STEINBERG GROUPS FOR JORDAN PAIRS – AN INTRODUCTION WITH OPEN PROBLEMS

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ABSTRACT. The paper gives an introduction to Steinberg groups for root graded Jordan pairs, a theory developed in the book [LN2] by Ottmar Loos and the author.

Dedicated to Vyjayanthi Chari at the occasion of her 60th birthday

INTRODUCTION

The connection between Jordan structures (Jordan algebras, Jordan pairs) and Lie algebras and groups has a long and successful history, starting with the work of Chevalley-Schafer [CS] and continued by Jacobson [Ja1, Ja2, Ja3], Kantor [Ka1, Ka2], Koecher [Ko1, Ko2, Ko3, Ko4], Loos [Lo2, Lo3, Lo5], Springer [Sp], Springer-Veldkamp [SV] and Tits [Ti1, Ti2].

The book [LN2] by Loos and the author is a further contribution to the theme "Groups and Jordan Structures". It contains a detailed study of Steinberg groups associated with certain types of Jordan pairs. These groups generalize the classical linear and unitary Steinberg groups of a ring by, roughly speaking, replacing associative coordinates with Jordan algebras or Jordan pairs. We are able to prove the basic results on Steinberg groups (central closedness, universal central extension in the stable case) in our setting, thereby recovering all previous results, except those on groups of type E_8, F_4 and G_2, and in addition deal with new types, not considered before. The main novelty however is our approach based on 3-graded root systems and Jordan pairs.

The present paper is an introduction to the theory developed in [LN2]. In §1 we describe the linear Steinberg group $\mathrm{St}(A)$ of a ring A from the point of view of Jordan pairs. This is motivation for §2 where we define the Steinberg group of a root graded Jordan pair and state the main results of [LN2] regarding these groups. The final section §3 discusses some open research problems in the area of Steinberg groups and Jordan pairs.

The paper does not assume any prior knowledge of linear Steinberg groups or Jordan pairs: all relevant definitions are given in the paper. We demonstrate their scope by many examples and refer the reader to [LN1] and [LN2] for most proofs. But we include the details of our discussion of the linear Steinberg group and the elementary linear group of a ring from the point of view of Jordan theory (1.7 – 1.10 and 1.12 respectively). We also give all details of our description of the Tits-Kantor-Koecher algebra and the projective elementary group of a rectangular Jordan pair (2.12, 2.13).

Notation. Throughout k is a unital commutative associative ring and A is a not necessarily commutative, but unital associative k-algebra. Its identity element and zero element are denoted 1_A and 0_A respectively. We will often simply write 1 for 1_A if A is clear from

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the context, and analogously for $0 \in A$. We use $A^\times$ to denote the invertible elements of A. If $k = \mathbb{Z}$ we will refer to A as a ring.

For non-empty sets I and J we denote by $\mathrm{Mat}_{IJ}(A)$ the k-module of $I \times J$-matrices over A, i.e., maps $x : I \times J \to A$ with only finitely many values different from 0. As usual, we write a matrix in the form $x = (x_{ij})_{(i,j)\in I\times J}$. In case $I = J$ we abbreviate $\mathrm{Mat}_I(A) = \mathrm{Mat}_{IJ}(A)$. This is an associative k-algebra with respect to ordinary matrix multiplication which is unital if and only if I is finite. We put $\mathrm{Mat}_n(A) = \mathrm{Mat}_I(A)$ if $|I| = n < \infty$. Here and in general $|I|$ denotes the cardinality of the set I. The identity element of $\mathrm{Mat}_n(A)$ is denoted $\mathbf{1}_n$, and the group $\mathrm{Mat}_n(A)^\times$ by $\mathrm{GL}_n(A)$.

The group commutator of elements g, h in a group G is $\big(g, h\big) = ghg^{-1}h^{-1}$.

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1. Elementary linear groups and their Steinberg groups

In this section we give an introduction to elementary linear groups over a ring A (1.1) and their associated Steinberg groups (1.3). After a review of central extensions in 1.4 we state the Kervaire-Milnor-Steinberg Theorem (1.6) which says, for example, that the stable Steinberg group is the universal central extension of the stable elementary group. We also exhibit a new set of generators and relations for the Steinberg groups considered in this section (1.7 – 1.9), which we take as axioms for a new Steinberg group defined in 1.10. The main result is Theorem 1.11: the classical and the new Steinberg groups are isomorphic.

1.1. **Elementary linear groups.** Let $n \in \mathbb{N}$, $n \geq 2$. As usual, $E_{ij} \in \mathrm{Mat}_n(A)$ is the $n \times n$-matrix with entry 1_A at the position (ij) and 0_A elsewhere. For $1 \leq i \neq j \leq n$ and $a \in A$ we put

$$\mathrm{e}_{ij}(a) = \mathbf{1}_n + aE_{ij}, \quad (a \in A)$$

The well-known multiplication rules $aE_{ij}\, bE_{kl} = \delta_{jk}\, abE_{il}$ for $a, b \in A$ imply

$$\text{(E1)} \qquad \mathrm{e}_{ij}(a)\,\mathrm{e}_{ij}(b) = \mathrm{e}_{ij}(a + b).$$

Hence $\mathrm{e}_{ij}(a)\,\mathrm{e}_{ij}(-a) = \mathbf{1}_n = \mathrm{e}_{ij}(-a)\,\mathrm{e}_{ij}(a)$, which shows that $\mathrm{e}_{ij}(a) \in \mathrm{GL}_n(A)$. The *elementary linear group (of rank n)* is the subgroup

$$\mathrm{E}_n(A) = \big\langle \mathrm{e}_{ij}(a) : 1 \leq i \neq j \leq n, a \in A \big\rangle$$

of $\mathrm{Mat}_n(A)^\times$ generated by all $\mathrm{e}_{ij}(a)$.

One easily verifies two further relations of the $\mathrm{e}_{ij}(a)$:

$$\text{(E2)} \qquad \big(\mathrm{e}_{ij}(a),\, \mathrm{e}_{kl}(b)\big) = \mathbf{1}_n \qquad (j \neq k, i \neq l),$$

$$\text{(E3)} \qquad \big(\mathrm{e}_{ij}(a),\, \mathrm{e}_{jl}(b)\big) = \mathrm{e}_{il}(ab) \qquad (i, j, l \neq).$$

Taking the inverse of (E3) and using $\big(g, h\big)^{-1} = \big(h, g\big)$ yields the equivalent relation

$$\text{(E4)} \qquad \big(\mathrm{e}_{ij}(a),\, \mathrm{e}_{ki}(b)\big) = \mathrm{e}_{kj}(-ba) \qquad (i, j, k \neq).$$

We will also need an infinite variant of $\mathrm{Mat}_n(A)$ and the group $\mathrm{E}_n(A)$. Let

$$\mathrm{Mat}_{\mathbb{N}}(A)$$

be the set of all $\mathbb{N} \times \mathbb{N}$-matrices $x = (x_{ij})_{i,j\in\mathbb{N}}$ with entries from A. Since $\mathrm{Mat}_{\mathbb{N}}(A)$ is a non-unital k-algebra,

, but with only finitely many $x_{ij} \neq 0$. The usual addition and multiplication of matrices are well-defined operations on $\mathrm{Mat}_{\mathbb{N}}(A)$ satisfying all axioms of a ring, except the existence of an identity element. To remedy this, let $\mathbf{1}_{\mathbb{N}} = \operatorname{diag}(1_A, 1_A, \dots)$ be the diagonal matrix of size $\mathbb{N} \times \mathbb{N}$ with every diagonal entry being 1_A. Then

$$\operatorname{Mat}_{\mathbb{N}}(A)_{\mathrm{ex}} := k\mathbf{1}_{\mathbb{N}} + \operatorname{Mat}_{\mathbb{N}}(A)$$

is a ring with the usual addition and multiplication of matrices. Its identity element is $\mathbf{1}_{\mathbb{N}}$ and its zero element is the zero matrix, see for example [HO, 1.2B] where this ring is denoted $\operatorname{Mat}_\infty(A)$ (its elements are the $\mathbb{N} \times \mathbb{N}$-matrices with entries from A which have only finitely many non-zero entries off the diagonal and whose diagonal elements become eventually constant).

We associate with $x \in \operatorname{Mat}_n(A)$ the matrix $\iota_n(x) \in \operatorname{Mat}_{\mathbb{N}}(A)_{\mathrm{ex}}$ by putting x in the upper left corner and filling the diagonal outside x with 1_A:

$$\iota_n(x) = \begin{pmatrix} x & 0 \\ 0 & \operatorname{diag}(1_A, \dots) \end{pmatrix}$$

Then ι_n maps invertible matrices in $\operatorname{Mat}_n(A)$ to invertible matrices of $\operatorname{Mat}_{\mathbb{N}}(A)_{\mathrm{ex}}$, in particular $\iota_n\big(\mathrm{e}_{ij}(a)\big) \in \operatorname{Mat}_{\mathbb{N}}(A)_{\mathrm{ex}}^\times$. Since $\iota_n\big(\mathrm{e}_{ij}(a)\big) = \iota_p\big(\mathrm{e}_{ij}(a)\big)$ for $p \ge n$, we can take the maps ι_n as identification and view all $\mathrm{e}_{ij}(a)$, $i, j \in \mathbb{N}$ with $i \neq j$, as elements of $\operatorname{Mat}_{\mathbb{N}}(A)_{\mathrm{ex}}^\times$. The *(stable) elementary linear group* is the subgroup $\mathrm{E}(A)$ of $\operatorname{Mat}_{\mathbb{N}}(A)_{\mathrm{ex}}^\times$ generated by all the $\mathrm{e}_{ij}(a)$:

$$\mathrm{E}(A) = \big\langle \mathrm{e}_{ij}(a) : i, j \in \mathbb{N}, i \neq j \big\rangle.$$

It is immediate that the relations (E1) – (E4) also hold in $\mathrm{E}(A)$. The group $\mathrm{E}(A)$ is canonically isomorphic to the limit of the inductive system $(\mathrm{E}_n(A), \iota_{pn})$ where $\iota_{pn} \colon \mathrm{E}_n(A) \to \mathrm{E}_p(A)$ for $p \ge n$ is defined by taking the left upper $(p \times p)$-corner of $\iota_n(x)$.

1.2. **Why is $\mathrm{E}_n(A)$ important?** One reason is that $\mathrm{E}_n(F) = \mathrm{SL}_n(F)$ in case of $A = F$ is a field – in other words, every matrix of determinant 1 can be reduced to the identity matrix by elementary row and column reductions. The equality $\mathrm{E}_n(A) = \mathrm{SL}_n(A)$ even holds for a noncommutative local ring, for example a division ring, if one uses the Dieudonné determinant ([HO, 2.2.2] or [Ro, 2.2.2–2.2.6]). If A is commutative then obviously $\mathrm{E}_n(A) \subset \mathrm{SL}_n(A)$. Equality holds for example if A is a Euclidean ring [HO, 1.2.11].

While all of this is interesting, the real interest in $\mathrm{E}_n(A)$ and $\mathrm{E}(A)$ stems from their connection to Steinberg groups of A and to the K-group $K_2(A)$, defined in (1.3.2).

1.3. **The Steinberg groups $\mathrm{St}_n(A)$ and $\mathrm{St}(A)$.** We assume $n \in \mathbb{N}$, $n \ge 3$ (the case $n = 2$ is uninteresting since then the definitions below yield free products of A. The group $\mathrm{St}_2(A)$ is defined differently, see e.g. [Mi]; it will not play a role in this paper).

The *Steinberg group* $\mathrm{St}_n(A)$ is the group presented by

- generators $\mathrm{x}_{ij}(a)$, $1 \le i \neq j \le n$ and $a \in A$, and
- relations (E1) – (E3) of §1.1:

$$\begin{aligned} \mathrm{x}_{ij}(a)\, \mathrm{x}_{ij}(b) &= \mathrm{x}_{ij}(a+b) \quad \text{ for all } 1 \le i \neq j \le n \text{ and } a, b \in A, \\ \big(\mathrm{x}_{ij}(a), \mathrm{x}_{kl}(b)\big) &= 1 \quad \text{if } j \neq k \text{ and } l \neq i, \\ \big(\mathrm{x}_{ij}(a), \mathrm{x}_{jl}(b)\big) &= \mathrm{x}_{il}(ab) \quad \text{if } i, j, l \neq. \end{aligned}$$

The *(stable) Steinberg group* $\mathrm{St}(A)$ is the group presented by

- generators $\mathrm{x}_{ij}(a)$, $i, j \in \mathbb{N}$ distinct, $a \in A$, and
- relations (E1) – (E3) for $i, j \in \mathbb{N}$.

Since the defining relations (E1) – (E3) hold in $\mathrm{E}_n(A)$ and $\mathrm{E}(A)$ we get surjective group homomorphisms

$$\wp_n \colon \mathrm{St}_n(A) \to \mathrm{E}_n(A) \qquad \text{and} \qquad \wp \colon \mathrm{St}(A) \to \mathrm{E}(A) \tag{1.3.1}$$

determined by $\mathrm{x}_{ij}(a) \mapsto \mathrm{e}_{ij}(a)$. The second K-group of A is then defined as

$$K_2(A) := \mathrm{Ker}(\wp). \tag{1.3.2}$$

This is an important but also mysterious group, even for fields. The reader can find more about this group in the classic [Mi], and in [HO, Ch. 1], [Ma, Parts IV and V], [Ro, Ch. 4], or [We, III] (the list is incomplete).

To put all of this in a bigger picture, we make a short detour on central extensions of groups.

1.4. **Central extensions.** Let G be a group. An *extension of G* is a surjective group homomorphism $p\colon E \to G$. An extension is called *central* if $\mathrm{Ker}(p)$ is contained in the centre of the group G. A central extension $q\colon X \to G$ is a *universal central extension* if for all central extensions $p\colon E \to G$ there exists a unique homomorphism $f\colon X \to E$ such that $q = p \circ f$:

$$\begin{tikzcd} X \arrow[rr, dotted, "\exists!\, f"] \arrow[dr, "q"'] & & E \arrow[dl, "p"] \\ & G & \end{tikzcd}$$

A group X is called *centrally closed* if $\mathrm{Id}_X\colon X \to X$ is a universal central extension. Thus, X is centrally closed if and only if every central extension $p\colon E \to X$ splits uniquely in the sense that there exists a unique group homomorphism $f\colon X \to E$ satisfying $p \circ f = \mathrm{Id}_X$. The concepts defined above are related by the following facts, proved for example in [HO, 1.4C], [Mi, §5], [Ro, 4.1] or [St2, §7].

(a) A group G has a universal central extension if and only if it is *perfect*, i.e., generated by all commutator (g, h) of $g, h \in G$. In particular, a centrally closed group is perfect.

(b) For two universal central extensions of a group G, say $q\colon X \to G$ and $q'\colon X' \to G$, there exists an isomorphism $f\colon X \to X'$ of groups, uniquely determined by the condition $q = q' \circ f$.

(c) Let $q\colon X \to G$ be a universal central extension, whence G is perfect by (a). Then X is centrally closed and thus also perfect, again by (a). Obviously, G is a central quotient of X. The following fact (d) says that every universal central extension of G is obtained as a central quotient of a centrally closed group.

(d) A surjective group homomorphism $q\colon X \to G$ is a universal central extension if and only if (i) X is centrally closed and (ii) $\mathrm{Ker}(q)$ is central.

(e) Let $f\colon X \to G$ and $g\colon G \to \bar{G}$ be central extensions. Then f is a universal central extension if and only if $g \circ f$ is a universal central extension.

To describe a universal central extension of a group G we have, by (d) and (e), two approaches:

(I) Find successive central extensions $G_1 \to G_0 = G, \dots, G_n \to G_{n-1}, \dots$ until one of them, say $G_n \to G_{n-1}$, becomes universal, and then take the composition $G_n \to G$ of these central extensions, or

(II) find an extension $q\colon X \to G$ with X centrally closed and then find conditions for $\mathrm{Ker}(q)$ to be central.

