\subseteq Z(\mathcal{L}_0)$, and χ is uniquely determined by its restriction to $\mathcal{L}_0$.*

(b) *Let $\mathcal{J}$ be a $\mathrm{Cent}(\mathcal{L})$-invariant ideal, and suppose there exists $0\neq\alpha\in\mathfrak{h}^*$ such that $\dim(\mathcal{J}\cap\mathcal{L}_\alpha)=1$ and the ideal of $\mathcal{L}$ generated by $\mathcal{J}\cap\mathcal{L}_\alpha$ is $\mathcal{J}$. Then*

$$\begin{aligned}\mathrm{Cent}(\mathcal{L}) &= \mathbb{F}\,\mathrm{id}\oplus\{\psi\in\mathrm{Cent}(\mathcal{L})\mid\psi(\mathcal{J})=0\}\\ &\cong \mathbb{F}\,\mathrm{id}\oplus\mathrm{Hom}_{\mathcal{L}/\mathcal{J}}(\mathcal{L}/\mathcal{J},\mathcal{C}_{\mathcal{L}}(\mathcal{J}))\end{aligned}\tag{3.2}$$

where $\mathcal{C}_{\mathcal{L}}(\mathcal{J})$ is the centralizer of $\mathcal{J}$ in $\mathcal{L}$. In particular, if $\mathcal{J}=\mathcal{L}^{(1)}$ satisfies the assumptions above, then

$$\begin{aligned}\mathrm{Cent}(\mathcal{L}) &= \mathbb{F}\,\mathrm{id}\oplus\{\psi\in\mathrm{End}_{\mathbb{F}}(\mathcal{L})\mid[\psi(\mathcal{L}),\mathcal{L}]=0=\psi(\mathcal{L}^{(1)})\} \qquad (3.3)\\ &= \mathbb{F}\,\mathrm{id}\oplus\mathcal{V}(\mathcal{L}^{(1)})\\ &\cong \mathbb{F}\,\mathrm{id}\oplus\mathrm{Hom}_{\mathbb{F}}\big(\mathcal{L}/\mathcal{L}^{(1)},\mathcal{C}_{\mathcal{L}}(\mathcal{L}^{(1)})\big).\end{aligned}$$

Proof. (a) For all $h \in \mathfrak{h}$ and $x_\alpha \in \mathcal{L}_\alpha$ we have $\alpha(h)\chi(x_\alpha) = \chi[h, x_\alpha] = [h, \chi(x_\alpha)]$ which implies $\chi(\mathcal{L}_\alpha) \subseteq \mathcal{L}_\alpha$. Also $[\chi(\mathfrak{h}), \mathcal{L}_0] = \chi[\mathfrak{h}, \mathcal{L}_0] = 0$, proving $\chi(\mathfrak{h})$ is central in $\mathcal{L}_0$. For $0 \neq \alpha$ there exists $t_\alpha \in \mathfrak{h}$ such that $\alpha(t_\alpha) = 1$. The last claim then follows from $\chi(x_\alpha) = \chi[t_\alpha, x_\alpha] = [\chi(t_\alpha), x_\alpha]$.

(b) Suppose $\chi \in \mathrm{Cent}(\mathcal{L})$. Our assumptions imply that there exists a scalar ξ such that $\chi\,|_{\mathcal{J}\cap\mathcal{L}_\alpha} = \xi\,\mathrm{id}$. Thus $\mathcal{J} \cap \mathcal{L}_\alpha$ is contained in the kernel of $\psi = \chi - \xi\,\mathrm{id}$, which is an ideal of $\mathcal{L}$ since $\psi \in \mathrm{Cent}(\mathcal{L})$. As $\mathcal{J} \cap \mathcal{L}_\alpha$ generates $\mathcal{J}$, we have $\psi\,|_{\mathcal{J}} = 0$. This proves $\mathrm{Cent}(\mathcal{L})$ is contained in the right-hand side. The other direction is obvious. The second part of (3.2) and the statement concerning $\mathcal{L}^{(1)}$ follow from Lemma 2.1. □

Corollary 3.4. *Let $\mathcal{L}$ be a Lie algebra with a toral subalgebra $\mathfrak{h}$ and suppose that $\mathcal{L}^{(1)}$ is generated by elements $e_i, f_i, (1 \leq i \leq n)$ such that the following conditions hold:*

(i) *$\alpha_i^\vee = [e_i, f_i] \in \mathfrak{h}$, and these elements act on the generators e_j, f_j by*

$$[\alpha_i^\vee, e_j] = a_{i,j} e_j \quad \textit{and} \quad [\alpha_i^\vee, f_j] = -a_{i,j} f_j \quad (1 \leq i, j \leq n)$$

where $a_{i,j} \in \mathbb{F}$.

(ii) *The matrix $\mathfrak{A} = (a_{i,j})$ is indecomposable in the sense that after possibly renumbering the indices we have $a_{1,2}a_{2,3}\cdots a_{n-1,n} \neq 0$. Moreover, $a_{i,i} \neq 0$ for all $i = 1, \ldots, n$.*

(iii) *For some i and some $\alpha \in \mathfrak{h}^*$, we have $\mathcal{L}_\alpha = \mathbb{F}e_i$.*

Then $\mathrm{Cent}(\mathcal{L}) \cong \mathbb{F}\,\mathrm{id} \oplus \mathrm{Hom}_{\mathbb{F}}\big(\mathcal{L}/\mathcal{L}^{(1)}, \mathcal{C}_{\mathcal{L}}(\mathcal{L}^{(1)})\big)$. *In particular, $\mathcal{L}$ is central if $\mathcal{L} = \mathcal{L}^{(1)}$ or $\mathcal{C}_{\mathcal{L}}(\mathcal{L}^{(1)}) = 0$.*

Proof. Let i be as in (iii), and note that $\alpha \neq 0$ since $\alpha(\alpha_i^\vee) = a_{i,i} \neq 0$. By Proposition 3.1 (b), it suffices to show that the ideal $\mathcal{J}$ of $\mathcal{L}$ generated by e_i contains all the generators e_j, f_j of $\mathcal{L}^{(1)}$. This follows from the indecomposibility of the matrix $\mathfrak{A}$ by upward and downward induction starting from i. Indeed, assume $e_j \in \mathcal{J}$. Relation (i) implies that $\alpha_j^\vee \in \mathcal{J}$ and then, since $a_{j,j} \neq 0$, that $f_j \in \mathcal{J}$. Moreover, $e_{j+1} = a_{j,j+1}^{-1}[\alpha_j^\vee, e_{j+1}] \in \mathcal{J}$ and similarly $f_{j+1} \in \mathcal{J}$. □

3.2. **Centroids of Kac-Moody algebras.** From now on we assume $\mathbb{F}$ is a field of characteristic 0. Corollary 3.4 applies to contragredient Lie algebras over $\mathbb{F}$ in the sense of [MP, Ch. 4]. In particular, it applies to Kac-Moody algebras. We will elaborate on this important special case.

Assume $\mathfrak{A} := (a_{i,j})_{i,j=1}^n$ is a generalized Cartan matrix of rank ℓ. Thus, $a_{i,i} = 2$ for $i = 1, \dots, n$; the entries $a_{i,j}$ for $i \neq j$ are nonpositive integers; and $a_{i,j} = 0$ if and only if $a_{j,i} = 0$. We will assume that the matrix $\mathfrak{A}$ is indecomposable as in Corollary 3.4(ii). A realization of $\mathfrak{A}$ is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ consisting of an $\mathbb{F}$-vector space $\mathfrak{h}$ and linearly independent subsets $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ such that $\langle \alpha_i \,|\, \alpha_j^\vee \rangle = a_{j,i}$ $(1 \leq i, j \leq n)$, and $\dim \mathfrak{h} = 2n - \ell$. The *Kac-Moody Lie algebra* $\mathfrak{g} := \mathfrak{g}(\mathfrak{A})$ associated to $\mathfrak{A}$ is the Lie algebra over $\mathbb{F}$ with generators $e_i, f_i, (1 \leq i \leq n)$ and $\mathfrak{h}$, which satisfy the defining relations,

(a) $[e_i, f_j] = \delta_{i,j} \alpha_i^\vee \qquad 1 \leq i, j \leq n$;
(b) $[h, h'] = 0 \qquad h, h' \in \mathfrak{h}$;
(c) $[h, e_i] = \langle \alpha_i \,|\, h \rangle e_i$ and $[h, f_i] = -\langle \alpha_i \,|\, h \rangle f_i \qquad h \in \mathfrak{h}, \ i = 1, \dots, n$;
(d) $(\operatorname{ad} e_i)^{1-a_{i,j}} e_j = 0 = (\operatorname{ad} f_i)^{1-a_{i,j}} f_j \qquad 1 \leq i \neq j \leq n$.

The Lie algebra $\mathfrak{g}$ is graded by the root lattice $Q := \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$, so that $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \langle \alpha \,|\, h \rangle x \text{ for all } h \in \mathfrak{h}\}$ are the root (weight) spaces of the toral subalgebra $\mathfrak{h}$. Condition (i) of Corollary 3.4 follows from relations (a) and (c) above, while (ii) holds since $a_{i,i} = 2$ for all i and since the Cartan matrix $\mathfrak{A}$ is assumed to be indecomposable. Finally, (iii) is well-known, see e.g. [K, (1.3.3)]. Therefore, we obtain

Corollary 3.5. *Let $\mathfrak{g} = \mathfrak{g}(\mathfrak{A})$ be the Kac-Moody Lie algebra corresponding to the indecomposable generalized Cartan matrix $\mathfrak{A}$. Then $\mathfrak{g}^{(1)}$ is central and* $\operatorname{Cent}(\mathfrak{g}) = \mathbb{F}\,\mathrm{id} \oplus \operatorname{Hom}_{\mathbb{F}}\big(\mathfrak{g}/\mathfrak{g}^{(1)}, \mathcal{C}_{\mathfrak{g}}(\mathfrak{g}^{(1)})\big)$.

Remarks 3.6. When $\mathfrak{A}$ is invertible, $\dim \mathfrak{h} = n$, $\mathfrak{h} = \bigoplus_{i=1}^n \mathbb{F}\alpha_i^\vee$, and $\mathfrak{g} = \mathfrak{g}^{(1)}$, so that $\operatorname{Cent}(\mathfrak{g}) = \mathbb{F}\,\mathrm{id}$ in that case.

