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Steinberg groups for Jordan pairs

Groupes de Steinberg pour les paires de Jordan

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ABSTRACT

We announce results on projective elementary groups and on Steinberg groups associated to Jordan pairs V with a grading by a locally finite 3-graded root system Φ: The projective elementary group $\mathrm{PE}(V)$ of V is a group with Φ-commutator relations with respect to appropriately defined root subgroups. Under some mild additional conditions, the Steinberg group associated to $\mathrm{PE}(V)$ uniquely covers all central extensions of $\mathrm{PE}(V)$ and is the universal central extension of $\mathrm{PE}(V)$ if Φ is irreducible and has infinite rank.

RÉSUMÉ

Nous annonçons les résultats suivants relatifs aux groupes élémentaires projectifs et aux groupes de Steinberg associés aux paires de Jordan V munies d'une graduation par un système de racines Φ localement fini : Le groupe élémentaire projectif $\mathrm{PE}(V)$ est un groupe avec des relations de commutateurs de type Φ par rapport à certains sous-groupes radiciels. Sous des conditions additionnelles faibles, le groupe de Steinberg associé à $\mathrm{PE}(V)$ couvre de manière unique chaque extension centrale de $\mathrm{PE}(V)$ et il est l'extension centrale universelle de $\mathrm{PE}(V)$ si Φ est irréductible et de rang infini.

1. Introduction

It is known that over algebraically closed fields simple algebraic groups of type $\neq \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$ can be described in terms of simple finite-dimensional Jordan pairs [5,7]. One way of associating a group to a Jordan pair V is to consider the adjoint group of the Tits–Kantor–Koecher algebra $\mathfrak{g}(V)$ of V. This construction makes sense for any Jordan pair V over a ring and leads to the projective elementary group $\mathrm{PE}(V)$ of V [3,8,9]. The Jordan pair approach provides a unifying framework for classical groups over rings, like the linear elementary groups, the unitary elementary groups and the orthogonal elementary groups, which avoids case-by-case considerations usually employed in this area, for example in [4]. In this vein, we present here a type-independent theory of Steinberg groups associated to Jordan pairs.

More specifically, we start from a Jordan pair V graded by a 3-graded root system Φ and show that the projective elementary group $\mathrm{PE}(V)$ has Φ-commutator relations in the sense of [2]. Since our root systems are allowed to be infinite (but locally finite, as in [10]), we are able to deal with the infinite elementary groups and Steinberg groups directly, without having to pass to the limit. To $\mathrm{PE}(V)$ we associate a Steinberg group $\mathrm{St}(V)$, following a method of J. Tits for Kac–Moody

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groups [12]. Our main result asserts that $\mathrm{St}(V)$ is the universal central extension of the projective elementary group in case Φ is irreducible of infinite rank.

Our approach to Steinberg groups is substantially less computational than those in the existing literature. It introduces two completely new techniques in the area of Steinberg groups: the combinatorics of 3-graded root systems and the methods of Jordan pairs.

Throughout we work over an associative commutative unital ring k. Proofs will appear elsewhere.

2. The projective elementary group of a Jordan pair

Let V be a Jordan pair over k [6]. Thus, $V = (V^+, V^-)$ is a pair of k-modules equipped with a pair (Q_+, Q_-) of quadratic maps $Q_\sigma : V^\sigma \to \mathrm{Hom}_k(V^\sigma, V^{-\sigma})$, $\sigma = \pm$, satisfying certain identities in all base ring extensions of V.

We say that a Lie algebra L is 3-*graded* if $L = \bigoplus_{n\in\mathbb{Z}} L_n$ is $\mathbb{Z}$-graded with $L_n = \{0\}$ for $|n| > 1$. Every Jordan pair V gives rise to a 3-graded Lie algebra, the Tits–Kantor–Koecher algebra $\mathfrak{g}(V) = \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. We have $\mathfrak{g}_{\pm 1} = V^\pm$ as k-modules and $\mathfrak{g}_0 = k \cdot (\mathrm{Id}_{V^+}, -\mathrm{Id}_{V^-}) + \mathrm{InDer}(V)$, where $\mathrm{InDer}(V)$ is the inner derivation algebra of V [6, 3.12]. Since $(\mathrm{ad}\, x)^3 = 0$ for $x \in \mathfrak{g}_{\pm 1}$, one can "exponentiate" $\mathrm{ad}\, x$ for $x \in \mathfrak{g}_{\pm 1}$ (even when 2 is not invertible in k), by defining for $x, z \in \mathfrak{g}_{\pm 1}$, $h \in \mathfrak{g}_0$, $y \in \mathfrak{g}_{\mp 1}$:

$$e^{\mathrm{ad}\, x} \cdot z = z, \quad e^{\mathrm{ad}\, x} \cdot h = h + [x, h], \quad e^{\mathrm{ad}\, x} \cdot y = y + [x, y] + Q_\sigma(x) y.$$

