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CONJUGACY OF CARTAN SUBALGEBRAS IN EALAS WITH A NON-FGC CENTRELESS CORE

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Dedicated to E. B. Vinberg
on the occasion of his 80th birthday

Abstract. We establish the conjugacy of Cartan subalgebras for extended affine Lie algebras whose centreless core is "of type A", i.e., $\ell \times \ell$-matrices over a quantum torus $\mathcal{Q}$ whose trace lies in the commutator space of $\mathcal{Q}$. This settles the last outstanding part of the conjugacy problem for Extended Affine Lie Algebras that remained open.

Introduction

This work is the last of a series of papers [CGP14, CNP16, CNPY16] devoted to proving the Conjugacy Theorem for Extended Affine Lie Algebras.

Conjugacy Theorem. *Let (E, H) and (E, H') be two extended affine Lie algebras, both defined on the same underlying Lie algebra E over an algebraically closed field of characteristic* 0. *Then there exists an automorphism f of E such that $f(H) = H'$.*

Conjugacy has been established for all but one family of EALAs, and it is this remaining case that our paper settles. Below we give a brief historical account of the "Conjugacy problem".

Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra over a field k of characteristic 0, and let $\mathbf{G}$ be the simply connected Chevalley–Demazure algebraic group associated to $\mathfrak{g}$. Chevalley's theorem [Bou75, VIII, § 3.3, Cor. de la Prop. 10] asserts that all split Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ are conjugate under the adjoint action of $\mathbf{G}(k)$ on $\mathfrak{g}$. This is one of the central results of classical Lie theory. One of its immediate consequences is that the corresponding root system is an invariant of the Lie algebra (i.e., it does not depend on the choice of Cartan subalgebra).

We now look at the analogous question in the infinite-dimensional set up as it relates to extended affine Lie algebras (EALAs for short). Even if the field k is assumed to be algebraically closed, the reader should keep in mind that our results are more akin to the setting of Chevalley's theorem for general k than to conjugacy of Cartan subalgebras in finite-dimensional simple Lie algebras over algebraically closed fields. The role of $(\mathfrak{g}, \mathfrak{h})$ is now played by a pair (E, H) consisting of a Lie algebra E and a "Cartan subalgebra" H.

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There are other Cartan subalgebras in E, and the question is whether they are conjugate and, if so, under the action of which group.

The first example is that of untwisted affine Kac–Moody Lie algebras. Let $R = k[t^{\pm 1}]$. Then

$$E = \mathfrak{g} \otimes_k R \oplus kc \oplus kd \tag{0.0.1}$$

and

$$H = \mathfrak{h} \otimes 1 \oplus kc \oplus kd. \tag{0.0.2}$$

The relevant information is as follows. The k-Lie algebra $\mathfrak{g} \otimes_k R \oplus kc$ is a central extension (in fact the universal central extension) of the k-Lie algebra $\mathfrak{g} \otimes_k R$. The derivation d of $\mathfrak{g} \otimes_k R$ corresponds to the degree derivation td/dt acting on R. Finally $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{g}$. The nature of H is that it is abelian, it acts k-diagonalizably on E, and it is maximal with respect to these properties. Correspondingly, these subalgebras are called MADs (Maximal Abelian Diagonalizable) subalgebras. A celebrated theorem of Peterson and Kac [PK83] states that all MADs of E are conjugate (under the action of a group that they construct which is the analogue of the simply connected group in the finite-dimensional case). Similar results hold for the twisted affine Lie algebras. These algebras are of the form

$$E = L \oplus kc \oplus kd.$$

The Lie algebra L is a loop algebra $L = L(\mathfrak{g}, \sigma)$ for some finite order automorphism σ of $\mathfrak{g}$ (see [Kac85] for details). If σ is the identity, we are in the untwisted case. The ring R can be recovered as the centroid of L.

Extended affine Lie algebras can be thought of as multivariable generalizations of finite-dimensional simple Lie algebras and affine Kac–Moody algebras. For example, taking $R = k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ in (0.0.1) and increasing kc and kd correspondingly in the obvious way leads to toroidal algebras, an important class of examples of EALAs. But as is already the case for affine Kac–Moody algebras, there are many interesting examples of EALAs where $\mathfrak{g} \otimes_k R$ is replaced by a more general algebra, a so-called Lie torus (see 2.1).

In the EALA set up, the Lie algebras $\mathfrak{g}$ as above are the case of nullity $n = 0$, while the affine Lie algebras are the case of nullity $n = 1$. In higher nullity n we have $R = k[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ for some $m \le n$, where again R is the centroid of the centreless core E_{cc} of the given EALA. The theory of EALAs divides naturally into two cases:

(a) $m = n$. In this case E_{cc} is a module of finite type over the centroid R. It is referred to as the "fgc case" (short for finitely generated over the centroid). If k is algebraically closed,[1] the R-Lie algebra E_{cc} is a multiloop algebra based on a (unique) $\mathfrak{g}$ as above. In particular, E_{cc} is a twisted form of $\mathfrak{g} \otimes_k R$. This fact allows the powerful methods of descent theory and reductive group schemes to be used. Conjugacy at the level of E_{cc} was established in [CGP14]. The lift of this conjugacy theorem from E_{cc} to E is the main result of [CNPY16].

(b) $m < n$. This is the so-called non-fgc case. Now E_{cc} is not a module of finite type over its centroid and E_{cc} is not a twisted form of $\mathfrak{g} \otimes_k R$. The non-abelian Galois cohomology methods used in (a) are not available. Fortunately, in the non-fgc case the nature of E_{cc} is fully understood. Indeed $E_{cc} = \mathfrak{sl}_\ell(\mathcal{Q})$ for some quantum torus $\mathcal{Q}$ and positive integer ℓ (see below for details). Conjugacy at the level of E_{cc} was established in [CNP16] by means of a "specialization" trick of its own interest. The main result of the present paper is the lift of conjugacy for E_{cc} to E in the non-fgc case. This completes the proof that "Conjugacy of Cartan subalgebras" holds for all EALAs.

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[1] See Remark 0.1 below.

The canonical procedure that associates to an EALA E its core E_c and centreless core $E_{cc} = E_c/\mathcal{Z}(E_c)$ can be reversed in the sense that one can reconstruct E from its centreless core E_{cc} by a special type of a 2-fold extension (in this paper we generalize this to so-called "interlaced extensions"). Moreover, going from E to E_{cc} is also a well-behaved procedure at the level of the Cartan subalgebras: consider $H_c = H \cap E_c$ and let $\pi\colon E_c \to E_{cc}$ be the canonical map, then $H_{cc} = \pi(H_c)$ and the analogously defined H'_{cc} are special types of MADs in E_{cc}. Even more, every automorphism f of E leaves E_c and hence also $\mathcal{Z}(E_c)$ invariant and so gives rise to an automorphism f_{cc} of E_{cc}. Thus, if our Main Theorem holds, then necessarily there exists some automorphism $f_{cc} \in \mathrm{Aut}_k(E_{cc})$ such that $f_{cc}(H_{cc}) = H'_{cc}$. From this perspective, our approach of proving conjugacy "upstairs" on the EALA level is the most natural one: we want to show that

(A) there exists $f_{cc} \in \mathrm{Aut}_k(E_{cc})$ satisfying $f_{cc}(H_{cc}) = H'_{cc}$, and
(B) the automorphism f_{cc} of (A) can be "lifted" to an automorphism f of E such that $f(H) = H'$.

Problem (A) has been solved in the two papers [CGP14] (the fgc case) and [CNP16] (the non-fgc case).

This leaves us with problem (B). Its difficulty lies in the fact that a lift $f \in \mathrm{Aut}(E)$ of f_{cc} (if it exists at all) will not necessarily satisfy $f(H) = H'$. However, for any EALA (E, H) and automorphism f of E it is easily seen that $(E, f(H))$ is an EALA which satisfies $(f(H))_{cc} = f_{cc}(H_{cc})$. We can therefore split a solution of problem (B) into two steps:

(B1) [CNPY16, Th. 7.1] If $H_{cc} = H'_{cc}$, then there exists $f \in \mathrm{Aut}_k(E)$ such that $f(H) = H'$.
(B2) The automorphism used to solve problem (A) can be lifted to an automorphism of E.

We have solved Problem (B2) and thus established the Conjugacy Theorem for extended affine Lie algebras in the fgc case in [CNPY16, Th. 7.6]. Thus the Conjugacy Theorem for extended affine Lie algebras is reduced to proving (B2) in the non-fgc case.

As explained in 2.2(d), in the non-fgc case $E_{cc} \simeq \mathfrak{sl}_\ell(\mathcal{Q})$ for some $\ell \geq 2$, and $\mathcal{Q}$ a quantum torus which is not finitely generated over its centre. But as in [CNP16] we will deal here with the Lie algebra $\mathfrak{sl}_\ell(\mathcal{Q})$ for an *arbitrary* quantum torus $\mathcal{Q}$.[2] The conjugacy theorem of [CNP16] for $L = \mathfrak{sl}_\ell(\mathcal{Q})$, i.e., the solution of problem (A) in the non-fgc case, uses an interior automorphism $\mathrm{Int}(g)$ for some $g \in \mathrm{GL}_\ell(\mathcal{Q})$. The final step in the proof of the Conjugacy Theorem for EALAs is therefore that such g can be suitably chosen. More precisely, we have the theorem below.

Main Theorem. *Let $L = \mathfrak{sl}_\ell(\mathcal{Q})$, $\ell \geq 2$ with $\mathcal{Q}$ a quantum torus, then problem* (A) *can be solved with a $g \in \mathrm{GL}_\ell(\mathcal{Q})$ such that* $\mathrm{Int}(g)$ *can be lifted to an automorphism of any extended affine Lie algebra E with $E_{cc} = L$.*

The somewhat curious formulation of our result refers to the fact that we are not claiming that all automorphisms $\mathrm{Int}(g)$ can be lifted to the EALA level.

Remark 0.1. A word on the nature of our base field k. The solution of problem (A) in the fgc case [CGP14] assumes k algebraically closed (and of course of characteristic 0). The reason for this assumption is the Realization Theorem of [ABFP09]. More precisely, [CGP14] holds as long as one knows that E_{cc} is a multiloop algebra, while [ABFP09] shows that this holds in the fgc case under the assumption that k be algebraically closed.

[2]Assuming that $\mathcal{Q}$ is fgc would not simplify our arguments. The additional generality may be of future independent interest.

In the non-fgc case k there is no need not to assume that k be algebraically closed to solve problem (A) (see [CNP16]). The lifting result (B1) works for any field of characteristic 0. In the remainder of this paper we will assume that our base field k has characteristic 0, but need not be algebraically closed. It is in this setting that we will prove our Main Theorem in the non-fgc case, namely the Conjugacy Theorem for EALAs with a non-fgc centreless core.

Notation. For elements g, h of a group G we denote by

$$[\![g,h]\!] = ghg^{-1}h^{-1}$$

the commutator of g and h, and by $\mathcal{D}(G) = (G,G)$ the commutator subgroup of G. As usual $\mathrm{Int}(g)(h) = ghg^{-1}$. We use $D < L$ to indicate that D is a subalgebra of the algebra L. For any (associative or Lie) algebra A we denote by $\mathrm{Der}_k(A)$ the Lie algebra of k-linear derivations of A, and by $\mathcal{Z}(A)$ its centre.

1. Interlaced extensions

In this section we introduce a general construction of Lie algebras, so-called interlaced extensions. We will see in § 2 that extended affine Lie algebras are examples of interlaced extensions. In addition, one of the principal components of our proof of the Main Theorem can and will be done in the setting of interlaced extensions (Theorem 3.2).

1.1. **Cocycles.** Let L be a Lie algebra and let V be an L-module. A 2-*cocycle with coefficients in* V is an alternating map $\sigma\colon L\times L \to V$ satisfying for $l_i \in L$,

$$\text{(1.1.1)}\quad l_1\cdot\sigma(l_2,l_3) + l_2\cdot\sigma(l_3,l_1) + l_3\cdot\sigma(l_1,l_2) = \sigma([l_1,l_2],l_3) + \sigma([l_2,l_3],l_1) + \sigma([l_3,l_1],l_2).$$

Given such a 2-cocycle σ, the vector space $L\oplus V$ becomes a Lie algebra with respect to the product

$$[l_1+v_1,\, l_2+v_2] = [l_1,l_2]_L + (l_1\cdot v_2 - l_2\cdot v_1 + \sigma(l_1,l_2)).$$

We will denote this Lie algebra by $L\oplus_\sigma V$. Note that the projection onto the first factor $\mathrm{pr}_L\colon L\oplus_\sigma V \to L$ is an epimorphism of Lie algebras whose kernel is the abelian ideal V. We refer to such an extension as an *abelian extension*.