When $\mathfrak{A}$ is an affine Cartan matrix (associated to an affine Dynkin diagram), then $\mathfrak{A}$ has rank $n - 1$. The center is one-dimensional, spanned by c say. We may suppose $\mathfrak{h} = \mathfrak{h}' \oplus \mathbb{C}d$ in this case, where $\mathfrak{h}' = \bigoplus_{i=1}^n \mathbb{F}\alpha_i^\vee$ and $\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathbb{F}d$. (Readers familiar with affine algebras will recognize d as the degree derivation.) Corollary 3.5 then says that $\mathfrak{g}^{(1)}$ is central and $\operatorname{Cent}(\mathfrak{g}) = \mathbb{F}\,\mathrm{id} \oplus \operatorname{Hom}_{\mathbb{F}}\big(\mathbb{F}d, \mathbb{F}c\big)$ for each affine Kac-Moody Lie algebra.

3.3. **Centroids of Lie tori.** First we introduce the notion of a root graded Lie algebra. We prove a result concerning the centroid of a special class of root graded Lie algebras but postpone giving the precise description of

the centroid for general root graded Lie algebras until Section 5. We then specialize considerations to certain root graded Lie algebras called Lie tori and describe their centroids. Our rationale for doing this is that Lie tori play a critical role in the theory of extended affine Lie algebras (EALAs) – they are precisely the cores of EALAs. The centroid of the core is another key ingredient in the structure of the EALA.

Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra over a field $\mathbb{F}$ of characteristic 0 with root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_{\mathfrak{g}}} \mathfrak{g}_\alpha$ relative to a split Cartan subalgebra $\mathfrak{h}$. Such a Lie algebra is the $\mathbb{F}$-analogue of a finite-dimensional complex Lie algebra, and the set of roots $\Delta_{\mathfrak{g}}$ is the finite reduced root system. Every finite irreducible root system Δ is one of the reduced root systems $\Delta_{\mathfrak{g}}$ or is a nonreduced root system BC_r (see for example, [Bo3, Sec. 1.1]).

Definition 3.7. Let $\mathcal{L}$ be a Lie algebra over a field $\mathbb{F}$ of characteristic 0, and let Δ be a finite irreducible root system. Then $\mathcal{L}$ is said to be *graded by the root system* Δ or to be Δ-*graded* if

(i) $\mathcal{L}$ contains as a subalgebra a finite-dimensional split "simple" Lie algebra $\mathfrak{g}$, called the *grading subalgebra*, with split Cartan subalgebra $\mathfrak{h}$;

(ii) $\mathfrak{h}$ is a toral subalgebra of $\mathcal{L}$, and the weights of $\mathcal{L}$ relative to $\mathfrak{h}$ are in $\Delta \cup \{0\}$,
$$\mathcal{L} = \bigoplus_{\alpha\in\Delta\cup\{0\}} \mathcal{L}_\alpha$$

(iii) $\mathcal{L}_0 = \sum_{\alpha\in\Delta} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$;

(iv) either Δ is reduced and equals the root system $\Delta_{\mathfrak{g}}$ of $(\mathfrak{g}, \mathfrak{h})$ or $\Delta = \mathrm{BC}_r$ and $\Delta_{\mathfrak{g}}$ is of type B_r, C_r, or D_r.

The word simple is in quotes, because in all instances except two, $\mathfrak{g}$ is a simple Lie algebra. The sole exceptions are when Δ is of type BC_2, $\Delta_{\mathfrak{g}}$ is of type $\mathrm{D}_2 = \mathrm{A}_1 \times \mathrm{A}_1$, and $\mathfrak{g}$ is the direct sum of two copies of $\mathfrak{sl}_2$; or when Δ is of type BC_1, $\Delta_{\mathfrak{g}}$ is of type D_1, and $\mathfrak{g} = \mathfrak{h}$ is one-dimensional.

The definition above is due to Berman-Moody [BM] for the case $\Delta = \Delta_{\mathfrak{g}}$. The extension to the nonreduced root systems was developed by Allison-Benkart-Gao in [ABG2]. The Δ-graded Lie algebras for Δ reduced have been classified up to central extensions by Tits [T] for $\Delta = \mathrm{A}_1$ (see also [BZ]); by Berman-Moody [BeM] for $\Delta = \mathrm{A}_r$, $(r \geq 2)$, D_r, E_6, E_7, and E_8;

by Benkart-Zelmanov [BZ] for $\Delta = \mathrm{B}_r$, C_r, F_4 and G_2 (see also[ABG1] for C_r); and by Neher [N1], who studied Lie algebras 3-graded by a locally finite root system Δ, which in our setting means $\Delta \neq \mathrm{E}_8$, F_4, G_2, or BC_r. Central extensions of Lie algebras graded by reduced root systems have been described by Allison-Benkart-Gao in [ABG1]. The Lie algebras graded by the root systems BC_r have been classified in [ABG2] for $r \geq 2$ and in [BS] for $r = 1$. As a result, the Δ-graded Lie algebras are determined completely up to isomorphism.

Examples of Lie algebras graded by a (not necessarily reduced) Δ include the affine Kac-Moody Lie algebras, the toroidal Lie algebras, the intersection matrix Lie algebras introduced by Slodowy in his study of singularities, and the cores of EALAs, to name just a few (see [ABG2, Exs. 1.16-1.23] for further discussion of examples). Any finite-dimensional simple Lie algebra over a field $\mathbb{F}$ of characteristic 0 which has an ad-nilpotent element (or equivalently, by the Jacobson-Morozov theorem has a copy of $\mathfrak{sl}_2$) is graded by a finite root system (see [Se]). Thus, the notion encompasses a diverse array of important Lie algebras.

Definition 3.8. Let $\mathcal{L}$ be a Δ-graded Lie algebra with grading subalgebra $\mathfrak{g}$, and let Λ be an abelian group. We say that $\mathcal{L}$ is (Δ, Λ)-*graded* if $\mathcal{L}$ has a Λ-grading

$$\mathcal{L} = \bigoplus_{\lambda \in \Lambda} \mathcal{L}^\lambda \quad \text{with} \quad \mathfrak{g} \subseteq \mathcal{L}^0. \tag{3.9}$$

This notion was introduced and studied by Yoshii in [Y1] where it was termed a *refined root grading of type* (Δ, Λ). In any (Δ, Λ)-graded Lie algebra, each space $\mathcal{L}^\lambda$ is an $\mathfrak{h}$-submodule, so $\mathcal{L}^\lambda = \bigoplus_{\alpha \in \Delta \cup \{0\}} \mathcal{L}^\lambda_\alpha$ for $\mathcal{L}^\lambda_\alpha = \mathcal{L}_\alpha \cap \mathcal{L}^\lambda$. Thus, $\mathcal{L}$ has a grading by $\Lambda \oplus \mathcal{Q}(\Delta)$, where $\mathcal{Q}(\Delta)$ is the root lattice of Δ,

$$\mathcal{L} = \bigoplus_{\alpha \in \Delta \cup \{0\},\, \lambda \in \Lambda} \mathcal{L}^\lambda_\alpha \quad \text{with} \quad [\mathcal{L}^\lambda_\alpha\,,\, \mathcal{L}^\mu_\beta] \subseteq \mathcal{L}^{\lambda+\mu}_{\alpha+\beta} \tag{3.10}$$

for $\alpha, \beta \in \Delta \cup \{0\}$ and $\lambda, \mu \in \Lambda$. Since the centre of any graded Lie algebra is a graded subspace, $Z(\mathcal{L}) = \bigoplus_{\lambda \in \Lambda} Z(\mathcal{L})^\lambda \subseteq \bigoplus_{\lambda \in \Lambda} \mathcal{L}^\lambda_0$.

Lemma 3.11. (1) *Let $\mathcal{L}$ be a (Δ, Λ)-graded Lie algebra. Then*

$$\mathrm{Cent}(\mathcal{L}) = \bigoplus_{\lambda \in \Lambda} \mathrm{Cent}(\mathcal{L})^\lambda$$

is a Λ-graded commutative associative algebra, where $\mathrm{Cent}(\mathcal{L})^\lambda$ *is the subspace of centroidal transformations that are homogeneous of degree λ with respect to the Λ-grading* (3.9).

(2) *If the Λ-graded Lie algebra $\mathcal{L}$ is graded-simple, its centroid* $\mathrm{Cent}(\mathcal{L})$ *is a commutative associative division-graded algebra, hence it is isomorphic to a twisted group ring $\mathbb{E}^t[\Gamma]$ for the extension field $\mathbb{E} = \mathrm{Cent}(\mathcal{L})^0$ of $\mathbb{F}$ and for the subgroup $\Gamma = \mathrm{supp}_{\,\mathrm{Cent}(\mathcal{L})}$ of Λ.*

Proof. Any Δ-graded Lie algebra $\mathcal{L}$ is perfect, so $\mathrm{Cent}(\mathcal{L})$ is commutative. As a $\mathrm{Mult}(\mathcal{L})$-module, $\mathcal{L}$ is generated by $\mathfrak{h}$. Indeed, $[\mathcal{L}_\alpha, \mathfrak{h}] = \mathcal{L}_\alpha$ for $0 \neq \alpha \in \Delta$ and then $\mathcal{L}_0 = \sum_{\alpha\in\Delta} [\mathcal{L}_{-\alpha}, [\mathcal{L}_\alpha, \mathfrak{h}]]$ by (iv) in Definition 3.7. Since $\mathfrak{h}$ is finite-dimensional, it follows from (2.15) that $\mathrm{Cent}(\mathcal{L}) = \mathrm{grCent}(\mathcal{L})$ with respect to the $\Lambda \oplus \mathcal{Q}(\Delta)$-grading (3.10). However, by Proposition 3.1 (a), every χ has degree 0 with respect to the $\mathcal{Q}(\Delta)$-grading of $\mathcal{L}$, which proves the first part of the lemma. The last part now follows from Proposition 2.16. $\square$

Definition 3.12. A (Δ, Λ)-graded Lie algebra $\mathcal{L}$ is called a *Lie torus* of type (Δ, Λ), or simply a *Lie torus*, if

(i) whenever $\mathcal{L}_\alpha^\lambda \neq 0$ for $\alpha \in \Delta$ and $\lambda \in \Lambda$, then there exists an $\mathfrak{sl}_2$-triple $(e, h, f) \in \mathcal{L}_\alpha^\lambda \times \mathcal{L}_0^0 \times \mathcal{L}_{-\alpha}^{-\lambda}$ such that
  (a) $\mathcal{L}_\alpha^\lambda = \mathbb{F}e$ and $\mathcal{L}_{-\alpha}^{-\lambda} = \mathbb{F}f$, and
  (b) $[h, x] = \langle \beta, \alpha^\vee \rangle x$ for all $x \in \mathcal{L}_\beta^\mu$, $\beta \in \Delta \cup \{0\}$, $\mu \in \Lambda$;
(ii) the group Λ is generated by $\mathrm{supp}_{\,\mathcal{L}} = \{\lambda \in \Lambda \mid \mathcal{L}^\lambda \neq 0\}$; and
(iii) if $\Delta = \mathrm{BC}_r$ for $r = 1, 2$, then $\Delta_{\mathfrak{g}} \neq \mathrm{D}_r$.