Although (I) sees to be the more natural approach, we will actually follow (II). But first back to Steinberg groups.

In [St1] Steinberg gave a very elegant description of the universal central extension of a perfect Chevalley group over a field. “Most” Chevalley groups are perfect by [St2, Lemma 32]. In particular, for $n \geq 2$ and F a field, the group $\mathrm{SL}_n(F) = \mathrm{E}_n(F)$ (equality by 1.2) is a Chevalley group, and it is perfect if $n \geq 4$ or if $n = 3$ and $|F| \geq 3$ or if $n = 2$ and $|F| \geq 4$. A special case of Steinberg's result in [St1] is the following theorem.

1.5. **Theorem** ([St1], [St2, Thm. 10], [St3, Thm. 1.1])**.** *Let $n \in \mathbb{N}$, $n \geq 2$ and let F be a field satisfying $|F| > 4$ if $n = 3$ and $|F| \notin \{2, 3, 4, 9\}$ if $n = 2$. Then the map $\wp_n \colon \mathrm{St}_n(F) \to \mathrm{E}_n(F)$ of (1.3.1) is a universal central extension.*

We have defined the maps $\wp_n \colon \mathrm{St}_n(A) \to \mathrm{E}_n(A)$ and $\wp \colon \mathrm{St}(A) \to \mathrm{E}(A)$ for any ring A. It is therefore natural to ask if Theorem 1.5 even holds for rings. An answer is given in the following Kervaire-Milnor-Steinberg Theorem.

1.6. **Kervaire-Milnor-Steinberg Theorem** ([Ke, Mi, St2])**.** *For an arbitrary ring A,*

(a) *the group $\mathrm{St}_n(A)$, $n \geq 5$, is centrally closed.*

(b) *The map $\wp \colon \mathrm{St}(A) \to \mathrm{E}(A)$ is a universal central extension.*

An indication of the proof of (a) can be found in see [HO, 1.4.12] or [St2, Cor. 1]. The attribution of part (b) of this theorem is somewhat complicated. Kervaire cites a preliminary version of [Mi], Milnor attributes it to Steinberg and Kervaire ([Mi, p. 43]), and Steinberg says that (b), proved in [St2, Thm. 14], is "based in part on a letter from J. Milnor".

In view of (a), the map $\wp_n \colon \mathrm{St}_n(A) \to \mathrm{E}_n(A)$, $n \geq 5$, is a universal central extension if and only if it is a central extension. It is known that this is not always the case, see [HO, 4.2.20]. However, by 1.4(c), both $\mathrm{St}_n(A)$ and $\mathrm{St}(A)$ are centrally closed. It is this result that we will be concentrating on, following the strategy 1.4(II).

1.7. **Another look at $\mathrm{St}_n(A)$ and $\mathrm{St}(A)$: using root systems.** To treat $\mathrm{St}_n(A)$, $n \in \mathbb{N}$, $n \geq 3$, and $\mathrm{St}(A)$ at the same time we use the subset $N \subset \mathbb{N}$, which is the finite integer interval $N = [1, n]$ or $N = \mathbb{N}$. We can then put

$$\mathrm{St}_N(A) = \begin{cases} \mathrm{St}_n(A) & \text{if } N = [1, n], \\ \mathrm{St}(A) & \text{if } N = \mathbb{N}. \end{cases}$$

By definition in 1.3, $\mathrm{St}_N(A)$ is generated by $\mathrm{x}_{ij}(a)$, $(i, j) \in N \times N$, $i \neq j$, and $a \in A$. We replace this indexing set by the root system $\dot{\mathrm{A}}_N$ (notation of 2.2), which we realize in the Euclidean space $X = \bigoplus_{i \in N} \mathbb{R}\varepsilon_i$ with basis $(\varepsilon_i)_{i \in N}$ and inner product $(\,|\,)$ defined by $(\varepsilon_i|\varepsilon_j) = \delta_{ij}$:

$$R = \dot{\mathrm{A}}_N = \{\varepsilon_i - \varepsilon_j : i, j \in N\}, \qquad R^\times = R \setminus \{0\}.$$

Thus $R^\times = \mathrm{A}_{n-1}$ for $N = [1, n]$ in the traditional notation, while for $N = \mathbb{N}$ we get an infinite locally finite root system – a concept that we will review later in 2.1. For the moment, it suffices to use the concretely given R above.

For α, $\beta \in R$ one easily checks that $(\alpha|\beta) \in \{0, \pm 1, \pm 2\}$ with $(\alpha|\beta) = \pm 2 \iff \alpha = \pm\beta$. To conveniently describe the remaining cases we use the symbols

$$\alpha \perp \beta \iff (\alpha|\beta) = 0 \qquad \text{and} \qquad \alpha \text{---} \beta \iff (\alpha|\beta) = 1. \tag{1.7.1}$$

A straightforward analysis of the indices shows for $\alpha = \varepsilon_i - \varepsilon_j$ and $\beta = \varepsilon_k - \varepsilon_l \in R$ that

$$\begin{aligned} \alpha \perp \beta \ \text{ or } \ \alpha \text{---} \beta \quad &\iff \quad j \neq k \ \text{ and } \ l \neq i, \\ \alpha \text{---} (-\beta) \quad &\iff \quad j = k, i, j, l \neq \ \text{ or } \ i = l, i, j, k \neq . \end{aligned} \tag{1.7.2}$$

Hence, putting

$$\mathrm{x}_\alpha(a) = \mathrm{x}_{ij}(a) \qquad \text{for } \alpha = \varepsilon_i - \varepsilon_j \in R^\times,$$

the relations (E1) – (E4) can be rewritten in terms of roots as follows. Let $\alpha,\,\beta\in R^\times$ and denote the relations corresponding to (Ei) by (ERi). Then the previous relations read

(ER1) $\quad \big(\mathrm{x}_\alpha(a),\, \mathrm{x}_\alpha(b)\big) = 1,$

(ER2) $\quad \big(\mathrm{x}_\alpha(a),\, \mathrm{x}_\beta(b)\big) = 1, \qquad$ if $\alpha \perp \beta$ or $\alpha \mathrel{—} \beta$,

(ER3) $\quad \big(\mathrm{x}_\alpha(a),\, \mathrm{x}_\beta(b)\big) = \mathrm{x}_{\alpha+\beta}(ab) \qquad$ if $\alpha = \varepsilon_i - \varepsilon_j,\ \beta = \varepsilon_j - \varepsilon_l,\ i, j, l \neq,$

(ER4) $\quad \big(\mathrm{x}_\alpha(a),\, \mathrm{x}_\beta(b)\big) = \mathrm{x}_{\alpha+\beta}(-ba) \qquad$ if $\alpha = \varepsilon_i - \varepsilon_j,\ \beta = \varepsilon_k - \varepsilon_i,\ i, j, k \neq.$

In particular, the two cases for $\alpha \mathrel{—} (-\beta)$ in (1.7.2) correspond precisely to the relations (ER3) and (ER4).

1.8. **Another look at $\mathrm{St}_N(A)$: fewer generators.** We continue with N and R as in 1.7. In addition we choose a nontrivial partition

$$N = I \mathbin{\dot\cup} J, \qquad \emptyset \neq I \neq N,$$

which we fix in the following. It induces a non-trivial partition

$$R = R_1 \mathbin{\dot\cup} R_0 \mathbin{\dot\cup} R_{-1}, \tag{1.8.1}$$

whose parts are

$$\begin{aligned} R_1 &= \{\varepsilon_i - \varepsilon_j : i \in I,\, j \in J\}, \\ R_{-1} &= \{\varepsilon_j - \varepsilon_i : i \in I, j \in J\} = -R_1, \\ R_0 &= \{\varepsilon_k - \varepsilon_l : (k,l) \in I \times I \text{ or } (k,l) \in J \times J\} = \dot{\mathrm{A}}_I \times \dot{\mathrm{A}}_J. \end{aligned}$$

The partition $R = R_1 \mathbin{\dot\cup} R_0 \mathbin{\dot\cup} R_{-1}$ will later be seen to be an example of a 3-grading of R, but we do not need this now. Observe that every $\mu \in R_0$ can be written (not uniquely) as $\mu = \alpha - \beta$ with α and $\beta \in R_1$ satisfying $\alpha \mathrel{—} \beta$. Indeed,

(i) if $\mu = \varepsilon_k - \varepsilon_l$ with $k, l \in I$ then $\mu = (\varepsilon_k - \varepsilon_j) - (\varepsilon_l - \varepsilon_j)$ for any $j \in J$, hence

$$\mathrm{x}_\mu(a) = \mathrm{x}_{kl}(a) = \big(\mathrm{x}_{kj}(a),\, \mathrm{x}_{jl}(1)\big) = \big(\mathrm{x}_\alpha(a),\, \mathrm{x}_{-\beta}(1)\big) \tag{1.8.2}$$

by (ER3) for $\alpha = \varepsilon_k - \varepsilon_j$ and $\beta = \varepsilon_l - \varepsilon_j \in R_1$, and

(ii) if $\mu = \varepsilon_k - \varepsilon_l$ with $k, l \in J$ then $\mu = (\varepsilon_i - \varepsilon_l) - (\varepsilon_i - \varepsilon_k)$ for any $i \in J$, hence

$$\mathrm{x}_\mu(a) = \mathrm{x}_{kl}(a) = \big(\mathrm{x}_{il}(-a),\, \mathrm{x}_{ki}(1)\big) = \big(\mathrm{x}_\alpha(-a),\, \mathrm{x}_{-\beta}(1)\big) \tag{1.8.3}$$

by (ER4) for $\alpha = \varepsilon_i - \varepsilon_l$ and $\beta = \varepsilon_i - \varepsilon_k \in R_1$.

The equations (1.8.2) and (1.8.3) show that $\mathrm{St}_N(A)$ is already generated by

$$\{\mathrm{x}_\alpha(a) : \alpha \in R_1 \cup R_{-1},\, a \in A\}. \tag{1.8.4}$$

1.9. **Another look at $\mathrm{St}_N(A)$: fewer relations.** Our next goal is to rewrite some of the relations (ER1) – (ER4) in terms of the smaller generating set (1.8.4). Each of these relations depend on two roots $\xi,\, \tau \in R$. Because of $\big(g,\, h\big)^{-1} = \big(h,\, g\big)$ we only need to consider the relations involving (ξ, τ) lying in one of the following subsets of $R \times R$:

$$R_1 \times R_1, \quad R_{-1} \times R_{-1}, \quad R_1 \times R_{-1}, \quad R_0 \times R_1, \quad R_0 \times R_{-1}, \quad R_0 \times R_0.$$

(a) Case $(\xi, \tau) = (\alpha, \beta) \in R_1 \times R_1$: Given $\alpha, \beta \in R_1$, exactly one of the relations $\alpha = \beta$, $\alpha \mathrel{—} \beta$, $\alpha \perp \beta$ holds. Hence only (ER1) and (ER2) apply in this case and yield

$$\big(\mathrm{x}_\alpha(a),\, \mathrm{x}_\beta(b)\big) = 1 \qquad \text{for } \alpha,\, \beta \in R_1 \text{ and } a, b \in A. \tag{1.9.1}$$

It will now be more convenient to change notation (again) and put for $\alpha = \varepsilon_i - \varepsilon_j \in R_1$ and $u_\alpha = aE^+_\alpha$

$$E^+_\alpha = E_{ij}, \qquad \mathrm{x}'_+(u_\alpha) = \mathrm{x}_\alpha(a) = \mathrm{x}_{ij}(a) \tag{1.9.2}$$

$$V^+_\alpha = AE^+_\alpha, \qquad V^+ = \textstyle\bigoplus_{\alpha\in R_1} V^+_\alpha = \bigoplus_{(ij)\in I\times J} AE_{ij}. \tag{1.9.3}$$

Because of (1.9.1) the map

$$\mathrm{x}'_+ \colon V^+ \longrightarrow \mathrm{St}_N(A), \quad u = \textstyle\sum_{\alpha\in R_1} u_\alpha \;\mapsto\; \prod_{\alpha\in R_1} \mathrm{x}'_+(u_\alpha) \tag{1.9.4}$$

is well-defined (independent of the chosen order for $\prod_{\alpha\in R_1}$) and satisfies

$$\mathrm{x}'_+(u+u') = \mathrm{x}'_+(u)\,\mathrm{x}'_+(u') \qquad \text{for } u, u' \in V^+. \tag{1.9.5}$$

It is clear that conversely (1.9.5) implies (1.9.1).

(b) Case $(\xi, \tau) = (-\alpha, -\beta) \in R_{-1} \times R_{-1}$: This case is analogous to Case (a). Given $\alpha = \varepsilon_i - \varepsilon_j \in R_1$ and $v_\alpha = aE^-_\alpha$ we define

$$\begin{aligned} &E^-_\alpha = E_{ji}, && \mathrm{x}_-(v_\alpha) = \mathrm{x}_{-\alpha}(-a) = \mathrm{x}_{ji}(-a),\\ &V^-_\alpha = AE^-_\alpha, && V^- = \textstyle\bigoplus_{\alpha\in R_1} V^-_\alpha = \bigoplus_{(ji)\in J\times I} AE_{ji}. \end{aligned} \tag{1.9.6}$$

(The minus sign in the definition of $\mathrm{x}_-(v_\alpha)$ is not significant, but has been included so that the relations below are precisely those used later on. It avoids a minus sign in the formula (1.12.3).) As in Case (a) the relations (ER1) – (ER4) for $(-\alpha, -\beta) \in R_{-1} \times R_{-1}$ yield $\big(\mathrm{x}'_-(v_\alpha),\, \mathrm{x}'_-(v'\beta)\big) = 1$ and thus give rise to a well-defined map

$$\mathrm{x}'_- \colon V^- \longrightarrow \mathrm{St}_N(A), \quad v = \textstyle\sum_{\alpha\in R_1} a_\alpha E^-_\alpha \;\mapsto\; \prod_{\alpha\in R_1} \mathrm{x}'_-(-a_\alpha E^-_\alpha) \tag{1.9.7}$$

satisfying

$$\mathrm{x}'_-(v+v') = \mathrm{x}'_-(v)\,\mathrm{x}'_-(v') \qquad \text{for } v, v' \in V^-. \tag{1.9.8}$$

At this point we obtain a new generating set of $\mathrm{St}_N(A)$,

$$\mathrm{St}_N(A) = \big\langle \mathrm{x}'_+(V^+) \cup \mathrm{x}'_-(V^-) \big\rangle, \tag{1.9.9}$$

(c) Case $(\xi, \tau) = (\alpha, -\beta) \in R_1 \times R_{-1}$: From this case we will only explicitly keep the relation (ER2), which in our new notation says

$$\big(\mathrm{x}'_+(u),\, \mathrm{x}'_-(v)\big) = 1 \qquad \text{for } (u, v) \in V^+_\alpha \times V^-_\beta \text{ with } \alpha \perp \beta. \tag{1.9.10}$$

In the following Case (d) we use the relations (ER3) and (ER4) for $(\alpha, -\beta) \in R_1 \times R_{-1}$ in double commutators.

(d) Case $(\xi, \tau) = (\mu, \gamma) \in R_0 \times R_1$: To deal with this case, we view the elements of V^+ as $I \times J$-matrices with only finitely many non-zero entries, as in (1.9.2). Similarly, elements in V^- are $J \times I$-matrices with finitely many non-zero entries. Matrix multiplication of matrices in $V^+ \times V^- \times V^+$ is then well-defined and yields the *Jordan triple product* $\{\cdots\}$, i.e., the map

$$\{\cdots\} \colon V^+ \times V^- \times V^+ \longrightarrow V^+, \quad (x, y, x) \mapsto \{x\,y\,z\} := xyz + zyx.$$

We claim that (ER2) – (ER4) imply

$$\begin{aligned} &\big(\big(\mathrm{x}'_+(u_\alpha),\, \mathrm{x}'_-(v_\beta)\big),\, \mathrm{x}'_+(z_\gamma)\big) = \mathrm{x}'_+(-\{u_\alpha\, v_\beta\, z_\gamma\})\\ &\qquad \text{for } \alpha, \beta, \gamma \in R_1 \text{ with } \alpha \,\text{—}\, \beta \text{ and all } u_\alpha \in V^+_\alpha,\ v_\beta \in V^-_\beta,\ z_\gamma \in V^+_\gamma \end{aligned} \tag{1.9.11}$$

We prove this by evaluating all possibilities for $\mu = \alpha - \beta$ with $\alpha, \beta \in R_1$ satisfying $\alpha \text{ — } \beta$ and $\gamma = \varepsilon_r - \varepsilon_s \in R_1$. By 1.8(i) and 1.8(ii) there are two cases for such a representation of μ, discussed below as (I) and (II).

