Invariant bilinear forms of algebras given by faithfully flat descent

Erhard Neher

University of Ottawa

December 2013

Reference

E. Neher, A. Pianzola, D. Prelat and C. Sepp,

Invariant bilinear forms of algebras given by faithfully flat decent,

Communications in Contemporary Mathematics 2014

ArXiv:1311.2094

Example: Lie algebras

L Lie algebra, bilinear form $(.|.)$ is invariant ($=$ associative) if

$$([x,y] \mid z) = (x \mid [y,z]) \quad \text{all } x,y,z \in L$$

Basic example:
L finite-dimensional over a field F,
Killing form κ is invariant, $\kappa(x,y) = \mathrm{tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y)$.

Humphreys, Exercise 6.6

L finite-dimensional simple over an algebraically closed field of characteristic 0. Show: Any two invariant nondegenerate symmetric bilinear forms on L are proportional.

Solution: β any invariant form, $\beta^* : L \to L^*$, $l \mapsto \beta(l, -)$ L-module homomorphism, $(\kappa^*)^{-1} \circ \beta^* \in \mathrm{End}_L(L) = \mathbb{C}\, \mathrm{Id}_L$.

Why invariant forms?

- easier structure theory
  e.g. semisimple Lie algebras
  symmetrizable Kac-Moody algebras
- consequences for representation theory
  Casimir
- applications:
  central extensions (many papers)
  Lie bialgebra structures (Montaner-Stolin-Zelmanov 2010)

Questions:

- Existence
- How many? Structure of

$$\mathrm{IBF}(L) = \{\beta : \beta \text{ invariant bilinear form}\}$$

E.g. $\mathrm{IBF}(L) = \mathbb{C}\kappa$ for L finite-dimensional simple over $\mathbb{C}$.

Generalization

- R associative commutative ring
- B R-algebra: B R-module + R-bilinear map $B \times B \to B$

Definition (nondegenerate/nonsingular)

$\beta \colon B \times B \to R$ bilinear, $\beta^* \colon B \to B^*$, $b \mapsto \beta(b, -)$
β nondegenerate (nonsingular) if β^* is injective (bijective)

Examples:
B finite-dimensional over field R: nondegenerate = nonsingular

Definition (Central)

Centroid $\mathrm{Ctd}(B) = \{\chi \in \mathrm{End}_R(B) : \chi(ab) = \chi(a)b = a\chi(b) \text{ all } a, b \in B\} \supset R\,\mathrm{Id}_B$

B central if $R \to \mathrm{Ctd}(B)$, $r \mapsto r\,\mathrm{Id}_L$ isomorphism

Examples:
B unital associative $\mathrm{Ctd}(B) \simeq C(B)$ (= centre of B),
but: $\mathrm{Ctd}(\mathrm{Lie}) \not\cong \mathrm{Centre}$.
B finite-dimensional simple over an algebraically closed field: B central

Humphreys' exercise revisited

Recall setting: B algebra over a commutative associative ring R,
$\mathrm{IBF}(B) = \{\beta\colon B \times B \to R, \beta \text{ invariant }\}$, R-module

Goal: Describe invariant bilinear forms $\beta\colon B \times B \to R$

$$\beta(ab, c) = \beta(a, bc) = \beta(b, ca)$$

$\mathrm{IBF}(B) = \{\beta\colon B \times B \to R : \beta \text{ invariant, } R\text{-bilinear}\}$

Lemma

B central algebra over R, β non-singular invariant bilinear form on B
$\Longrightarrow$ $\mathrm{IBF}(B)$ *free of rank 1 with R-basis β.*

Questions:
How do we get central algebras?
A nonsingular β?

Answer: Ascent and descent!

Setting

- k associative commutative unital ring
  k-**alg** category of associative commutative unital k-algebras
- $\mathfrak{a}$ an algebra over k ($=$ k-module $+$ k-bilinear map $\mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$)
- $R \in k$-**alg** flat,
- B an algebra over R, twisted form of $\mathfrak{a} \otimes_k R$
  split by some faithfully flat extension $S \in R$-**alg**;

$$B \otimes_R S \simeq (\mathfrak{a} \otimes_k R) \otimes_R S \equiv \mathfrak{a} \otimes_k S \quad \text{(as } S\text{-algebras)}$$

Examples:

(1) Kac-Moody: $k = \mathbb{C}$, $\mathfrak{a} = \mathfrak{g}$ finite-dimensional simple, $R = \mathbb{C}[t^{\pm 1}]$, B twisted loop algebra, split by $S = \mathbb{C}[s^{\pm 1}]$, $s^r = t$ ($r =$ order of diagram automorphism.

(2) Associative: $\mathfrak{a} = M_n(k)$
B Azumaya algebra over R of constant rank n^2.