The Jordan identities guarantee that $\exp_\pm(x) := e^{\mathrm{ad}\, x} \in \mathrm{Aut}(\mathfrak{g})$.

Finally one defines the *projective elementary group of* V by $\mathrm{PE}(V) = \langle \exp_+(V^+) \cup \exp_-(V^-) \rangle$, a subgroup of the automorphism group of $\mathfrak{g}$ [3,8,9].

3. Groups with commutator relations

A subset Φ of a real vector space X is a *locally finite root system* [10] if it satisfies the same axioms as finite root systems (see e.g. [1]), except that the finiteness condition is replaced by *local finiteness*: The intersection of Φ with every finite-dimensional subspace of X is finite. In particular, any finite root system is a locally finite root system. Locally finite root systems are direct sums of irreducible locally finite root systems, and irreducible locally finite root systems are either finite or the infinite analogues of the classical root systems of type A, B, C, D and BC.

Let Φ be a locally finite root system. For $\alpha, \beta \in \Phi$ put $(\!(\alpha, \beta)\!) = \Phi \cap (\mathbb{N}_+\alpha + \mathbb{N}_+\beta)$ where $\mathbb{N}_+$ denotes the positive natural numbers. A pair of roots (α, β) is called *nilpotent* if $p\alpha + q\beta \neq 0$ for all $p, q \in \mathbb{N}_+$. A *group with commutator relations of type* Φ is a group G together with a family $(U_\alpha)_{\alpha\in\Phi}$ of subgroups, called *root groups*, which generate G and for which the commutator relations $(\!(U_\alpha, U_\beta)\!) \subset U_{(\!(\alpha,\beta)\!)}$ hold for all nilpotent pairs (α, β). Here $U_{(\!(\alpha,\beta)\!)}$ is the subgroup of G generated by all U_γ, $\gamma \in (\!(\alpha, \beta)\!)$. In case Φ is not reduced, we also require that $\beta = n\alpha$ for $n \in \mathbb{N}$ implies $U_\beta \subset U_\alpha$.

Example. The elementary linear group $\mathrm{E}_n(R)$ and its stable version $\mathrm{E}(R)$ are groups with commutator relations for root systems of type A. The elementary unitary groups in the sense of [4, 5.3] are groups with commutator relations for root systems of type C.

4. Steinberg groups defined by groups with commutator relations

Let $(\bar{G}, (\bar{U}_\alpha)_{\alpha\in\Phi})$ be a group with Φ-commutator relations with respect to the family $(\bar{U}_\alpha)$, and let $(G, (U_\alpha))$ be another such group. Naturally, a morphism $\phi : (G, U_\alpha) \to (\bar{G}, \bar{U}_\alpha)$ is just a group homomorphism preserving root groups: $\phi(U_\alpha) \subset \bar{U}_\alpha$ for all $\alpha \in \Phi$. A *strong morphism* is a morphism $\phi : (G, U_\alpha) \to (\bar{G}, \bar{U}_\alpha)$ with the property that $\phi : U_{[\![\alpha,\beta]\!]} \to \bar{U}_{[\![\alpha,\beta]\!]}$ is bijective, for all nilpotent pairs (α, β), where we put $[\![\alpha, \beta]\!] := \{\alpha\} \cup (\!(\alpha, \beta)\!) \cup \{\beta\}$. Since the pair (α, α) is nilpotent and $[\![\alpha, \alpha]\!] = \{\alpha\}$ or $\{\alpha, 2\alpha\}$, a strong morphism satisfies that $\phi : U_\alpha \to \bar{U}_\alpha$ is bijective for all $\alpha \in \Phi$. Roughly speaking, this means that G has "the same" generators and commutator relations as $\bar{G}$. Adapting an argument of Tits [12], one proves that there is a largest such group G, more precisely:

Theorem 1. *Let Φ be a locally finite root system and let $(\bar{G}, (\bar{U}_\alpha)_{\alpha\in\Phi})$ be a group with Φ-commutator relations. Then there exists a group $(\hat{G}, \hat{U}_\alpha)_{\alpha\in\Phi}$ with Φ-commutator relations and a strong morphism $\pi : \hat{G} \to \bar{G}$ such that every strong morphism $\phi : G \to \bar{G}$ is obtained from π by taking a quotient: There exists a unique strong morphism $\psi : \hat{G} \to G$ such that $\pi = \phi \circ \psi$. The pair $(\hat{G}, \pi)$ is uniquely determined up to unique isomorphism by this property.*

We call $\hat{G}$ the *Steinberg group of* $\bar{G}$. This is justified since for example the classical Steinberg group $\mathrm{St}_n(R) = \hat{G}$ is obtained in this way from the elementary group $\bar{G} = \mathrm{E}_n(R)$. Similarly, the stable Steinberg group $\mathrm{St}(R)$ is $\hat{G}$ for $\bar{G} = \mathrm{E}(R)$, the infinite elementary group. The same approach works for the unitary Steinberg groups, as for example defined in [4, 5.5].

5. Commutator relations for PE(*V*)

A 3-*grading* of a locally finite root system Φ is a partition $\Phi = \Phi_{-1} \dot\cup \Phi_0 \dot\cup \Phi_1$ such that $(\Phi_i + \Phi_j) \cap \Phi \subset \Phi_{i+j}$ ($= \emptyset$ if $|i+j| > 1$), $\Phi_0 \subset \Phi_1 + \Phi_{-1}$ and $\Phi_{-1} = -\Phi_1$ ([11], [10, §§17, 18]). Since a 3-grading is uniquely determined by Φ_1, it makes sense to denote a 3-graded root system by (Φ, Φ_1). We note that 3-gradings exist for all locally finite irreducible reduced root systems except for the finite root systems of type G_2, F_4 or E_8.

Given a 3-graded root system (Φ, Φ_1) and a Jordan pair V, we define a (Φ, Φ_1)-*grading* Γ *of* V as a decomposition $\Gamma : V = \bigoplus_{\alpha \in \Phi_1} V_\alpha$ where the $V_\alpha = (V_\alpha^+, V_\alpha^-)$ are pairs of submodules and the direct sum is to be understood componentwise, such that for all $\alpha, \beta, \gamma \in \Phi_1$ and $\sigma \in \{+, -\}$ the following multiplication rules hold:

$$Q_\sigma(V_\alpha^\sigma)V_\beta^{-\sigma} \subset V_{2\alpha-\beta}^\sigma, \qquad \{V_\alpha^\sigma, V_\beta^{-\sigma}, V_\gamma^\sigma\}_\sigma \subset V_{\alpha-\beta+\gamma}^\sigma, \qquad \{V_\alpha^\sigma, V_\alpha^{-\sigma}, V_\beta^\sigma\}_\sigma = 0 \quad \text{for } \alpha \perp \beta.$$

Here $\{x, y, z\}_\sigma = Q_\sigma(x+z)y - Q_\sigma(x)y - Q_\sigma(z)(y)$.

Theorem 2. *Assume the Jordan pair V has a (Φ, Φ_1)-grading with homogeneous subspaces V_α. Then the projective elementary group $\bar G = \mathrm{PE}(V)$ has Φ-commutator relations with the following root groups*:

$$\bar U_{\pm\alpha} = \exp_\pm(V_\alpha^\pm) \quad \text{for } \alpha \in \Phi_1,$$
$$\bar U_\mu = \Big\langle \bigcup \{\beta(V_\alpha^+, V_\beta^-) \colon \alpha - \beta = \mu,\ \alpha, \beta \in \Phi_1\} \Big\rangle \quad \text{for } 0 \neq \mu \in \Phi_0.$$

Here $\beta(x, y)$ is the inner automorphism associated to the quasi-invertible pair (x, y), see [6, 3.9].

6. Steinberg groups for Jordan pairs

Combining Theorem 1 and Theorem 2, we now define: The *Steinberg group* $\mathrm{St}(V, \Gamma)$ of a Jordan pair V with a (Φ, Φ_1)-grading Γ is the Steinberg group of $\mathrm{PE}(V)$, considered as a group with Φ-commutator relations.

