Graded-simple Lie algebras of type B_2 and Jordan systems covered by a triangle

Erhard Neher[1] Maribel Tocón[2]

Abstract

We announce a classification of graded-simple Jordan systems covered by a compatible triangle, under some natural assumptions on the abelian group, in order to get the corresponding classification of graded-simple Lie algebras of type B_2.

1 Introduction

Graded-simple Lie algebras which also have second compatible grading by a root system appear in the structure theory of extended affine Lie algebras, which generalize affine Lie algebras and toroidal Lie algebras. If the root system in question is 3-graded, these Lie algebras are Tits-Kantor-Koecher algebras of Jordan pairs covered by a grid.

In this note we will consider the case of the root system B_2. A centreless B_2-graded Lie algebra is the Tits-Kantor-Koecher algebra of a Jordan pair covered by a triangle. Such a Lie algebra is graded-simple with respect to a compatible Λ-grading if and only if the Jordan pair is graded-simple with respect to a Λ-grading which is compatible with the covering triangle

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[1] Address: Department of Mathematics and Statistics, University of Ottawa, 585 King Edward Ave., Ottawa, Ontario K1N 6N5, Canada. Email: neher@uottawa.ca. Supported in part by Discovery Grant #8836 of the Natural Sciences and Research Council of Canada

[2] This paper was written during the second author's stay at Department of Mathematics and Statistics, University of Ottawa, Canada. Current address: Departamento de Estadística, Econometría, Inv. Operativa y Org. Empresa, Facultad de Derecho, Puerta Nueva, 14071, Córdoba, Spain. Email: td1tobam@uco.es. Support from Discovery Grant #8836 of the NSRC of Canada, MEC and fondos FEDER (Becas posdoctorales and MTM2004-03845), and Junta de Andalucía (FQM264) is gratefully acknowledged.

[8]. In [10] we give a classification of graded-simple Jordan systems covered by a triangle that is compatible with the grading, under some natural assumptions on the abelian group, as well as the corresponding classification of graded-simple Lie algebras of type B_2. Our work generalizes earlier results of Allison-Gao [1] and Benkart-Yoshii [2], and is an extension of the structure theory of simple Jordan pairs and Jordan triple systems covered by a triangle due to McCrimmon-Neher [7].

The aim of this note is to provide an outline of our results for Jordan systems. The details of proofs will appear in [10]. For unexplained notation we refer the reader to [3] and [4].

2 Graded-simple Lie algebras of type B_2

The motivation for our research is two-fold. On the one hand, we would like to advance the theory of graded Jordan structures, and on the other hand we are interested in certain types of root-graded Lie algebras. In this section we will describe the second part of our motivation and how it is related to the first.

Let R be a reduced root system. In the following only the case $R = \mathrm{B}_2$ will be of interest, but the definition below works for any finite, even locally finite reduced root system. We will assume that $0 \in R$ and denote by $\mathcal{Q}(R) = \mathbb{Z}[R]$ the root lattice of R. We will consider Lie algebras defined over a ring of scalars k containing $\frac{1}{2}$ and $\frac{1}{3}$. Let Λ be an arbitrary abelian group.

Definition 2.1. ([9]) A Lie algebra L over k is called (R, Λ)-*graded* if

(1) L has a compatible $\mathcal{Q}(R)$- and Λ-gradings,

$$L = \oplus_{\lambda\in\Lambda} L^\lambda \quad \text{and} \quad L = \oplus_{\alpha\in\mathcal{Q}(R)} L_\alpha,$$

i.e., using the notation $L^\lambda_\alpha = L^\lambda \cap L_\alpha$ we have

$$L_\alpha = \oplus_{\lambda\in\Lambda} L^\lambda_\alpha, \quad L^\lambda = \oplus_{\alpha\in\mathcal{Q}(R)} L^\lambda_\alpha, \quad \text{and} \quad [L^\lambda_\alpha, L^\kappa_\beta] \subseteq L^{\lambda+\kappa}_{\alpha+\beta},$$