Ascent

$\mathfrak{a}$ k-algebra, $R \in k$-**alg**

$\mathfrak{a}_R = \mathfrak{a} \otimes_k R$ is R-algebra

$\kappa \colon \mathfrak{a} \times \mathfrak{a} \to k$ bilinear

$\kappa_R \colon \mathfrak{a}_R \times \mathfrak{a}_R \to R$, $\kappa_R(a_1 \otimes r_1, a_2 \otimes r_2) = \kappa(a_1, a_2) r_1 r_2$

B finitely presented (k-module), R flat,

κ nondegenerate (-singular) $\implies$ κ_R nondegenerate (-singular)

$\Longleftarrow$ if $\mathfrak{a}$ finitely presented k-module

Descent?

$\kappa \colon \mathfrak{a} \times \mathfrak{a} \to k$ is $\mathrm{Aut}_k(\mathfrak{a})$-invariant:

$\kappa\big(f(a_1), f(a_2)\big) = \kappa(a_1, a_2)$ for all automorphisms f and $a_i \in \mathfrak{a}$

$\kappa \colon \mathfrak{a} \times \mathfrak{a} \to k$ is $\mathbf{Aut}(\mathfrak{a})$-invariant

κ_S is $\mathrm{Aut}_S(\mathfrak{a} \otimes_k S)$-invariant for all $S \in k$-**alg**

Setting: k base ring, $\mathfrak{a}$ k-algebra, $R \in k$-**alg** flat, B R-algebra, twisted from of $\mathfrak{a}_R$, split by some $S \in R$-**alg** faithfully flat

Theorem (Descent)

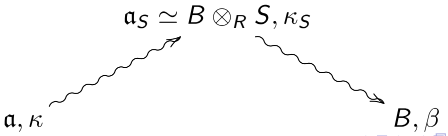
Assume $\kappa\colon \mathfrak{a} \times \mathfrak{a} \to k$ *is* $\mathbf{Aut}(\mathfrak{a})$*-invariant.*

Then $\exists$ *unique* R*-bilinear* $\beta\colon B \times B \to R$ *such that*

$$\beta_T \simeq \kappa_T \qquad \text{whenever } B \otimes_R T \simeq \mathfrak{a} \otimes_k T \text{ for } T \in R\text{-}\mathbf{alg} \text{ f.f.}$$

Moreover:

- β $\mathbf{Aut}(B)$*-invariant*
- κ *invariant* $\implies$ β *invariant*
- $\mathfrak{a}$ *finitely presented (as k-module) and* κ *nondegenerate (-singular)* $\implies$ β *same*



Construction of **Aut**-invariant and invariant forms

Trace forms!

Examples

- Lie
  $\mathfrak{a}$ Lie algebra, finitely generated projective (k-module)
  Killing form $\kappa = \kappa_{\mathfrak{a}}$, $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y)$
  invariant and $\mathbf{Aut}(\mathfrak{g})$-invariant

  B twisted form:
  $\implies$ B finitely generated projective
  $\kappa_B = \beta$ (of Theorem)

- Azumaya
  $\mathfrak{a} = M_n(k)$, $\kappa(x, y) = \mathrm{tr}(xy)$ invariant and $\mathbf{Aut}(\mathfrak{a})$-invariant

- Others
  Octonion, alternative, Albert, Jordan ... privatissimum

More general maps

We now know (for example)

$\mathfrak{a}$ finite-dimensional simple Lie algebra over $\mathbb{C}$,
B twisted loop algebra, Lie algebra over $R = \mathbb{C}[t^{\pm 1}]$,
split by $S = \mathbb{C}[s^{\pm 1}]$, $s^r = t$ (r = order of diagram automorphism).

$$\begin{aligned} \mathrm{IBF}(B) &= \{\beta\colon B \times B \to \mathbb{C}[t^{\pm 1}] : \beta \text{ invariant}\} \\ &= (\mathbb{C}[t^{\pm 1}])\, \kappa_B, \quad \kappa_B \text{ Killing form of the } R\text{-algebra } B \end{aligned}$$

But we are interested in invariant bilinear forms

$$\beta\colon B \times B \to \mathbb{C},$$

i.e. consider B as $\mathbb{C}$-algebra.

Invariant maps

Add V to the general setting:

- k associative commutative unital ring, $\mathfrak{a}$ an algebra over k
- B an algebra over flat $R \in k$-**alg**, twisted form of $\mathfrak{a} \otimes_k R$ split by some faithfully flat extension $S \in R$-**alg**;
- V k-module

Definition (Invariant maps)

$\beta \colon B \times B \to V$ k-bilinear and invariant: $\beta(ab, c) = \beta(a, bc) = \beta(b, ca)$ for $a, b, c \in B$. If $B \neq BB$ add: $\beta(ar, b) = \beta(a, rb)$ for $a, b \in B$, $r \in R$.