A special case of this construction is the situation when V is a trivial L-module. In this case a 2-cocycle will be called a *central* 2-*cocycle*. Note that all terms on the left hand side of (1.1.1) vanish. For a central 2-cocycle, $\mathrm{pr}_L\colon L\oplus_\sigma V\to L$ is an epimorphism whose kernel V is the central ideal V of $L\oplus_\sigma V$, i.e., pr_L is a *central extension*.

A basic construction of a central 2-cocycle goes as follows. We assume that β is a bilinear form on L which is *invariant* in the sense that $\beta([l_1,l_2],l_3) = \beta(l_1,[l_2,l_3])$ holds for all $l_i\in L$. We denote by

$$\mathrm{SDer}_{k,\beta}(L)$$

(or simply $\mathrm{SDer}_k(L)$ if β is fixed within our context) the subalgebra of $\mathrm{Der}_k(L)$ consisting of *skew derivations*, i.e., those derivations d satisfying $\beta\big(d(l),l\big) = 0$ for all $l\in L$. We further suppose that D is a Lie algebra acting on L by skew derivations. It is well-known and easy to check that

$$\text{(1.1.2)}\qquad \sigma_{D,\beta}\colon L\times L\to D^* := \mathrm{Hom}_k(D,k),\quad \sigma_{D,\beta}(l_1,l_2)\,(d) = \beta(d(l_1),l_2)$$

is a central 2-cocycle.

1.2. **Interlaced extensions.** As we explained in the introduction, one of the main problems solved in this paper is lifting an automorphism from the centreless core $\mathfrak{sl}_\ell(Q)$ of an EALA E to E. We will see that this can be done without additional work in a more general setting than extended affine Lie algebras. By working on this more general edifice not only do we strip the lifting process from unnecessary assumptions, but we also suggest the possibility of recasting EALA theory in a more general cadre. In this subsection we will introduce this general framework. It uses the following data:

(i) a Lie algebra L equipped with an invariant bilinear form β;
(ii) a Lie algebra D acting on L by skew derivations of (L, β); we write this action as $d \cdot l$ or sometimes $d(l)$ for $d \in D$ and $l \in L$;
(iii) a subspace $C \subset D^*$ which is invariant under the coadjoint action of D on D^*, defined by $(d \cdot c)(d') = c([d', d])$, and satisfies

$$\sigma_{D,\beta}(l_1, l_2) \in C \quad (l_i \in L) \tag{1.2.1}$$

for $\sigma_{D,\beta}$ as in (1.1.2);
(iv) a 2-cocycle $\tau\colon D \times D \to C$.

Given these data, we define a product on the vector space

$$E = L \oplus C \oplus D \tag{1.2.2}$$

by ($l_i \in L$, $c_i \in C$ and $d_i \in D$)

$$\begin{aligned}[l_1 + c_1 + d_1,\, l_2 + c_2 + d_2] &= ([l_1, l_2]_L + d_1 \cdot l_2 - d_2 \cdot l_1) \\ &\oplus \big(\sigma_{D,\beta}(l_1, l_2) + d_1 \cdot c_2 - d_2 \cdot c_1 + \tau(d_1, d_2)\big) \oplus [d_1, d_2]_D.\end{aligned} \tag{1.2.3}$$

In this formula $[.,.]_L$ and $[.,.]_D$ denote the Lie algebra products of L and D, respectively. We use $\oplus$ on the right hand side of (1.2.3) as a mnemonic device to indicate the components of the product with respect to the decomposition (1.2.2). To avoid any possible confusion we will sometimes indicate the product of E by $[.,.]_E$. We often abbreviate $\sigma = \sigma_{D,\beta}$.

Our construction is a special case of [CNPY16, 1.4]. Thus, by [CNPY16, 1.5], the vector space E together with the product (1.2.3) is a Lie algebra. Since it is obtained by interlacing the central extension $0 \to C \to L \oplus C \to L \to 0$ (obvious maps) with the abelian extension $0 \to C \to C \oplus D \to D \to 0$ (again obvious maps) we call this Lie algebra the *interlaced extension given by the data* (L, β, D, C, τ) and denote it $\mathrm{IE}(L, D, C)$ or $\mathrm{IE}(L, \beta, D, C, \tau)$ if more precision is helpful.

Later on the bilinear form β on L will be unique, up to a scalar in $k^\times$. In general, we have for $s \in k^\times$,

$$\sigma_{D,s\beta} = s\sigma_{D,\beta} \quad \text{and} \quad \mathrm{IE}(L, \beta, D, C, \tau) \simeq \mathrm{IE}(L, s\beta, D, C, s\tau) \tag{1.2.4}$$

via the isomorphism $l \oplus c \oplus d \mapsto l \oplus sc \oplus d$.

Lemma 1.1.[3] Let $E = \mathrm{IE}(L, D, C) = L \oplus C \oplus D$ be an interlaced extension, and let $f\colon E \to E$ be a linear map of the form

$$f(l \oplus c \oplus d) = \big(f_L(l) + \eta(d)\big) \oplus \big(\psi(l) + c + \varphi(d)\big) \oplus d, \tag{1.2.5}$$

where $l \in L$, $c \in C$, $d \in D$ and

$$f_L\colon L \to L, \quad \eta\colon D \to L, \quad \psi\colon L \to C, \quad \varphi\colon D \to C \tag{1.2.6}$$

are linear maps. Then f is an automorphism of the Lie algebra E if and only if the following conditions hold for all $l, l_1, l_2 \in L$ and $d, d_1, d_2 \in D$:

[3]This lemma holds in the more general setting of [CNPY16, 1.4]. But we have no use for this generality.

(a) f_L is an automorphism of the Lie algebra L,
(b) $\sigma(f_L(l_1), f_L(l_2)) = \psi([l_1, l_2]) + \sigma(l_1, l_2)$ for $\sigma = \sigma_{D,\beta}$,
(c) $f_L(d \cdot l) = [\eta(d), f_L(l)]_L + d \cdot f_L(l)$,
(d) $\psi(d \cdot l) = \sigma(\eta(d), f_L(l)) + d \cdot \psi(l)$,
(e) $\eta([d_1, d_2]_D) = [\eta(d_1), \eta(d_2)]_L + d_1 \cdot \eta(d_2) - d_2 \cdot \eta(d_1)$,
(f) $\varphi([d_1, d_2]) = \sigma(\eta(d_1), \eta(d_2)) + d_1 \cdot \varphi(d_2) - d_2 \cdot \varphi(d_1)$.

Proof. The map f is bijective if and only if f_L is also. Moreover, the definition of the product of E in (1.2.3) and the definition of f in (1.2.5) show that f is a homomorphism of the Lie algebra E if and only if it respects the products $[l_1, l_2]_E$, $[d, l]_E$ and $[d_1, d_2]_E$. This leads to the conditions (a)–(f). □

We will call an automorphism of type (1.2.5) a *special automorphism*. Not all automorphisms of E are special, but we have the following result.

Proposition 1.2 ([CNPY16, Prop. 1.6])**.** *Let $E = \mathrm{IE}(L, D, C)$ be an interlaced extension. Every elementary automorphism of L lifts to a special automorphism of E.*

We recall that an elementary automorphism of a Lie algebra M is a product of automorphisms of type $\exp \mathrm{ad}_M x$ with $\mathrm{ad}_M x \in \mathrm{End}_k(M)$ (locally) nilpotent. The reader can easily verify that for $f_L = \exp \mathrm{ad}_L x$, the maps η, ψ and φ of (1.2.6) are given by

$$\psi(l) = \sum_{n \geq 1} \frac{1}{n!} \sigma\big(x, (\mathrm{ad}_L x)^{n-1}(l)\big) \quad \text{for } \sigma = \sigma_{D,\beta},$$

$$\eta(d) = -\sum_{n \geq 1} \frac{1}{n!} (\mathrm{ad}_L x)^{n-1}(d \cdot x),$$

$$\varphi(d) = -\sum_{n \geq 2} \frac{1}{n!} \sigma\big(x, \ (\mathrm{ad}_L x)^{n-2}(d \cdot x)\big).$$

These formulas indicate that the maps ψ, η and φ are in general not zero.

1.3. **Enlarging interlaced extensions.** In the process of lifting an automorphism from L to an interlaced extension E, we will enlarge E to a bigger interlaced extension using the following construction:

(i) $E = \mathrm{IE}(L, \beta, D, C, \tau)$ is an interlaced extension;
(ii) L is a subalgebra of a Lie algebra L' equipped with an invariant bilinear form β' such that $\beta' \mid_{L \times L} = \beta$;
(iii) the action of D on L extends to an action of D on L' by skew derivations, and
(iv) $\sigma_{D,\beta'}(l'_1, l'_2) \in C$ for $l'_1, l'_2 \in L$.[4]

The data (L', β', D, C, τ) satisfy the assumptions (i)–(iv) of 1.2, so that we can form the interlaced extension

$$E' = \mathrm{IE}(L', \beta', D, C, \tau) = L' \oplus C \oplus D.$$

Since for $l_1, l_2 \in L$ we have

$$\sigma_{D,\beta'}(l_1, l_2)(d) = \beta'(d \cdot l_1, l_2) = \beta(d \cdot l_1, l_2) = \sigma_{D,\beta}(l_1, l_2)(d),$$

i.e.,

$$\sigma_{D,\beta'}\big|_{L \times L} = \sigma_{D,\beta},$$

it is immediate that E *is a subalgebra of* E'.

[4]Note that because of assumption (iii) we necessarily have that $\sigma_{D,\beta'} : L' \times L' \to C \subset D^*$ coincides with the central 2-cocycle of (1.1.2) when restricted to $L \times L$.

In this setting suppose that f' is a special automorphism of E', thus given by the data

$$f'_{L'}\colon L' \to L', \quad \eta'\colon D \to L', \quad \psi'\colon L' \to C, \quad \varphi'\colon D \to C$$

as in (1.2.6), satisfying the conditions (a)–(f) of Lemma 1.1. It is then immediate that

$$f'(E) = E \iff f'_{L'}(L) = L \text{ and } \eta'(D) \subset L. \tag{1.3.1}$$

In this case $f' \mid_E$ is obviously an automorphism of E, in fact a special automorphism given by the data

$$f_L = f'_{L'}\big|_L, \quad \eta = \eta', \quad \psi = \psi', \quad \varphi = \varphi'. \tag{1.3.2}$$

2. Review: Lie tori and extended affine Lie algebras (EALAs)

In this section we review the theory of extended affine Lie algebras, in order to give the reader a perspective about the achievements of this paper. The structure of extended affine Lie algebra is intimately connected to Lie tori. We therefore start with a short summary of the pertinent facts from the theory of Lie tori. We then introduce EALAs and describe their construction as a special case of an interlaced extension (1.2) based on a Lie torus.

2.1. **Lie tori.** We use the term "root system" to mean a finite, not necessarily reduced root system Δ in the usual sense, except that we will assume $0 \in \Delta$, as for example in [AAB$^+$97]. We denote by

$$\Delta_{\text{ind}} = \{0\} \cup \left\{\alpha \in \Delta\colon \frac{1}{2}\alpha \notin \Delta\right\}$$

the subsystem of indivisible roots and by $\mathcal{Q}(\Delta) = \operatorname{span}_{\mathbb{Z}}(\Delta)$ the root lattice of Δ. To avoid some degeneracies we will always assume that $\Delta \neq \{0\}$.

Let Δ be a finite irreducible root system, and let Λ be free abelian group of finite type.[5] A *Lie torus of type* (Δ, Λ) is a Lie algebra L satisfying the following conditions (LT1)–(LT4).

(LT1) (a) L is graded by $\mathcal{Q}(\Delta) \oplus \Lambda$. We write this grading as

$$L = \bigoplus_{\alpha \in \mathcal{Q}(\Delta), \lambda \in \Lambda} L_\alpha^\lambda$$

and thus have $[L_\alpha^\lambda, L_\beta^\mu] \subset L_{\alpha+\beta}^{\lambda+\mu}$. It is convenient to define

$$L_\alpha = \bigoplus_{\lambda \in \Lambda} L_\alpha^\lambda \quad \text{and} \quad L^\lambda = \bigoplus_{\alpha \in \mathcal{Q}(\Delta)} L_\alpha^\lambda. \tag{2.1.1}$$

(b) We further assume that $\operatorname{supp}_{\mathcal{Q}(\Delta)} L = \{\alpha \in \mathcal{Q}(\Delta); L_\alpha \neq 0\} = \Delta$, so that

$$L = \bigoplus_{\alpha \in \Delta} L_\alpha.$$

(LT2) (a) If $L_\alpha^\lambda \neq 0$ and $\alpha \neq 0$, then there exist $e_\alpha^\lambda \in L_\alpha^\lambda$ and $f_\alpha^\lambda \in L_{-\alpha}^{-\lambda}$ such that

$$L_\alpha^\lambda = k e_\alpha^\lambda, \quad L_{-\alpha}^{-\lambda} = k f_\alpha^\lambda$$

and

$$[[e_\alpha^\lambda, f_\alpha^\lambda], x_\beta] = \langle \beta, \alpha^\vee \rangle x_\beta$$

for all $\beta \in \Delta$ and $x_\beta \in L_\beta$.[6]

(b) $L_\alpha^0 \neq 0$ for all $0 \neq \alpha \in \Delta_{\text{ind}}$.