In (b), $\alpha^\vee$ is the usual *coroot*, and $\langle \beta, \alpha^\vee \rangle$ comes from the usual inner product on the real span of the root system Δ. An $\mathfrak{sl}_2$-triple in (i) is assumed to satisfy the canonical relations, $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$. We point out that no condition is imposed on $\dim \mathcal{L}_0^\lambda$; however, when Λ is free of finite rank, one knows that $\dim \mathcal{L}_0^\lambda \leq m$ for some positive integer m that does not depend on λ by [N2, Thm. 5(a)]. In [N2], Neher considers Lie tori only for $\Lambda = \mathbb{Z}^n$; while in [Y3], the above notion is referred to as a Λ-torus, and the term Lie torus is reserved for the case $\Lambda = \mathbb{Z}^n$. Here we consider Lie tori for an arbitrary Λ, since the determination of the centroid is exactly the same as for $\Lambda = \mathbb{Z}^n$.

A centreless Lie torus is graded-simple by [Y3, Lem. 4.4]. It is assumed in [Y3] that $\Delta_{\mathfrak{g}}$ is of type B_r when Δ is of type BC_r; however, the same proof as given there works in our more general setting.

The following result provides a proof of [N2, Thm. 7(a)]:

Proposition 3.13. *Let $\mathcal{L}$ be a centreless Lie torus of type (Δ, Λ). Then $\mathrm{Cent}(\mathcal{L})$ is a twisted group ring $\mathbb{F}^t[\Gamma]$ for some subgroup Γ of Λ. In particular,*

(i) $\mathrm{Cent}(\mathcal{L})^0 = \mathbb{F}\,\mathrm{id}$, *and*
(ii) *if $\Lambda \cong \mathbb{Z}^n$, then $\mathrm{Cent}(\mathcal{L})$ is isomorphic to a Laurent polynomial ring in r variables, where $0 \le r \le n$.*

Proof. We know from Lemma 3.11 and [Y3, Lem. 3.4] that $\mathrm{Cent}(\mathcal{L}) = \bigoplus_{\lambda\in\Lambda} \mathrm{Cent}(\mathcal{L})^\lambda$ is a commutative associative division-graded algebra. Since $\mathrm{Cent}(\mathcal{L})^\lambda \to \mathcal{L}^\lambda_\alpha$ is injective for each λ and α by Proposition 2.16, it follows that all homogeneous spaces $\mathrm{Cent}(\mathcal{L})^\lambda$ are at most one-dimensional. In particular (i) holds. It also follows that $\mathrm{Cent}(\mathcal{L})$ is a twisted group ring for a subgroup Γ of Λ. When $\Lambda = \mathbb{Z}^n$, this subgroup is isomorphic to $\mathbb{Z}^r$ for some $0 \le r \le n$. But any twisted group ring over $\mathbb{Z}^r$ is in fact a group ring, and hence isomorphic to a Laurent polynomial ring, (compare [BGKN, Lem. 1.8]). □

4. The centroid of an EALA and its core

We will prove in Corollary 4.13 below that the core of a tame EALA is central, and from this result, the centroid of the EALA itself can easily be determined. Since the core of an EALA is a central extension of a centreless Lie torus, and since we know the centroid of centreless Lie tori by Proposition 3.13, it is natural to base our investigation on a general result which describes the behavior of the centroid under a central extension, see Lemma 4.3 below. We start by recalling some facts about central extensions and 2-cocycles, which also serves to establish our notation.

Throughout we consider Lie algebras over an arbitrary field $\mathbb{F}$. We recall that a *central extension* of a Lie algebra $\mathcal{L}$ is a pair $(\mathcal{K}, \pi)$ consisting of a Lie algebra $\mathcal{K}$ and a surjective Lie algebra homomorphism $\pi : \mathcal{K} \to \mathcal{L}$ whose kernel lies in the center $Z(\mathcal{K})$ of $\mathcal{K}$. If $\mathcal{K}$ is perfect, then $\mathcal{K}$ is said to be a *cover* or *covering* of $\mathcal{L}$. In this case $\mathcal{L}$ is necessarily perfect also. A *homomorphism* (resp. *isomorphism*) from a central extension $f : \mathcal{K} \to \mathcal{L}$

to a central extension $f' : \mathcal{K}' \to \mathcal{L}$ is a Lie algebra homomorphism (resp. isomorphism) $g : \mathcal{K} \to \mathcal{K}'$ satisfying $f = f' \circ g$.

A central extension $\mathfrak{u} : \widehat{\mathcal{L}} \to \mathcal{L}$ is a *universal central extension* if there exists a unique homomorphism from $\widehat{\mathcal{L}}$ to any other central extension $\mathcal{K}$ of $\mathcal{L}$. This universal property implies that any two universal central extensions of $\mathcal{L}$ are isomorphic as central extensions. A Lie algebra $\mathcal{L}$ has a universal central extension if and only if $\mathcal{L}$ is perfect. In this case, the universal central extension $\widehat{\mathcal{L}}$ is perfect, and $\widehat{\mathcal{L}}$ is a covering of every covering of $\mathcal{L}$.

Examples of central extensions can be constructed in terms of 2-*cocycles* which are bilinear maps $\sigma : \mathcal{L} \times \mathcal{L} \to C$ into some vector space C satisfying

$$\sigma(x, x) = 0 \quad \text{and} \quad \sigma([x, y], z) + \sigma([y, z], x) + \sigma([z, x], y) = 0 \tag{4.1}$$

for all $x, y, z \in \mathcal{L}$. Given such a 2-cocycle, the vector space $E = \mathcal{L} \oplus C$ becomes a Lie algebra with product

$$[x \oplus c\,,\, y \oplus c']_E = [x, y] \oplus \sigma(x, y). \tag{4.2}$$

(Here we are using the notation $x \oplus c$ to designate that $x \in \mathcal{L}$ and $c \in C$.) This is a central extension of $\mathcal{L}$, which we denote $E(\mathcal{L}, \sigma)$, with respect to the projection map $E(\mathcal{L}, \sigma) \to \mathcal{L}, \quad x \oplus c \mapsto x$. Since we are considering Lie algebras over a field, every central extension $f : \mathcal{K} \to \mathcal{L}$ is isomorphic as central extension to some $E(\mathcal{L}, \sigma)$, see e.g. [MP, 1.9].

Lemma 4.3. *Let $\pi : \mathcal{K} \to \mathcal{L}$ be a central extension of the Lie algebra $\mathcal{L}$ written in the form* (4.2)*, and suppose that $Z(\mathcal{L}) = 0$. Then $Z(\mathcal{K}) = C$. Moreover, $\Psi \in \mathrm{End}_{\mathbb{F}}(\mathcal{K})$ lies in the centroid $\mathrm{Cent}(\mathcal{K})$ if and only if there exist $\chi \in \mathrm{Cent}(\mathcal{L})$, $\psi \in \mathrm{Hom}_{\mathbb{F}}(\mathcal{L}, C)$ and $\eta \in \mathrm{End}_{\mathbb{F}}(C)$ such that*

$$\Psi(x \oplus c) = \chi(x) \oplus \big(\psi(x) + \eta(c)\big) \quad \textit{and} \tag{4.4}$$

$$\sigma(x, \chi(y)) = \psi([x, y]) + \eta\big(\sigma(x, y)\big) \tag{4.5}$$

for all $x, y \in \mathcal{L}$ and $c \in C$. In this case, $\sigma(x, \chi(y)) = \sigma(\chi(x), y)$.

Proof. If $x \oplus c \in Z(\mathcal{K})$, then (4.2) implies that $x \in Z(\mathcal{L}) = 0$, hence $Z(\mathcal{K}) \subseteq C$. The other inclusion is obvious.

Now assume $\Psi \in \mathrm{Cent}(\mathcal{K})$. By Lemma 2.7 (a), Ψ leaves $Z(\mathcal{K}) = C$ invariant, hence has the form (4.4) for ψ, η as in the statement of the lemma and some $\chi \in \mathrm{End}_{\mathbb{F}}(\mathcal{L})$. Since $\pi_{\mathrm{Cent}}(\Psi) = \chi$, it follows from Lemma 2.7 (a) that $\chi \in \mathrm{Cent}(\mathfrak{L})$ (this is also immediate from the computation below). It

now remains to characterize when a map of the form (4.4) belongs to the centroid of $\mathcal{K}$. For $x, y \in \mathfrak{L}$ and $c, c' \in C$ we have

$$\begin{aligned}\Psi\left([x \oplus c, y \oplus c']_{\mathcal{K}}\right) &= \Psi\big([x, y] \oplus \sigma(x, y)\big) \\ &= \chi([x, y]) \oplus \Big(\psi([x, y]) + \eta\big(\sigma(x, y)\big)\Big), \\ [x \oplus c, \Psi(y \oplus c')]_{\mathcal{K}} &= [x, \chi(y)] \oplus \sigma(x, \chi(y)).\end{aligned}$$

Combined they show that $\Psi \in \mathrm{Cent}(\mathcal{K})$ if and only if $\chi \in \mathrm{Cent}(\mathcal{L})$ and (4.5) holds. □

Next we will construct a special class of 2-cocycles for Lie algebras with a nondegenerate invariant bilinear form and describe the centroid of the corresponding central extension in Proposition 4.11. Later this will be applied to determine the centroid of the core of an EALA.

Let Λ be an abelian group. We say $\pi : \mathcal{K} \to \mathcal{L}$ is a Λ-*graded central extension* if both $\mathcal{L}$ and $\mathcal{K}$ are graded by Λ, say $\mathcal{L} = \bigoplus_{\lambda \in \Lambda} \mathcal{L}^\lambda$ and $\mathcal{K} = \bigoplus_{\lambda \in \Lambda} \mathcal{K}^\lambda$, and if π is homogeneous of degree 0, i.e., $\pi(\mathcal{K}^\lambda) \subseteq \mathcal{L}^\lambda$. Every Λ-graded central extension $\pi : \mathcal{K} \to \mathcal{L}$ is isomorphic to a central extension $E(\mathcal{L}, \sigma)$ where $\sigma : \mathcal{L} \times \mathcal{L} \to C$ is a Λ-*graded* 2-*cocycle*, i.e., $C = \bigoplus_{\lambda \in \Lambda} C^\lambda$ is a Λ-graded vector space and σ is a 2-cocycle satisfying $\sigma(\mathcal{L}^\lambda, \mathcal{L}^\mu) \subseteq C^{\lambda+\mu}$ for all $\lambda, \mu \in \Lambda$. When $\mathcal{K}$ is perfect, such a Λ-graded central extension $\pi : \mathcal{K} \to \mathcal{L}$ is called a Λ-*covering*. In particular, if $\mathcal{L}$ is perfect, the universal central extension $\widehat{\mathcal{L}}$ is a Λ-covering of $\mathcal{L}$.