(I) $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = \varepsilon_i - \varepsilon_l$ for $i \in I$ and $j, l \in J$ distinct. Thus $\mu = \alpha - \beta = \varepsilon_l - \varepsilon_j$. We let $u_\alpha = aE_{ij}$, $v_\beta = bE_{li}$, $z_\gamma = cE_{rs}$. Then, by (ER4) – (E4),

$$\begin{aligned}
&\big(\big(\mathrm{x}'_+(u_\alpha),\, \mathrm{x}'_-(v_\beta)\big),\, \mathrm{x}'_+(z_\gamma)\big) = \big(\big(\mathrm{x}_{ij}(a),\, \mathrm{x}_{li}(-b)\big),\, \mathrm{x}_{rs}(c)\big) = \big(\mathrm{x}_{lj}(ba),\, \mathrm{x}_{rs}(c)\big) =: A,\\
&\{u_\alpha\, v_\beta\, z_\gamma\} = \{aE_{ij}\, bE_{li}\, cE_{rs}\} = \delta_{sl}\, cbaE_{rj} =: B.
\end{aligned}$$

If $l = s$ then, again by (E4), $A = \mathrm{x}_{rj}(-cba)$, while $B = cba\, E_{rj}$. Otherwise $l \neq s$, whence $A = 1$ by (E2) and clearly $B = 0$. This finishes the proof of (1.9.11) in case (I).

(II) $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = \varepsilon_k - \varepsilon_j$ for distinct $i, k \in I$ and $j \in J$. This can be shown in the same way as (I).

To obtain a slightly simpler version of (1.9.11) we apply the commutator formula

$$\big(g,\, h_1 h_2\big) = \big(g,\, h_1\big) \cdot \big(g,\, h_2\big) \cdot \big(\big(h_2,\, g\big),\, h_1\big)$$

with $g = \big(\mathrm{x}'_+(u_\alpha),\, \mathrm{x}'_-(v_\beta)\big)$, $h_1 = \mathrm{x}'_+(z_\gamma)$ and $h_2 = \mathrm{x}'_+(z_\delta)$ for arbitrary $\delta \in R_1$. We obtain $\big(g,\, h_1 h_2\big) = \big(g,\, h_1\big) \cdot \big(g,\, h_2\big)$, which allows us to rewrite (1.9.11) in the form

$$\begin{aligned}
&\big(\big(\mathrm{x}'_+(u_\alpha),\, \mathrm{x}'_-(v_\beta)\big),\, \mathrm{x}'_+(z)\big) = \mathrm{x}'_+(-\{u_\alpha\, v_\beta\, z\})\\
&\qquad \text{for } \alpha, \beta \in R_1 \text{ with } \alpha \text{ — } \beta \text{ and arbitrary } u_\alpha \in V^+_\alpha,\ v_\beta \in V^-_\beta,\ z \in V^+.
\end{aligned} \tag{1.9.12}$$

(e) Case $(\xi, \tau) = (\mu, -\gamma) \in R_0 \times R_{-1}$. We proceed as in Case (d) and define the Jordan triple product

$$\{\cdots\}\colon V^- \times V^+ \times V^- \longrightarrow V^-, \quad (x, y, x) \mapsto \{x\, y\, z\} := xyz + zyx,$$

using matrix multiplication in the definition of $\{\cdots\}$. As in Case (d) one then proves the relation

$$\begin{aligned}
&\big(\big(\mathrm{x}'_-(v_\alpha),\, \mathrm{x}'_+(u_\beta)\big),\, \mathrm{x}'_-(w)\big) = \mathrm{x}'_-(-\{v_\alpha\, u_\beta\, w\})\\
&\qquad \text{for } \alpha, \beta \in R_1 \text{ with } \alpha \text{ — } \beta \text{ and arbitrary } v_\alpha \in V^-_\alpha,\ u_\beta \in V^+_\beta,\ w \in V^-.
\end{aligned} \tag{1.9.13}$$

(f) Case $(\xi, \tau) \in R_0 \times R_0$: As we will see below, the relations involving these (ξ, τ) are not needed for presenting $\mathrm{St}_N(A)$.

1.10. **The Steinberg group $\mathrm{St}(\mathbb{M}_{IJ}(A), \mathfrak{R})$.** We keep the setting of (1.7) – (1.9). In 1.9 we defined a pair of matrix spaces,

$$(V^+, V^-) = \big(\mathrm{Mat}_{IJ}(A),\, \mathrm{Mat}_{JI}(A)\big) =: \mathbb{M}_{IJ}(A),$$

and Jordan triple products

$$\{\cdots\}\colon V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma, \quad (x, y, z) \mapsto \{x\, y\, z\} = xyz + zyx$$

for $\sigma \in \{+, -\}$. In (1.9.3) and (1.9.6) we also introduced a family

$$\mathfrak{R} = (V_\alpha)_{\alpha \in R_1}, \quad V_\alpha = (V^+_\alpha, V^-_\alpha)$$

of pairs of subgroups with the property that $V^\sigma = \bigoplus_{\alpha \in R_1} V^\sigma_\alpha$. Furthermore, in (1.9.9) we found a new generating set for $\mathrm{St}_N(A)$, and we rewrote some of the relations defining $\mathrm{St}_N(A)$ in terms of this new generating set. It is then natural to define a new Steinberg group using these new generators and relations.

The *Steinberg group* $\mathrm{St}(\mathbb{M}_{IJ}(A), \mathfrak{R})$ is the group presented by

- generators $\mathrm{x}_+(u)$, $u \in V^+$, and $\mathrm{x}_-(v)$, $v \in V^-$, and

- the relations (1.9.5), (1.9.8), (1.9.10), (1.9.12) and (1.9.13). Taking $\sigma \in \{+,-\}$ these are

(EJ1) $$\mathrm{x}_\sigma(u+u') = \mathrm{x}_\sigma(u)\,\mathrm{x}_\sigma(u') \quad \text{for } u, u' \in V^\sigma,$$

(EJ2) $$\big(\mathrm{x}_+(u),\, \mathrm{x}_-(v)\big) = 1 \quad \text{for } (u,v) \in V^+_\alpha \times V^-_\beta,\ \alpha \perp \beta,$$

(EJ3) $$\big(\big(\mathrm{x}_\sigma(u),\, \mathrm{x}_{-\sigma}(v)\big),\, \mathrm{x}_-(z)\big) = \mathrm{x}_\sigma(-\{u\,v\,z\})$$
$$\text{for } u_\alpha \in V^\sigma_\alpha,\ v \in V^{-\sigma}_\beta,\ z \in V^\sigma \text{ with } \alpha \,—\, \beta.$$

(The letter "J" in (EJi) stands for "Jordan", to be explained in the next section.)

From the review above, it is clear that we have a surjective homomorphism of groups

(1.10.1) $$\Phi\colon\ \mathrm{St}(\mathbb{M}_{IJ}(A), \mathfrak{R}) \to \mathrm{St}_N(A), \quad \mathrm{x}_\sigma(u) \mapsto \mathrm{x}'_\sigma(u)$$

where x'_σ is defined in (1.9.4) and (1.9.7). Moreover, composing Φ with the surjective group homomorphisms $\wp_N\colon \mathrm{St}_N(A) \to \mathrm{E}_N(A)$ of (1.3.1) yields another surjective group homomorphism

(1.10.2) $$\mathfrak{p}_N\colon\ \mathrm{St}(\mathbb{M}_{IJ}(A), \mathfrak{R}) \to \mathrm{E}_N(A), \qquad \mathrm{x}_+(aE_{ij}) \mapsto \mathrm{e}_{ij}(a), \quad \mathrm{x}_-(aE_{ji}) \mapsto \mathrm{e}_{ji}(-a)$$

and hence a commutative diagram

(1.10.3) $$\begin{tikzcd} \mathrm{St}(\mathbb{M}_{IJ}(A), \mathfrak{R}) \arrow[rr, "\Phi"] \arrow[dr, "\mathfrak{p}_N"'] & & \mathrm{St}_N(A) \arrow[dl, "\wp_N"] \\ & \mathrm{E}_N(A) & \end{tikzcd}$$

1.11. **Theorem** ([LN2]). *The map Φ of (1.10.1) is an isomorphism of groups.*

In particular, $\mathrm{St}(\mathbb{M}_{IJ}(A), \mathfrak{R})$ *is centrally closed if* $|N| \ge 5$ *and* $\mathfrak{p}_N$ *is a universal central extension of* $\mathrm{E}(A)$ *if* $N = \mathbb{N}$.

Proof. To prove bijectivity of Φ is bijective, a canonical approach is to show that the family of $\mathrm{x}_+(aE_{ij})$ and $\mathrm{x}_-(-bE_{ij}) \in \mathrm{St}(\mathbb{M}_{IJ}(A), \mathfrak{R})$ can be extended to a family of elements satisfying the defining relations (E1) – (E3) of $\mathrm{St}_N(A)$. As a consequence, this yields a group homomorphism $\Psi\colon\ \mathrm{St}_N(A) \to \mathrm{St}(\mathbb{M}_{IJ}(A), \mathfrak{R})$ such that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the identity on the respective generators and therefore also on the corresponding groups. Another proof of the bijectivity of Φ is given in [LN2, 24.18], based on the interpretation of both groups as initial objects in an appropriate category of groups mapping onto $\mathrm{E}_N(A)$.

The second part of the theorem follows from the Kervaire-Milnor-Steinberg Theorem 1.6. □

1.12. **Another look at $\mathrm{E}_N(A)$.** It follows from the existence of the surjective group homomorphism $\mathfrak{p}_N$ of (1.10.2) that $\mathrm{E}_N(A)$ is generated by $\mathfrak{p}_N(\mathrm{x}_+(V^+) \cup \mathrm{x}_-(V^-))$ and that the matrices in this image satisfy the relations (EJ1) – (EJ3). It is instructive to verify this directly.

For $(u,v) \in \mathbb{M}_{IJ}(A)$ we define elements $\mathrm{e}_+(u)$ and $\mathrm{e}_-(v)$ of the ring $\mathrm{Mat}_N(A)_{\mathrm{ex}}$ of 1.1 by

(1.12.1) $$\mathrm{e}_+(u) = \begin{pmatrix} \mathbf{1}_I & u \\ 0 & \mathbf{1}_J \end{pmatrix}, \qquad \mathrm{e}_-(v) = \begin{pmatrix} \mathbf{1}_I & 0 \\ -v & \mathbf{1}_J \end{pmatrix}.$$

Then clearly

(1.12.2) $$\mathrm{e}_+(u+u') = \mathrm{e}_+(u)\,\mathrm{e}_+(u') \quad \text{and} \quad \mathrm{e}_-(v+v') = \mathrm{e}_-(v)\,\mathrm{e}_-(v').$$

In particular, the matrices $\mathrm{e}_+(u)$ and $\mathrm{e}_-(v)$ are invertible with inverses $\mathrm{e}_+(u)^{-1} = \mathrm{e}_+(-u)$ and $\mathrm{e}_-(v)^{-1} = \mathrm{e}_-(-v)$. Since $\mathrm{e}_+(aE_{ij}) = \mathrm{e}_{ij}(a)$ and $\mathrm{e}_-(vE_{ji}) = \mathrm{e}_{ji}(-v)$ for $(ij) \in I \times J$, the equations (1.12.2) also show that $\mathrm{e}_+(u) \in \mathrm{E}_N(A)$ and $\mathrm{e}_-(v) \in \mathrm{E}_N(A)$. By straightforward matrix multiplication one obtains

$$\big(\mathrm{e}_+(u),\, \mathrm{e}_-(v)\big) = \begin{pmatrix} \mathbf{1}_I - uv + uvuv & uvu \\ vuv & \mathbf{1}_J + vu \end{pmatrix}. \tag{1.12.3}$$

In particular, taking (u, v) with $vu = 0$ or $uv = 0$, this proves

$$\begin{pmatrix} \mathbf{1}_I - uv & 0 \\ 0 & \mathbf{1}_J \end{pmatrix} \in \mathrm{E}_N(A) \quad \text{and} \quad \begin{pmatrix} \mathbf{1}_I & 0 \\ 0 & \mathbf{1}_J + vu \end{pmatrix} \in \mathrm{E}_N(A).$$

Specifying (u, v) even more, one then easily sees that all elementary matrices $\mathrm{e}_{kl}(a)$ with $(k, l) \in I \times I$ or $(k, l) \in J \times J$ lie in the subgroup of $\mathrm{E}_N(A)$ generated by $\mathrm{e}_+(V^+) \cup \mathrm{e}_-(V^-)$. Therefore, this subgroup equals $\mathrm{E}_N(A)$.

The relation (EJ1) is (1.12.2), and the relation (EJ2) follows from (1.12.3) since for $(u, v) \in V_\alpha^+ \times V_\alpha^-$ with $\alpha \perp \beta$ we have $uv = 0$ and $vu = 0$. In order to prove (EJ3) in case $\sigma = +$, let $(u, v) \in V_\alpha^+ \times V_\beta^-$ with $\alpha \perp \beta$ and let $z \in V^+$ arbitrary. Then $uvu = 0 = uvzvu$, $vuv = 0$ and $(\mathbf{1}_J + vu)^{-1} = \mathbf{1}_J - vu$. Hence, by (1.12.3),

$$\begin{aligned} \big(\big(\mathrm{e}_+(u),\, \mathrm{e}_-(v)\big),\, \mathrm{e}_+(z)\big) &= \left(\begin{pmatrix} \mathbf{1}_I - uv & 0 \\ 0 & (\mathbf{1}_J - vu)^{-1} \end{pmatrix}, \begin{pmatrix} \mathbf{1}_I & z \\ 0 & \mathbf{1}_J \end{pmatrix} \right) \\ &= \begin{pmatrix} \mathbf{1}_I & -z + (1 - uv)z(1 - vu) \\ 0 & \mathbf{1}_J \end{pmatrix} = \mathrm{e}_+(-\{u\, v\, z\}). \end{aligned}$$

The relation (EJ3) for $\sigma = -$ can be verified in the same way.

To put this example in the general setting of the following section §2 we point out that the calculations above are not only valid for matrices of finite or countable size $|N|$, but hold for N of arbitrary cardinality.

2. Generalizations

In this section we generalize the Steinberg groups considered in §1. The generalization has a combinatorial aspect, 3-graded root systems, and an algebraic aspect, root graded Jordan pairs. They are presented in 2.1 – 2.3 and 2.4 – 2.6 respectively. We define the Steinberg group of a root graded Jordan pair (2.7) and state the Jordan pair version of the Kervaire-Milnor-Steinberg Theorem (2.8 and 2.11). Since the elementary linear group only makes sense for special Jordan pairs, we replac it by its central quotient which can be defined for any Jordan pair: the projective elementary group $\mathrm{PE}(V)$ of a Jordan pair V defined in terms of the Tits-Kantor-Koecher algebra $\mathfrak{L}(V)$ (2.10). We discuss $\mathfrak{L}(V)$ and $\mathrm{PE}(V)$ for the Jordan pair of rectangular matrices over a ring in 2.12 and 2.13.