(LT3) As a Lie algebra, L is generated by $\bigcup_{0 \neq \alpha \in \Delta} L_\alpha$.

[5] Thus $\Lambda \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$. But it is not helpful to assume $\Lambda = \mathbb{Z}^n$.

[6] Here and elsewhere $\alpha^\vee$ denotes the coroot corresponding to α in the sense of [Bou75].

(LT4) As an abelian group, Λ is generated by $\operatorname{supp}_\Lambda L = \{\lambda \in \Lambda\colon L^\lambda \neq 0\}$.

We define the *nullity* of a Lie torus L of type (Δ, Λ) as the rank of Λ. We will say that L is a *Lie torus* (without qualifiers) if L is a Lie torus of type (Δ, Λ) for some pair (Δ, Λ). A Lie torus is called *centreless* if its centre $\mathcal{Z}(L) = \{0\}$. If L is an arbitrary Lie torus, its centre $\mathcal{Z}(L)$ is contained in L_0 from which it easily follows that $L/\mathcal{Z}(L)$ is in a natural way a centreless Lie torus of the same type as L and nullity (see [Yos06, Lemma 1.4]).

The structure of Lie tori is known, see [All12] for a recent survey. Some more background on Lie tori is given in the papers [ABFP09, Neh11a, Neh11b]. Lie tori can of course be defined for any abelian group Λ (see for example [Yos06]), but only the case of a free abelian group of finite rank is of interest for EALAs.

An obvious example of a Lie torus of type $(\Delta, \mathbb{Z}^n)$ is the Lie k-algebra $\mathfrak{g} \otimes R$, where $\mathfrak{g}$ is a finite-dimensional split simple Lie algebra of type Δ and $R = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is the Laurent polynomial ring in n-variables with coefficients in k equipped with the natural $\mathbb{Z}^n$-grading. Another important example, first studied in [BGK96], is the Lie algebra $\mathfrak{sl}_\ell(\mathcal{Q})$ for $\mathcal{Q}$ a quantum torus (see 3.5 and 3.6).

2.2. **Some known properties of centreless Lie tori.** We review some of the properties of Lie tori needed in the following. We assume that L is a centreless Lie torus of type (Δ, Λ) and nullity n.

(a) For e_α^λ and f_α^λ as in (LT2) we put

$$h_\alpha^\lambda = [e_\alpha^\lambda, f_\alpha^\lambda] \in L_0^0$$

and observe that $(e_\alpha^\lambda, h_\alpha^\lambda, f_\alpha^\lambda)$ is an $\mathfrak{sl}_2$-triple. Then

$$\mathfrak{h} = \operatorname{span}_k\{h_\alpha^\lambda\} = L_0^0 \tag{2.2.1}$$

is a toral[7] subalgebra of L whose root spaces are the L_α, $\alpha \in \Delta$.

(b) Up to scalars, L has a unique nondegenerate symmetric bilinear form $(\cdot|\cdot)$ which is Λ-*graded* in the sense that $(L^\lambda \mid L^\mu) = 0$ if $\lambda + \mu \neq 0$; see [NPPS15, Yos06]. Since the subspaces L_α are the root spaces of the toral subalgebra $\mathfrak{h}$ we also know $(L_\alpha \mid L_\tau) = 0$ if $\alpha + \tau \neq 0$.

(c) Let $\operatorname{Ctd}_k(L) = \{\chi \in \operatorname{End}_k(L)\colon \chi([l_1, l_2]) = [l_1, \chi(l_2)]\ \forall\, l_1, l_2 \in L\}$ be the centroid of L (see for example [BN06] for general facts about centroids). Since L is perfect, $\operatorname{Ctd}_k(L)$ is a commutative associative unital subalgebra of $\operatorname{End}_k(L)$. It is graded with respect to the Λ-grading (2.1.1) of L:

$$\operatorname{Ctd}(L) = \bigoplus_{\lambda \in \Lambda} \operatorname{Ctd}(L)^\lambda,$$

where $\operatorname{Ctd}(L)^\lambda$ consists of those centroidal transformations χ satisfying $\chi(L^\mu) \subset L^{\lambda+\mu}$ for all $\mu \in \Lambda$. One knows that $\operatorname{Ctd}_k(L)$ is graded-isomorphic to the group ring $k[\Xi]$ for a subgroup Ξ of Λ, the so-called *central grading group*. Hence $\operatorname{Ctd}_k(L)$ is a Laurent polynomial ring in ν variables, $0 \le \nu \le n$ [Neh04a, 7], [BN06, Prop. 3.13]. (All possibilities for ν do in fact occur, for example, for $L = \mathfrak{sl}_\ell(\mathcal{Q})$, see 3.5 and 3.6.)

(d) The space L is naturally a $\operatorname{Ctd}_k(L)$-module via $\chi \cdot a = \chi(a)$. As a $\operatorname{Ctd}_k(L)$-module, L is free. If L is *fgc*, i.e., finitely generated as a module over its centroid, then L is a multiloop algebra [ABFP09]. If L is not fgc, equivalently $\nu < n$, one knows [Neh04a, Th. 7] that L has root-grading type A. Lie tori with this root-grading type are classified in [BGK96, BGKN95, Yos00]. It follows from this classification together with

[7] A subalgebra T of a Lie algebra L is toral, sometimes also called ad-diagonalizable, if $L = \bigoplus_{\alpha \in T^*} L_\alpha(T)$ for $L_\alpha(T) = \{l \in L\colon [t, l] = \alpha(t)l \text{ for all } t \in T\}$. In this case $\{\operatorname{ad} t\colon t \in T\}$ is a commuting family of ad-diagonalizable endomorphisms. Conversely, if $\{\operatorname{ad} t\colon t \in T\}$ is a commuting family of ad-diagonalizable endomorphisms and T is a finite-dimensional subalgebra, then T is a toral.

[NY03, 4.9] that $L \simeq \mathfrak{sl}_l(\mathcal{Q})$ for $\mathcal{Q}$ a quantum torus in n variables and structure matrix $q = (q_{ij})$ an $n \times n$ quantum matrix with at least one q_{ij} not a root of unity (3.5).

(e) Any $\theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, k)$ induces a so-called *degree derivation* ∂_θ of L defined by $\partial_\theta(l^\lambda) = \theta(\lambda) l^\lambda$ for $l^\lambda \in L^\lambda$. We put $\mathcal{D} = \{\partial_\theta : \theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, k)\}$ and note that $\theta \mapsto \partial_\theta$ is a vector space isomorphism from $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, k)$ to $\mathcal{D}$, whence $\mathcal{D} \simeq k^n$. We define $\mathrm{ev}_\lambda \in \mathcal{D}^*$ by $\mathrm{ev}_\lambda(\partial_\theta) = \theta(\lambda)$. One knows [Neh04a, 8] that $\mathcal{D}$ induces the Λ-grading of L in the sense that

$$L^\lambda = \{l \in L \colon \partial_\theta(l) = \mathrm{ev}_\lambda(\partial_\theta) l \text{ for all } \theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, k)\}$$

holds for all $\lambda \in \Lambda$.

(f) If $\chi \in \mathrm{Ctd}_k(L)$, then $\chi d \in \mathrm{Der}_k(L)$ for any derivation $d \in \mathrm{Der}_k(L)$. We call

$$\mathrm{CDer}_k(L) := \mathrm{Ctd}_k(L)\mathcal{D} = \bigoplus_{\xi \in \Xi} \mathrm{Ctd}(L)^\xi \mathcal{D} \tag{2.2.2}$$

the *centroidal derivations* of L. It is easily seen that $\mathrm{CDer}(L)$ is a Ξ-graded subalgebra of $\mathrm{Der}_k(L)$, a generalized Witt algebra. Note that $\mathcal{D}$ is a toral subalgebra of $\mathrm{CDer}_k(L)$ whose root spaces are the $\mathrm{Ctd}(L)^\xi \mathcal{D} = \{d \in \mathrm{CDer}(L) \colon [t, d] = \mathrm{ev}_\xi(t) d \text{ for all } t \in \mathcal{D}\}$. One also knows [Neh04a, 9] that

$$\mathrm{Der}_k(L) = \mathrm{IDer}(L) \rtimes \mathrm{CDer}_k(L) \quad \text{(semidirect product)}, \tag{2.2.3}$$

where $\mathrm{IDer}(L)$ is the ideal of inner derivations of L.

(g) For the construction of EALAs, the Ξ-graded subalgebra $\mathrm{SCDer}_k(L)$ of *skew-centroidal derivations* is important:

$$\mathrm{SCDer}_k(L) = \{d \in \mathrm{CDer}_k(L) \colon (d \cdot l \mid l) = 0 \text{ for all } l \in L\} = \bigoplus_{\xi \in \Xi} \mathrm{SCDer}_k(L)^\xi,$$
$$\mathrm{SCDer}_k(L)^\xi = \mathrm{Ctd}(L)^\xi \{\partial_\theta \colon \theta(\xi) = 0\}.$$

2.3. **Extended affine Lie algebras (EALAs).** An *extended affine Lie algebra* or EALA for short, is a pair (E, H) consisting of a Lie algebra E over k and a subalgebra H of E satisfying the axioms (EA0)–(EA5) below.

(EA0) E has an invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$.

(EA1) H is a nontrivial finite-dimensional toral and self-centralizing subalgebra of E.

Thus $E = \bigoplus_{\alpha \in H^*} E_\alpha$ for $E_\alpha = \{e \in E \colon [h, e] = \alpha(h) e \text{ for all } h \in H\}$ and $E_0 = H$. We denote by $\Psi = \{\alpha \in H^* \colon E_\alpha \neq 0\}$ the set of roots of (E, H) — note that $0 \in \Psi$! Because the restriction of $(\cdot|\cdot)$ to $H \times H$ is nondegenerate, one can in the usual way transfer this bilinear form to H^* and then introduce anisotropic roots $\Psi^{\mathrm{an}} = \{\alpha \in \Psi \colon (\alpha \mid \alpha) \neq 0\}$ and isotropic (= null) roots $\Psi^0 = \{\alpha \in \Psi \colon (\alpha \mid \alpha) = 0\}$. The *core of* $(E, H, (\cdot|\cdot))$ is by definition the subalgebra generated by $\bigcup_{\alpha \in \Psi^{\mathrm{an}}} E_\alpha$. It will be henceforth denoted by E_c.

(EA2) For every $\alpha \in \Psi^{\mathrm{an}}$ and $x_\alpha \in E_\alpha$, the operator $\mathrm{ad}\, x_\alpha$ is locally nilpotent on E.

(EA3) Ψ^{an} is connected in the sense that for any decomposition $\Psi^{\mathrm{an}} = \Psi_1 \cup \Psi_2$ with $\Psi_1 \neq \emptyset$ and $\Psi_2 \neq \emptyset$ we have $(\Psi_1 \mid \Psi_2) \neq 0$.

(EA4) The centralizer of the *core* E_c of E is contained in E_c, i.e.,

$$\{e \in E \colon [e, E_c] = 0\} \subset E_c.$$

(EA5) The subgroup $\Lambda = \mathrm{span}_{\mathbb{Z}}(\Psi^0) \subset H^*$ generated by Ψ^0 in $(H^*, +)$ is a free abelian group of finite rank.

The attentive reader will have noticed that the choice of invariant nondegenerate symmetric bilinear form in (EA0) is part of the structural data defining an EALA. However, one can show that another choice of an invariant nondegenerate symmetric bilinear form leads to the same set of anisotropic and isotropic roots Ψ^{an} and Ψ^0, and thus also to the same core E_c and *centreless core* $E_{cc} = E_c/\mathcal{Z}(E_c)$; see [CNPY16, Rem. 2.4 and Cor. 3.3]. The core E_c of an EALA E is always an ideal of E.

Some references for EALAs are [AAB$^+$97, BGK96, Neh04b, Neh11a, Neh11b]. It is immediate that any finite-dimensional split simple Lie algebra $\mathfrak{g}$ is an EALA of nullity 0 and $\mathfrak{g} = \mathfrak{g}_c = \mathfrak{g}_{cc}$. The converse is also true, [Neh11b, Prop. 5.3.24]. It is also known that any affine Kac–Moody algebra over $\mathbb{C}$ is an EALA — in fact, by [ABGP97], the affine Kac–Moody algebras $\mathfrak{g}$ are precisely the EALAs over $\mathbb{C}$ of nullity 1. For those, $\mathfrak{g}_c = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}_{cc}$ is a (twisted or untwisted) loop algebra.