for $\lambda, \kappa \in \Lambda$, $\alpha, \beta \in \mathcal{Q}(R)$,

(2) $\{\alpha \in \mathcal{Q}(R) : L_\alpha \neq 0\} \subseteq R$,

(3) $L_0 = \sum_{0\neq\alpha\in R} [L_\alpha, L_{-\alpha}]$, and

(4) for every $0 \neq \alpha \in R$ the homogeneous space L^0_α contains an element e_α that is *invertible*, i.e., there exists $f_{-\alpha} \in L^0_{-\alpha}$ such that $h_\alpha := [e_\alpha, f_{-\alpha}]$ acts on L_β, $\beta \in R$, by

$$[h_\alpha, x_\beta] = \langle \beta, \alpha^\vee \rangle x_\beta, \quad x_\beta \in L_\beta.$$

In particular, $(e_\alpha, h_\alpha, f_\alpha)$ is an $\mathfrak{sl}_2$-triple.

An (R, Λ)-graded Lie algebra is said to be *graded-simple* if it does not contain proper nontrivial Λ-graded ideals and *graded-division* if every nonzero element in L^λ_α, $\alpha \neq 0$, is invertible.

Let now L be a centerless $(B_2, \{0\})$-graded Lie algebra. It then follows from [8] that L is the Tits-Kantor-Koecher algebra of a Jordan pair V *covered by a triangle*: i.e.,

$$V = V_1 \oplus M \oplus V_2,$$

where $V_i = V_2(e_i)$, $i = 1, 2$, and $M = V_1(e_1) \cap V_1(e_2)$, for a triangle $(u; e_1, e_2)$. Recall that a triple $(u; e_1, e_2)$ of nonzero idempotents of V is a *triangle* if

$$e_i \in V_0(e_j),\ i \neq j, \quad e_i \in V_2(u),\ i = 1, 2, \quad \text{and} \quad u \in V_1(e_1) \cap V_1(e_2),$$

and the following multiplication rules hold for $\sigma = \pm$:

$$Q(u^\sigma)e_i^{-\sigma} = e_j^\sigma,\ i \neq j, \quad \text{and} \quad Q(e_1^\sigma, e_2^\sigma)u^{-\sigma} = u^\sigma.$$

If moreover, L is (B_2, Λ)-graded, then V is also Λ-graded, i.e., as k-module $V^\sigma = \bigoplus_{\lambda \in \Lambda} V^\sigma[\lambda]$, $\sigma = \pm$, with

$$Q(V^\sigma[\lambda])V^{-\sigma}[\mu] \subseteq V^\sigma[2\lambda+\mu] \quad \text{and} \quad \{V^\sigma[\lambda], V^{-\sigma}[\mu], V^\sigma[\nu]\} \subseteq V^\sigma[\lambda+\mu+\nu]$$

for all $\lambda, \mu, \nu \in \Lambda$, $\sigma = \pm$. A Jordan pair that is Λ-graded and covered by a triangle which lies in the homogeneous 0-space $V[0]$ is called Λ-*triangulated*. It therefore follows from the above that if L is a centerless (B_2, Λ)-graded Lie algebra, then L is the Tits-Kantor-Koecher algebra of a Λ-triangulated Jordan pair V. Moreover, L is graded-simple if and only if V is graded-simple.

Therefore, one can get a description of graded-simple (B_2, Λ)-graded Lie algebras from the corresponding classification of graded-simple Λ-triangulated Jordan pairs. However, the classification of graded-simple Λ-triangulated Jordan pairs is only known for $\Lambda = \{0\}$ [7]. In what follows, we extend this classification to more general Λ. In doing so, we work with Jordan structures over arbitrary rings of scalars k. This generality is of independent interest from the point of view of Jordan theory. Moreover, the simplifications that would arise from assuming $\frac{1}{2}$ and $\frac{1}{3} \in k$ are minimal, e.g., we could avoid working with ample subspaces in our two basic examples 3.1 and 3.2.