2.1. **Locally finite root systems** [LN1]**.** We use $\langle \cdot, \cdot \rangle$ to denote the canonical pairing between a real vector space X of arbitrary dimension and its dual space X^*, thus $\langle x, \varphi \rangle = \varphi(x)$ for $x \in X$ and $\varphi \in X^*$. If $\varphi \in X^*$ satisfies $\langle \alpha, \varphi \rangle = 2$, we define the *reflection* $s_{\alpha,\varphi} \in \mathrm{GL}(X)$ by

$$s_{\alpha,\varphi}(x) = x - \langle x, \varphi \rangle \alpha.$$

A *locally finite root system* is a pair (R, X) consisting of a real vector space X and a subset $R \subset X$ satisfying the axioms (i) – (iv) below.

(i) R spans X as a real vector space and $0 \in R$,
(ii) for every $\alpha \in R^\times = R \setminus \{0\}$ there exists $\alpha^\vee \in X^*$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and $s_{\alpha,\alpha^\vee}(R) = R$.

(iii) $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in R^\times$.
(iv) R is locally finite in the sense that $R \cap Y$ is finite for every finite-dimensional subspace of X.

Locally finite root systems form a category **RS**, in which a morphism $f\colon (R,X) \to (S,Y)$ is an $\mathbb{R}$-linear map with $f(R) \subset S$. In this category, an isomorphism $f\colon (R,X) \to (S,Y)$ is a vector space isomorphism $f\colon X \to Y$ with $f(R) = S$. Such an isomorphism necessarily satisfies $\langle f(\alpha), f(\beta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle$ for all $\alpha, \beta \in R^\times$.

Remarks, facts and more definitions. (a) The linear form $\alpha^\vee$ in (ii) is uniquely determined by the two conditions in (ii). Therefore, we simply write s_α instead of $s_{\alpha,\alpha^\vee}$ in the future.

(b) Our standard reference for locally finite root systems is [LN1]. As in [LN1] we will also here abbreviate the term "locally finite root system" by *root system*. Then a *finite root system* is a root system (R,X) with R a finite set, equivalently $\dim X < \infty$. Finite root systems are the root systems studied for example in [Bo, Ch. VI]. That [Bo] assumes $0 \notin R$ does not pose any problem in applying the results developed there.

The real vector space X of a root system (R,X) is usually not important. We will therefore often just refer to R rather than to (R,X) as a root system.

(c) As in [LN1] and again in [LN2, §2] we assume here that $0 \in R$, which is more natural from a categorical point of view. In [LN2, §2] the real vector space X is replaced by a free abelian group X and condition (iv) becomes that $R \cap Y$ be finite for every finitely generated subgroup Y of X. With the obvious concept of a morphism, this defines a category of root systems over the integers, which is equivalent to the category **RS** [LN2, Prop. 2.9].

(d) A locally finite root system need not be *reduced* in the sense that $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for every $\alpha \in R^\times$. The *rank* of a root system (R,X) is defined as the dimension of the real vector space X.

(e) The *direct sum* of a family $(R^{(j)}, X^{(j)})_{j\in J}$ of root systems is the pair

$$\big(\textstyle\bigcup_{j\in J} R^{(j)},\ \bigoplus_{j\in J} X^{(j)}\big),$$

which is again a root system [LN1, 3.10], traditionally written as $R = \bigoplus_{j\in J} R^{(j)}$. A non-empty root system is called *irreducible* if it is not isomorphic to a direct sum of two non-empty root systems. Every root system uniquely decomposes as a direct sum of irreducible root systems, called its *irreducible components* [LN1, 3.13].

(f) Every root system (R,X) admits an inner product $(\,|\,)\colon X \times X \to \mathbb{R}$, which is *invariant* in the sense that $(s_\alpha(x) \mid s_\alpha(y)) = (x \mid y)$ holds for all $\alpha \in R^\times$ and $x, y \in X$, equivalently

$$\langle \beta, \alpha^\vee \rangle = 2\,\frac{(\beta \mid \alpha)}{(\alpha \mid \alpha)} \qquad \text{for all } \alpha, \beta \in R^\times \tag{2.1.1}$$

[LN1, 4.2]. If R is irreducible, $(\,|\,)$ is unique up to a non-zero scalar. It follows that the definition of a root system given in [Ne2] is equivalent to the definition above, and that a finite reduced root system is the same as a "root system" in [Hu], again up to $0 \notin R$.

2.2. **Classification of root systems.** We first present, as examples, the *classical root systems* $\dot{\mathrm{A}}_I, \ldots, \mathrm{BC}_I$. Let I be a set of cardinality $|I| \geq 2$ and let $X = \bigoplus_{i\in I} \mathbb{R}\varepsilon_i$ be the

$\mathbb{R}$-vector space with basis $(\varepsilon_i)_{i\in I}$. Define

$$\dot{\mathrm{A}}_I = \{\varepsilon_i - \varepsilon_j : i,j \in I\}, \tag{2.2.1}$$

$$\mathrm{D}_I = \dot{\mathrm{A}}_I \cup \{\pm(\varepsilon_i + \varepsilon_j) : i \neq j\}, \tag{2.2.2}$$

$$\mathrm{B}_I = \mathrm{D}_I \cup \{\pm\varepsilon_i : i \in I\}, \tag{2.2.3}$$

$$\mathrm{C}_I = \mathrm{D}_I \cup \{\pm 2\varepsilon_i : i \in I\}, \tag{2.2.4}$$

$$\mathrm{BC}_I = \mathrm{B}_I \cup \mathrm{C}_I. \tag{2.2.5}$$

Then $\dot{\mathrm{A}}_I$ is a root system in $\dot{X} = \mathrm{Ker}(t)$ where $t \in X^*$ is defined by $t(\varepsilon_i) = 1$, $i \in I$. Its rank is therefore $|I| - 1$. The notation $\dot{\mathrm{A}}$ instead of the traditional A is meant to indicate this fact. Observe that $\dot{\mathrm{A}}_{\mathbb{N}}$ is the root system R of 1.7. All other sets $\mathrm{D}_I, \dots, \mathrm{BC}_I$, are root systems in X, whence of rank $|I|$. The root systems $\dot{\mathrm{A}}_I$, B_I, C_I and D_I are reduced, while BC_I is not.

The isomorphism class of a classical root system only depends on the cardinality of the set I. In particular, when I is finite of cardinality n we will use the index n instead of I. Thus, $\mathrm{D}_n = \mathrm{D}_{\{1,\dots,n\}}$ etc. Note $\dot{\mathrm{A}}_{n+1} = \dot{\mathrm{A}}_{\{0,1,\dots,n\}} = \mathrm{A}_n$ in the traditional notation.

The standard inner product $(\,|\,)$, defined by $(\varepsilon_i|\varepsilon_j) = \delta_{ij}$, is an invariant inner product in the sense of 2.1(f). With the exception of $\mathrm{D}_2 = \mathrm{A}_1 \oplus \mathrm{A}_1$, the root systems $\dot{\mathrm{A}}_I, \dots, \mathrm{BC}_I$ are irreducible. Apart from the low-rank isomorphisms $\mathrm{B}_2 \cong \mathrm{C}_2$, $\mathrm{D}_3 \cong \mathrm{A}_3$, they are pairwise non-isomorphic.

The classification of root systems [LN1, Thm. 8.4] says that *an irreducible root system is either isomorphic to a classical root system or to an exceptional finite root system.*

2.3. **3-graded root systems.** A 3-*grading* of a root system (R, X) is a partition $R = R_1 \,\dot\cup\, R_0 \,\dot\cup\, R_{-1}$ satisfying the following conditions (i) – (iii) below:

(i) $R_{-1} = -R_1$;
(ii) $(R_i + R_j) \cap R \subset R_{i+j}$ for $i, j \in \{1, 0, -1\}$, with the understanding that $R_k = \emptyset$ for $k \notin \{1, 0, -1\}$
(iii) $R_1 + R_{-1} = R_0$, i.e., every root in R_0 is a difference of two roots in R_1.

In particular (ii) says that the sum of two roots in R_1 is never a root and $(R_1 + R_{-1}) \cap R \subset R_0$, a condition which is strengthened in (iii).

Since a 3-grading of a root system (R, X) is determined by the subset R_1 of R, we will denote a 3-graded root system by (R, R_1, X) or simply by (R, R_1). A 3-*graded root system* is a root system equipped with a 3-grading. An *isomorphism* $f\colon (R, R_1, X) \to (S, S_1, Y)$ between 3-graded root systems is a vector space isomorphism $f\colon X \to Y$ satisfying $f(R_1) = S_1$, hence also $f(R_i) = S_i$ for $i \in \{1, 0, -1\}$, and is therefore an isomorphism $f\colon (R, X) \to (S, Y)$ of the underlying root systems. References for 3-graded root systems are [LN1, §17, §18], [LN2, Ch. IV] and [Ne2].

Some facts and examples. (a) The decomposition (1.8.1) of the root system $R = \dot{\mathrm{A}}_{\mathbb{N}}$ is a 3-grading. The restrictions on N imposed in 1.7 are not necessary for the definition of a 3-graded root system, as we have seen in 2.2 for the root system $\dot{\mathrm{A}}_N$. Any non-trivial partition $N = I \,\dot\cup\, J$ induces a 3-grading of $\dot{\mathrm{A}}_N$ as in 1.8, denoted $\dot{\mathrm{A}}_N^I$. Thus, the 1-part of the 3-graded root system $\dot{\mathrm{A}}_N^I$ is

$$(\dot{\mathrm{A}}_N^I)_1 = \{\varepsilon_i - \varepsilon_j : i \in I, j \in N \setminus I\}. \tag{2.3.1}$$

Every 3-grading of $\dot{\mathrm{A}}_N$ is obtained in this way for a non-empty proper subset $I \subset N$.

(b) A 3-grading $\mathrm{B}_I^{\mathrm{qf}}$ of the root system B_I is obtained by choosing a distinguished element of I, say $0 \in I$, and putting $R_1 = \{\varepsilon_0\} \cup \{\varepsilon_0 \pm \varepsilon_i : 0 \neq i \in I\}$.

(c) The root system $R = \mathrm{C}_I$ has a 3-grading, denoted $\mathrm{C}_I^{\mathrm{her}}$, whose 1-part is $R_1 = \{\varepsilon_i + \varepsilon_j : i, j \in I\}$. Note $R_0 = \{\varepsilon_i - \varepsilon_j : i, j \in I\} \cong \dot{\mathrm{A}}_I$.

(d) The root system D_I, $|I| \geq 4$, is a subsystem of B_I and C_I. The 3-gradings of these two root systems, defined in (b) and (c), induce 3-gradings $\mathrm{D}_I^{\mathrm{qf}}$ and $\mathrm{D}_I^{\mathrm{alt}}$ of D_I. The first of these is determined by $R_1 = \{\varepsilon_0 \pm \varepsilon_i : 0 \neq i \in I\}$ and the second by $R_1 = \{\varepsilon_i + \varepsilon_j : i, j \in I, i \neq j\}$. It is known that $\mathrm{D}_I^{\mathrm{qf}} \cong \mathrm{D}_I^{\mathrm{alt}}$ if $|I| = 4$, but $\mathrm{D}_I^{\mathrm{qf}} \not\cong \mathrm{D}_I^{\mathrm{alt}}$ if $|I| \geq 5$.

(e) The 3-gradings of a root system R are determined by the 3-gradings of its irreducible components $(R^{(j)})_{j \in J}$ as follows.

If (R, R_1) is a 3-grading, then $(R^{(j)}, R_1 \cap R^{(j)})$ is a 3-grading for every $j \in J$. Conversely, given 3-gradings $(R^{(j)}, R_1^{(j)})$ for every j, the set $R_1 = \bigcup_j R_1^{(j)}$ defines a 3-grading of R.

These easy observations reduce the classification of 3-graded root systems to the case of irreducible root systems. Their classification is given in [LN1, 17.8, 17.9]. It turns out that an irreducible root system has a 3-grading if and only if it is not isomorphic to E_8, F_4 and G_2. Some irreducible root systems have several non-isomorphic 3-gradings, such as $\dot{\mathrm{A}}_N$ or D_I. But every 3-grading of C_I is isomorphic to the 3-grading $\mathrm{C}_I^{\mathrm{her}}$ of (c).

(f) The relations $\perp$ and $\frac{\quad}{}$ introduced in (1.7.1) in case $R = \dot{\mathrm{A}}_{\mathbb{N}}$ can be defined for any root system R without using an invariant inner product. For $\alpha, \beta \in R^\times$ we put

$$\begin{aligned} \alpha \perp \beta \quad &\Longleftrightarrow \quad \langle \alpha, \beta^\vee \rangle = 0, \quad \text{equivalently } \langle \beta, \alpha^\vee \rangle = 0, \\ \alpha \frac{\quad}{} \beta \quad &\Longleftrightarrow \quad \langle \alpha, \beta^\vee \rangle = 1 = \langle \beta, \alpha^\vee \rangle \\ \alpha \to \beta \quad &\Longleftrightarrow \quad \langle \alpha, \beta^\vee \rangle = 2,\ \langle \beta, \alpha^\vee \rangle = 1. \end{aligned}$$

The formula (2.1.1) shows that the definitions of $\perp$ and $\frac{\quad}{}$ above generalize (1.7.1). The relation $\to$ occurs for example in the root system C_I: we have $2\varepsilon_i \to \varepsilon_i + \varepsilon_j$ for $\neq j$. (In [LN1] the notation $\top$ and $\dashv$ is used in place of $\frac{\quad}{}$ and $\to$, respectively.)

(g) Given α, $\beta \in R_1$, exactly one of these relations holds:

$$\alpha = \beta \quad \text{or} \quad \alpha \to \beta \quad \text{or} \quad \alpha \leftarrow \beta \quad \text{or} \quad \alpha \frac{\quad}{} \beta \quad \text{or} \quad \alpha \perp \beta. \tag{2.3.2}$$

Moreover, again for $\alpha, \beta \in R_1$,

$$2\alpha - \beta \in R \quad \Longleftrightarrow \quad 2\alpha - \beta \in R_1 \quad \Longleftrightarrow \quad \alpha \leftarrow \beta \text{ or } \alpha = \beta. \tag{2.3.3}$$

2.4. **Jordan pairs.** This subsection contains a very short introduction to Jordan pairs over a commutative ring k, although for the purpose of this paper $k = \mathbb{Z}$ is completely sufficient. We will only present what is needed to understand this paper. A more detailed but still concise introduction to Jordan pairs is given in [LN2, §6]; the standard reference for Jordan pairs is [Lo1].

We have already seen an example of a Jordan pair in 1.9: the rectangular matrix pair $\mathbb{M}_{IJ}(A) = (\mathrm{Mat}_{IJ}(A), \mathrm{Mat}_{JI}(A))$ equipped with the Jordan triple product $\{x\,y\,z\} = xyz + zyx$ for $(x, y, z) \in V^\sigma \times V^{-\sigma} \times V^\sigma$ and $\sigma = \pm$. The Jordan triple product is the linearization with respect to x of the expression $Q(x)y = xyx$, which did not play any role in §1, but which is the basic structure underlying Jordan pairs.