2.4. **The roots of an EALA.** The set Ψ of roots of an EALA (E, H) is an extended affine root system in the sense of [AAB$^+$97, Ch. I] (see also the surveys [Neh11a, § 2, § 3] and [Neh11b, § 5.3]). Thus, there exists an irreducible finite (but possibly nonreduced) root system $\Delta \subset H^*$, an embedding $\Delta_{\rm ind} \subset \Psi$ and a family $(\Lambda_\alpha \colon \alpha \in \Delta)$ of subsets $\Lambda_\alpha \subset \Lambda$ such that

$$\text{(2.4.1)} \qquad \operatorname{span}_k(\Psi) = \operatorname{span}_k(\Delta) \oplus \operatorname{span}_k(\Lambda) \quad \text{and} \quad \Psi = \bigcup_{\alpha\in\Delta} (\alpha + \Lambda_\alpha).$$

Using this (nonunique) decomposition of Ψ, we write any $\psi \in \Psi$ as $\psi = \alpha+\lambda$ with $\alpha \in \Delta$ and $\lambda \in \Lambda_\alpha \subset \Lambda$ and define $(E_c)^\lambda_\alpha = E_c \cap E_\psi$. Then the core

$$E_c = \bigoplus_{\alpha\in\Delta,\lambda\in\Lambda} (E_c)^\lambda_\alpha$$

is a Lie torus of type (Δ, Λ), and the centreless core $E_{cc} = E_c/\mathcal{Z}(E_c)$ is a centreless Lie torus.

2.5. **Construction of EALAs.** To construct an EALA one reverses the process described in 2.4. We will use data (L, σ_D, τ) described below. Some more background material can be found in [Neh11a, § 6] and [Neh11b, § 5.5]:

- L is a centreless Lie torus of type (Δ, Λ). We fix a Λ-graded invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$ (see 2.2(b)) and let Ξ be the central grading group of L (see 2.2(c)).
- $D = \bigoplus_{\xi\in\Xi} D^\xi$ is a graded subalgebra of $\operatorname{SCDer}_k(L)$ (see 2.2(g)) such that the evaluation map $\operatorname{ev}_{D^0} \colon \Lambda \to {D^0}^*$, $\lambda \to \operatorname{ev}_\lambda |_{D^0}$, defined in 2.2(e), is injective.

  Since $(L^\lambda \mid L^\mu) = 0$ if $\lambda + \mu \neq 0$ and since $D^\xi(L^\lambda) \subset L^{\xi+\lambda}$ it follows that the central cocycle σ_D of (1.1.2) has values in the graded dual $D^{\mathrm{gr}*} =: C$ of D. Recall $C = \bigoplus_{\xi\in\Xi} C^\xi$ with $C^\xi = (D^{-\xi})^* \subset D^*$. The contragredient action of D on D^* leaves C invariant.
- $\tau \colon D\times D \to C$ is an *affine cocycle* defined to be a 2-cocycle satisfying $\tau(d^0, d) = 0$ and $\tau(d_1, d_2)(d_3) = \tau(d_2, d_3)(d_1)$ for all $d, d_i \in D$ and $d^0 \in D^0$.

  It is important to point out that there do exist nontrivial affine cocycles; see [BGK96, Rem. 3.71].

The data (L, σ_D, τ) with β the unique invariant bilinear form $(\cdot|\cdot)$ of 2.2(b) satisfy all the axioms of our general construction 1.2. Thus the interlaced extension

$$\text{(2.5.1)} \qquad E = L \oplus C \oplus D$$

is a Lie algebra with respect to the product (1.2.3). Note that E contains the toral subalgebra

$$H = \mathfrak{h} \oplus C^0 \oplus D^0$$

for $\mathfrak{h}$ as in (2.2.1). The symmetric bilinear form $(\cdot|\cdot)$ on E, defined by

$$(l_1 + c_1 + d_1 \mid l_2 + c_2 + d_2) = (l_1 \mid l_2)_L + c_1(d_2) + c_2(d_1),$$

is nondegenerate and invariant, thus fulfilling the axiom (E0).

Examples.

(a) In case $L = \mathfrak{g}$ is a finite-dimensional split simple Lie algebra, $\mathrm{Ctd}(\mathfrak{g}) = 0$, $\Xi = \{0\}$, and so also $\mathrm{SCDer}(\mathfrak{g}) = 0$. The construction above therefore yields $E = \mathfrak{g}$.

(b) In case L is a twisted or untwisted loop algebra based on $\mathfrak{g}$ as in (a) over $\mathbb{C}$, the centroid $\mathrm{Ctd}(L)$ is isomorphic to a Laurent polynomial ring R, $\mathrm{CDer}(L)$ is a free R-module of rank 1, but $\mathrm{SCDer}(\mathfrak{g})$ is 1-dimensional over $\mathbb{C}$. The only non-trivial choice is therefore $D \simeq \mathbb{C}$. In this case necessarily $\tau = 0$. Thus the construction of affine Kac–Moody algebras is a special case of our construction above.

Theorem 2.1 ([Neh04b, Th. 6])**.** (a) *The triple $(E, H, (\cdot|\cdot))$ constructed above is an extended affine Lie algebra, denoted $\mathrm{EA}(L, D, \tau)$. Its core is $L \oplus D^{\mathrm{gr}\,*}$ and its centreless core is L.*

(b) *Conversely, let $(E, H, (\cdot|\cdot))$ be an extended affine Lie algebra, and let $L = E_c/\mathcal{Z}(E_c)$ be its centreless core. Then there exists a subalgebra $D \subset \mathrm{SCDer}_k(L)$ and an affine cocycle τ satisfying the conditions in 2.5 such that*

$$(E, H, (\cdot|\cdot)) \simeq \mathrm{EA}(L, (\cdot|\cdot)_L, D, \tau)$$

for some Λ-graded invariant nondegenerate bilinear form $(\cdot|\cdot)_L$ on L.

3. Lifting automorphisms from $\mathfrak{sl}_\ell(\mathcal{A})$ to $\mathrm{IE}(\mathfrak{sl}_\ell(\mathcal{A}), D, C)$

3.1. **The Lie algebras $\mathfrak{gl}_\ell(\mathcal{A})$ and $\mathfrak{sl}_\ell(\mathcal{A})$.** We assume throughout that $\ell \geq 2$. The letter $\mathcal{A}$ will always denote a unital associative k-algebra. It becomes a Lie algebra $\mathrm{Lie}(\mathcal{A})$ with respect to the commutator. We denote by $[\mathcal{A}, \mathcal{A}]$ the commutator subalgebra of $\mathrm{Lie}(\mathcal{A})$,

$$[\mathcal{A}, \mathcal{A}] = \mathrm{span}_{\mathbb{Z}}\{ab - ba \colon a, b \in \mathcal{A}\}$$

and by $\mathcal{Z}(\mathcal{A}) = \{z \in \mathcal{A} \colon [z, \mathcal{A}] = 0\}$ the centre of $\mathcal{A}$, which is also the centre of $\mathrm{Lie}(\mathcal{A})$.

We denote by $\mathrm{M}_\ell(\mathcal{A})$ the unital associative algebra of $\ell \times \ell$-matrices with coefficients in $\mathcal{A}$, and by $\mathfrak{gl}_\ell(\mathcal{A})$ its associated Lie algebra: $\mathfrak{gl}_\ell(\mathcal{A}) = \mathrm{Lie}(\mathrm{M}_\ell(\mathcal{A}))$.

The derived algebra of $\mathfrak{gl}_\ell(\mathcal{A})$ is the *special linear Lie algebra $\mathfrak{sl}_\ell(\mathcal{A})$ with coefficients from $\mathcal{A}$*:

$$\mathfrak{sl}_\ell(\mathcal{A}) = [\mathfrak{gl}_\ell(\mathcal{A}),\, \mathfrak{gl}_\ell(\mathcal{A})]. \tag{3.1.1}$$

We let Tr be the trace of a matrix in $\mathrm{M}_\ell(\mathcal{A})$. The reader should be warned that $\mathrm{Tr}(xy) \neq \mathrm{Tr}(yx)$ in general, rather we have the well-known fact

$$\mathfrak{sl}_\ell(\mathcal{A}) = \{x \in \mathfrak{gl}_\ell(\mathcal{A}) \colon \mathrm{Tr}(x) \in [\mathcal{A}, \mathcal{A}]\}.\text{[8]} \tag{3.1.2}$$

Moreover, we will need

$$\mathcal{C}_{\mathfrak{gl}_\ell(\mathcal{A})}(\mathfrak{sl}_\ell(\mathcal{A})) = \mathcal{Z}(\mathcal{A})\,\mathrm{E}_\ell = \mathcal{Z}(\mathfrak{gl}_\ell(\mathcal{A})), \tag{3.1.3}$$

where $\mathcal{C}$ denotes the centralizer and E_ℓ the $\ell \times \ell$ identity matrix.

Any $d \in \mathrm{Der}_k(\mathcal{A})$ stabilizes $\mathcal{Z}(\mathcal{A})$ and $[\mathcal{A}, \mathcal{A}]$, and induces a derivation of the associative algebra $\mathrm{M}_\ell(\mathcal{A})$ by

$$d \cdot x = (d(x_{ij})) \quad \text{for } x = (x_{ij}) \in \mathrm{M}_\ell(\mathcal{A}). \tag{3.1.4}$$

It is then also a derivation of $\mathfrak{gl}_\ell(\mathcal{A})$, stabilizing $\mathcal{Z}(\mathfrak{gl}_\ell(\mathcal{A})) = \mathcal{Z}(\mathcal{A})E_\ell$ and $\mathfrak{sl}_\ell(\mathcal{A})$. In the following, a subalgebra $D < \mathrm{Der}_k(\mathcal{A})$ will be a standard feature of our work. We will always use the action of $\mathrm{Der}_k(\mathcal{A})$ and hence of D described in (3.1.4) without further explanation. Also, we will sometimes write dx or $d(x)$ for $d \cdot x$.

[8]Of course if $\mathcal{A}$ is commutative, then $[\mathcal{A}, \mathcal{A}] = 0$ and we recover the "usual" definition of $\mathfrak{sl}_\ell(\mathcal{A})$.

3.2. **The groups** $\mathrm{GL}_\ell(\mathcal{A})$ **and** $\mathrm{EL}_\ell(\mathcal{A})$**.** We denote by $\mathrm{GL}_\ell(\mathcal{A})$ the group of invertible $\ell\times\ell$-matrices with coefficients from the unital associative k-algebra $\mathcal{A}$. Every $g\in \mathrm{GL}_\ell(\mathcal{A})$ gives rise to an automorphism $\mathrm{Int}(g)$ of the associative algebra $\mathrm{M}_\ell(\mathcal{A})$, defined by $\mathrm{Int}(g)(a)=gag^{-1}$. A fortiori, $\mathrm{Int}(g)$ is an automorphism of $\mathfrak{gl}_\ell(\mathcal{A})$. It stabilizes $\mathfrak{sl}_\ell(\mathcal{A})$, whence is by restriction an automorphism of $\mathfrak{sl}_\ell(\mathcal{A})$, again denoted $\mathrm{Int}(g)$. Moreover, $\mathrm{Int}(g)$ induces the identity on $\mathcal{Z}(\mathfrak{gl}_\ell(\mathcal{A}))$ as can be seen for the last equality of (3.1.3).