Let $\mathcal{L} = \bigoplus_{\lambda \in \Lambda} \mathcal{L}^\lambda$ be a Λ-graded Lie algebra over $\mathbb{F}$. We denote by $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{F})$ the $\mathbb{F}$-vector space of additive maps $\theta : \Lambda \to \mathbb{F}$. For any $\theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{F})$, the corresponding *degree derivation* ∂_θ of $\mathcal{L}$ is defined by

$$\partial_\theta(x) = \theta(\lambda)x \qquad (x \in \mathcal{L}^\lambda), \tag{4.6}$$

There is a linear map

$$\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{F}) \to \mathcal{D} := \{\partial_\theta \mid \theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{F})\} \qquad \theta \mapsto \partial_\theta,$$

into the space $\mathcal{D}$ of degree derivations, which is an isomorphism if Λ is spanned by the support of $\mathcal{L}$, i.e., if

$$\Lambda = \mathrm{span}_{\mathbb{Z}}\{\lambda \in \Lambda \mid \mathcal{L}^\lambda \neq 0\}. \tag{4.7}$$

Indeed, if $\partial_\theta = 0$ then $\theta(\lambda) = 0$ for all $\lambda \in \Lambda$ with $\mathcal{L}^\lambda \neq 0$ whence $\theta = 0$. We note that (4.7) is essentially a notational convenience; if it is not fulfilled,

one can always replace Λ by the subgroup generated by the support of $\mathcal{L}$. Assuming (4.7), we have a well-defined linear map

$$\mathtt{ev} : \Lambda \to \mathcal{D}^*, \quad \lambda \mapsto \mathtt{ev}(\lambda)$$

into the dual space $\mathcal{D}^*$ of $\mathcal{D}$ given by

$$\mathtt{ev}(\lambda)(\partial_\theta) = \theta(\lambda), \quad \lambda \in \Lambda,\ \partial_\theta \in \mathcal{D}. \tag{4.8}$$

We denote by $\mathrm{Der}(\mathcal{L})^\lambda$, the vector space of $\mathbb{F}$-linear derivations of $\mathcal{L}$ of degree λ, and we set

$$\mathrm{grDer}(\mathcal{L}) = \bigoplus_{\lambda\in\Lambda} \mathrm{Der}(\mathcal{L})^\lambda,$$

which is obviously a graded subalgebra of $\mathrm{Der}(\mathcal{L})$. It is well-known (see for example [F, Prop. 1]) that $\mathrm{grDer}(\mathcal{L}) = \mathrm{Der}(\mathcal{L})$ if $\mathcal{L}$ is finitely generated as Lie algebra.

Let $(\,.\mid .\,)$ be an invariant Λ-graded bilinear form on $\mathcal{L}$, i.e., $(\mathcal{L}^\lambda \mid \mathcal{L}^\mu) = 0$ for all $\lambda + \mu \neq 0$. We denote by $\mathrm{SDer}(\mathcal{L}) = \mathrm{SDer}_{\mathbb{F}}(\mathcal{L})$ the subalgebra of $\mathrm{Der}(\mathcal{L})$ consisting of skew derivations ∂ of $\mathcal{L}$: $(\partial x \mid y) = -(x \mid \partial y)$ for all $x, y \in \mathcal{L}$. It is easily seen that $\partial \in \mathrm{grDer}(\mathcal{L})$ is skew if and only if every homogeneous component of ∂ is, so that

$$\mathrm{grSDer}(\mathcal{L}) := \mathrm{SDer}(\mathcal{L}) \cap \mathrm{grDer}(\mathcal{L}) = \bigoplus_{\lambda\in\Lambda} \mathrm{SDer}(\mathcal{L})^\lambda.$$

where $\mathrm{SDer}(\mathcal{L})^\lambda = \mathrm{SDer}(\mathcal{L}) \cap \mathrm{Der}(\mathcal{L})^\lambda$. Moreover

$$\mathcal{D} \subseteq \mathrm{SDer}(\mathcal{L})^0 \quad \text{and} \quad \mathrm{IDer}(\mathcal{L}) \subseteq \mathrm{grSDer}(\mathcal{L})$$

where $\mathrm{IDer}(\mathcal{L}) = \mathrm{IDer}_{\mathbb{F}}(\mathcal{L}) = \{\mathrm{ad}\, x \mid x \in \mathcal{L}\}$ denotes the ideal of inner derivations of $\mathcal{L}$.

Let $\mathcal{S} = \bigoplus_{\lambda\in\Lambda} \mathcal{S}^\lambda$ be a graded subspace of $\mathrm{grSDer}(\mathcal{L})$, and let $\mathcal{S}^{\mathrm{gr}*}$ be the graded dual space. Thus,

$$\mathcal{S}^{\mathrm{gr}*} = \bigoplus_{\lambda\in\Lambda} (\mathcal{S}^{\mathrm{gr}*})^\lambda, \quad \text{where} \quad (\mathcal{S}^{\mathrm{gr}*})^\lambda = (\mathcal{S}^{-\lambda})^*. \tag{4.9}$$

We may assume $(\mathcal{S}^{-\lambda})^* \subseteq \mathcal{S}^*$ by defining $f\mid_{\mathcal{S}^\mu} = 0$ for $f \in (\mathcal{S}^{-\lambda})^*$ and $\mu \neq -\lambda$. Then it is easy to verify that

$$\sigma_{\mathcal{S}} : \mathcal{L} \times \mathcal{L} \to \mathcal{S}^{\mathrm{gr}*}, \quad \sigma_{\mathcal{S}}(x, y)(d) = (d(x) \mid y) \tag{4.10}$$

for $x, y \in \mathcal{L}$ and $d \in \mathcal{S}$ is a Λ-graded 2-cocycle. Thus $E(\mathcal{L}, \sigma_{\mathcal{S}}) = \mathcal{L} \oplus \mathcal{S}^{\mathrm{gr}*}$ with product $[x \oplus c, y \oplus c']_E = [x, y] \oplus \sigma_{\mathcal{S}}(x, y)$ for all $x, y \in \mathcal{L}, c, c' \in \mathcal{S}^{\mathrm{gr}*}$ is a Λ-graded central extension of $\mathcal{L}$.

Proposition 4.11. *Let $\mathcal{L} = \bigoplus_{\lambda\in\Lambda} \mathcal{L}^\lambda$ be a perfect Λ-graded Lie algebra with a nondegenerate invariant graded bilinear form such that* (4.7) *holds. Let $\mathcal{S} \subseteq \mathrm{grSDer}(\mathcal{L})$ be a Λ-graded subspace such that*

$$\mathtt{ev}_{\mathcal{S}} : \Lambda \to (\mathcal{D} \cap \mathcal{S})^*, \quad \lambda \mapsto \mathtt{ev}(\lambda)\big|_{\mathcal{D}\cap\mathcal{S}} \tag{4.12}$$

is injective, where the evaluation map $\mathtt{ev}$ *is as in* (4.8)*. Let $\mathcal{K} = E(\mathcal{L}, \sigma_{\mathcal{S}}) = \mathcal{L} \oplus C$ where $C = \mathcal{S}^{\mathrm{gr}*}$.*

(i) *Suppose $\Psi \in \mathrm{Cent}(\mathcal{K})$ is homogeneous of degree λ. Then there exists $\chi \in \mathrm{Cent}(\mathcal{L})$, $\psi \in \mathrm{Hom}_{\mathbb{F}}(\mathcal{L}, C)$, and $\eta \in \mathrm{End}_{\mathbb{F}}(C)$ all of degree λ such that*
   (a) $\Psi\big(x \oplus c\big) = \chi(x) \oplus \big(\psi(x) \oplus \eta(c)\big)$ *for all $x \in \mathcal{L}, c \in C$.*
   (b) $\chi = 0$ *if $\lambda \neq 0$.*

(ii) *If $\mathcal{K}$ is perfect, then $\mathrm{Cent}(\mathcal{K})^\lambda = 0$ for all $\lambda \neq 0$. In particular, if $\mathcal{K}$ is a Lie torus, then $\mathcal{K}$ is central.*

Proof. (i) Observe first that $Z(\mathcal{L}) = 0$. This follows from the computation $([x, y] \mid z) = (x \mid [y, z]) = 0$ for all $x, y \in \mathcal{L}$, $z \in Z(\mathcal{L})$ and the fact that $\mathcal{L}$ is perfect and the form is nondegenerate.

Now suppose $\Psi \in \mathrm{Cent}(\mathcal{K})$ has degree λ, and apply Lemma 4.3 to conclude that $\Psi(x \oplus c) = \chi(x) \oplus \big(\psi(x) + \eta(c)\big)$ where $\chi \in \mathrm{Cent}(\mathcal{L})$, $\psi \in \mathrm{Hom}_{\mathbb{F}}(\mathcal{L}, C)$, and $\eta \in \mathrm{End}_{\mathbb{F}}(C)$, and all have degree λ. For $x^\mu \in \mathcal{L}^\mu$, $y \in \mathcal{L}$, and $\partial_\theta \in \mathcal{D}\cap\mathcal{S}$ we have by (4.5),

$$\begin{aligned}
\sigma_{\mathcal{S}}(\chi(x^\mu)\,,\, y)(\partial_\theta) &= (\partial_\theta \chi(x^\mu) \,|\, y) = \theta(\mu + \lambda)(\chi(x^\mu) \,|\, y) \\
&= \sigma_{\mathcal{S}}(x^\mu, \chi(y))(\partial_\theta) = (\partial_\theta(x^\mu) \,|\, \chi(y)) = \theta(\mu)(x^\mu \,|\, \chi(y)) \\
&= \theta(\mu)(\chi(x^\mu) \,|\, y),
\end{aligned}$$

where in the last equality we used Lemma 2.1 (f). Hence $\theta(\lambda)(\chi(x^\mu) \,|\, y) = 0$ for all $x^\mu \in \mathcal{L}^\mu$ and $\mu \in \Lambda$. Suppose $\lambda \neq 0$. Then there exists a $\theta \in \mathcal{D} \cap \mathcal{S}$ such that $\theta(\lambda) \neq 0$. From this we see $(\chi(x^\mu), y) = 0$ and nondegeneracy forces $\chi = 0$. Thus, (b) holds.