A *Jordan pair* is a pair $V = (V^+, V^-)$ of k-modules together with maps

$$Q^\sigma \colon V^\sigma \times V^{-\sigma} \to V^\sigma, \quad (x, y) \mapsto Q^\sigma(x)y, \qquad (\sigma = \pm),$$

which are quadratic in x and linear in y and which satisfy the identities (JP1) – (JP3) below in all base ring extensions. To define these identities, we will simplify the notation and omit

σ, thus writing $Q(x)y$ or simply $Q_x y$. This does not lead to any confusion, as long as one takes care that the expressions make sense. Linearizing $Q_x y$ in x gives

$$Q_{x,z}y = Q(x,z)y = Q_{x+z}y - Q_x y - Q_z y,$$

which we use to define the *Jordan triple product*

$$\{\cdots\} \colon V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma, \quad (x,y,z) \mapsto \{x\,y\,z\} = Q_{x,z}y.$$

To improve readability we will sometimes write $\{x,\, y,\, z\}$ instead of $\{x\,y\,z\}$. If K is a commutative associative unital k-algebra we let $V^\sigma_K = V^\sigma \otimes_k K$ and observe that there exist unique extensions of the Q^σ to quadratic-linear maps $Q\colon V_K \times V_K \to V_K$. The identities required to hold for $x,z \in V^\sigma_K$, $y,v \in V^{-\sigma}_K$, $\sigma \in \{+,-\}$ and any K as above are

$$\{x,\, y,\, Q_x v\} = Q_x\,\{y,\, x,\, v\}, \tag{JP1}$$

$$\{Q_x y,\, y,\, z\} = \{x,\, Q_y x,\, z\}, \tag{JP2}$$

$$Q_{Q_x y} v = Q_x\, Q_y\, Q_z v. \tag{JP3}$$

A *homomorphism* $f\colon V \to W$ of Jordan pairs is a pair $f = (f_+, f_-)$ of k-linear maps $f_\sigma \colon V^\sigma \to W^\sigma$ satisfying $f_\sigma(Q(x)y) = Q\big(f_\sigma(x)\big)f_{-\sigma}(y)$ for all $(x,y) \in V^\sigma \times V^{-\sigma}$ and $\sigma = \pm$.

Remarks and more definitions. (a) Instead of requiring that (JP1) – (JP3) hold for all extensions K, one can demand that (JP1) – (JP3) as well as all their linearizations hold in V. For example, linearizing the identity (JP1) with respect to x gives the identity

$$\{z,\, y,\, Q_x v\} + \{x,\, y,\, Q_{x,z} v\} = Q_{x,z}\,\{y,\, x,\, v\} + Q_x\,\{y,\, z,\, v\}$$

(b) If $V = (V^+, V^-)$ is a Jordan pair and $S = (S^+, S^-)$ is a pair of submodules of V satisfying $Q(S^\sigma)S^{-\sigma} \subset S^\sigma$ for $\sigma = \pm$, then S is a Jordan pair with the induced operations, called a *subpair* of V.

(c) An *idempotent* in a Jordan pair V is a pair $e = (e_+, e_-) \in V$ satisfying $Q(e_+)e_- = e_+$ and $Q(e_-)e_+ = e_-$. An idempotent e gives rise to the *Peirce decomposition* of V,

$$V^\sigma = V^\sigma_2(e) \oplus V^\sigma_1(e) \oplus V^\sigma_0(e), \qquad \sigma = \pm,$$

where the *Peirce spaces* $V_i(e) = (V_i^+(e), V_i^-(e))$, $i = 0,1,2$, are given by

$$\begin{aligned}
V^\sigma_2(e) &= \{x \in V^\sigma : Q(e_\sigma)Q(e_{-\sigma})x = x\},\\
V^\sigma_1(e) &= \{x \in V^\sigma : \{e_\sigma\, e_{-\sigma}\, x\} = x\},\\
V^\sigma_0(e) &= \{x \in V^\sigma : Q(e_\sigma)x = 0 = \{e_\sigma\, e_{-\sigma}\, x\}\}.
\end{aligned}$$

The Peirce spaces $V_i^\pm = V_i^\pm(e)$ satisfy the multiplication rules

$$Q(V_i^\sigma)V_j^{-\sigma} \subset V^\sigma_{2i-j}, \qquad \{V_i^\sigma\, V_j^{-\sigma}\, V_l^\sigma\} \subset V^\sigma_{i-j+l}, \tag{2.4.1}$$

$$\{V_2^\sigma\, V_0^{-\sigma}\, V^\sigma\} = 0 = \{V_0^\sigma\, V_2^{-\sigma}\, V^\sigma\}, \tag{2.4.2}$$

where $i,j,l \in \{0,1,2\}$, with the understanding that $V^\sigma_m = 0$ if $m \notin \{0,1,2\}$. In particular, the $V_i = V_i(e)$ are subpairs of V. If $2 \in k^\times$ we have $V^\sigma_i(e) = \{x \in V : \{e^\sigma\, e^{-\sigma}\, x\} = ix\}$ for $i = 0,1,2$.

2.5. **Examples of Jordan pairs.** We now give concrete examples of Jordan pairs, to illustrate the abstract definition of 2.4.

(i) Any associative, not necessarily unital or commutative k-algebra A gives rise to a Jordan pair $V = (A, A)$ with respect to the operations $Q_x y = xyx$.

Indeed, since a base ring extension of A is again associative, it suffices to verify the identities in V, where they follows from the following calculations.

$$\begin{aligned} \{x, y, Q_x v\} &= xy(xvx) + (xvx)yx = x(yxv + vxy)x = Q_x\{y, v, x\}, \\ \{Q_x y, y, z\} &= (xyx)yz + zy(xyx) = x(yxy)z + z(yxy)x = \{x, Q_y x, z\}, \\ Q_{Q_x y} v &= (xyx)v(xyx) = x(y(xvx)y)x = Q_x Q_y Q_x v. \end{aligned}$$

An idempotent c of the associative algebra A, defined by $c^2 = c$, induces an associative Peirce decomposition $A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}$ with $A_{ij} = \{a \in A : ca = ia,\, ac = ja\}$. The pair $e = (c, c)$ is an idempotent of the Jordan pair V whose Peirce spaces are $V_2^\sigma(e) = A_{11}$, $V_1^\sigma(e) = A_{10} \oplus A_{01}$ and $V_0^\sigma(e) = A_{00}$. Not only $V_1(e)$ but also (A_{10}, A_{01}) and (A_{01}, A_{10}) are subpairs of V.

Not every idempotent of V has the form (c, c), c an idempotent of A. For example, if $u \in A^\times$ and c is an idempotent of A, then (uc, cu^{-1}) is an idempotent of V which, however, has the same Peirce spaces as (c, c).

(ii) By (b) and (i), any pair $(S^+, S^-) \subset (A, A)$ of k-submodules closed under the operation $(x, y) \mapsto xyx$ is also a Jordan pair. Jordan pairs of this form are called *special*. Their Jordan triple product is

$$\{x\,y\,z\} = xyz + zyx \qquad (x, z \in S^\sigma, y \in S^{-\sigma}). \tag{2.5.1}$$

We next describe some important cases of special Jordan pairs.

(iii) Let I and J be non-empty sets. Let $N = I \,\dot\cup\, J'$ where J' is a set disjoint from I and in bijection with J under $j \mapsto j'$, and embed $\mathrm{Mat}_{IJ}(A)$ into the right upper corner of the associative algebra $\mathrm{Mat}_N(A)$. Similarly, we identify $\mathrm{Mat}_{JI}(A)$ with the left lower corner of $\mathrm{Mat}_N(A)$. Then

$$\mathbb{M}_{IJ}(A) = \big(\mathrm{Mat}_{IJ}(A), \mathrm{Mat}_{JI}(A)\big)$$

is a subpair of the Jordan pair $\big(\mathrm{Mat}_N(A), \mathrm{Mat}_N(A)\big)$ and is therefore a Jordan pair, as claimed at the beginning of this subsection. If N is finite, $\mathbb{M}_{IJ}(A)$ is of type (A_{10}, A_{01}) for $A = \mathrm{Mat}_N(A)$ and the idempotent $c = \mathbf{1}_I$, see (i).

(iv) Let $a \mapsto a^J$ be an involution of the associative k-algebra A. Then $\mathrm{H}(A, J) = \{a \in A : a^J = a\}$ is closed under the Jordan pair product, whence $\mathbb{H}(A, J) = (\mathrm{H}(A, J), \mathrm{H}(A, J))$ is a Jordan pair.

More generally, extend J to an involution of the associative k-algebra $\mathrm{Mat}_I(A)$, $|I| \ge 2$, defined by $(x_{ij})^J = (x_{ji}^J)$ and again denoted by J. Then the *hermitian matrix pair*

$$\mathbb{H}_I(A, J) = \big(\mathrm{H}(\mathrm{Mat}_I(A), J),\, \mathrm{H}(\mathrm{Mat}_I(A), J)\big)$$

is a special Jordan pair. In particular, taking $A = k$, $J = \mathrm{Id}_k$ and $I = \{1, \dots, n\}$ we get the *symmetric matrix pair* $\mathbb{H}_n(k) = (\mathrm{H}_n(k), \mathrm{H}_n(k))$.

(v) Let $\mathrm{Alt}_I(k)$ be the alternating $I \times I$-matrices over k, where $x = (x_{ij})$ is called alternating if $x_{ii} = 0 = x_{ij} + x_{ji}$ for $i, j \in I$. Then the *alternating matrix pair* $\mathbb{A}_I(k) = (\mathrm{Alt}_I(k), \mathrm{Alt}_I(k))$ is a subpair of $\mathbb{M}_I(k)$, whence a special Jordan pair.

(vi) Let M be a k-module and let $q\colon M \to k$ be a quadratic form with polar form b, defined by $b(x, y) = q(x + y) - q(x) - q(y)$. Then $\mathbb{J}(M, q) = (M, M)$ is a Jordan pair with quadratic operators $Q_x y = b(x, y)x - q(x)y$.

(vii) Let J be a unital quadratic Jordan algebra with quadratic operators U_x, $x \in J$ ([Ja4, Ja5]). Then (J, J) is a Jordan pair with quadratic maps $Q_x = U_x$. For example, if k is a field, the rectangular matrix pair $\mathbb{M}_{pp}(k)$ is of this form, but $\mathbb{M}_{pq}(k)$ for $p \neq q$ is not. Thus, there are "more" Jordan pairs than Jordan algebras.

Let $\mathcal{C}$ be an octonion k-algebra, see for example [SV] in case k is a field or [LPR] in general, and let $J = \mathrm{H}_3(\mathcal{C})$ be the exceptional Jordan algebra of 3×3 matrices over $\mathcal{C}$ which are hermitian with respect to the standard involution of $\mathcal{C}$. Then (J, J) is a Jordan pair, which is not special in the sense of (ii). Such Jordan pairs are called *exceptional*.

2.6. **Root graded Jordan pairs.** Let us first recast the Peirce decomposition 2.4(c) of an idempotent e in a Jordan pair V from the point of view of a grading.

We use the 3-graded root system $\mathrm{C}_I^{\mathrm{her}}$ of 2.3(c) with $I = \{0, 1\}$. Its 1-part is $R_1 = \{\varepsilon_i + \varepsilon_j : i, j \in \{0, 1\}\} = \{2\varepsilon_1, \varepsilon_1 + \varepsilon_0, 2\varepsilon_0\}$. Putting

$$V_\alpha^\sigma = V_{i+j}^\sigma(e) \qquad (\alpha = \varepsilon_i + \varepsilon_j \in R_1)$$

we have the decomposition $V^\sigma = \bigoplus_{\alpha \in R_1} V_\alpha^\sigma$ which satisfies

(RG1) $$Q(V_\alpha^\sigma)V_\beta^{-\sigma} \subset V_{2\alpha-\beta}^\sigma, \qquad \{V_\alpha^\sigma\, V_\beta^{-\sigma}\, V_\gamma^\sigma\} \subset V_{\alpha-\beta+\gamma}^\sigma,$$

(RG2) $$\{V_\alpha^\sigma\, V_\beta^{-\sigma}\, V^\sigma\} = 0 \qquad \text{if } \alpha \perp \beta.$$

Here $2\alpha - \beta$ and $\alpha - \beta + \gamma$ in (RG1) are calculated in $X = \mathbb{R} \cdot \varepsilon_0 \oplus \mathbb{R} \cdot \varepsilon_1 \cong \mathbb{R}^2$, and $V_{2\alpha-\beta}^\sigma = 0$ if $2\alpha - \beta \notin R_1$ or $V_{\alpha-\beta+\gamma}^\sigma = 0$ if $\alpha - \beta + \gamma \notin R_1$. We see that, apart from the actual definition of the Peirce spaces, the rules governing the Peirce decomposition can be completely described in terms of R_1. The following generalisation is then natural.

Given a 3-graded root system (R, R_1) and a Jordan pair V, an (R, R_1)*-grading of* V is a decomposition $V^\sigma = \bigoplus_{\alpha \in R_1} V_\alpha^\sigma$, $\sigma = \pm$, satisfying (RG1) and (RG2). We will use $\mathfrak{R} = (V_\alpha)_{\alpha \in R_1}$ to denote such a grading. A *root graded Jordan pair* is a Jordan pair equipped with an (R, R_1)-grading for some 3-graded root system. In view of (2.3.3) we can make the first inclusion in (RG1) more precise:

(2.6.1) $$\begin{gathered} Q(V_\alpha^\sigma)V_\beta^{-\sigma} = 0 \quad \textit{unless } \alpha \to \beta, \textit{ in which case} \\ 2\alpha - \beta \in R_1 \textit{ and } Q(V_\alpha^\sigma)V_\beta^{-\sigma} \subset V_{2\alpha-\beta}^\sigma. \end{gathered}$$

We call $\mathfrak{R}$ an *idempotent root grading* if there exists a subset $\Delta \subset R_1$ and a family $(e_\alpha)_{\alpha \in \Delta}$ of non-zero idempotents $e_\alpha \in V_\alpha$ such that the V_β are given by

(2.6.2) $$V_\beta = \bigcap_{\alpha \in \Delta} V_{\langle \beta, \alpha^\vee \rangle}(e_\alpha)$$

Observe that (2.6.2) makes sense since $\langle \alpha, \beta^\vee \rangle \in \{0, 1, 2\}$ by (2.3.2). Neither the idempotents nor the subset $\Delta \subset R_1$ are uniquely determined by an idempotent root grading, see for example (iii) below.

To avoid some technicalities, we will often assume that $\mathfrak{R}$ is a *fully idempotent root grading*, i.e., $\mathfrak{R}$ is idempotent with respect to a family of idempotents with $\Delta = R_1$. In the terminology of [Ne1] this means that V is covered by the cog $(e_\alpha)_{\alpha \in R_1}$.

Examples. (i) Let $R = \mathrm{A}_1 = \{\alpha, -\alpha\}$ equipped with the 3-grading defined by $R_1 = \{\alpha\}$. An (R, R_1)-graded Jordan pair is simply a Jordan pair $V = (V^+, V^-)$ for which $V^\sigma = V_\alpha^\sigma$. This root grading is idempotent if and only if $V \cong (J, J)$ where J is a unital Jordan algebra. To see sufficiency in case J is a Jordan algebra with identity element 1_J, one uses $e_\alpha = (1_J, 1_J)$ and observes $(J, J) = V_2(e_\alpha)$.

(ii) Let $(R,R_1)=\mathrm{C}_2^{\mathrm{her}}$. We have seen above that the Peirce decomposition of an idempotent $e\in V$ can be viewed as a $\mathrm{C}_2^{\mathrm{her}}$-grading, which is idempotent with respect to $e=e_\alpha$, $\alpha=2\varepsilon_1$. Thus here $\Delta=\{\alpha\}$.

(iii) Let $(R,R_1)=\dot{\mathrm{A}}_N^I$ be the 3-graded root system of 2.3(a). Put $J=N\setminus I$. An $\dot{\mathrm{A}}_I^N$-grading of a Jordan pair V is a decomposition

$$V=\bigoplus\nolimits_{(ij)\in I\times J} V_{(ij)}$$

such that for all (ij) and $(lm)\in I\times J$ and $\sigma=\pm$ we have, defining $V_{(ij)}=V_{\varepsilon_i-\varepsilon_j}$ for $\varepsilon_i-\varepsilon_j\in R_1$,

$$\begin{aligned} Q(V^\sigma_{(ij)})V^{-\sigma}_{(ij)} &\subset V^\sigma_{(ij)}, &\qquad \{V^\sigma_{(ij)}\,V^{-\sigma}_{(ij)}\,V^\sigma_{(im)}\} &\subset V^\sigma_{(im)},\\ \{V^\sigma_{(ij)}\,V^{-\sigma}_{(ij)}\,V^\sigma_{(lj)}\} &\subset V^\sigma_{(lj)}, &\qquad \{V^\sigma_{(ij)}\,V^{-\sigma}_{(lj)}\,V^\sigma_{(lm)}\} &\subset V^\sigma_{(im)}, \end{aligned}$$

and all other types of products vanish.