The *elementary linear group* $\mathrm{EL}_\ell(\mathcal{A})$ is the subgroup of $\mathrm{GL}_\ell(\mathcal{A})$ generated by the matrices $\mathrm{E}_\ell+aE_{ij}$ for arbitrary $a\in\mathcal{A}$ and $i\neq j$. Since $(aE_{ij})^2=0$ in $\mathrm{M}_\ell(\mathcal{A})$ the derivation $\mathrm{ad}\,aE_{ij}\in\mathrm{Der}\big(\mathfrak{sl}_\ell(\mathcal{A})\big)$ is nilpotent, in fact $(\mathrm{ad}\,aE_{ij})^3=0$, and the inner automorphism $\mathrm{Int}(E_\ell+aE_{ij})\in\mathrm{Aut}_k\big(\mathfrak{sl}_\ell(\mathcal{A})\big)$ is elementary in the sense of 1.2:

$$\mathrm{Int}(\mathrm{E}_\ell+aE_{ij})=\exp(\mathrm{ad}\,aE_{ij}).$$

It follows that

$$\text{(3.2.1)}\qquad \mathrm{Int}(g)\in\mathrm{EAut}_k(\mathfrak{sl}_\ell(\mathcal{A}))\quad \textit{for every } g\in\mathrm{EL}_\ell(\mathcal{A}),$$

where $\mathrm{EAut}(\mathfrak{sl}_\ell(\mathcal{A}))$ is the group of elementary automorphisms of $\mathfrak{sl}_\ell(\mathcal{A})$. Moreover, the commutator relation

$$[\![\mathrm{E}_\ell+aE_{ij},\mathrm{E}_\ell+E_{j\ell}]\!]=E_\ell+aE_{i\ell}\quad(i\neq j)$$

shows that

$$\text{(3.2.2)}\qquad \mathrm{EL}_\ell(\mathcal{A})\subset\mathcal{D}(\mathrm{GL}_\ell(\mathcal{A})).$$

Lemma 3.1. *Let $\mathcal{A}$ be a unital associative k-algebra satisfying*

$$\text{(3.2.3)}\qquad \mathcal{A}=\mathcal{Z}(\mathcal{A})\oplus[\mathcal{A},\mathcal{A}].$$

Then

$$\text{(3.2.4)}\qquad \mathfrak{gl}_\ell(\mathcal{A})=\mathcal{Z}(\mathcal{A})\,\mathrm{E}_\ell\oplus\mathfrak{sl}_\ell(\mathcal{A})\quad\textit{and}$$

$$\text{(3.2.5)}\qquad (dg)g^{-1}\in\mathfrak{sl}_\ell(\mathcal{A})\iff g^{-1}dg\in\mathfrak{sl}_\ell(\mathcal{A})$$

for any $d\in\mathrm{Der}_k(\mathcal{A})$ and $g\in\mathrm{GL}_\ell(\mathcal{A})$. Moreover, for $D\subset\mathrm{Der}_k(\mathcal{A})$ the set

$$H=H_D=\big\{g\in\mathrm{GL}_\ell(\mathcal{A})\colon (dg)g^{-1}\in\mathfrak{sl}_\ell(\mathcal{A})\ \textit{for all } d\in D\big\}$$

is a normal subgroup of $\mathrm{GL}_\ell(\mathcal{A})$ containing the commutator subgroup $\mathcal{D}(\mathrm{GL}_\ell(\mathcal{A}))$ of $\mathrm{GL}_\ell(\mathcal{A})$.

Proof. Our assumption (3.2.3) implies $\mathcal{A}\,\mathrm{E}_\ell=\mathcal{Z}(\mathcal{A})\,\mathrm{E}_\ell\oplus[\mathcal{A},\mathcal{A}]\,\mathrm{E}_\ell$. Since $[\mathcal{A},\mathcal{A}]\,\mathrm{E}_\ell=\mathcal{A}\,\mathrm{E}_\ell\cap\mathfrak{sl}_\ell(\mathcal{A})$ by (3.1.2), the equation (3.2.4) follows from the decomposition

$$x=(\frac{1}{\ell}\,\mathrm{Tr}(x)\,\mathrm{E}_\ell)+(x-\frac{1}{\ell}\,\mathrm{Tr}(x)\,\mathrm{E}_\ell)$$

with $\mathrm{Tr}(x-\frac{1}{\ell}\,\mathrm{Tr}(x)\,\mathrm{E}_\ell)=0$ for arbitrary $x\in\mathfrak{gl}_\ell(\mathcal{A})$.

The equivalence (3.2.5) is a consequence of $g^{-1}dg=(dg)g^{-1}+[g^{-1},dg]$ and $[g^{-1},dg]\in\mathfrak{sl}_\ell(\mathcal{A})$. This shows that $g\in H\implies g^{-1}\in H$ since

$$\text{(3.2.6)}\qquad 0=d(g^{-1}g)=(dg^{-1})g+g^{-1}(dg).$$

Given that $\mathrm{E}_\ell\in H$, for H to be a subgroup it suffices to show that

$$g_1,g_2\in H\implies g_1g_2\in H.$$

But this follows from

$$\begin{aligned}(d(g_1g_2))(g_1g_2)^{-1}&=(dg_1)g_2g_2^{-1}g_1^{-1}+g_1(dg_2)g_2^{-1}g_1^{-1}\\&=(dg_1)g_1^{-1}+\mathrm{Int}(g_1)((dg_2)g_2^{-1})\end{aligned}$$

since $\operatorname{Int}(g_1)$ stabilizes $\mathfrak{sl}_\ell(\mathcal{A})$. Thus H is a subgroup, and it will be a normal subgroup as soon as we have shown that H contains any commutator $[\![g_1, g_2]\!]$, where $g_1, g_2 \in \mathrm{GL}_\ell(\mathcal{A})$. We have

$$\begin{aligned}(d[\![g_1, g_2]\!])[\![g_1, g_2]\!]^{-1} &= d(g_1 g_2 g_1^{-1} g_2^{-1}) g_2 g_1 g_2^{-1} g_1^{-1} \\ &= (dg_1)(g_2 g_1^{-1} g_2^{-1} g_2 g_1 g_2^{-1} g_1^{-1}) + g_1((dg_2) g_1^{-1} g_2^{-1} g_2 g_1 g_2^{-1}) g_1^{-1} \\ &\quad + g_1 g_2 (d(g^{-1}) g_2^{-1} g_2 g_1) g_2^{-1} g_1^{-1} + g_1 g_2 g_1^{-1} (d(g_2^{-1} g_2)) g_1 g_2^{-1} g_1^{-1} \\ &= (dg_1) g_1^{-1} + \operatorname{Int}(g_1)((dg_2) g_2^{-1}) + \operatorname{Int}(g_1 g_2)((dg_1^{-1}) g_1) + \operatorname{Int}(g_1 g_2 g_1^{-1})((dg_2^{-1}) g_2).\end{aligned}$$

To proceed, we use (3.2.4), thus uniquely writing any $x \in \mathfrak{gl}_\ell(\mathcal{A})$ as $x = x_z + x_s$ with $x_z \in \mathcal{Z}\big(\mathfrak{gl}_\ell(\mathcal{A})\big)$ and $x_s \in \mathfrak{sl}_\ell(\mathcal{A})$. Decomposing $y \in \mathfrak{gl}_\ell(\mathcal{A})$ in the same way, we have

$$(xy)_z = (yx)_z \tag{3.2.7}$$

since $xy = yx + [x, y]$ with $[x, y] \in \mathfrak{sl}_\ell(\mathcal{A})$. Because $\operatorname{Int}(g)$ stabilizes $\mathfrak{sl}_\ell(\mathcal{A})$ and satisfies $\operatorname{Int}(g)(zE_\ell) = zE_\ell$ for $z \in \mathcal{Z}(\mathcal{A})$ we now get

$$\begin{aligned}((dg_1)g_1^{-1})_z + \big(\operatorname{Int}(g_1 g_2)((dg_1^{-1})g_1)\big)_z &= ((dg_1)g_1^{-1})_z + \operatorname{Int}(g_1 g_2)((dg_1^{-1})g_1)_z \\ = ((dg_1)g_1^{-1})_z + ((dg_1^{-1})g_1)_z &= ((dg_1)g_1^{-1})_z + (g_1(dg_1^{-1}))_z = (d(g_1 g_1^{-1}))_z = 0,\end{aligned}$$

thus proving that

$$(dg_1)g_1^{-1} + \operatorname{Int}(g_1 g_2)((dg_1^{-1})g_1) \in \mathfrak{sl}_\ell(\mathcal{A}).$$

Similarly,

$$\operatorname{Int}(g_1)((dg_2)g_2^{-1}) + \operatorname{Int}(g_1 g_2 g_1^{-1})((dg_2^{-1})g_2) \in \mathfrak{sl}_\ell(\mathcal{A}).$$

Hence $[\![g_1, g_2]\!] \in H$, and therefore $\mathcal{D}(\mathrm{GL}_\ell(\mathcal{A})) \subset H$. □

3.3. **Interlaced extensions based on $\mathfrak{sl}_\ell(\mathcal{A})$.** We specialize the setting of 1.2 to $L = \mathfrak{sl}_\ell(\mathcal{A})$ with the aim of constructing a suitable interlaced extension that will allow us to lift the automorphisms used in conjugacy. Being an interlaced extension, we need to specify data (β, D, C, τ).

(i) We fix a linear form

$$\varepsilon\colon \mathcal{A} \to k, \quad \varepsilon([\mathcal{A}, \mathcal{A}]) = 0 \tag{3.3.1}$$

and define $\beta = \beta_\varepsilon \colon L \times L \to k$ by

$$\beta_\varepsilon(x, y) = \varepsilon(\operatorname{Tr}(xy)) = \sum_{i,j=1}^{\ell} \varepsilon(x_{ij} y_{ji}) \tag{3.3.2}$$

for $x = (x_{ij})$ and $y = (y_{ij})$. Then β is an invariant bilinear form on L, and every invariant bilinear form on L is of the type β_ε for a unique linear form ε satisfying (3.3.1) [Neh11a, 7.10].

(ii) We let D be a subalgebra of derivations of $\mathcal{A}$, which are skew with respect to the bilinear form $(a, b) \mapsto \varepsilon(ab)$,

$$D < \mathrm{SDer}_k(\mathcal{A}),$$

and let D act on L as in (3.1.4). Then D acts on L by skew derivations with respect to β.

(iii) We choose $C \subset D^*$ and τ as in (iii) and (iv) of 1.2. Using these data we form the interlaced extension

$$\mathrm{IE}(L, \beta_\varepsilon, D, C, \tau) = E = L \oplus C \oplus D.$$

3.4. **Enlarging interlaced extensions.** To suitably enlarge an interlaced extension $E = \mathrm{IE}(L, D, C)$ with $L = \mathfrak{sl}_\ell(\mathcal{A})$ as in 3.3, we embed L into $L' = \mathfrak{sl}_{\ell+m}(\mathcal{A})$, $m \in \mathbb{N}$ arbitrary, via

$$\mathfrak{sl}_\ell(\mathcal{A}) \to \mathfrak{sl}_{\ell+m}(\mathcal{A}), \quad l \mapsto \begin{pmatrix} l & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.4.1}$$

Following the outline of 1.3 we next need an invariant bilinear form β' on L'. We take $\beta' = \beta'_\varepsilon$ as defined in (3.3.2):

$$\beta'(x', y') = \varepsilon(\mathrm{Tr}(x'y')) = \sum_{i,j=1}^{\ell+m} \varepsilon(x'_{ij} y'_{ji})$$

for $x' = (x'_{ij})$ and $y' = (y'_{ij}) \in \mathfrak{sl}_{\ell+m}(\mathcal{A})$. Then the condition (ii) of 1.3 is fulfilled: $\beta'(l_1, l_2) = \beta(l_1, l_2)$ for $l_1, l_2 \in L$.

We also have condition (iii) of 1.3, i.e., D acts on L' by skew derivations extending the action of D on L. Finally, 1.3(iv) also holds. Indeed, for x', $y' \in L'$ as before and $d \in D$, we have

$$\begin{aligned} \sigma_{D,\beta'}(x', y')(d) &= \beta'(d \cdot x', y') = \sum_{i,j=1}^{\ell+m} \varepsilon((d \cdot x'_{ij}) y'_{ji}) \\ &= \sum_{i,j=1}^{\ell+m} \beta\big(d \cdot (x'_{ij} E_{12}), y'_{21} E_{21}\big) = \Big(\sum_{i,j=1}^{\ell+m} \sigma_{D,\beta}(x'_{ij} E_{12}, y'_{ji} E_{12}) \Big)(d) \end{aligned}$$

which shows $\sigma_{D,\beta'}(x', y') \in C$. In sum, we have shown that *for any $m \in \mathbb{N}$ the interlaced extension $E = \mathrm{IE}(L, \beta_\varepsilon, D, C, \tau)$ is a subalgebra of $E' = \mathrm{IE}(\mathfrak{sl}_{\ell+m}(\mathcal{A}), \beta'_\varepsilon, D, C, \tau)$.*

We are now ready to prove the main result of this section.

Theorem 3.2. *Let $\mathcal{A}$ be a unital associative k-algebra satisfying $\mathcal{A} = \mathcal{Z}(\mathcal{A}) \oplus [\mathcal{A}, \mathcal{A}]$, and let $E = \mathrm{IE}(L, D, C)$ be an interlaced extension based on $L = \mathfrak{sl}_\ell(\mathcal{A})$ as specified in* 3.3. *Assume that $g \in \mathrm{GL}_\ell(\mathcal{A})$ is* stably elementary *in the sense that there exists $m \in \mathbb{N}$ such that*

$$g' = \begin{pmatrix} g & 0 \\ 0 & \mathrm{E}_m \end{pmatrix} \in \mathrm{EL}_{\ell+m}(\mathcal{A}).$$

Then the automorphism $\mathrm{Int}(g)$ of $\mathfrak{sl}_\ell(\mathcal{A})$ lifts to an automorphism of E.