(ii) In the preceding paragraph we have shown that $Z(\mathcal{L}) = 0$, and $\chi = 0$ whenever $\lambda \neq 0$. If $\pi : \mathcal{K} \to \mathcal{L}$ is the cover map and $\mathcal{K}$ is perfect, it follows from Lemma 2.7 (c) that $\pi_{\mathrm{Cent}} : \mathrm{Cent}(\mathcal{K}) \to \mathrm{Cent}(\mathcal{L})$ is an algebra monomorphism which takes Ψ to χ. In particular, if $\lambda \neq 0$, then χ has degree λ, forcing both χ and Ψ to be 0. Therefore, $\pi_{\mathrm{Cent}} : \mathrm{Cent}(\mathcal{K}) = \mathrm{Cent}(\mathcal{K})^0 \to \mathrm{Cent}(\mathcal{L})^0 = \mathbb{F}\,\mathrm{id}$ by Proposition 3.13 (i). Consequently, $\mathcal{K}$ is central. $\square$

As an application of Proposition 4.11, we can now determine the centroid of a tame EALA $\mathcal{E}$ and of its core $\mathcal{K}$, which is the ideal of $\mathcal{E}$ generated by the root spaces corresponding to the nonisotropic roots. The reader is referred to [AABGP, Ch. I] for the precise definition of a tame EALA over $\mathbb{F} = \mathbb{C}$ and to [N3] for arbitrary fields $\mathbb{F}$ of characteristic 0. An EALA has an invariant nondegenerate bilinear form, which when restricted to the core $\mathcal{K}$ is Λ-graded for $\Lambda = \mathbb{Z}^n$ for some $n \geq 0$. Tameness says that the ideal $\mathcal{K}$ satisfies $\mathcal{C}_{\mathcal{E}}(\mathcal{K}) = Z(\mathcal{K})$.

Corollary 4.13. *Let $\mathcal{E}$ be a tame extended affine Lie algebra, let $\mathcal{K}$ be its core and set $\mathcal{D} = \mathcal{E}/\mathcal{K}$. Then $\mathcal{K}$ is central and*

$$\mathrm{Cent}(\mathcal{E}) = \mathbb{F}\,\mathrm{id} \oplus \mathcal{V}(\mathcal{K}) \cong \mathbb{F}\,\mathrm{id} \oplus \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}, Z(\mathcal{K})).$$

Proof. It is known that $\mathcal{K}$ is a Lie torus with $\Lambda \cong \mathbb{Z}^n$ (see [Y4, Cor. 7.3] for $\mathbb{F} = \mathbb{C}$ or [N3, Prop. 3(a)] for arbitrary $\mathbb{F}$). Moreover, by [N3, Thm. 6], $\mathcal{K}$ is obtained from the centreless Lie torus $\mathcal{L} = \mathcal{K}/Z(\mathcal{K})$ by the construction of Proposition 4.11. Thus $\mathcal{K}$ is central by part (ii) of that proposition. Since $\mathcal{K}$ is a $\mathrm{Cent}(\mathcal{E})$-invariant ideal of $\mathcal{E}$, $\mathrm{Cent}(\mathcal{E}) = \mathbb{F}\,\mathrm{id} \oplus \mathcal{V}(\mathcal{K})$ follows from (2.2). Finally, $\mathcal{V}(\mathcal{K}) \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}, Z(\mathcal{K}))$ by Lemma 2.1 (b) and the fact that $\mathrm{Ann}_{\mathcal{E}}(\mathcal{K}) = \mathcal{C}_{\mathcal{E}}(\mathcal{K}) = Z(\mathcal{K})$ because of tameness. $\square$

Example 4.14. Finite-dimensional split simple Lie algebras are examples of tame EALAs. In this case $\mathcal{E} = \mathcal{K}$ and $Z(\mathcal{K}) = 0$, so the result above simply says that $\mathcal{E}$ is central – which is of course well-known.

Another class of examples of tame EALAs are the affine Lie algebras (see [ABGP]). In this case, $\mathcal{K} = \mathcal{E}^{(1)}$ and $\mathcal{D}$ and $Z(\mathcal{K})$ are both one-dimensional, so that our result recovers Corollary 3.5.

However there are many other examples of extended affine Lie algebras besides the two just mentioned, e.g. toroidal algebras extended by some derivations.

5. Centroids of Lie Algebras Graded by Finite Root Systems

In this section, we will describe the centroid of Lie algebras graded by finite root systems. The case of reduced root systems will be treated in Subsection 5.1, while the nonreduced case will be done in 5.2. As a prelude to that, we begin with a general result about Lie algebras $\mathcal{L}$ which are completely reducible relative to the adjoint action of a subalgebra $\mathfrak{g}$. By

gathering together isomorphic summands, we may assume that such Lie algebras are written in the form

$$\mathcal{L} = \bigoplus_k (V_k \otimes A_k)\,,$$

where the V_k are nonisomorphic irreducible $\mathfrak{g}$-modules; the subspace A_k indexes the copies of V_k; and the $\mathfrak{g}$-action is given by $x.(v_k \otimes a_k) = [x, v_k \otimes a_k)] = x.v_k \otimes a_k$ for $x \in \mathfrak{g}$, $v_k \in V_k$, $a_k \in A_k$.

Lemma 5.1. *Assume $\mathcal{L}$ is a Lie algebra which is completely reducible relative to the adjoint action of a subalgebra $\mathfrak{g}$, and let $\mathcal{L} = \bigoplus_k (V_k \otimes A_k)$ be its decomposition relative to $\mathfrak{g}$. Assume $\mathrm{End}_{\mathfrak{g}}(V_k) = \mathbb{F}\,\mathrm{id}$ for each irreducible $\mathfrak{g}$-module V_k. If $\Psi \in \mathrm{Cent}(\mathcal{L})$, then there exist transformations $\psi_k : A_k \to A_k$ such that $\Psi(v_k \otimes a_k) = v_k \otimes \psi_k(a_k)$ for all $v_k \in V_k$, $a_k \in A_k$.*

Proof. Assume $\{a_k^i \mid i \in \mathfrak{I}_k\}$ is a basis for A_k, and let π_k^i denote the projection of $\mathcal{L}$ onto the summand $V_k \otimes a_k^i$. Fix $j \in \mathfrak{I}_k$. Then for any $\Psi \in \mathrm{Cent}(\mathcal{L})$ and any $i \in \mathfrak{I}_k$, we have $(\pi_k^i \circ \Psi) : V_k \otimes a_k^j \to V_k \otimes a_k^i$ is a $\mathfrak{g}$-module homomorphism. Thus, it determines an element of $\mathrm{End}_{\mathfrak{g}}(V_k) = \mathbb{F}\,\mathrm{id}$, and there exists a scalar $\xi_{i,j} \in \mathbb{F}$, so that $(\pi_k^i \circ \Psi)(v_k \otimes a_k^j) = \xi_{i,j} v_k \otimes a_k^i$ for all $v_k \in V_k$. When $\ell \neq k$, $(\pi_k^i \circ \Psi) : V_\ell \otimes a_\ell^j \to V_k \otimes a_k^i$ is a $\mathfrak{g}$-module homomorphism determining an element of $\mathrm{Hom}_{\mathfrak{g}}(V_\ell, V_k)$. Such a homomorphism must be the zero map since V_k and V_ℓ are irreducible and nonisomorphic. Consequently, $\Psi(v_k \otimes a_k^j) \in V_k \otimes A_k$ for all $v_k \in V_k$, and

$$\Psi(v_k \otimes a_k^j) = \sum_{i \in \mathfrak{I}_k} \xi_{i,j} v_k \otimes a_k^i = v_k \otimes \left(\sum_{i \in \mathfrak{I}_k} \xi_{i,j} a_k^i \right).$$

Define $\psi_k(a_k^j) = \sum_{i \in \mathfrak{I}_k} \xi_{i,j} a_k^i$ for each $j \in \mathfrak{I}_k$ and extend this linearly to all of A_k. Then

$$\Psi(v_k \otimes a_k) = v_k \otimes \psi_k(a_k) \qquad \text{for all } \ v_k \in V_k,\ a_k \in A_k. \quad \square$$

5.1. **Lie algebras graded by reduced root systems.** Suppose that $\mathcal{L}$ is a Lie algebra graded by the reduced root system Δ as in Definition 3.7. Then $\mathcal{L}$ is completely reducible relative to the adjoint action of the grading subalgebra $\mathfrak{g}$, and by results in ([BeM], [BZ], [N1]) we know that

$$\mathcal{L} \cong (\mathfrak{g} \otimes A) \oplus (W \otimes B) \oplus D,$$

where the following hold:

(1) W is the irreducible $\mathfrak{g}$-module with highest weight the highest short root of $\mathfrak{g}$; thus W and B are zero when Δ is simply laced.

(2) The sum $\mathfrak{a} = A \oplus B$ is a unital algebra called the *coordinate algebra* of $\mathcal{L}$. In all cases except for type C_2, $\mathfrak{a}$ is an associative, alternative, or Jordan algebra depending on Δ. The unit element 1 of $\mathfrak{a}$ lives in A, and $\mathfrak{g}$ is identified with $\mathfrak{g} \otimes 1$.

(3) D is the sum of the trivial one-dimensional $\mathfrak{g}$-modules. Moreover, D can be identified with a quotient space, $D = \langle \mathfrak{a}, \mathfrak{a} \rangle = \langle A, A \rangle + \langle B, B \rangle$, of the skew-dihedral homology of $\mathfrak{a}$, and D acts by inner derivations on $\mathfrak{a}$. Thus, $\langle \alpha, \beta \rangle(\gamma) = D_{\alpha,\beta}(\gamma)$ for all $\alpha, \beta, \gamma \in \mathfrak{a}$, where $D_{\alpha,\beta}$ is the inner derivation determined by α, β.