An example of an $\dot{\mathrm{A}}_I^N$-graded Jordan pair V is the rectangular matrix pair $\mathbb{M}_{IJ}(A)$ of an associative unital k-algebra $A\neq 0$, see 2.5(iii), with respect to the subpairs $V_{(ij)}=(AE_{ij},AE_{ji})$. This $\dot{\mathrm{A}}_N^I$-grading of $\mathbb{M}_{IJ}(A)$ is fully idempotent with respect to the family $(e_\alpha)_{\alpha\in R_1}$, $e_\alpha=(a_{ij}E_{ij},a_{ij}^{-1}E_{ji})$, where $\alpha=\varepsilon_i-\varepsilon_j$ and $a_{ij}\in A^\times$. It is also idempotent with respect to the following smaller family: fix $i_0\in I$ and $j_0\in J$ and consider $(e_\alpha)_{\alpha\in\Delta}$ where $\Delta=\{\varepsilon_i-\varepsilon_{j_0}: i\in I\}\cup\{\varepsilon_{i_0}-\varepsilon_j: j\in J\}$.

Let $\mathfrak{a},\mathfrak{b}\subset A$ be k-submodules with $\mathfrak{a}\mathfrak{b}\mathfrak{a}\subset\mathfrak{a}$ and $\mathfrak{b}\mathfrak{a}\mathfrak{b}\subset\mathfrak{b}$. Then $S=(\mathrm{Mat}_{IJ}(\mathfrak{a}),\mathrm{Mat}_{JI}(\mathfrak{b}))$ is a Jordan subpair of $\mathbb{M}_{IJ}(A)$. It inherits the $\dot{\mathrm{A}}_N^I$-grading from $\mathbb{M}_{IJ}(A)$ by putting $S_{(ij)}=(\mathfrak{a}E_{ij},\mathfrak{b}E_{ji})$. This $\dot{\mathrm{A}}_N^I$-grading of S is in general not idempotent, e.g., if $\mathfrak{a}$ is a nil ideal.

(iv) The hermitian matrix pair $V=\mathbb{H}_I(A,J)$ of 2.5(iv) has an idempotent $\mathrm{C}_I^{\mathrm{her}}$-grading. Indeed, define

$$h_{ij}(a)=\begin{cases} aE_{ij}+a^JE_{ji} & \text{if } a\in A \text{ and } i\neq j,\\ aE_{ii} & \text{if } i=j \text{ and } a\in \mathrm{H}(A,J).\end{cases}$$

Then $V=\bigoplus_{\alpha\in R_1}V_\alpha$ with

$$V_\alpha=V_{\varepsilon_i+\varepsilon_j}=\begin{cases}\big(h_{ij}(A),h_{ij}(A)\big), & \text{if } i\neq j\\ \big(h_{ii}(\mathrm{H}(A,J)),\,h_{ii}(\mathrm{H}(A,J))\big), & \text{if } i=j.\end{cases}$$

is a $\mathrm{C}_I^{\mathrm{her}}$-grading of V. It is fully idempotent, for example with respect to the family $(e_\alpha)_{\alpha\in R_1}$ for which $e_\alpha=(h_{ij}(1),h_{ij}(1))$, $\alpha=\varepsilon_i+\varepsilon_j$.

(v) The remaining examples in 2.5 all have idempotent root gradings. The alternating matrix pair $\mathbb{A}_I(k)$ of 2.5(v) has an idempotent $\mathrm{D}_I^{\mathrm{alt}}$-grading ([LN2, 23.24]). The Jordan pair $\mathbb{J}(M,q)$ associated with a quadratic form q in 2.5(v) has an idempotent $\mathrm{B}_I^{\mathrm{qf}}$ if q contains a hyperbolic plane, or even a $\mathrm{D}_I^{\mathrm{qf}}$-grading if q is hyperbolic ([LN2, 23.25]). If $\mathcal{C}$ is a split octonion algebra in the sense of [SV] or [LPR] the exceptional Jordan pair $(\mathrm{H}_3(\mathcal{C}),\mathrm{H}_3(\mathcal{C}))$ has an idempotent root grading with R of type E_7 ([Ne1, III, §3]).

2.7. **The Steinberg group** $\mathrm{St}(V,\mathfrak{R})$**.** Let (R,R_1) be a 3-graded root system and let V be a Jordan pair with a root grading $\mathfrak{R}=(V_\alpha)_{\alpha\in R_1}$, not necessarily idempotent. The *Steinberg group* $\mathrm{St}(V,\mathfrak{R})$ is the group with the following presentation:

- The generators are $\mathrm{x}_+(u)$, $u \in V^+$, and $\mathrm{x}_-(v)$, $v \in V^-$.

  To formulate the relations, we first introduce, for $\alpha \neq \beta \in R_1$ and $(u,v) \in V^+_\alpha \times V^-_\beta$, the element $\mathrm{b}(u,v)$ in the free group with the above generators by the equation

$$\mathrm{x}_+(u)\,\mathrm{x}_-(v) = \mathrm{x}_-(v + Q_v u)\,\mathrm{b}(u,v)\,\mathrm{x}_+(u + Q_u v) \tag{2.7.1}$$

- Then the relations are

$$\mathrm{x}_\sigma(u+u') = \mathrm{x}_\sigma(u)\,\mathrm{x}_\sigma(u') \qquad \text{for } u,u' \in V^\sigma, \tag{St1}$$

$$\big(\mathrm{x}_+(u),\, \mathrm{x}_-(v)\big) = 1 \qquad \text{for } (u,v) \in V^+_\alpha \times V^-_\beta,\ \alpha \perp \beta, \tag{St2}$$

$$\begin{cases} \big(\mathrm{b}(u,v),\, \mathrm{x}_+(z)\big) = \mathrm{x}_+(-\{u\,v\,z\} + Q_uQ_v z), \\ \big(\mathrm{b}(u,v)^{-1},\, \mathrm{x}_-(y)\big) = \mathrm{x}_-(-\{v\,u\,y\} + Q_vQ_u y) \end{cases} \tag{St3}$$

for all $(u,v) \in V^+_\alpha \times V^-_\beta$ with $\alpha \neq \beta$ and all $(z,y) \in V$.

Remarks. (a) Let us have a closer look at the element $\mathrm{b}(u,v)$ in (2.7.1). By (2.3.2) the possibilities for α, β are $\alpha \perp \beta$, $\alpha \,—\, \beta$, $\alpha \to \beta$ and $\alpha \leftarrow \beta$ and by (2.6.1), $Q_v u = 0$ unless $\beta \to \alpha$, and $Q_u v = 0$ unless $\alpha \to \beta$. Therefore, by (St2),

$$\mathrm{b}(u,v) = \begin{cases} 1, & \text{if } \alpha \perp \beta, \\ \big(\mathrm{x}_-(-v),\, \mathrm{x}_+(u)\big), & \text{if } \alpha \,—\, \beta, \\ \mathrm{x}_-(-Q_v u)\,\big(\mathrm{x}_-(-v),\, \mathrm{x}_+(u)\big), & \text{if } \alpha \to \beta, \\ \big(\mathrm{x}_-(-v),\, \mathrm{x}_+(u)\big)\,\mathrm{x}_+(-Q_u v), & \text{if } \alpha \leftarrow \beta. \end{cases} \tag{2.7.2}$$

In general, the factors $\mathrm{x}_-(-Q_v u)$ and $\mathrm{x}_+(-Q_u v)$ in the last two cases are $\neq 1$.

The reader may be puzzled by the definition of $\mathrm{b}(u,v)$: why not take "$\mathrm{b}(u,v) = \big(\mathrm{x}_-(-v)\,,\ \mathrm{x}_+(u)\big)$"? We will give a justification for this in 2.11.

(b) We claim that (St3) follows from (St2) in case $\alpha \perp \beta$. Indeed, the left hand sides of the two equations (St3) are 1 because $\mathrm{b}(u,v) = 1$, but also the right hand sides are 1, since, say for $\sigma = +$, we have $\{u\,v\,z\} = 0$ by (RG2) and $Q_uQ_vz = 0$ by (2.7.3) below, :

$$Q(V^\sigma_\alpha)Q(V^{-\sigma}_\beta)V^\sigma_\gamma \neq 0 \implies \alpha = \beta \text{ or } \alpha \leftarrow \beta = \gamma \text{ or } \alpha \,—\, \beta \leftarrow \gamma \perp \alpha, \tag{2.7.3}$$

which can be shown by repeated application of (2.3.3).

(c) Let $(V, \mathfrak{R}) = (\mathbb{M}_{IJ}(A), \mathfrak{R})$. Comparing the definition of $\mathrm{St}(\mathbb{M}_{IJ}(A), \mathfrak{R})$ with the one in 1.10, it is clear that the first two relations coincide: (EJ1) = (St1) and (EJ2) = (St2). We claim that the relations (EJ3) coincide with the two relations in (St3). Indeed, since we do not have a relation $\alpha \leftarrow \beta$ in $\dot{\mathrm{A}}^I_N$, it follows from (b) and the assumption $\alpha \neq \beta$ in (St3) that we only need to consider the case $\alpha \,—\, \beta$. Then the map $V^-_\beta \to \mathrm{St}(V, \mathfrak{R})$, $v \mapsto \big(\mathrm{x}_+(u),\, \mathrm{x}_-(v)\big)$ is homomorphism of groups, whence, by (2.7.2),

$$\mathrm{b}(u,v) = \big(\mathrm{x}_-(-v),\, \mathrm{x}_+(u)\big) = \big(\mathrm{x}_+(u),\, \mathrm{x}_-(-v)\big)^{-1} = \big(\mathrm{x}_+(u),\, \mathrm{x}_-(v)\big).$$

Thus (EJ3) = (St3) for $\sigma = +$ because $Q_uQ_xz = 0$ by (2.7.3). The equality of the two relations for $\sigma = -$ can be established in the same way.

We can now state the generalization of part (a) of the Kervaire-Milnor-Steinberg Theorem 1.6 in the setting of this section.

2.8. **Theorem A.** *Let (R, R_1) be a 3-graded root system whose irreducible components all have rank ≥ 5, and let V be a Jordan pair with a fully idempotent root grading $\mathfrak{R}$. Then the Steinberg group $\mathrm{St}(V, \mathfrak{R})$ is centrally closed.*

This theorem is one of the main results of [LN2]; its proof takes up all of Chapter VI of [LN2]. It is shown there in greater generality. First, it also true when R has connected components of rank 4, but not of type D_4. Moreover, it is not necessary to assume that the root grading $\mathfrak{R}$ is fully idempotent. For the irreducible components of type B_I or C_I, $|I| \geq 5$, one only needs idempotents $e_\alpha \in V_\alpha$ in case α is a long root in type B and α a short root in type C. This generality allows us to consider groups defined in terms of hermitian matrices associated with form rings in the sense of [Ba].

2.9. **Highlights of our approach.** The novel aspect of our approach is the consistent use of the theory of 3-graded root systems and Jordan pairs, which introduces new methods in the theory of elementary and Steinberg groups. For example, instead of first dealing with the case of finite root systems and then taking a limit to get the stable (= infinite rank) case, we deal with both cases at the same time. Moreover, our approach avoids having to deal with concrete matrix realizations of the groups in question, as is traditionally done, see e.g. [Ba] or [HO]. It allows for a concise description of the defining relations, independent of the types of root systems involved. Finally, as the discussion of the linear Steinberg group in 1.9 – 1.11 shows, we need fewer relations than in previous work, for example no relations involving two roots in R_0.

With the exception of groups defined in terms of root systems of type E_8, F_4 and G_2, which are not amenable to a Jordan approach, cf. 2.3(e), our Theorem 2.8 covers all types of Steinberg groups considered before. In addition, it also presents some new types, e.g., for elementary orthogonal groups. A detailed comparison of our Theorem 2.8 with previously known results is given in [LN2, 27.11].

At this point it is natural to ask if there also exists a generalization of part (b) of Theorem 1.6, stating that the map $\wp\colon \mathrm{St}(A) \to \mathrm{E}(A)$ is a universal central extension. While the group $\mathrm{St}(V, \mathfrak{R})$ gives a satisfactory replacement for the linear Steinberg group $\mathrm{St}(A)$, recasting the elementary linear group $\mathrm{E}(A)$ in the framework of Jordan pairs is limited to special Jordan pairs in the sense of 2.5(ii). While this can be done, see [Lo4], we will instead replace the elementary group $\mathrm{E}(A)$ by the projective elementary group $\mathrm{PE}(V)$, see 2.10, that can be defined for any Jordan pair V. From the point of view of universal central extensions, this is harmless since, as we will see in 2.13, the group $\mathrm{PE}(V)$ is isomorphic to the central quotient $\mathrm{PE}(A) = \mathrm{E}(A)/\mathcal{Z}(\mathrm{E}(A))$ and universal central extensions of a group and its central quotients are essentially the same by 1.4(e).

2.10. **The Tits-Kantor-Koecher algebra and the projective elementary group of a Jordan pair.** Let V be a Jordan pair, defined over a commutative ring k of scalars. It is fundamental (and well-known) that V gives rise to a $\mathbb{Z}$-graded Lie k-algebra

$$\mathfrak{L}(V) = \mathfrak{L}(V)_1 \oplus \mathfrak{L}(V)_0 \oplus \mathfrak{L}(V)_{-1}, \tag{2.10.1}$$

introduced at about the same time by Tits, Kantor and Koecher in [Ti1, Ti2, Ka1, Ka2, Ka3, Ko1, Ko3] and called the *Tits-Kantor-Koecher algebra of* V. Various versions of $\mathfrak{L}(V)$ exist, but all agree that $\big(\mathfrak{L}(V)_1, \mathfrak{L}(V)_{-1}\big) = (V^+, V^-)$ as k-modules. For our purposes, the most appropriate choice for $\mathfrak{L}(V)_0$ is

$$\mathfrak{L}(V)_0 = k\zeta + \mathrm{Span}_k\{\delta(x, y) : (x, y) \in V\}, \tag{2.10.2}$$

where $\zeta = (\mathrm{Id}_{V^+}, \mathrm{Id}_{V^-})$ and $\delta(x, y) = (D(x, y), -D(y, x)) \in \mathrm{End}(V^+) \times \mathrm{End}(V^-)$, defined by $D(x, y)z = \{x\, y\, z\}$. We let $\mathfrak{gl}(V^\sigma)$ be the Lie algebra defined by $\mathrm{End}(V^\sigma)$ with the commutator as the Lie product. By definition, the Lie product of $\mathfrak{L}(V)$ is determined by the conditions that it be alternating, that $\mathfrak{L}(V)_0$ be a subalgebra of the Lie algebra $\mathfrak{gl}(V^+) \times \mathfrak{gl}(V^-)$ and that

$$[V^\sigma, V^\sigma] = 0, \qquad [D, z] = D_\sigma(z), \qquad [x, y] = -\delta(x, y)$$

for $D = (D_+, D_-) \in \mathfrak{L}(V)_0$, $z \in V^\sigma$ and $(x, y) \in V$. It follows from the identity (JP15) in [Lo1],

$$[D(x, y),\, D(u, v)] = D(\{x\, y\, u\},\, v) - D(u,\, \{y\, x\, v\}).$$

that $\mathfrak{L}(V)_0$ is indeed a subalgebra. As a k-Lie algebra, $\mathfrak{L}(V)$ is generated by ζ, V^+ and V^-, and it has trivial centre.

For a Jordan pair V with a fully idempotent root grading $\mathfrak{R}$ a description of the derived algebra $[\mathfrak{L}(V), \mathfrak{L}(V)]$ is given in [Ne3]. The Tits-Kantor-Koecher algebra of a special Jordan pair is described in [Lo4, §2]. We will work out $\mathfrak{L}(V)$ for a rectangular matrix pair in 2.12.