It is a classical result that the stable Steinberg group $\mathrm{St}(R)$ of a ring R is the universal central extension of the stable elementary group $\mathrm{E}(R)$. Hence it is natural to ask whether similar results are true for $\mathrm{St}(V, \Gamma)$. This is indeed the case, but one needs to make stronger assumptions on V than just having a (Φ, Φ_1)-grading. In the classical cases of linear and unitary Steinberg groups, R is always a ring with unit element. This yields so-called Weyl elements in the elementary groups. In the Jordan pair case, one uses idempotents [6, 5.1] to construct Weyl elements.

A (Φ, Φ_1)-grading Γ of V is called an *idempotent grading* if there exists a family $\mathcal{E} = (e_\alpha)_{\alpha \in \Phi_1}$ of idempotents with the following properties: (i) $e_\alpha \in V_\alpha$, and (ii) for all $\alpha, \beta \in \Phi_1$, we have $V_\beta \subset V_{\langle \beta, \alpha^\vee \rangle}(e_\alpha)$, where $\langle \beta, \alpha^\vee \rangle$ is the usual Cartan integer. For $\alpha, \beta \in \Phi_1$ we always have $\langle \beta, \alpha^\vee \rangle \in \{0, 1, 2\}$ so that $V_{\langle \beta, \alpha^\vee \rangle}(e_\alpha)$ makes sense (and is the corresponding Peirce space of e_α).

Actually, it suffices to require the existence of idempotents only for roots $\alpha \in \Phi_1^{id}$ where, for Φ irreducible, we define $\Phi_1^{id} = \Phi_1$ if Φ is simply-laced, $\Phi_1^{id} = \{\text{long roots}\}$ if Φ is of type B, and $\Phi_1^{id} = \{\text{short roots}\}$ if Φ is of type C. This generalization allows us to realize the unitary groups of [4] as Steinberg groups of appropriate Jordan pairs. The link to Weyl elements comes from the fact that an idempotent $e_\alpha = (e_\alpha^+, e_\alpha^-)$ gives rise to the Weyl element $w_\alpha = \exp_+(e_\alpha^+)\exp_-(e_\alpha^-)\exp_+(e_\alpha^+)$ in $\bar G = \mathrm{PE}(V)$ and with a similar definition in any group G over $\bar G$.

Theorem 3. *Let Γ be an idempotent (Φ, Φ_1)-grading of a Jordan pair V with Φ irreducible and different from* A_1 *and* B_2.

(a) $\mathrm{St}(V, \Gamma)$ *is perfect, i.e., equal to its commutator group.*
(b) *If Φ has rank* $\geqslant 5$ *then* $\mathrm{St}(V, \Gamma)$ *covers uniquely all central extensions of* $\mathrm{PE}(V)$.
(c) *If Φ has infinite rank then* $\mathrm{St}(V, \Gamma)$ *is the universal central extension of* $\mathrm{PE}(V)$.

Remarks. (a) The term "covers" means the following: If $\phi : G \to \mathrm{PE}(V)$ is any central extension of abstract groups (G need not have commutator relations!) then there exists a unique group homomorphism $\psi : \mathrm{St}(V, \Gamma) \to G$ satisfying $\pi = \phi \circ \psi$, where $\pi : \mathrm{St}(V) \to \mathrm{PE}(V)$ is the canonical strong morphism of Theorem 1. Observe that "covering" does not yet mean that $\mathrm{St}(V, \Gamma)$ is itself a central extension of $\mathrm{PE}(V)$.

(b) Parts (b) and (c) of this theorem were known before in the following cases:

(i) Φ of type A: Then V is a rectangular matrix pair over an associative algebra R, $\mathrm{PE}(V)$ is the usual projective elementary group of R, and $\mathrm{St}(V, \Gamma)$ coincides with the usual linear Steinberg group $\mathrm{St}_n(R)$ for $n = 1 + \operatorname{rank}\Phi$. In this case, the theorem follows from [4, 1.4.12 and 1.4.13].
(ii) Φ is of type C and V is a hermitian matrix pair: In this case, the projective elementary group of V coincides with the usual projective elementary unitary group, while $\mathrm{St}(V, \Gamma)$ is the usual unitary Steinberg group. In case $\operatorname{rank}\Phi = \infty$, the theorem is due to Sharpe and Bak [4, 5.5.10]. The case $5 \leqslant \operatorname{rank}\Phi < \infty$ does not seem to be explicitly stated in [4], but it follows from a suitably modified version of the proof of [4, 5.5.10], see [4, 5.5.11].

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