(4) the multiplication in $\mathcal{L}$ is given in terms of the product on $\mathfrak{a}$ as follows (note here we do not use $\oplus$ to separate summands to simplify the expressions):

$$\big(\Delta = \mathrm{A}_1,\ \mathrm{B}_r, (r \geq 3),\ \mathrm{D}_r, (r \geq 4),\ \mathrm{E}_6,\ \mathrm{E}_7,\ \mathrm{E}_8,\ \mathrm{F}_4,\ \mathrm{G}_2\big) \tag{5.2}$$

$$\begin{aligned}
&[x \otimes a, y \otimes a'] = [x, y] \otimes aa' + (x|y)\langle a, a' \rangle \\
&[d, x \otimes a] = x \otimes da = -[x \otimes a, d] \\
&[x \otimes a, u \otimes b] = xu \otimes ab = -[u \otimes b, x \otimes a] \\
&[d, u \otimes b] = u \otimes db = -[u \otimes b, d] \\
&[u \otimes b, v \otimes b'] = \partial_{u,v} \otimes (b, b') + (u * v) \otimes (b * b') + (u|v)\langle b, b' \rangle,
\end{aligned}$$

$$\big(\Delta = \mathrm{A}_r, (r \geq 2),\ \mathrm{C}_r, (r \geq 2)\big) \tag{5.3}$$

$$\begin{aligned}
&[x \otimes a, y \otimes a'] = [x, y] \otimes \frac{1}{2}(aa' + a'a) + (x \circ y) \otimes \frac{1}{2}(aa' - a'a) + (x|y)\langle a, a' \rangle, \\
&[d, x \otimes a] = x \otimes da = -[x \otimes a, d] \\
&[x \otimes a, u \otimes b] = (x \circ u) \otimes \frac{1}{2}(ab - ba) + [x, u] \otimes \frac{1}{2}(ab + ba) = -[u \otimes b, x \otimes a] \\
&[u \otimes b, v \otimes b'] = [u, v] \otimes \frac{1}{2}(bb' + b'b) + (u \circ v) \otimes \frac{1}{2}(bb' - b'b) + (u|v)\langle b, b' \rangle \\
&[d, u \otimes b] = u \otimes db = -[u \otimes b, d]
\end{aligned}$$

for all $a, a' \in A$, $b, b' \in B$, $x, y \in \mathfrak{g}$, $u, v \in W$, and $d \in D$.

(5) In (4), $(\,|\,)$ denotes the Killing form when applied to $\mathfrak{g}$, and the unique $\mathfrak{g}$-invariant bilinear form when applied to W. The maps $\partial \in \mathrm{Hom}_{\mathfrak{g}}(W \otimes W, \mathfrak{g})$ and $* \in \mathrm{Hom}_{\mathfrak{g}}(W \otimes W, W)$ in (5.2) are unique up

to scalars. In (5.2), the product in the coordinate algebra $\mathfrak{a} = A \oplus B$ is given by $(a+b)(a'+b') = \Big(aa' + (b,b')\Big) + \Big(ab' + a'b + b * b'\Big)$. The algebra $\mathfrak{g}$ in (5.3) can be realized as a matrix Lie algebra $\mathfrak{sl}_{r+1}$ or $\mathfrak{sp}_{2r}$ respectively. So for any two matrices w, z, we have $[w,z] = wz - zw$ and $w \circ z = wz + zw - (2/n)\mathfrak{tr}(wz)$, where $n = r+1$ or $2r$ and $\mathfrak{tr}$ denotes the trace.

Proposition 5.4. *Assume $\mathcal{L} \cong (\mathfrak{g} \otimes A) \oplus (W \otimes B) \oplus D$ is a Lie algebra graded by a finite reduced root system. Then $\Psi \in \mathrm{Cent}(\mathcal{L})$ if and only if there exist maps $\psi_{\mathfrak{a}} \in \mathrm{Cent}(\mathfrak{a})$ and $\psi_{_D} \in \mathrm{Cent}(D)$ such that $\psi_{\mathfrak{a}}(A) \subseteq A$, $\psi_{\mathfrak{a}}(B) \subseteq B$, and*

$$\Psi\Big((x \otimes a) + (u \otimes b) + d\Big) = \big(x \otimes \psi_{\mathfrak{a}}(a)\big) + \big(u \otimes \psi_{\mathfrak{a}}(b)\big) + \psi_{_D}(d) \tag{5.5}$$

$$\psi_{_D}\big(\langle \alpha, \alpha' \rangle\big) = \langle \alpha, \psi_{\mathfrak{a}}(\alpha') \rangle = \langle \psi_{\mathfrak{a}}(\alpha), \alpha' \rangle \tag{5.6}$$

$$\big(\psi_{\mathfrak{a}} \circ d\big)(\alpha) = \big(d \circ \psi_{\mathfrak{a}}\big)(\alpha) \tag{5.7}$$

$$\psi_{_D}(d)(\alpha) = \big(\psi_{\mathfrak{a}} \circ d\big)(\alpha) \tag{5.8}$$

for all $\alpha, \alpha' \in \mathfrak{a}$ and $d \in D$.

Proof. Applying Lemma 5.1, we see that corresponding to $\Psi \in \mathrm{Cent}(\mathcal{L})$ are maps $\psi_{_A} \in \mathrm{End}_{\mathbb{F}}(A)$, $\psi_{_B} \in \mathrm{End}_{\mathbb{F}}(B)$, and $\psi_{_D} \in \mathrm{End}_{\mathbb{F}}(D)$ such that $\Psi(x \otimes a) = x \otimes \psi_{_A}(a)$, $\Psi(u \otimes b) = u \otimes \psi_{_B}(b)$ and $\Psi(d) = \psi_{_D}(d)$ for $x \otimes a \in \mathfrak{g} \otimes A$, $u \otimes b \in W \otimes B$, and $d \in D$. Set $\psi_{\mathfrak{a}}(a \oplus b) = \psi_{_A}(a) + \psi_{_B}(b)$ and observe that $\psi_{\mathfrak{a}}(A) \subseteq A$ and $\psi_{\mathfrak{a}}(B) \subseteq B$ clearly hold.

Now suppose $x \otimes a, y \otimes a' \in \mathfrak{g} \otimes A$, and consider $\Psi([x \otimes a, y \otimes a'])$. When Δ is as in (5.2), then

$$\begin{aligned} \Psi([x \otimes a, y \otimes a']) &= [x \otimes a, \Psi(y \otimes a')] \qquad \Longleftrightarrow \\ [x,y] \otimes \psi_{\mathfrak{a}}(aa') + (x|y)\psi_{_D}\big(\langle a, a' \rangle\big) &= [x \otimes a, y \otimes \psi_{\mathfrak{a}}(a')] \\ &= [x,y] \otimes a\psi_{\mathfrak{a}}(a') + (x|y)\langle a, \psi_{\mathfrak{a}}(a') \rangle. \end{aligned}$$

Equating components shows that $\psi_{\mathfrak{a}}(aa') = a\psi_{\mathfrak{a}}(a')$, and $\psi_{_D}\big(\langle a, a' \rangle\big) = \langle a, \psi_{\mathfrak{a}}(a') \rangle$ hold for all $a, a' \in A$. Similarly, using $\Psi([x \otimes a, y \otimes a']) = [\Psi(x \otimes a), y \otimes a']$, we obtain $\psi_{\mathfrak{a}}(aa') = \psi_{\mathfrak{a}}(a)a'$ and the second equality in (5.6) for all $a, a' \in A$.

Now when Δ is as in (5.3), then

$$\begin{aligned}
\Psi([x\otimes a, y\otimes a']) &= [x,y]\otimes \frac{1}{2}\psi_{\mathfrak{a}}(aa'+a'a) \qquad (5.9)\\
&\quad +(x\circ y)\otimes \frac{1}{2}\psi_{\mathfrak{a}}(aa'-a'a) + (x|y)\psi_D\big(\langle a,a'\rangle\big) \quad \text{while}\\
[x\otimes a, \Psi(y\otimes a')] &= [x\otimes a, y\otimes \psi_{\mathfrak{a}}(a')]\\
&= [x,y]\otimes \frac{1}{2}\big(a\psi_{\mathfrak{a}}(a')+\psi_{\mathfrak{a}}(a')a\big)\\
&\quad +(x\circ y)\otimes \frac{1}{2}\big(a\psi_{\mathfrak{a}}(a')-\psi_{\mathfrak{a}}(a')a\big) + (x|y)\langle a,\psi_{\mathfrak{a}}(a')\rangle.
\end{aligned}$$

In particular, if $\Delta = \mathrm{A}_r$ for $r\geq 2$, we may set $x = e_{1,1}-e_{2,2} = y$ (matrix units) and get $[x,y]=0$, but $x\circ y\neq 0$. Then equating components in these expressions, we obtain that

$$\psi_{\mathfrak{a}}(aa'-a'a) = a\psi_{\mathfrak{a}}(a') - \psi_{\mathfrak{a}}(a')a$$

holds for all $a,a'\in A$, (as does the first equality in (5.6)). Then putting that relation back in, we determine that

$$\psi_{\mathfrak{a}}(aa'+a'a) = a\psi_{\mathfrak{a}}(a') + \psi_{\mathfrak{a}}(a')a$$

for all $a,a'\in A$. Combining these gives $\psi_{\mathfrak{a}}(aa') = a\psi_{\mathfrak{a}}(a')$ for all $a,a'\in A$.

When $\Delta = \mathrm{C}_r$ for $r\geq 2$, then the summands on the right side of (5.9) lie in different components, so equating them gives the same information as obtained in the A case.

Applying Ψ to all the various other products in (5.2) and (5.3) and arguing similarly will complete the proof of the proposition. $\square$

For the centroidal transformation $\psi_{\mathfrak{a}}\in \mathrm{Cent}(\mathfrak{a})$ coming from an element $\Psi\in\mathrm{Cent}(\mathcal{L})$ as in Proposition 5.4, it follows that $\psi_{\mathfrak{a}}(1)\in A\cap Z(\mathfrak{a})$, as $\psi_{\mathfrak{a}}$ preserves the space A and the unit element of the coordinate algebra $\mathfrak{a}$ belongs to A.

The inner derivations of the coordinate algebra $\mathfrak{a}$ involve certain expressions in the left multiplication and right multiplication operators which can be found in [ABG1, (2.41)]. For any $\Psi\in\mathrm{Cent}(\mathcal{L})$, the associated transformation $\psi_{\mathfrak{a}}$ belongs to $\mathrm{Cent}(\mathfrak{a})$, so it commutes with the left and right multiplication operators of $\mathfrak{a}$. It also commutes with the involution σ on $\mathfrak{a}$ when Δ is of type C_r, as $\psi_{\mathfrak{a}}$ preserves the spaces A and B, which are the symmetric elements and skew-symmetric elements respectively relative to σ.