Proof. We embed $L = \mathfrak{sl}_\ell(\mathcal{A})$ into $L' = \mathfrak{sl}_{\ell+m}(\mathcal{A})$ as in (3.4.1). We then know that E can be enlarged to an interlaced extension $E' = \mathrm{IE}(L', D, C)$. Moreover, by (3.2.1) and Proposition 1.2 the elementary automorphisms $\mathrm{Int}(g')$ of L' lifts to a special automorphism f' of E', determined by maps $\mathrm{Int}(g') = f_{L'} \in \mathrm{Aut}_k(L')$ and linear maps $\eta' \colon D \to L'$, $\psi' \colon L' \to C$ and $\varphi' \colon D \to C$ as in Lemma 1.1. It will be sufficient to show $f'(E) = E$. Since $f'_{L'}(L) = \mathrm{Int}(g)(L) = L$, it is in view of (1.3.1) enough to prove $\eta'(d) \in L$ for all $d \in D$.

By 1.1(c) we have

$$d \cdot f_{L'}(l') - f_{L'}(d \cdot l') = [f_{L'}(l'), \eta'(d)] \tag{3.4.2}$$

for all $d \in D$ and $l' \in L'$. For $l \in L$ we know $f_{L'}(l) = \mathrm{Int}(g)(l) \in L$ and also

$$d \cdot f_{L'}(l) - f_{L'}(d \cdot l) \in L.$$

It thus follows from (3.4.2) for $l' = l \in L$ that $\eta'(d)$ normalizes L. One easily calculates that then $\eta'(d)$ has the form

$$\eta'(d) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha \in \mathfrak{gl}_\ell(\mathcal{A}), \beta \in \mathfrak{gl}_m(\mathcal{A})$$

(we have suppressed in our notation that α and β depend linearly on d). Employing the obvious subdivision for matrices $l' \in L'$,

$$l' = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \quad x_1 \in \mathfrak{gl}_\ell(\mathcal{A}),$$

we get

$$f_{L'}(l') = \operatorname{Int}(g')(l') = \begin{pmatrix} gx_1g^{-1} & gx_2 \\ x_3g^{-1} & x_4 \end{pmatrix},$$

whence for $d \in D$ the left hand side of (3.4.2) becomes

$$d \cdot f_{L'}(l') - f_{L'}(d \cdot l') = \begin{pmatrix} (dg)x_1g^{-1} + gx_1d(g^{-1}) & (dg)x_2 \\ x_3d(g^{-1}) & 0 \end{pmatrix},$$

while the right hand side of (3.4.2) is

$$[f_{L'}(l'),\, \eta'(d)] = \begin{pmatrix} [gx_1g^{-1}, \alpha] & gx_2\beta - \alpha gx_2 \\ x_3g^{-1}\alpha - \beta x_3g^{-1} & [x_4, \beta] \end{pmatrix}.$$

Thus

$$(dg)x_2 = gx_2\beta - \alpha gx_2, \tag{3.4.3}$$

$$0 = [x_4, \beta]. \tag{3.4.4}$$

Since every $x_4 \in \mathfrak{gl}_m(\mathcal{A})$ is part of some matrix $l' \in L'$, it follows that (3.4.4) holds for all $x_4 \in \mathfrak{gl}_m(\mathcal{A})$. Therefore, by (3.1.3),

$$\beta = z\,\mathrm{E}_m$$

for some $x \in \mathcal{Z}(\mathcal{A})$. We substitute this expression for β into (3.4.3) and obtain $(dg)x_2 = (zg - \alpha g)x_2$. Since this holds for all $x_2 \in \mathrm{M}_{mn}(\mathcal{A})$ we get $dg = zg - \alpha g$ or

$$\alpha = z\,\mathrm{E}_\ell - (dg)g^{-1}.$$

Because $g' \in \mathrm{EL}_{\ell+m}(\mathcal{A})$ it follows from (3.2.2) and Lemma 3.1 that $(dg')(g')^{-1} \in \mathfrak{sl}_{\ell+m}(\mathcal{A})$ for all $d \in D$. But

$$(dg')(g')^{-1} = \begin{pmatrix} (dg)g^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

so that $(dg)g^{-1} \in \mathfrak{sl}_\ell(\mathcal{A})$ follows. Since $\eta'(d) \in \mathfrak{sl}_{\ell+m}(\mathcal{A})$ we now get

$$\begin{aligned} \operatorname{Tr}(\eta'(d)) = \operatorname{Tr}(\alpha) + \operatorname{Tr}(\beta) &= \ell z - \operatorname{Tr}((dg)g^{-1}) + mz \\ &= (\ell + m)z - \operatorname{Tr}((dg)g^{-1}) \in [\mathcal{A}, \mathcal{A}]. \end{aligned}$$

As $\mathcal{A} = \mathcal{Z}(\mathcal{A}) \oplus [\mathcal{A}, \mathcal{A}]$ by assumption and $\operatorname{Tr}((dg)g^{-1}) \in [\mathcal{A}, \mathcal{A}]$, this forces $(\ell + m)z = 0$ so that $z = 0$ and finally $\beta = 0$, i.e., $\eta'(d) \in L$ follows. □

3.5. **Quantum tori (review).** We will later specialize $\mathcal{A} = \mathcal{Q}$ to be a quantum torus. Why we do so, is explained in 3.6: $\mathfrak{sl}_\ell(\mathcal{Q})$ is then a centreless Lie torus. In this subsection we review some properties of quantum tori that we will use. Contrary to the standing assumption for this paper, in this subsection our base field k can have arbitrary characteristic. We let Λ be a free abelian group of rank n.

(a) (Definition) By definition, a *quantum torus (with grading group Λ)* is an associative unital Λ-graded k-algebra $\mathcal{Q} = \bigoplus_{\lambda \in \Lambda} \mathcal{Q}^\lambda$ such that

(QT1) $\dim \mathcal{Q}^\lambda \le 1$ for all $\lambda \in \Lambda$,

(QT2) every $0 \ne a \in \mathcal{Q}^\lambda$ is invertible, and

(QT3) Λ is generated as abelian group by $\{\lambda \in \Lambda \colon \mathcal{Q}^\lambda \ne 0\}$.

Since the invertible elements of an associative algebra form a group,

$$\{\lambda \in \Lambda \colon \mathcal{Q}^\lambda \neq 0\}$$

is a subgroup of Λ, whence equals Λ by (QT3).

(b) After fixing a basis $\boldsymbol{\varepsilon} = (\varepsilon_i)$ of Λ, we can choose $0 \neq x_i \in \mathcal{Q}^{\varepsilon_i}$ and then get a quantum matrix $q = (q_{ij}) \in \mathrm{M}_n(k)$ defined by $x_i x_j = q_{ij} x_j x_i$. We recall that $q = (q_{ij}) \in \mathrm{M}_n(k)$ is called a *quantum matrix* if $q_{ij} = q_{ji}^{-1}$ and $q_{ii} = 1$ for all $1 \le i, j \le n$.

Then, using x_i^{-1} = the inverse of x_i, we define

$$x^\lambda = x_1^{\ell_1} \dots x_n^{\ell_n}$$

for $\lambda = \ell_1 \varepsilon_1 + \cdots + \ell_n \varepsilon_n \in \Lambda$:

$$\mathcal{Q} = \bigoplus_{\lambda \in \Lambda} k x^\lambda. \tag{3.5.1}$$

One can then also realize a quantum torus as the unital associative k-algebra presented by generators $x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}$ and relations

$$x_i x_i^{-1} = 1_{\mathcal{Q}} = x_i^{-1} x_i, \quad x_i x_j = q_{ij} x_j x_i.$$

We will refer to this view of $\mathcal{Q}$ as a *coordinatization*.

(c) The centre of $\mathcal{Q}$ is a Λ-graded subalgebra,

$$\mathcal{Z}(\mathcal{Q}) = \bigoplus_{\xi \in \Xi} \mathcal{Q}^\xi$$

where Ξ is the so-called *central grading group*:

$$\Xi = \{\lambda \in \Lambda \colon \mathcal{Q}^\lambda \subset \mathcal{Z}(\mathcal{Q})\}.$$

This is a free abelian group of rank $z \le n$. Hence $\mathcal{Z}(\mathcal{Q})$ is a Laurent polynomial ring in z variables, which we may take as $t_1, \dots, t_z$ (these can be taken to be of the form x^λ for suitable λ's).

(d) The grading properties of a quantum torus $\mathcal{Q}$ show that $\mathcal{Q}$ is *fgc* in the sense that $\mathcal{Q}$ is finitely generated as a module over $\mathcal{Z}(\mathcal{Q})$ if and only if Ξ has finite index in Λ. Equivalently, for some (hence all) coordinatization all entries q_{ij} of the quantum matrix q have finite order. If this holds, then for every coordinatization the q_{ij} have finite order.

(e) We define $[\mathcal{Q}, \mathcal{Q}] = \mathrm{span}_k\{[a, b] \colon a, b \in \mathcal{Q}\}$, a graded subspace of $\mathcal{Q}$. One knows (see e.g. [BGK96, Prop. 2.44(iii)] for $k = \mathbb{C}$ or [NY03, (3.3.2)] in general)

$$\mathcal{Q} = \mathcal{Z}(\mathcal{Q}) \oplus [\mathcal{Q}, \mathcal{Q}]. \tag{3.5.2}$$

(f) An element u of $\mathcal{Q}$ is invertible if and only if $0 \neq u \in \mathcal{Q}^\lambda$ for some $\lambda \in \Lambda$.

(g) The derivation Lie algebra $\mathrm{Der}_k(\mathcal{Q})$ is graded: $\mathrm{Der}_k(\mathcal{Q}) = \bigoplus_{\lambda \in \Lambda} \mathrm{Der}_k(\mathcal{Q})^\lambda$, where $\mathrm{Der}_k(\mathcal{Q})^\lambda$ consists of those derivations d satisfying $d(\mathcal{Q}^\mu) \subset \mathcal{Q}^{\lambda + \mu}$ for all $\mu \in \Lambda$. The inner derivations of $\mathcal{Q}$ are the maps $\mathrm{ad}\, q$, given by $\mathrm{ad}(q)(q') = qq' - q'q$ for $q, q' \in \mathcal{Q}$. They form a graded ideal $\mathrm{IDer}\, \mathcal{Q} = \{\mathrm{ad}\, q \colon q \in \mathcal{Q}\}$ of $\mathrm{Der}_k(\mathcal{Q})$. As in 2.2(e), the grading $\mathcal{Q} = \bigoplus_{\lambda \in \Lambda} \mathcal{Q}^\lambda$ gives rise to *degree derivations* ∂_θ of $\mathcal{Q}$, defined by $\partial_\theta(q) = \theta(\lambda) q$ for $\theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, k)$ and $q \in \mathcal{Q}^\lambda$. We put

$$\mathcal{D}_{\mathcal{Q}} = \{\partial_\theta \colon \theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, k)\}$$

and define

$$\mathrm{CDer}(\mathcal{Q}) = \mathcal{Z}(\mathcal{Q})\, \mathcal{D}_{\mathcal{Q}} = \bigoplus_{\xi \in \Xi} \mathcal{Q}^\xi\, \mathcal{D}_{\mathcal{Q}},$$

the graded subalgebra of *centroidal derivations*. Then [OP95, Cor. 2.3]

$$\mathrm{Der}_k(\mathcal{Q}) = \mathrm{IDer}(\mathcal{Q}) \times \mathrm{CDer}(\mathcal{Q}), \quad \text{so } \mathrm{IDer}(\mathcal{Q}) = \bigoplus_{\lambda \notin \Xi} \mathrm{Der}_k(\mathcal{Q})^\lambda.$$

Let $\varepsilon\colon \mathcal{Q} \to k$ be the linear form defined by $\varepsilon(1_{\mathcal{Q}}) = 1$ and $\varepsilon(\mathcal{Q}^\lambda) = 0$ for $\lambda \neq 0$. The skew-symmetric derivations with respect to the bilinear form $(q, q') \mapsto \varepsilon(qq')$ have the following description:

$$\mathrm{SDer}(\mathcal{Q}) = \mathrm{SCDer}(\mathcal{Q}) \oplus \mathrm{IDer}(\mathcal{Q}),$$

$$\mathrm{SCDer}(\mathcal{Q}) = \mathrm{SDer}(\mathcal{Q}) \cap \mathrm{CDer}(\mathcal{Q}) = \bigoplus_{\xi\in\Xi} \mathrm{SCDer}(\mathcal{Q})^\xi, \quad \text{where}$$

$$\mathrm{SCDer}(\mathcal{Q})^\xi = \mathcal{Q}^\xi \,\{\partial_\theta\colon \theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, k), \theta(\xi) = 0\}. \tag{3.5.3}$$

3.6. **$\mathfrak{sl}_\ell(\mathcal{A})$ as a Lie torus.** In this subsection we describe for which algebras $\mathcal{A}$ the Lie algebra $\mathfrak{sl}_\ell(\mathcal{A})$ is a Lie torus as defined in 2.1 and identify the data 2.5 necessary to construct an EALA with centreless core $\mathfrak{sl}_\ell(\mathcal{A})$. All unattributed result can be found in [Neh11a, § 7] or are easily verified by the reader. We assume $\mathcal{A} \neq 0$ throughout.