WANT:

$$\mathrm{IBF}_{(R,k)}(B; V) = \{\beta \colon B \times B \to V : \beta \text{ } k\text{-bilinear and invariant}\}$$

Easy construction: $\beta \in \mathrm{IBF}(B)$, $\phi \in \mathrm{Hom}_k(R, V)$

$$B \times B \xrightarrow{\beta} R \xrightarrow{\phi} V \text{ is in } \mathrm{IBF}_{(R,k)}(B; V).$$

IBF is representable

$$\mathbf{IBF}(B) = (B \otimes_R B) \big/ \operatorname{span}_R\{ab \otimes c - a \otimes bc, ab \otimes c - b \otimes ca\}$$
$$\beta_{\text{uni}} \colon B \times B \to \mathbf{IBF}(B), \quad \beta_{\text{uni}}(a,b) = \overline{a \otimes b}$$

Crucial:

$$\operatorname{Hom}_k(\mathbf{IBF}(B), V) \xrightarrow{\simeq} \operatorname{IBF}_{(R,k)}(B; V), \quad \phi \mapsto \phi \circ \beta_{\text{uni}}$$

Goal: $\mathbf{IBF}(B) \cong R$ via $\overline{a \otimes b} \mapsto \beta(a,b)$ for a suitable $\beta \in \operatorname{IBF}(B)$. Then

$$\operatorname{Hom}_k(R, V) \xrightarrow{\simeq} \operatorname{IBF}_{(R,k)}(B; V), \quad \phi \mapsto \phi \circ \beta$$

E.g. $R^* = \operatorname{Hom}_k(R, k) \xrightarrow{\simeq} \operatorname{IBF}_{(R,k)}(B; k)$

Theorem

$B \mapsto \mathbf{IBF}(B)$ functor stable under base change: $\mathbf{IBF}(B \otimes_R S) \simeq \mathbf{IBF}(B) \otimes_R S$.

Theorem (Lie algebras)

Assume

- k *field of characteristic* 0
- $\mathfrak{a} = \mathfrak{g}$ *central-simple finite-dimensional Lie algebra over* k
- $R \in k$-**alg**
- B *Lie algebra over* R, $B \otimes_R S \simeq \mathfrak{g} \otimes_k S$ *for some f.f.* $S \in R$-**alg**
- V k-*module*

Then

$$Hom_k(R, V) \xrightarrow{\simeq} \mathrm{IBF}_{(R,k)}(B; V), \quad \phi \mapsto \phi \circ \kappa_B$$

($\kappa_B \colon B \times B \to R$ *Killing form of* B).

Example ($B = \mathfrak{g} \otimes_k R$, $V = k \implies B^* \simeq \mathrm{IBF}_{(R,k)}(B; k)$)

- Zusmanovich 1994 Lie algebra homology
- Montaner-Stolin-Zelmanov 2010 structure theory (construction of Lie bialgebras)

Kac, Exercise 2.5

A indecomposable (generalized Cartan) matrix, $\mathfrak{g}(A)$ associated Kac-Moody algebra. Show that any two invariant bilinear forms on $[\mathfrak{g}(A), \mathfrak{g}(A)]$ are proportional.

just do $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}]$ (any invariant form vanishes on the centre of $[\mathfrak{g}(A), \mathfrak{g}(A)]$ for A untwisted affine.)

$\mathrm{IBF}_{\mathbb{C}}\, B = \{\gamma\colon B \times B \to \mathbb{C}, \mathbb{C}\text{-bilinear}\} = \{\phi \circ \kappa_B : \phi \in \mathbb{C}[t^{\pm 1}]^*\} \simeq \mathbb{C}[t^{\pm 1}]^*$.
Proportional?

Definition (Graded objects)

Λ abelian group,

Graded algebra: $B = \bigoplus_{\lambda \in \Lambda} B_\lambda$, $[B_\lambda, B_\mu] \subset B_{\lambda+\mu}$ and $R_\lambda B_\mu \subset B_{\lambda+\mu}$.

Graded twisted form (everything graded): R Λ-graded, B Λ-graded, $S \in R$-**alg** is Λ-graded R-algebra, $B \otimes_R S \simeq \mathfrak{g} \otimes S$ graded isomorphism

Graded bilinear form $\beta\colon L \times L \to k$: $\beta(L_\lambda, L_\mu) = 0$ if $\lambda + \mu \neq 0$.

Theorem (Graded version)

Assume

- k *field of characteristic* 0
- $\mathfrak{g}$ *central-simple finite-dimensional Lie algebra over* k
- $R \in k$-**alg** *graded*
- B *graded Lie algebra over* R, $B \otimes_R S \simeq \mathfrak{g} \otimes_k S$ *for some graded f.f.* $S \in R$-**alg**

Then Killing form κ_B *is graded and every graded invariant bilinear form* γ *can be uniquely written as*

$$\gamma = \phi \circ \kappa_B, \quad \phi \in B_0^*$$

Examples

Graded invariant bilinear forms are proportional if $\dim B_0 = 1$

- B twisted loop algebra (Kac' exercise)
- Multiloop algebra: Graded invariant bilinear forms are proportional
  Yoshii 2006 for multiloop Lie tori using classification