Thus, $\psi_{\mathfrak{a}}$ commutes with the inner derivation $D_{\alpha,\beta}$ for all $\alpha, \beta \in \mathfrak{a}$, and

$$\psi_{\mathfrak{a}} \circ D_{\alpha,\beta} = D_{\psi_{\mathfrak{a}}(\alpha),\beta} = D_{\alpha,\psi_{\mathfrak{a}}(\beta)}. \tag{5.10}$$

As

$$\langle \alpha, \beta \rangle(\gamma) = D_{\alpha,\beta}(\gamma), \tag{5.11}$$

we have

$$\Big(\psi_{\mathfrak{a}} \circ \langle \alpha, \beta \rangle\Big)(\gamma) = \langle \psi_{\mathfrak{a}}(\alpha), \beta \rangle(\gamma) = \langle \alpha, \psi_{\mathfrak{a}}(\beta) \rangle(\gamma) = \psi_{D}(\langle \alpha, \beta \rangle)(\gamma), \tag{5.12}$$

so that for $\mathfrak{z} := \psi_{\mathfrak{a}}(1)$,

$$\mathfrak{z}\langle \alpha, \beta \rangle(\gamma) = \langle \mathfrak{z}\alpha, \beta \rangle(\gamma) = \langle \alpha, \mathfrak{z}\beta \rangle(\gamma) = \psi_{D}(\langle \alpha, \beta \rangle)(\gamma)$$

for all $\alpha, \beta, \gamma \in \mathfrak{a}$. Combining this with Proposition 5.4, we obtain the following:

Corollary 5.13. *Assume $\mathcal{L}$ is a Δ-graded Lie algebra as in Proposition 5.4, and let $\Psi \in \mathrm{Cent}(\mathcal{L})$. Then there exists an element $\mathfrak{z}$ $(= \psi_{\mathfrak{a}}(1) \in A)$ in the center $Z(\mathfrak{a})$ of the coordinate algebra $\mathfrak{a}$ of $\mathcal{L}$ such that*

$$\begin{aligned} \Psi(x \otimes a) &= x \otimes \mathfrak{z}a \\ \Psi(w \otimes b) &= w \otimes \mathfrak{z}b \\ \Psi\big(\langle \alpha, \beta \rangle\big) &= \langle \mathfrak{z}\alpha, \beta \rangle = \langle \alpha, \mathfrak{z}\beta \rangle \end{aligned} \tag{5.14}$$

for all $x \otimes a \in \mathfrak{g} \otimes A$, $w \otimes b \in W \otimes B$, $\alpha, \beta \in \mathfrak{a}$.

Theorem 5.15. *Let $\mathcal{L} = (\mathfrak{g} \otimes A) \oplus (W \otimes B) \oplus \langle \mathfrak{a}, \mathfrak{a} \rangle$ denote a Lie algebra graded by a finite reduced root system Δ with coordinate algebra $\mathfrak{a} = A \oplus B$. Then $\mathrm{Cent}(\mathcal{L}) \cong \mathfrak{Z}_{\mathfrak{a}}$, where $\mathfrak{Z}_{\mathfrak{a}}$ is the set of elements $\mathfrak{z}$ in $Z(\mathfrak{a}) \cap A$ which satisfy the following properties:*

(a) *$\langle \mathfrak{z}\alpha, \beta \rangle = \langle \alpha, \mathfrak{z}\beta \rangle$ for all $\alpha, \beta \in \mathfrak{a}$;*
(b) *$\sum_t \langle \alpha_t, \beta_t \rangle = 0$ implies $\sum_t \langle \mathfrak{z}\alpha_t, \beta_t \rangle = 0$.*

More specifically, if $\mathfrak{z} \in Z(\mathfrak{a}) \cap A$ satisfies (a) and (b), and if $\Psi \in \mathrm{End}_{\mathbb{F}}(\mathcal{L})$ is given by (5.14) above, then $\Psi \in \mathrm{Cent}(\mathcal{L})$; and every element of $\mathrm{Cent}(\mathcal{L})$ has this form.

Proof. Suppose $\mathfrak{z} \in \mathfrak{Z}_{\mathfrak{a}}$, and define Ψ as in (5.14). We need to know that the action of Ψ on $\langle \mathfrak{a}, \mathfrak{a} \rangle$ is well-defined. But that is apparent from condition (b). Also, to make sense of the definition, we must have $\mathfrak{z}a \in A$ and $\mathfrak{z}b \in B$

for $a \in A$, $b \in B$, which is of course obvious in case $\mathfrak{a} = A$, i.e., Δ is simply laced. If Δ is not simply laced and $\Delta \neq \mathrm{F}_4$ or G_2, the condition follows from the fact that the subspaces A and B are the symmetric and skew-symmetric elements with respect to an involution of $\mathfrak{a}$ ([BZ]).

In case $\Delta = \mathrm{F}_4$ or G_2, the algebra $\mathfrak{a}$ is a unital algebra over the commutative associative subalgebra $A = A.1$, while the subspace B is the kernel of an A-linear trace functional $\mathfrak{a} \to A$, and hence $AB \subset B$. The fact that $\Psi \in \mathrm{Cent}(\mathcal{L})$ can be verified directly using Proposition 5.4. Now for the other direction, apply Corollary 5.13 to deduce Ψ has the form in (5.14). If $\sum_t \langle \alpha_t, \beta_t \rangle = 0$, then Ψ must map that expression to 0, so that $\sum_t \langle \mathfrak{z}\alpha_t, \beta_t \rangle = 0$. $\square$

A Δ-graded Lie algebra $\mathcal{L}$ with trivial centre has the following form $\mathcal{L} = (\mathfrak{g} \otimes A) \oplus (W \otimes B) \oplus D_{\mathfrak{a},\mathfrak{a}}$, where $D_{\mathfrak{a},\mathfrak{a}}$ is the space of all inner derivations. In this particular case, $D_{\mathfrak{z}\alpha,\beta} = D_{\alpha,\mathfrak{z}\beta}$ for all $\alpha, \beta \in \mathfrak{a}$ and all $\mathfrak{z} \in Z(\mathfrak{a}) \cap A$. Moreover, $\mathfrak{z}D_{\alpha,\beta} = D_{\mathfrak{z}\alpha,\beta}$, so that (b) holds as well. Therefore, Theorem 5.15 implies:

Corollary 5.16. *Let $\mathcal{L} = (\mathfrak{g} \otimes A) \oplus (W \otimes B) \oplus D_{\mathfrak{a},\mathfrak{a}}$ denote the centerless Lie algebra graded by a finite reduced root system Δ with coordinate algebra $\mathfrak{a} = A \oplus B$. Then* $\mathrm{Cent}(\mathcal{L}) \cong Z(\mathfrak{a}) \cap A$.

Any Δ-graded Lie algebra $\mathcal{K} = (\mathfrak{g} \otimes A) \oplus (W \otimes B) \oplus \langle \mathfrak{a}, \mathfrak{a} \rangle$ with coordinate algebra $\mathfrak{a} = A \oplus B$ is a cover of the centreless Δ-graded Lie algebra $\mathcal{L} = (\mathfrak{g} \otimes A) \oplus (W \otimes B) \oplus D_{\mathfrak{a},\mathfrak{a}}$ with that same coordinate algebra via the map which is the identity on $(\mathfrak{g} \otimes A) \oplus (W \otimes B)$ and sends $\langle \alpha, \beta \rangle$ to $D_{\alpha,\beta}$. By Theorem 5.15 and Corollary 5.16, there is always an embedding

$$\mathrm{Cent}(\mathcal{K}) \to Z(\mathfrak{a}) \cap A \cong \mathrm{Cent}(\mathcal{L}).$$

Of course, we already knew that from Lemma 2.7(c).

Example 5.17. Let $\mathcal{K} = \big(\mathfrak{g} \otimes \mathbb{F}[t, t^{-1}]\big) \oplus \mathbb{F}c$ be the derived algebra of an untwisted affine algebra. As an application of general results, we have seen in Corollary 3.6 and then again in Example 4.14 that $\mathcal{K}$ is central. An alternate proof of this fact comes from specializing Theorem 5.15.

Indeed, $\mathcal{K}$ is a Δ-graded Lie algebra with grading subalgebra the split simple Lie algebra $\mathfrak{g}$. The element c is central, and $[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + (x \,|\, y)\langle t^m, t^n \rangle$ for all $x, y \in \mathfrak{g}$, where $\langle t^m, t^n \rangle = m\delta_{m,-n}c$. The

centreless Δ-graded Lie algebra $\mathcal{L}$ with the same coordinate algebra is just the loop algebra $\mathcal{L} := \mathfrak{g}\otimes\mathbb{F}[t,t^{-1}]$, whose centroid according to Corollary 5.16 is $\mathrm{Cent}(\mathcal{L}) = \mathbb{F}[t,t^{-1}]$, since $\mathfrak{a} = A = \mathbb{F}[t,t^{-1}]$ is a commutative associative algebra (compare Remark 2.24 and Remark 2.27 with $\sigma = \mathrm{id}$). By Theorem 5.15, we know that each element $\chi \in \mathrm{Cent}(\mathcal{K})$ is determined by an element $\mathfrak{z} = \sum_p k_p t^p \in \mathfrak{Z}_{\mathfrak{a}} = \mathfrak{a}$, which satisfies $\langle \mathfrak{z}t^m, t^n\rangle = \langle t^m, \mathfrak{z}t^n\rangle$ for all $m,n \in \mathbb{Z}$. Thus,

$$\sum_p k_p(m+p)\delta_{m+p,-n} = \sum_p k_p m\delta_{m,-n-p}.$$

must hold. If $k_q \neq 0$ for some $q \neq 0$, then choosing m,n so that $m = -n-q \neq 0$, we obtain $m+q = m$, a contradiction. So it must be that $\mathfrak{z} = k_0 1$. Consequently, $\mathrm{Cent}(\mathcal{K}) = \mathbb{F}\,\mathrm{id}$.

5.2. **Lie algebras graded by nonreduced root systems (the BC_r case).** For simplicity, we will assume $r \geq 3$. Exceptional behavior occurs for small ranks, and somewhat different arguments need to be used for them. We will not address those cases here. In what follows, we will apply results from [ABG2] without specifically quoting chapter and verse.