(a) Let Δ be the root system of type $\mathrm{A}_{\ell-1}$, realized as $\Delta = \{\varepsilon_i - \varepsilon_j, 1 \le i, j \le \ell\}$ in standard notation. Then the Lie algebra $\mathfrak{sl}_\ell(A)$ has a canonical grading by the root lattice $\mathcal{Q}(\Delta)$,

$$\mathfrak{sl}_\ell(\mathcal{A}) = \bigoplus_{\alpha\in\Delta} \mathfrak{sl}_\ell(\mathcal{A})_\alpha, \qquad \text{for} \tag{3.6.1}$$

$$\mathfrak{sl}_\ell(\mathcal{A})_\alpha = \begin{cases} \mathcal{A}\, E_{ij}, & \alpha = \varepsilon_i - \varepsilon_j \neq 0, \\ \{x \in \mathfrak{sl}_\ell(\mathcal{A})\colon x \text{ diagonal}\}, & \alpha = 0. \end{cases}$$

(b) Let $e = aE_{ij} \in \mathfrak{sl}_\ell(\mathcal{A})$ for $i \neq j$. Then e is part of an $\mathfrak{sl}_2$-triple (e, h, f) satisfying $[h, x_\beta] = \langle \beta, \alpha^\vee\rangle x_\beta$ for all $\beta \in \Delta$ and $x_\beta \in L_\beta$ if and only if a is invertible in $\mathcal{A}$. In this case $f = a^{-1}E_{ji}$ and $h = E_{ii} - E_{jj}$.

(c) Let Λ be an abelian group, and let $\mathcal{A} = \bigoplus_{\lambda\in\Lambda} \mathcal{A}^\lambda$ be a Λ-graded unital associative k-algebra. Then the $\mathcal{Q}(\Delta)$ grading (3.6.1) of $\mathfrak{sl}_\ell(\mathcal{A})$ extends to a $(\mathcal{Q}(\Delta) \oplus \Lambda)$-grading of $\mathfrak{sl}_\ell(\mathcal{A})$,

$$\mathfrak{sl}_\ell(\mathcal{A}) = \bigoplus_{\alpha\in\Delta,\, \lambda\in\Lambda} \mathfrak{sl}_\ell(\mathcal{A})^\lambda_\alpha$$

by letting $\mathfrak{sl}_\ell(\mathcal{A})^\lambda_\alpha$ consist of those matrices, for which all entries lie in $\mathcal{A}^\lambda$. Conversely, a $\mathcal{Q}(\Delta) \oplus \Lambda$-grading of $\mathfrak{sl}_\ell(\mathcal{A})$ extending the $\mathcal{Q}(\Delta)$-grading (3.6.1) arises from a Λ-grading of the associative algebra $\mathcal{A}$ as described above.

(d) Because of (c), for $\mathfrak{sl}_\ell(\mathcal{A})$ to satisfy the axiom (LT1) of 2.1 with $\mathcal{Q}(\Delta)$-grading (3.6.1) it is necessary and sufficient for the associative k-algebra $\mathcal{A}$ to be Λ-graded. Observe that then also (LT2.b) holds since $0 \neq 1_{\mathcal{A}} \in \mathcal{A}^0$ and therefore $0 \neq 1_{\mathcal{A}}E_{ij} \in \mathfrak{sl}_\ell(\mathcal{A})^0_\alpha$ for $\alpha = \varepsilon_i - \varepsilon_j \neq 0$. Because of (b), the axiom (LT2.a) holds if and only if $\mathcal{A} = \mathcal{Q} = \bigoplus_{\lambda\in\Lambda} \mathcal{Q}^\lambda$ is a quantum torus.

Since (LT3) is clear, (LT4) says that $\mathfrak{sl}_\ell(\mathcal{Q})$ is a Lie torus of type (Δ, Λ) if and only if $\mathcal{Q}$ is a quantum torus of type Λ, as defined in 3.5(a). In this case, it follows from (g) that L is fgc as defined in 2.2 if and only if $\mathcal{Q}$ is an fgc quantum torus in the sense of 3.5(d) — but we will not assume this in the following.

(e) In the remainder of this subsection we let $L = \mathfrak{sl}_\ell(\mathcal{Q})$ for $\mathcal{Q}$ a quantum torus with grading group Λ. Because of (3.5.2), the assumption of Lemma 3.1 is fulfilled. Then (3.2.4) and (3.1.3) imply that L is a centreless Lie torus of type (Δ, Λ).

Hence, by Theorem 2.1, L is centreless core of an EALA obtained by the construction 2.5. We describe the bilinear forms $(\cdot|\cdot)$ and derivation algebras D allowed in this construction in the next two items.

(f) Every Λ-graded invariant symmetric bilinear form β on L has the form (3.3.2), where $\varepsilon\colon \mathcal{Q} \to k$ is a linear form vanishing on $\bigoplus_{0\neq\lambda} \mathcal{Q}^\lambda$ and is therefore given by the scalar $\varepsilon(1_\mathcal{Q})$ which we can assume to be $1 \in k$.

(g)[9] For $z \in \mathcal{Z}(\mathcal{Q})$ define $\chi_z \in \mathrm{End}_k(\mathrm{M}_\ell(\mathcal{Q}))$ by $\chi_z(x) = (zx_{ij})$ for $x = (x_{ij}) \in \mathrm{M}_\ell(\mathcal{Q})$. Then χ_z stabilizes $\mathfrak{sl}_\ell(\mathcal{Q})$ and defines by restriction a centroidal transformation of $\mathfrak{sl}_\ell(\mathcal{Q})$. The map $\mathcal{Z}(\mathcal{Q}) \to \mathrm{Ctd}(\mathfrak{sl}_\ell(\mathcal{Q}))$, $z \mapsto \chi_z$, is an isomorphism of k-algebras.

(h) For $d \in \mathrm{Der}(\mathcal{Q})$ we denote by $\mathrm{M}_\ell(d)$ the derivation of $\mathfrak{sl}_\ell(\mathcal{Q})$ defined in (3.1.4). The maps $d \mapsto \mathrm{M}_\ell(d)$ is clearly a monomorphism of Lie algebras. Moreover,

$$\begin{aligned}
\mathrm{Der}_k(\mathfrak{sl}_\ell(\mathcal{Q})) &= \mathrm{IDer}(\mathfrak{sl}_\ell(\mathcal{Q})) + \mathrm{M}_\ell(\mathrm{Der}(\mathcal{Q})),\\
\mathrm{M}_\ell(\mathrm{IDer}(\mathcal{Q})) &= \mathrm{IDer}(\mathfrak{sl}_\ell(\mathcal{Q})) \cap \mathrm{M}_\ell(\mathrm{Der}(\mathcal{Q})) \simeq \mathrm{IDer}(\mathcal{Q}),\\
\mathrm{CDer}(\mathfrak{sl}_\ell(\mathcal{Q})) &= \mathrm{M}_\ell(\mathrm{CDer}(\mathcal{Q})) \simeq \mathrm{CDer}(\mathcal{Q}),
\end{aligned}$$

$$\begin{aligned}
\mathrm{SDer}(\mathfrak{sl}_\ell(\mathcal{Q})) &= \mathrm{IDer}(\mathfrak{sl}_\ell(\mathcal{Q})) + \mathrm{M}_\ell(\mathrm{SDer}(\mathcal{Q})),\\
\mathrm{SCDer}(\mathfrak{sl}_\ell(\mathcal{Q})) &= \mathrm{M}_\ell(\mathrm{SCDer}(\mathcal{Q})) \simeq \mathrm{SCDer}(\mathcal{Q})
\end{aligned}$$

for $\mathrm{Der}(\mathcal{Q})$, $\mathrm{IDer}(\mathcal{Q})$, $\mathrm{CDer}(\mathcal{Q})$, $\mathrm{SDer}(\mathcal{Q})$ and $\mathrm{SCDer}(\mathcal{Q})$ described in 3.5(g). Note that the first three equations above together with 3.5(g) prove (2.2.3) for the case $L = \mathfrak{sl}_\ell(\mathcal{Q})$.

(i) The maximal possible choice for D in the construction 2.5 is $\mathrm{SCDer}(\mathfrak{sl}_\ell(\mathcal{Q}))$ which we identify with $\mathrm{SCDer}(\mathcal{Q})$ using the isomorphism M_ℓ of (h). For $\mathcal{Q} = k[x_1^{\pm}, \dots, x_n^{\pm 1}]$, $n \geq 2$, a nonzero affine cocycle τ has been exhibited in [BGK96, Rem. 3.71]. It can be described as follows.

Modulo the isomorphism M_ℓ of (h) we identify $\mathrm{SCDer}(\mathfrak{sl}_\ell(\mathcal{Q}))$ with $\mathrm{SCDer}(\mathcal{Q})$. Denoting by $\langle\cdot,\cdot\rangle$ the standard inner product of k^n and using the natural embedding $\mathbb{Z}^n \subset k^n$ we can further identify

$$\mathrm{SCDer}(\mathcal{Q}) = \bigoplus_{\lambda \in \Lambda = \mathbb{Z}^n} \mathrm{SCDer}(\mathcal{Q})^\lambda,$$

where for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$,

$$\mathrm{SCDer}(\mathcal{Q})^\lambda \equiv \left\{ u = (u_i) \in k^n : \sum_{i=1}^n u_i \lambda_i = 0 \right\} =: D^\lambda;$$

cf. (3.5.3). $u_\alpha \in D^\alpha$, $v_\beta \in D^\beta$ and $w_\gamma \in D^\gamma$ define

$$\tau(u_\alpha, v_\beta)\,(w_\gamma) = \begin{cases} \alpha(v)\,\beta(w)\,\gamma(u) & \text{if } \alpha + \beta + \gamma = 0,\\ 0 & \text{otherwise,} \end{cases}$$

Then τ is an affine cocycle. It is nontrivial in the sense that the EALAs associated with $L = \mathfrak{sl}_\ell(\mathcal{Q})$, $\ell \geq 3$, $D = \mathrm{SCDer}(L)$ and the two affine cocycles τ as above, respectively, $\tau = 0$ are not isomorphic [Kry00, Th. 5.76].

4. Proof of the main theorem

The proof of our main result will be based on the computation of K-Theory of noncommutative (twisted) Laurent polynomial rings due to D. Quillen. We first briefly recall functors K_0 and K_1. A nice introduction to the subject can be found in [Ros94] and [Wei13].

[9] The items (g) and (h) are true for any algebra $\mathcal{A}$ in place of $\mathcal{Q}$.

4.1. $K_0(\mathcal{A})$ **and** $K_1(\mathcal{A})$ **for a ring** $\mathcal{A}$. Let $\mathcal{A}$ be a ring (unital, but not necessarily commutative). If P is a (left) $\mathcal{A}$-module, we denote its isomorphism class by $[P]$. Consider the free abelian group $FK_0(\mathcal{A})$ generated by the set of isomorphism classes of projective $\mathcal{A}$-modules of finite type. Then $K_0(\mathcal{A})$ is the quotient of the group $FK_0(\mathcal{A})$ by the normal subgroup generated by the relation

$$[P] = [P'] + [P'']$$

whenever there exists an exact sequence of $\mathcal{A}$-modules $0 \to P' \to P \to P'' \to 0$.

As in 3.2 we denote by $\mathrm{GL}_\ell(\mathcal{A})$, $\ell \in \mathbb{N}_+$, the group of invertible $\ell \times \ell$-matrices with entries in $\mathcal{A}$. For each $m \in \mathbb{N}_+$ we have a natural embedding $\mathrm{GL}_\ell(\mathcal{A}) \hookrightarrow \mathrm{GL}_{\ell+m}(\mathcal{A})$ given by

$$X \longrightarrow \begin{pmatrix} X & 0 \\ 0 & \mathrm{E}_{\ell+m} \end{pmatrix}; \tag{4.1.1}$$

cf. (3.4.1) for the corresponding embedding on the level of Lie algebras. We let $\mathrm{GL}_\infty(\mathcal{A})$ be the direct limit of $\mathrm{GL}_\ell(\mathcal{A})$ with respect to the embeddings (4.1.1). Again as in 3.2, we let $\mathrm{EL}_\ell(\mathcal{A})$ be the elementary linear subgroup of $\mathrm{GL}_\ell(\mathcal{A})$ and let $\mathrm{EL}_\infty(\mathcal{A})$ be the direct limit of the $\mathrm{EL}_\ell(\mathcal{A})$. Then

$$K_1(\mathcal{A}) = \mathrm{GL}_\infty(\mathcal{A})/[\mathrm{GL}_\infty(\mathcal{A}), \mathrm{GL}_\infty(\mathcal{A})] = \mathrm{GL}_\infty(\mathcal{A})/\mathrm{EL}_\infty(\mathcal{A})$$

(the first equality is the standard definition of $K_1(\mathcal{A})$, while the second equality is a classical theorem of Whitehead).