An automorphism f of V gives rise to an automorphism $\mathfrak{L}(f)$ of $\mathfrak{L}$, defined by

$$x \oplus D \oplus y \quad \mapsto \quad f_+(x) \oplus (f \circ D \circ f^{-1}) \oplus f_-(y).$$

The map $f \mapsto \mathfrak{L}(f)$ is an embedding of the automorphism group $\mathrm{Aut}(V)$ of V into the automorphism group of $\mathfrak{L}(V)$.

Any $(x, y) \in V$ gives rise to automorphisms $\exp_+(x)$ and $\exp_-(y)$ of $\mathfrak{L}(V)$, defined in terms of the decomposition (2.10.1) by the formal 3×3-matrices

$$\exp_+(x) = \begin{pmatrix} 1 & \operatorname{ad} x & Q_x \\ 0 & 1 & \operatorname{ad} x \\ 0 & 0 & 1 \end{pmatrix}, \qquad \exp_-(y) = \begin{pmatrix} 1 & 0 & 0 \\ \operatorname{ad} y & 1 & 0 \\ Q_y & \operatorname{ad} y & 1 \end{pmatrix}. \tag{2.10.3}$$

The map $\exp_\sigma$, $\sigma = \pm$, is an injective homomorphism from the abelian group $(V^\sigma, +)$ to the automorphism group of $\mathfrak{L}(V)$, whose image is denoted U^σ. The *projective elementary group of* V is the subgroup $\mathrm{PE}(V)$ of $\mathrm{Aut}\big(\mathfrak{L}(V)\big)$ generated by $U^+ \cup U^-$, introduced in [Lo4] and studied further in [LN2, §7, §8].

We have now explained all the concepts used in the generalization of part (b) of the Kervaire-Milnor-Steinberg Theorem 1.6.

2.11. **Theorem B.** *Let (R, R_1) be a 3-graded root system and let V be a Jordan pair with a root grading $\mathfrak{R} = (V_\alpha)_{\alpha \in R_1}$.*

(a) *There exists a group homomorphism $\pi\colon \mathrm{St}(V, \mathfrak{R}) \to \mathrm{PE}(V)$, uniquely determined by*

$$\pi\big(\mathrm{x}_\sigma(u)\big) = \exp_\sigma(u), \qquad (u \in V^\sigma).$$

(b) *If all irreducible components of R have infinite rank and $\mathfrak{R}$ is fully idempotent with respect to a family $(e_\alpha)_{\alpha \in R_1}$, the homomorphism π is a universal central extension.*

Theorem B is established in [LN2]. Part (a) follows from [LN2, Cor. 21.12]. By Fact 1.4(d) and Theorem 2.8, the proof of (b) boils down to showing that $\operatorname{Ker} \pi$ is central, which we do in [LN2, Cor. 27.6]. As for Theorem 2.8, it is not necessary to assume that $\mathfrak{R}$ is fully idempotent.

In the setting of (a) let $(u, v) \in V_\alpha^+ \times V_\beta^-$ with $\alpha \neq \beta$ and let $\mathrm{b}(u, v)$ be the element of $\mathrm{St}(V, \mathfrak{R})$ defined in (2.7.1). Then $\pi(\mathrm{b}(u, v)) = \mathfrak{L}(f)$ for some $f \in \mathrm{Aut}(V)$ (for the experts: f is the inner automorphism $(B(u, v), B(v, u)^{-1})$ of [Lo1, 3.9]). That $\pi(\mathrm{b}(u, v)) \in \mathfrak{L}\big(\mathrm{Aut}(V)\big) \subset \mathrm{Aut}(\mathfrak{L}(V))$ is the motivation for the perhaps surprising definition of $\mathrm{b}(u, v)$.

We finish this section by describing $\mathfrak{L}(V)$ and $\mathrm{PE}(V)$ for $V = \mathbb{M}_{IJ}(A)$.

2.12. **The Tits-Kantor-Koecher algebra of a rectangular matrix pair.** Let $V = \mathbb{M}_{IJ}(A) = \big(\mathrm{Mat}_{IJ}(A), \mathrm{Mat}_{JI}(A)\big)$ be the rectangular matrix pair of 2.5(iii). In this subsection we present a model for the Tits-Kantor-Koecher algebra $\mathfrak{L} = \mathfrak{L}(V)$ in terms of elementary matrices which will be used in 2.13 to link the elementary group of V and the abstractly defined group $\mathrm{PE}(V)$.

Let $\mathbf{1}_I = \mathrm{diag}(1_A, \dots)$ be the diagonal matrix of size $I \times I$, define $\mathbf{1}_J$ analogously and let $\mathfrak{A}$ be the unital associative k-algebra

$$\mathfrak{A} = \mathfrak{A}(V) = \begin{pmatrix} k\,\mathbf{1}_I + \mathrm{Mat}_I(A) & \mathrm{Mat}_{IJ}(A) \\ \mathrm{Mat}_{JI}(A) & k\,\mathbf{1}_J + \mathrm{Mat}_J(A) \end{pmatrix} = \begin{pmatrix} \mathrm{Mat}_I(A)_{\mathrm{ex}} & \mathrm{Mat}_{IJ}(A) \\ \mathrm{Mat}_{JI}(A) & \mathrm{Mat}_J(A)_{\mathrm{ex}} \end{pmatrix}$$

whose operations are given by matrix addition and matrix multiplication. In particular,

$$e_1 = \begin{pmatrix} \mathbf{1}_I & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_J \end{pmatrix}$$

are orthogonal idempotents of $\mathfrak{A}$. We consider $\mathfrak{A}$ rather than its subalgebra $\mathrm{Mat}_N(A)_{\mathrm{ex}}$, $N = I \,\dot\cup\, J$, since this will allow us to model the element ζ of (2.10.2).

The Peirce decomposition of $\mathfrak{A}$ with respect to the idempotent e_1 is

$$\mathfrak{A}_{11} = \begin{pmatrix} \mathrm{Mat}_I(A)_{\mathrm{ex}} & 0 \\ 0 & 0 \end{pmatrix}, \qquad\qquad \mathfrak{A}_{10} = \begin{pmatrix} 0 & \mathrm{Mat}_{IJ}(A) \\ 0 & 0 \end{pmatrix},$$

$$\mathfrak{A}_{01} = \begin{pmatrix} 0 & 0 \\ \mathrm{Mat}_{JI}(A) & 0 \end{pmatrix}, \qquad\qquad \mathfrak{A}_{00} \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Mat}_J(A)_{\mathrm{ex}} \end{pmatrix}.$$

Let $\mathfrak{A}^{(-)}$ be the Lie algebra associated with $\mathfrak{A}$. Thus, $\mathfrak{A}^{(-)}$ is defined on the k-module underlying $\mathfrak{A}$ and its Lie algebra product is $[x, y] = xy - yx$ for $x, y \in \mathfrak{A}$. The Lie algebra $\mathfrak{A}^{(-)}$ is $\mathbb{Z}$-graded, $\mathfrak{A}^{(-)} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{A}^{(-)}_n$ with

$$\mathfrak{A}^{(-)}_1 = \mathfrak{A}_{10}, \quad \mathfrak{A}^{(-)}_0 = \mathfrak{A}_{11} \oplus \mathfrak{A}_{00}, \quad \mathfrak{A}^{(-)}_{-1} = \mathfrak{A}_{01}$$

and $\mathfrak{A}^{(-)}_n = 0$ for $n \notin \{1, 0, -1\}$. We define $\mathfrak{e} = \mathfrak{e}(V)$ as the subalgebra of $\mathfrak{A}^-$ generated by e_1, e_2 and

$$\mathfrak{e}_1 = \begin{pmatrix} 0 & \mathrm{Mat}_{IJ}(A) \\ 0 & 0 \end{pmatrix} = \mathfrak{A}^{(-)}_1 \quad \text{and} \quad \mathfrak{e}_{-1} = \begin{pmatrix} 0 & 0 \\ \mathrm{Mat}_{JI}(A) & 0 \end{pmatrix} = \mathfrak{A}^{(-)}_{-1}.$$

Put $\mathfrak{e}_0 = k\,e_1 + k\,e_2 + [\mathfrak{e}_1, \mathfrak{e}_{-1}]$ and $\mathfrak{e}_i = 0$ for $i \notin \{-1, 0, 1\}$. Then $\mathfrak{e} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{e}_i$ is a $\mathbb{Z}$-graded Lie algebra.

We now relate $\mathfrak{e}$ to the Tits-Kantor-Koecher algebra $\mathfrak{L} = \mathfrak{L}(V)$ of V. First, for $\mathsf{a} = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in \mathfrak{e}_0$ define $\Delta(\mathsf{a}) = (\Delta(\mathsf{a})_+, \Delta(\mathsf{a})_-) \in \mathrm{End}_k(V^+) \times \mathrm{End}_k(V^-)$ by

$$\Delta(\mathsf{a})_+(u) = au - ud, \qquad \Delta(\mathsf{a})_-(v) = dv - va,$$

so that

$$\text{(2.12.1)} \qquad \left[\mathsf{a}, \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & au - ud \\ dv - va & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Delta_+(\mathsf{a})(u) \\ \Delta_-(\mathsf{a})(v) & 0 \end{pmatrix}.$$

We claim: *the map*

$$\Psi\colon \mathfrak{e} \to \mathfrak{L}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \oplus \Delta(a, d) \oplus (-c)$$

is a surjective Lie algebra homomorphism whose kernel is $\mathfrak{z}(e)$, the centre of $\mathfrak{e}$, and thus induces an isomorphism

$$\text{(2.12.2)} \qquad \mathfrak{e}/\mathfrak{z}(\mathfrak{e}) \cong \mathfrak{L}$$

of Lie algebras ([Lo4, 2.6], [LN2, 7.2]). Indeed, Ψ is surjective since $\Delta(e_1) = \zeta = -\Delta(e_2)$ and for $(x,y) \in V$

$$\Delta\big(\big[\big(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\big),\ \big(\begin{smallmatrix} 0 & 0 \\ y & 0 \end{smallmatrix}\big)\big]\big) = \Delta\big(\begin{smallmatrix} xy & 0 \\ 0 & -yx \end{smallmatrix}\big) = \delta(x,y) \tag{2.12.3}$$

by (2.5.1). To see that $\operatorname{Ker}\Psi = \mathfrak{z}(\mathfrak{e})$, observe for $\mathsf{m} = \big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big) \in \mathfrak{A}$ that

$$[\mathsf{m}, e_1] = 0 \iff b = 0 = c \iff [\mathsf{m}, e_2] = 0, \tag{2.12.4}$$

whence by (2.12.1),

$$\begin{aligned} \mathsf{m} \in \operatorname{Ker}\Psi \quad &\iff \quad b = 0 = c,\ \Delta(a,d) = 0,\ \big(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\big) \in \mathfrak{e}_0, \\ &\iff \quad [\mathsf{m}, e_1] = 0 = [\mathsf{m}, e_2],\ [\mathsf{m},\ \mathfrak{e}_1] = 0 = [\mathsf{m},\ \mathfrak{e}_{-1}],\ \mathsf{m} \in \mathfrak{e}, \\ &\iff \quad \mathsf{m} \in \mathfrak{z}(\mathfrak{e}) \end{aligned}$$

because $e_1, e_2, \mathfrak{e}_1$ and $\mathfrak{e}_{-1}$ generate $\mathfrak{e}$ as Lie algebra. Finally, since both $\mathfrak{e}$ and $\mathfrak{L}$ are $\mathbb{Z}$-graded and Ψ preserves this grading, Ψ is a Lie algebra homomorphism as soon as Ψ preserves products of type $[\mathsf{x}_\mathsf{i}, \mathsf{y}_\mathsf{j}]$ for $(i,j) = (0, \pm 1), (1, -1)$ and $(0,0)$. For $(0, \pm 1)$ and $(1,-1)$ this follows from (2.12.1) and (2.12.3) respectively; the case $(0,0)$ is left to the reader.

In the remainder of this subsection we will give a more precise description of $\mathfrak{e}$ and its centre, see (2.12.5) and (2.12.7). Let $[A,A] = \operatorname{Span}\{ab - ba : a, b \in A\}$, the derived algebra of the Lie algebra A^-, and let

$$\mathfrak{sl}_N(A) := \{x = (x_{kl}) \in \operatorname{Mat}_N(A) \colon \textstyle\sum_{n \in N} x_{nn} \in [A,A]\}.$$

From $[aE_{kl}, bE_{rs}] = ab\delta_{lr}E_{ks} - ba\delta_{ks}E_{rl}$ one then gets

$$\begin{aligned} &\mathfrak{e}_1 \oplus [\mathfrak{e}_1, \mathfrak{e}_{-1}] \oplus \mathfrak{e}_{-1} = \mathfrak{sl}_N(A), \qquad \text{whence} \\ &k\, e_1 + k\, e_2 + \mathfrak{sl}_N(A) = \mathfrak{e}. \end{aligned} \tag{2.12.5}$$

The description of the centre $\mathfrak{z}(\mathfrak{e})$ depends on the cardinality of N because

$$\mathfrak{A}_0^{(-)} \cap A\mathbf{1}_N = \begin{cases} A\,\mathbf{1}_N & \text{if } |N| < \infty, \\ k\,\mathbf{1}_N & \text{if } |N| = \infty. \end{cases}$$

Denoting by $\mathrm{Z}(A) = \{z \in A : [z, A] = 0\}$ the centre of A (= centre of $A^{(-)}$), a straightforward calculation shows

$$\big\{x \in \mathfrak{A}_0^{(-)} : [x, \mathfrak{e}_1] = 0 = [x, \mathfrak{e}_{-1}]\big\} = \mathfrak{A}_0^{(-)} \cap \mathrm{Z}(A)\mathbf{1}_N = \begin{cases} \mathrm{Z}(A)\mathbf{1}_N, & |N| < \infty, \\ k\mathbf{1}_N, & |N| = \infty. \end{cases} \tag{2.12.6}$$

Since $\mathfrak{e}$ is generated by e_1, e_2, $\mathfrak{e}_1$ and $\mathfrak{e}_{-1}$, (2.12.4) and (2.12.6) imply

$$\mathfrak{z}(\mathfrak{e}) = \mathfrak{e}_0 \cap (\mathrm{Z}(A)\,\mathbf{1}_N) = \begin{cases} \mathfrak{e}_0 \cap (\mathrm{Z}(A)\mathbf{1}_N), & |N| < \infty, \\ k\mathbf{1}_N, & |N| = \infty. \end{cases} \tag{2.12.7}$$

For example, if $A = k$ is a field of characteristic 0 and $|N| = n$ is finite, we get $\mathfrak{e} = \mathfrak{gl}_n(k)$, $\mathfrak{z}(\mathfrak{e}) = k\mathbf{1}_n$ and $\mathfrak{L} \cong \mathfrak{sl}_n(k)$.

2.13. **The projective elementary group of a rectangular matrix pair.** We use the notation of 2.12 and let $V = \mathbb{M}_{IJ}(A)$. The goal of this subsection is to show that the group $\mathrm{PE}(V)$ is isomorphic to a central quotient of the elementary group $\mathrm{E}(A)$. We put

$$\mathrm{E}(V) = \mathrm{E}_N(A)$$

and call it the *elementary group of* V. Since by 1.12 the group $\mathrm{E}_N(A)$ is generated by $\mathrm{e}_+(V^+) \cup \mathrm{e}_-(V^-)$ this agrees with the definition of the elementary group of an arbitrary special Jordan pair in [Lo4, §2] or [LN2, 6.2]. We will identify the Tits-Kantor-Koecher algebra $\mathfrak{L}(V) = \mathfrak{L}$ with $\mathfrak{e}/\mathfrak{z}(\mathfrak{e})$ via the isomorphism (2.12.2) induced by $\Psi\colon \mathfrak{e} \to \mathfrak{L}$.