When $r \geq 3$, each BC_r-graded Lie algebra $\mathcal{L}$ admits a decomposition,

$$\mathcal{L} = (\mathfrak{g}\otimes A) \oplus (\mathfrak{s}\otimes B) \oplus (V\otimes C) \oplus D,$$

relative to the grading subalgebra $\mathfrak{g}$. The spaces $\mathfrak{s}$ and V are irreducible $\mathfrak{g}$-modules and D is the sum of the trivial $\mathfrak{g}$-modules. Moreover,

(a) The sum $\mathfrak{a} = A \oplus B$ is a unital algebra with involution η whose symmetric elements are A and skew-symmetric elements are B.
(b) The algebra $\mathfrak{a}$ is associative in all cases except when $\mathfrak{g}$ is of type C_3. In that exceptional case $\mathfrak{a}$ is an alternative algebra, and the set A of symmetric elements must be contained in the nucleus (associative center) of $\mathfrak{a}$.
(c) The space C is a left $\mathfrak{a}$-module, and it is equipped with a hermitian or skew-hermitian form $\chi(\,,\,)$ depending on whether $\Delta_{\mathfrak{g}}$ is of type B_r, D_r or C_r.
(d) $\mathfrak{b} = \mathfrak{a} \oplus C$ is an algebra (the *coordinate algebra* of $\mathcal{L}$) with product given by

$$(\alpha + c)\cdot(\alpha' + c') = \alpha\alpha' + \chi(c,c') + \alpha.c' + \alpha'^{\eta}.c.$$

Now suppose $\Psi \in \mathrm{Cent}(\mathcal{L})$. Then by Lemma 5.1 (compare also Proposition 5.4), it follows that there are transformations $\psi_{\mathfrak{a}}$, ψ_C, and ψ_D such that

$$\Psi\big((x\otimes a)+(s\otimes b)+(v\otimes c)+d\big)=(x\otimes\psi_{\mathfrak{a}}(a))+(s\otimes\psi_{\mathfrak{a}}(b))+(v\otimes\psi_C(c))+\psi_D(d)$$

Since $\mathfrak{M}=(\mathfrak{g}\otimes A)\oplus(\mathfrak{s}\otimes B)\oplus\langle\mathfrak{a},\mathfrak{a}\rangle$ is a subalgebra having exactly the same multiplication as in (5.3) (think of $\mathfrak{s}$ as playing the role of W in the reduced case), we obtain just as before that there exists an element $\mathfrak{z}\in\mathfrak{Z}_{\mathfrak{a}}$ such that $\Psi(x\otimes a)=x\otimes\mathfrak{z}a$, $\Psi(s\otimes b)=s\otimes\mathfrak{z}b$, and $\Psi(\langle\alpha,\alpha'\rangle)=\langle\mathfrak{z}\alpha,\alpha'\rangle=\langle\alpha,\mathfrak{z}\alpha'\rangle$ for all $a\in A$, $b\in B$, $\alpha,\alpha'\in\mathfrak{a}$, $x\in\mathfrak{g}$, and $s\in\mathfrak{s}$.

Note that $\Psi\big([x\otimes 1,u\otimes c]\big)=\Psi\big(x.u\otimes c\big)=x.u\otimes\psi_C(c)$, which must equal $[\Psi(x\otimes 1),u\otimes c]=[x\otimes\mathfrak{z},u\otimes c]=x.u\otimes\mathfrak{z}.c$ for all $x\in\mathfrak{g}$, $v\in V$, and $c\in C$. Thus, $\psi_C(c)=\mathfrak{z}.c$. Applying the formulas in [ABG2, (2.8)], we determine from the relation $\Psi\big([u\otimes c,v\otimes c']\big)=[\Psi(u\otimes c),v\otimes c']=[u\otimes c,\Psi(v\otimes c')]$ that $\Psi\big(\langle c,c'\rangle\big)=\langle\mathfrak{z}.c,c'\rangle=\langle c,\mathfrak{z}.c'\rangle$ for all $c,c'\in C$. Since $D=\langle\mathfrak{a},\mathfrak{a}\rangle+\langle C,C\rangle$, this determines Ψ completely.

Theorem 5.18. *Let $\mathcal{L}=(\mathfrak{g}\otimes A)\oplus(\mathfrak{s}\otimes B)\oplus(V\otimes C)\oplus\langle\mathfrak{b},\mathfrak{b}\rangle$ denote a Lie algebra graded by BC_r for $r\geq 3$ with coordinate algebra $\mathfrak{b}=\mathfrak{a}\oplus C$ where $\mathfrak{a}=A\oplus B$. Then $\mathrm{Cent}(\mathcal{L})\cong\mathfrak{Z}_{\mathfrak{a}}$, where $\mathfrak{Z}_{\mathfrak{a}}$ is the set of elements in $Z(\mathfrak{a})\cap A$ which satisfy the following properties:*

(a) $\langle\mathfrak{z}\beta,\beta'\rangle=\langle\beta,\mathfrak{z}\beta'\rangle$ *for all* $\beta,\beta'\in\mathfrak{b}$*;*

(b) $\sum_t\langle\beta_t,\beta_t'\rangle=0$ *implies* $\sum_t\langle\mathfrak{z}\beta_t,\beta_t'\rangle=0$ *for all* $\beta_t,\beta_t'\in\mathfrak{b}$.

More specifically, if $\mathfrak{z}\in Z(\mathfrak{a})\cap A$ satisfies (a) and (b), and if $\Psi\in\mathrm{End}_{\mathbb{F}}(\mathcal{L})$ is given by

$$\begin{aligned}\Psi(x\otimes a) &= x\otimes\mathfrak{z}a\\ \Psi(s\otimes b) &= s\otimes\mathfrak{z}b\\ \Psi(v\otimes c) &= v\otimes\mathfrak{z}.c\\ \Psi\big(\langle\beta,\beta'\rangle\big) &= \langle\mathfrak{z}\beta,\beta'\rangle=\langle\beta,\mathfrak{z}\beta'\rangle\end{aligned}\tag{5.19}$$

for all $x\otimes a\in\mathfrak{g}\otimes A$, $s\otimes b\in\mathfrak{s}\otimes B$, $v\otimes c\in V\otimes C$, and $\beta,\beta'\in\mathfrak{b}$, then $\Psi\in\mathrm{Cent}(\mathcal{L})$; and every element of $\mathrm{Cent}(\mathcal{L})$ has this form.

References

[AABGP] B.N. Allison, S. Azam, S. Berman, Y. Gao, and A. Pianzola, *Extended Affine Lie Algebras and Their Root Systems*, Mem. Amer. Math. Soc. **126** vol. 603, Providence, R.I. 1997.

[ABG1] B.N. Allison, G. Benkart, and Y. Gao, Central extensions of Lie algebras graded by finite root systems, Math. Ann. **316** (2000), 499-527.

[ABG2] B.N. Allison, G. Benkart, and Y. Gao, *Lie Algebras Graded by the Root Systems* BC_r, $r \geq 2$, Mem. Amer. Math. Soc. **158** vol. 751, Providence, R.I. 2002.

[ABGP] B.N. Allison, S. Berman, Y. Gao and A. Pianzola, A characterization of affine Kac-Moody Lie algebras, Comm. Math. Phys. **185** (1997), 671–688.

[ABP1] B.N. Allison, S. Berman and A. Pianzola, Covering algebras II: Isomorphism of loop algebras, J. Reine Angew. Math. **571** (2004), 39–71.

[ABP2] B.N. Allison, S. Berman and A. Pianzola, Iterated loop algebras, preprint February 2005, to appear in Pacific J. Math.

[BM] G. Benkart and R.V. Moody, Derivations, central extensions, and affine Lie algebras, Algebras, Groups and Geom. **3** (1986) 456–492.

[BO] G. Benkart and J.M. Osborn, Derivations and automorphisms of nonassociative matrix algebras, Trans. Amer. Math. Soc. **263** (1981), 411–430.

[BS] G. Benkart and O. Smirnov, Lie algebras graded by the root system BC_1, J. Lie Theory **13** (2003), 91–132.

[BZ] G. Benkart and E. Zelmanov, Lie algebras graded by finite root systems and intersection matrix algebras, Invent. Math. **126** (1996), 1–45.

[BGKN] S. Berman, Y. Gao, Y. Krylyuk, and E. Neher, The alternative torus and the structure of elliptic quasi-simple Lie algebras of type A_2, Trans. Amer. Math. Soc. **347** (1995), 4315–4363.

[BeM] S. Berman and R.V. Moody, Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy, Invent. Math. **108** (1992), 323–347.

[Bo1] N. Bourbaki, *Algèbre*, Ch. II, Diffusion C.C.L.C, Paris 1970.

[Bo2] N. Bourbaki, *Groupes et Algèbres de Lie*, Ch. I, Hermann, Paris 1971.

[Bo3] N. Bourbaki, *Groupes et Algèbres de Lie*, Ch. VI, Masson, Paris 1981.

[F] R. Farnsteiner, Derivations and central extensions of finitely generated graded Lie algebras. J. Algebra **118** (1988), 33–45.

[J] N. Jacobson, *Lie Algebras*, Interscience Publishers, 1962.

[K] V.G. Kac, *Infinite Dimensional Lie Algebras*, Third Ed., Cambridge U. Press, Cambridge 1990.

[L] T.Y. Lam, *A First Course in Noncommutative Rings*, Graduate Texts in Math. vol. **131**, Springer-Verlag New York, 1991.

[Mc] K. McCrimmon, *A Taste of Jordan Algebras*, Universitext, Springer-Verlag, New York 2004.

[Me] D. Melville, Centroids of nilpotent Lie algebras, Comm. Algebra **20(12)** (1992), 3649–3682.

[MP] R.V. Moody and A. Pianzola, *Lie Algebras With Triangular Decompositions*, Canad. Math. Soc. Series of Monographs and Advanced Texts, John Wiley & Sons, New York 1995.

[N1] E. Neher, Lie algebras graded by 3-graded root systems, Amer. J. Math. **118** (1996), 439–491.

[N2] E. Neher, Lie tori, C. R. Math. Acad. Sci. Soc. R. Can. **26** (2004), no. 3, 84–89.

[N3] E. Neher, Extended affine Lie algebras, C. R. Math. Acad. Sci. Soc. R. Can. **26** (2004), no. 3, 90–96.

[P] D. Passman, *Infinite Crossed Products*, Pure and Applied Mathematics, **135** Academic Press, Inc., Boston, MA, 1989.

[Pi] A. Pianzola, Automorphisms of toroidal Lie algebras and their central quotients, J. Algebra Appl. **1** (2002), 113–121.

[Po] K. Ponomarëv, Invariant Lie algebras and Lie algebras with a small centroid, Algebra Logic **40** (2001), 365–377.

[S1] K. Saito, Extended affine root systems 1 (Coxeter transformations), Publ. RIMS, Kyoto Univ. **21** (1985) 75-179.

[S2] K. Saito, Extended affine root systems 2 (flat invariants), Publ. RIMS, Kyoto Univ. **26** (1990) 15-78.

[Se] G.B. Seligman, *Rational Methods in Lie Algebras*, Lect. Notes in Pure and Applied Math. **17** Marcel Dekker, New York 1976.

[Sl] P. Slodowy, A character approach to Looijenga's invariant theory for generalized root systems, Compositio Math. **55** (1985), 3-32.

[T] J. Tits, Une classe d'algébres de Lie en relation avec les algébres de Jordan, Indag. Math. **24** (1962), 530–535.

[W] W.C. Waterhouse, *Introduction to Affine Group Schemes*, Graduate Texts in Mathematics vol. **66**, Springer-Verlag, New York 1979.

[Y1] Y. Yoshii, Root-graded Lie algebras with compatible grading, Comm. Algebra **29** (2001), 3365–3391.

[Y2] Y. Yoshii, Classification of division $\mathbb{Z}^n$-graded alternative algebras, J. Algebra **256** (2002), 28–50.

[Y3] Y. Yoshii, Root systems extended by an abelian group and their Lie algebras, J. Lie Theory **14** (2004), 371–394.

[Y4] Y. Yoshii, Lie tori – A simple characterization of extended affine Lie algebras, preprint 2003.