Remark 4.1. The construction of K_0 and K_1 is functorial on k-algebras. Given a k-algebra homomorphism $\eta\colon \mathcal{A} \to \mathcal{B}$ we will denote by η^* the induced group homomorphisms $K_0(\mathcal{A}) \to K_0(\mathcal{B})$ and $K_1(\mathcal{A}) \to K_1(\mathcal{B})$.

Next we recall the definition of noncommutative Laurent polynomial ring $\mathcal{A}_\phi[t^{\pm 1}]$. Consider an automorphism ϕ of a (unital, associative and not necessarily commutative) k-algebra $\mathcal{A}$. The multiplication in $\mathcal{A}$ will be denoted by juxtaposition. We define a new unital and associative k-algebra $\mathcal{A}_\phi[t^{\pm 1}]$ as follows. The underlying k-vector space structure is the free left $\mathcal{A}$-module with basis $\{t^m\}_{m\in\mathbb{Z}}$. The multiplication on $\mathcal{A}_\phi[t^{\pm 1}]$, which we will denote by $\cdot$, is given by

$$\sum_{i\in\mathbb{Z}} a_i t^i \cdot \sum_{j\in\mathbb{Z}} a'_j t^j = \sum_{i,\, j\in\mathbb{Z}} a_i \phi^i(a'_j) t^{i+j} \quad \text{for all } a_i, a'_j \in \mathcal{A}. \tag{4.1.2}$$

It is known that if $\mathcal{A}$ is noetherian (resp. regular), so is $\mathcal{A}_\phi[t^{\pm 1}]$ (see [Art98, Prop. 2.21]). We also observe that ϕ induces a natural action ϕ^* on $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$. Namely, if P is a projective $\mathcal{A}$-module, then $\phi^*([P]) := [P \otimes_\phi \mathcal{A}]$. It is obvious that if P is free or projective of finite type, so is $\phi^*(P)$. Also, for every matrix $X = (x_{ij})$ in $\mathrm{GL}_\ell(\mathcal{A})$ we let $\phi^*(X) = (\phi(x_{ij}))$. This of course induces an action ϕ^* on $\mathrm{GL}_\infty(\mathcal{A})$ stabilizing $\mathrm{EL}_\infty(\mathcal{A})$, and hence also an action on $K_1(\mathcal{A})$.

The following result is due to D. Quillen [Qui73, § 6, p. 122].

Theorem 4.2. *Let $\phi \in \mathrm{Aut}_k(\mathcal{A})$. Assume that $\mathcal{A}$ is noetherian and regular. Let $\eta\colon \mathcal{A} \to \mathcal{A}_\phi[t^{\pm 1}]$ be the canonical embedding of k-algebras. Then the following sequence of abelian groups is exact:*[10]

$$K_1(\mathcal{A}) \xrightarrow{1-\phi^*} K_1(\mathcal{A}) \xrightarrow{\eta^*} K_1(\mathcal{A}_\phi[t^{\pm 1}]) \xrightarrow{\partial} K_0(\mathcal{A}) \xrightarrow{1-\phi^*} K_0(\mathcal{A}) \xrightarrow{\eta^*} K_0(\mathcal{A}_\phi[t^{\pm 1}]) \to 0. \tag{4.1.3}$$

[10]The maps ϕ^* and η^* have been defined already. The nature of ∂ is explained in Quillen's paper.

We will apply Theorem 4.2 to a quantum torus $\mathcal{Q}$, Thus, as explained in 3.5, we can view $\mathcal{Q}$ as the unital associative k-algebra presented by generators $x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}$ and relations $x_i x_i^{-1} = 1_{\mathcal{Q}} = x_i^{-1} x_i$, $x_i x_j = q_{ij} x_j x_i$, where the q_{ij} are nonzero elements of k, $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$. For convenience in what follows we assume that the elements q_{ij} are fixed throughout our discussion, and we write

$$\mathcal{Q} = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

It is immediate from the defining relations that the k-vector space $\mathcal{Q}$ is a direct sum $\mathcal{Q} = \bigoplus_{i_1,\dots,i_n \in \mathbb{Z}} k x_1^{i_1} \dots x_n^{i_n}$.

The quantum torus $\mathcal{Q}$ contains a subring

$$\mathcal{Q}_{n-1} = k[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$$

generated by $x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}$. Obviously, the conjugation by x_n stabilizes $\mathcal{Q}_{n-1}$ and thus induces an automorphism ϕ on $\mathcal{Q}_{n-1}$ so that we may view $\mathcal{Q}$ as a noncommutative Laurent polynomial ring $\mathcal{Q} = \mathcal{A}_\phi[x_n^{\pm 1}]$, where $\mathcal{A} = \mathcal{Q}_{n-1}$. The advantage of realizing $\mathcal{Q}$ in this form is that it allows us to compute $K_0(\mathcal{Q})$ and $K_1(\mathcal{Q})$ by induction on n. We start with computing $K_0(\mathcal{Q})$.

Lemma 4.3. *The group $K_0(\mathcal{Q})$ is isomorphic to $\mathbb{Z}$. Its generator is the class of a free $\mathcal{Q}$-module of rank* 1*.*

This is [Art98, Th. 3.17]. We include a short proof for the sake of completeness.

Proof. We reason by induction on n. If $n = 1$, then $\mathcal{Q} = k[x^{\pm 1}]$ is a commutative Laurent polynomial ring. Since $\mathcal{Q}$ is then a principal ideal domain, every projective $\mathcal{Q}$-module is free. Our result is then clear.

Assume $n > 1$. Consider the natural k-algebra inclusion $\eta\colon \mathcal{Q}_{n-1} \to \mathcal{Q}_n$. By induction we may assume that $K_0(\mathcal{Q}_{n-1}) \simeq \mathbb{Z}$. Since ϕ^* acts trivially on its generator, it acts trivially on $K_0(\mathcal{Q}_{n-1})$. From Quillen's exact sequence (4.1.3) we see that the base change map $\eta\colon K_0(\mathcal{Q}_{n-1}) \to K_0(\mathcal{Q})$ is an isomorphism and the result follows. □

We now pass to the computation of the group $K_1(\mathcal{Q})$ for a quantum torus $\mathcal{Q}$. We first remark that for an arbitrary ring $\mathcal{A}$ and a unit $u \in \mathcal{A}^\times$ the 1×1-matrix (u) is an element of $\mathrm{GL}_1(\mathcal{A})$. Taking the composition of $\mathcal{A}^\times \to \mathrm{GL}_1(\mathcal{A})$ with $\mathrm{GL}_1(\mathcal{A}) \to \mathrm{GL}_\infty(\mathcal{A}) \to K_1(\mathcal{A})$ we obtain a canonical group homomorphism $\lambda_{\mathcal{A}}\colon \mathcal{A}^\times \to K_1(\mathcal{A})$. In general, $\lambda_{\mathcal{A}}$ is neither injective nor surjective, but we will show that $\lambda_{\mathcal{Q}}$ is surjective when $\mathcal{Q}$ is a quantum torus.

Proposition 4.4. *Let $\mathcal{Q}$ be a quantum torus. Then $\lambda_{\mathcal{Q}}\colon \mathcal{Q}^\times \to K_1(\mathcal{Q})$ is surjective.*

Proof. We argue by induction on $n \in \mathbb{N}$. In case $n = 0$ and $n = 1$ it is well-known that $\lambda_{\mathcal{Q}}$ is actually an isomorphism: $K_1(k) \simeq k^\times$ and $K_1(k[t^{\pm 1}]) \simeq k[t^{\pm 1}]^\times$ for any field k by (for example) [Ros94, Prop. 2.2.2] and [Ros94, Th. 2.3.2], respectively, where both isomorphisms are induced by the determinant. Thus we can assume $n \geq 2$ in the following.

Consider the sequence (4.1.3) with $\mathcal{A} = \mathcal{Q}_{n-1}$. We already know, by Lemma 4.3, that ϕ^* acts trivially on $K_0(\mathcal{Q}_{n-1})$ so that we have a commutative diagram with an exact horizontal row at the bottom:

$$\begin{array}{ccccccccc}
\mathcal{Q}_{n-1}^\times & \xrightarrow{1-\phi} & \mathcal{Q}_{n-1}^\times & \xrightarrow{\eta} & \mathcal{Q}^\times & & & & \\
\downarrow{\scriptstyle \lambda_{\mathcal{Q}_{n-1}}} & & \downarrow{\scriptstyle \lambda_{\mathcal{Q}_{n-1}}} & & \downarrow{\scriptstyle \lambda_{\mathcal{Q}}} & & & & \\
K_1(\mathcal{Q}_{n-1}) & \xrightarrow{1-\phi^*} & K_1(\mathcal{Q}_{n-1}) & \xrightarrow{\eta^*} & K_1(\mathcal{Q}) & \xrightarrow{\partial} & K_0(\mathcal{Q}_{n-1}) & \longrightarrow & 0.
\end{array}$$

By induction, $\lambda_{\mathcal{Q}_{n-1}}$ is surjective. By Lemma 4.3, $K_0(\mathcal{Q}_{n-1}) \simeq \mathbb{Z}$. Clearly $\mathcal{Q}^\times$ is generated by $k^\times$ and $x_1, \dots, x_n$; cf. 3.5(f). It is shown in the proof of Lemma 5.16 in [Qui73] that $\partial(\lambda_{\mathcal{Q}}(x_i))$ is a generator of $K_0(\mathcal{Q}) \simeq \mathbb{Z}$. To prove surjectivity of $\lambda_{\mathcal{Q}}$, let $a \in K_1(\mathcal{Q})$. Then $\partial(a) = m \in \mathbb{Z}$ and either the element $a - \lambda_Q(x_i^m)$ or $a + \lambda_Q(x_i^m)$ lies in the kernel of ∂. The claim now follows by a standard diagram chase. □

Remark 4.5. A further diagram chase yields more than surjectivity. In fact $K_1(\mathcal{Q}) = \mathcal{Q}^\times/[\mathcal{Q}^\times, \mathcal{Q}^\times]$. We do not need this more detailed result for our purposes.

Interpreted in terms of matrices, Proposition 4.4 yields the following corollary.

Corollary 4.6. *Let $\mathcal{Q}$ be a quantum torus. Let $h \in \mathrm{GL}_\ell(\mathcal{Q})$. Then there exists a non-negative integer m and a unit $u \in \mathcal{Q}^\times$ such that the matrix*

$$\begin{pmatrix} h & 0 \\ 0 & \mathrm{E}_m \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & \mathrm{E}_{\ell+m-1} \end{pmatrix} \tag{4.1.4}$$

is contained in $\mathrm{EL}_{\ell+m}(\mathcal{Q})$.

4.2. **Proof of the Main Theorem.** To prove the Main Theorem as stated in the introduction, we can assume that the Cartan subalgebra H of E is such that

$$H_{cc} = \left\{ \sum_{i=1}^{\ell} s_i E_{ii} \colon s_i \in k, \sum_i s_i = 0 \right\} =: \mathfrak{h}_{\mathrm{st}}$$

in the notation of [CNP16]. Let (E, H') be a second EALA structure, and set $H'_{cc} = \mathfrak{h}$. We then know by the main theorem of [CNP16] that there exists $h \in \mathrm{GL}_\ell(\mathcal{Q})$ such that $\mathrm{Int}(h)$ maps $\mathfrak{h}_{\mathrm{st}}$ to $\mathfrak{h}$. We now apply Corollary 4.6 and get $u \in \mathcal{Q}^\times$ such that the matrix of (4.1.4) is elementary. Put

$$g = h\begin{pmatrix} u & 0 \\ 0 & \mathrm{E}_{\ell-1} \end{pmatrix} \in \mathrm{GL}_\ell(\mathcal{Q}).$$

Then also $\mathrm{Int}(g)$ maps $\mathfrak{h}_{\mathrm{st}}$ to $\mathfrak{h}$ [CNP16, Lemma 2.10]. Moreover,

$$\begin{pmatrix} g & 0 \\ 0 & \mathrm{E}_m \end{pmatrix} = g'$$

is elementary. Because of (3.5.2) we can now apply Theorem 3.2 and obtain that $\mathrm{Int}(g)$ lifts to an automorphism of E. This finishes the proof. □

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