Any $g \in \mathfrak{A}^\times$ gives rise to an automorphism $\operatorname{Ad} g$ of $\mathfrak{A}^-$ defined by $(\operatorname{Ad} g)(x) = gxg^{-1}$. If $\operatorname{Ad} g$ stabilizes the subalgebra $\mathfrak{e}$ of $\mathfrak{A}^-$, it also stabilizes $\mathfrak{z}(\mathfrak{e})$, and therefore descends to an automorphism $\overline{\operatorname{Ad}}(g)$ of $\mathfrak{e}/\mathfrak{z}(\mathfrak{e}) = \mathfrak{L}$ satisfying $\Psi \circ (\operatorname{Ad} g|_{\mathfrak{e}}) = (\overline{\operatorname{Ad}}\, g) \circ \Psi$. The map

$$\overline{\operatorname{Ad}}\colon \{g \in \mathfrak{A}^\times : (\operatorname{Ad} g)(\mathfrak{e}) = \mathfrak{e}\} \to \operatorname{Aut}(\mathfrak{L}), \qquad g \mapsto \overline{\operatorname{Ad}}\, g$$

is a group homomorphism. For $g = \mathrm{e}_+(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{A}^\times$ as in (1.12.1), the automorphism $\operatorname{Ad} \mathrm{e}_+(x)$ acts as follows:

$$\begin{aligned}
\big(\operatorname{Ad} \mathrm{e}_+(x)\big) \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \\
\big(\operatorname{Ad} \mathrm{e}_+(x)\big) \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} &= \begin{pmatrix} a & -ax + xd \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \Big[\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Big], \\
\big(\operatorname{Ad} \mathrm{e}_+(x)\big) \begin{pmatrix} 0 & 0 \\ -c & 0 \end{pmatrix} &= \begin{pmatrix} -xc & xcx \\ -c & cx \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ -c & 0 \end{pmatrix} + \Big[\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -c & 0 \end{pmatrix} \Big] + \begin{pmatrix} 0 & Q_x c \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

These equations show that the automorphism $\operatorname{Ad} \mathrm{e}_+(x)$ stabilizes $\mathfrak{e}$ and, by comparison with (2.10.3), that $\overline{\operatorname{Ad}}\, \mathrm{e}_+(x) = \exp_+(x)$. One proves in the same way that $\operatorname{Ad} \mathrm{e}_-(y)$, $y \in V^-$, stabilizes $\mathfrak{e}$ and that $\overline{\operatorname{Ad}}\, \mathrm{e}_-(y) = \exp_-(y)$. Since $\mathrm{PE}(V)$ is generated by $\exp_+(V^+) \cup \exp_-(V^-)$, the homomorphism $\overline{\operatorname{Ad}}$ restricts to a surjective group homomorphism

$$\overline{\operatorname{Ad}}_{\mathrm{E}}\colon\ \mathrm{E}(V) \to \mathrm{PE}(V), \quad g \mapsto \overline{\operatorname{Ad}}\, g.$$

We claim that its kernel is the centre $\mathrm{Z}(\mathrm{E}(V))$ of $\mathrm{E}(V)$:

(2.13.1) $$\mathrm{Z}(\mathrm{E}(V)) = \operatorname{Ker} \operatorname{Ad}|_{\mathrm{E}(V)} = \operatorname{Ker} \overline{\operatorname{Ad}}_{\mathrm{E}}, \qquad \text{whence}$$

(2.13.2) $$\mathrm{E}(V)/\,\mathrm{Z}(\mathrm{E}(V)) \cong \mathrm{PE}(V).$$

Proof of (2.13.1): Clearly, $\mathrm{Z}(\mathrm{E}(V)) = \operatorname{Ker} \operatorname{Ad}|_{\mathrm{E}(V)} \subset \operatorname{Ker} \overline{\operatorname{Ad}}_{\mathrm{E}}$, so it remains to show that $g \in \operatorname{Ker} \overline{\operatorname{Ad}}_{\mathrm{E}}$ is central in $\mathrm{E}(V)$. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $g^{-1} = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$. Then

(2.13.3) $$g \begin{pmatrix} \mathbf{1}_I & 0 \\ 0 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} aa' & ab' \\ ca' & cb' \end{pmatrix},$$

(2.13.4) $$g \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_{\mathbf{J}} \end{pmatrix} g^{-1} = \begin{pmatrix} bc' & bd' \\ dc' & dd' \end{pmatrix},$$

(2.13.5) $$g^{-1} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} g = \begin{pmatrix} a'xc & a'xd \\ c'xc & c'xd \end{pmatrix},$$

(2.13.6) $$g^{-1} \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} g = \begin{pmatrix} b'ya & b'yb \\ d'ya & d'yp \end{pmatrix}.$$

Since $(\operatorname{Ad} g)\,(\mathsf{m}) \equiv \mathsf{m} \equiv (\operatorname{Ad} g^{-1})(\mathsf{m}) \mod \mathfrak{z}(\mathfrak{e})$ for all $\mathsf{m} \in \mathfrak{e}$ and since $\mathfrak{z}(\mathfrak{e})$ is diagonal by (2.12.7), it follows from (2.13.3), (2.13.4) and (2.13.6) that

$$ab' = 0 = ca' = bd', \qquad d'ya = y \text{ for } y \in V^-.$$

Applied to $c \in V^-$, this proves $cb' = (d'ca) \cdot b' = d'c \cdot ab' = 0$ and then $bc' = b \cdot (d'c'a) = bd' \cdot c'a = 0$. From $\mathbf{1}_N = gg^{-1}$ we obtain

$$\begin{pmatrix} \mathbf{1}_I & 0 \\ 0 & \mathbf{1}_J \end{pmatrix} = \begin{pmatrix} aa' + bc' & * \\ * & * \end{pmatrix} = \begin{pmatrix} aa' & * \\ * & * \end{pmatrix}.$$

Together with the already established equations this shows, using (2.13.3), that $(\operatorname{Ad} g)(e_1) = e_1$. Because $g \in \mathfrak{A}$, we get $b = 0 = c$ from (2.12.4). Thus also $b' = 0 = c'$, so that (2.13.5) and (2.13.6) prove that $\operatorname{Ad} g$ fixes $\mathfrak{e}_{\pm 1}$. Since $\mathrm{e}_\pm(V^\pm) = \mathbf{1}_N + \mathfrak{e}_{\pm 1}$, we see that $\operatorname{Ad} g$ fixes the generators $\mathrm{e}_\pm(V^\pm)$ of $\mathrm{E}(V)$, i.e. g is central. □

A similar result holds for any special Jordan pair V: there always exists a surjective group homomorphism from the elementary group $\mathrm{E}(V)$ (which we have not defined) onto the projective elementary group $\mathrm{PE}(V)$, whose kernel is central, but not necessarily the centre of $\mathrm{E}(V)$, see [Lo4, Thm. 2.8].

3. Some open problems

We describe some open problems for Steinberg and projective elementary groups of Jordan pairs. Our list is very much limited by the author's taste and knowledge. This section requires some expertise in Jordan pairs.

3.1. **The normal subgroup structure of** $\mathrm{PE}(V)$**.** The problem is quite easily stated: *Given a Jordan pair V, describe all normal subgroups of* $\mathrm{PE}(V)$. As stated, this may be too general. We therefore discuss some special cases.

(a) In view of the results of [Lo5] it is natural to ask: when is $\mathrm{PE}(V)$ a perfect group, when is it simple? Indeed, [Lo5, Thm. 2.6] says that, for a nondegenerate Jordan pair V with dcc on principal ideals, $\mathrm{PE}(V)$ is a perfect group if and only if V has no simple factors isomorphic to $(\mathbb{F}_2, \mathbb{F}_2)$, $(\mathbb{F}_3, \mathbb{F}_3)$ or $\big(\mathrm{H}_2(\mathbb{F}_2), \mathrm{H}_2(\mathbb{F}_2)\big)$. Here $\mathbb{F}_q$ is the field with q elements. Also, by [Lo5, Thm. 2.8], $\mathrm{PE}(V)$ is a simple (abstract) group if and only if V is simple and not isomorphic to $(\mathbb{F}_2, \mathbb{F}_2)$, $(\mathbb{F}_3, \mathbb{F}_3)$ or $\big(\mathrm{H}_2(\mathbb{F}_2), \mathrm{H}_2(\mathbb{F}_2)\big)$ ([Lo5, Thm. 2.8]). That the exceptional cases have to be excluded in these two theorems is evident from the isomorphisms $\mathrm{PE}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathfrak{S}_3$ (the symmetric group on three letters), $\mathrm{PE}(\mathbb{F}_3, \mathbb{F}_3) \cong \mathfrak{A}_4$ (the alternating group on four letters), and $\mathrm{PE}\big(\mathrm{H}_2(\mathbb{F}_2), \mathrm{H}_3(\mathbb{F}_2)\big) \cong \mathfrak{S}_6$. Thus, the problem is to find out if these two theorems of [Lo5] hold for more general Jordan pairs. It follows from (b) that is natural to assume simplicity of V for the second theorem.

(b) Every ideal I of the Jordan pair V gives rise to a normal subgroup of $\mathrm{PE}(V)$. Indeed, one can show ([LN2, 7.5]) that the canonical map $\mathrm{can}\colon V \to V/I$ induces a surjective group homomorphism $\mathrm{PE}(\mathrm{can})\colon \mathrm{PE}(V) \to \mathrm{PE}(V/I)$. We let $\mathrm{PE}(V, I)$ be its kernel:

$$1 \longrightarrow \mathrm{PE}(V, I) \longrightarrow \mathrm{PE}(V) \xrightarrow{\mathrm{PE}(\mathrm{can})} \mathrm{PE}(V/I) \longrightarrow 1$$

Problem: describe $\mathrm{PE}(V, I)$ by generators and relations. For elementary linear groups over rings this is a standard result, see for example [HVZ, Lemma 3]. The paper [CK] shows that even in case $\mathrm{SL}_2(A)$ one needs methods from Jordan algebras.

3.2. **Central closedness of** $\mathrm{St}(V, \mathfrak{R})$ **in low ranks.** We have excluded low rank cases in Theorem 2.8 for the simple reason that it is not true without further assumptions in low ranks. We discuss $2 \le \operatorname{rank} R \le 4$ in (a) and $\operatorname{rank} R = 1$ in (b).

(a) One knows ([LN2, 27.11]) that $\mathrm{St}(V, \mathfrak{R})$ is a classical linear or unitary Steinberg group. Let us first consider the case that V is defined over a field F and that $\dim_F V_\alpha = 1$ for all $\alpha \in R_1$. Then [St3, Thm. 1.1] applies and yields that $\mathrm{St}(V, \mathfrak{R})$ is not centrally closed if and only if (R, R_1) and F satisfy one of the following conditions.

(3.2.1)

(R, R_1)	A_2^1	A_3^1 or A_3^2	$\mathrm{C}_2^{\mathrm{her}}$	$\mathrm{B}_3^{\mathrm{qf}}$	$\mathrm{C}_3^{\mathrm{her}}$	$\mathrm{D}_4^{\mathrm{alt}}$
$\lvert F\rvert$	$2, 4$	2	2	3	2	2

In this table we use the abbreviation $\mathrm{A}_2^1 = \mathrm{A}_N^I$ for $|N| = 2$, $|I| = 1$ and analogously for $\mathrm{A}_3^1, \dots, \mathrm{D}_4^{\mathrm{alt}}$. The cases $R = \mathrm{A}_4, \mathrm{B}_4, \mathrm{C}_4$ do not appear in the table because in these cases $\mathrm{St}(V, \mathfrak{R})$ is centrally closed, as mentioned in 2.8.

Still assuming that V is defined over a field F, it is natural to replace the assumption $\dim_F V_\alpha = 1$ by the requirement that the fully idempotent root grading $\mathfrak{R}$ of V is a *division grading* in the sense that all root spaces V_α are Jordan division pairs, which means that for every non-zero $x \in V_\alpha^\sigma$ the endomorphism $Q(x)|V_\alpha^{-\sigma}$ is invertible. Preliminary investigations lead us to conjecture:

(C) *If* $\mathrm{St}(V, \mathfrak{R})$ *is not centrally closed then* $\dim_F V_\alpha = 1$ *for all* $\alpha \in R_1$ *and* F *satisfies the restrictions of table* (3.2.1).

(b) $R = \mathrm{A}_1$: As in (a) we assume that $\mathfrak{R}$ is a division grading, i.e., V is a division pair and is therefore isomorphic to the Jordan pair (J, J) of a division Jordan algebra J. By [LN2, 9.13] this is equivalent to $\mathrm{PE}(V)$ being a rank one group in the sense of [Lo6]. Since the grading is trivial, $\mathrm{St}(V, \mathfrak{R})$ is the free product of the abelian groups V^+ and V^-, which is not perfect in general, a necessary condition for a group to be centrally closed (1.4(a)). Following the example of Chevalley groups [St2], it seems more promising to consider the group $\mathrm{St}(J)$ defined by the following presentation:

- generators $\mathrm{x}_\sigma(a)$, $a \in J$, $\sigma = \pm$ and, putting
$$\mathrm{w}_b = \mathrm{x}_-(b^{-1})\,\mathrm{x}_+(b)\,\mathrm{x}_-(b^{-1})$$
for $0 \neq b \in J$,
- relations
$$\mathrm{x}_\sigma(a+b) = \mathrm{x}_\sigma(a)\,\mathrm{x}_\sigma(b) \text{ for } a, b \in J \text{ and}$$
$$\mathrm{w}_b\,\mathrm{x}_-(a)\,\mathrm{w}_b^{-1} = \mathrm{x}_+\big(U(b)a\big) \text{ for all } a \in J \text{ and all } 0 \neq b \in J.$$

We remark that $\mathrm{St}(J)$ is the Steinberg group $\mathrm{St}(V, \mathcal{S})$ of [LN2, 13.1], where $\mathcal{S}$ is the set of all non-zero idempotents of V. By [LN2, 13.6], $\mathrm{St}(J)$ is the classical Steinberg group $\mathrm{St}(A)$ in case $V = (A, A)$ and A an associative division algebra.

To motivate our conjecture in this case, let us first consider the special case $J = \mathbb{F}_q$. Since by 3.1(a) the group $\mathrm{PE}(V)$ is not perfect in case $J = \mathbb{F}_q$, $q = 2, 3$, these cases have to be excluded. Moreover, by [St3, Th. 1.1], $\mathrm{St}(J)$ is not centrally closed in case $V = (\mathbb{F}_q, \mathbb{F}_q)$ and $q \in \{4, 9\}$, but these values of q are the only exceptions for $V = (F, F)$, F a field. This leads us to ask:

(Q) *Is* $\mathrm{St}(J)$ *centrally closed whenever* $J \neq \mathbb{F}_q$ *with* $q \in \{2, 3, 4, 9\}$ *?*

There exists an example of an associative unital $\mathbb{F}_5$-algebra A for which $\mathrm{St}(A)$ is not centrally closed [Str, Ex. 4], but A is not a division algebra.

3.3. **Centrality of** $\mathrm{Ker}(\pi)$**.** Let $\pi\colon \mathrm{St}(V, \mathfrak{R}) \to \mathrm{PE}(V)$ be the homomorphism of Theorem 2.11(a). For simplicity, let us assume that R is irreducible. If R has infinite rank, part (b) of 2.11 says that π is a universal central extension. The problem here is: *find sufficient conditions for* $\mathrm{Ker}(\pi)$ *to be central if* R *has finite rank.*

Some special cases are known. For example, if V is split in the sense of [Ne4], centrality of $\mathrm{Ker}(\pi)$ is established in [vdK], [La], [LS] and [Si] for rank ≥ 3. The quoted papers all use the same method, pioneered by [vdK], namely a "basis-free presentation of $\mathrm{St}(V, \mathfrak{R})$". *Can the method of* [vdK] *be generalized to treat* $\mathrm{St}(V, \mathfrak{R})$*,* V *split root graded, in a case-free manner?*

For a slightly different type of Steinberg group and a unit regular V, centrality of $\mathrm{Ker}(\pi)$ is shown in [Lo5, Th. 1.12].

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