3 Graded-simple triangulated Jordan triple systems

Let k be an arbitrary ring of scalars and let J be a Jordan triple system over k. Recall that a triple of nonzero tripotents $(u; e_1, e_2)$ is called a *triangle* if $e_i \in J_0(e_j)$, $i \neq j$, $e_i \in J_2(u)$, $i = 1, 2$, $u \in J_1(e_1) \cap J_1(e_2)$, and the following multiplication rules hold: $P(u)e_i = e_j$, $i \neq j$, and $P(e_1, e_2)u = u$. In this case, $e := e_1+e_2$ is a tripotent such that e and u have the same Peirce spaces. A Jordan triple system with a triangle $(u; e_1, e_2)$ is said to be *triangulated* if $J = J_2(e_1) \oplus \big(J_1(e_1) \cap J_1(e_2)\big) \oplus J_2(e_2)$ which is equivalent to $J = J_2(e)$. In this case, we will use the notation $J_i = J_2(e_i)$ and $M = J_1(e_1) \cap J_1(e_2)$. Hence

$$J = J_1 \oplus M \oplus J_2.$$

Note that $^* := P(e)P(u) = P(u)P(e)$ is an automorphism of J of period 2 such that $u^* = u$, $e_i^* = e_j$, and so $J_i^* = J_j$.

Let Λ be an abelian group. We say that J is Λ-*graded* if the underlying module is Λ-graded, say $J = \bigoplus_{\lambda\in\Lambda} J^\lambda$, and the family $(J^\lambda : \lambda \in \Lambda)$ of k-submodules satisfies $P(J^\lambda)J^\mu \subseteq J^{2\lambda+\mu}$ and $\{J^\lambda, J^\mu, J^\nu\} \subseteq J^{\lambda+\mu+\nu}$ for all $\lambda, \mu, \nu \in \Lambda$. We call J Λ-*triangulated* if it is Λ-graded and triangulated by $(u; e_1, e_2) \subseteq J^0$ and *faithfully* Λ-*triangulated* if any $x_1 \in J_1$ with $x_1 \cdot u = 0$ vanishes, where the product $\cdot$ is defined as follows:

$$J_i \times M \to M : (x_i, m) \mapsto x_i \cdot m = L(x_i)m := \{x_i, e_i, m\}.$$

There are two basic models for Λ-triangulated Jordan triple systems:

Example 3.1. *Λ-triangulated hermitian matrix systems $H_2(A, A_0, \pi, ^-)$. A Λ-graded (associative) coordinate system $(A, A_0, \pi, ^-)$ consists of a unital associative Λ-graded k-algebra $A = \bigoplus_{\lambda\in\Lambda} A^\lambda$, a graded submodule $A_0 = \bigoplus_{\lambda\in\Lambda} A_0^\lambda$ for $A_0^\lambda = A_0 \cap A^\lambda$, an involution π and an automorphism $^-$ of period 2 of A. These data satisfy the following conditions: π and $^-$ commute and are both of degree 0, i.e., $(A^\lambda)^\pi = A^\lambda = \overline{A^\lambda}$ for all $\lambda \in \Lambda$, A_0 is $^-$-stable and π-ample in the sense that $\overline{A_0} = A_0 \subseteq H(A, \pi)$, $1 \in A_0$ and $aa_0a^\pi \subseteq A_0$ for all $a \in A$ and $a_0 \in A_0$.*

To a Λ-graded coordinate system $(A, A_0, \pi, ^-)$ we associate the Λ-triangulated hermitian matrix system $H = H_2(A, A_0, \pi, ^-)$ which, by definition, is the Jordan triple system of 2×2-matrices over A which are hermitian $(X = X^{\pi t})$ and have diagonal entries in A_0, with triple product $P(X)Y = X\overline{Y}^{\pi t}X = X\overline{Y}X$. This system is clearly Λ-graded: $H = \bigoplus_{\lambda\in\Lambda} H^\lambda$, where

$H^\lambda = \operatorname{span}\{a_0^\lambda E_{ii}, a^\lambda E_{12} + (a^\lambda)^\pi E_{21} : a_0^\lambda \in A_0^\lambda, a^\lambda \in A^\lambda, i = 1,2\}$, and Λ-triangulated by $(u = E_{12} + E_{21}; e_1 = E_{11}, e_2 = E_{22}) \subseteq H^0$.

One can prove that $H_2(A, A_0, \pi, ^-)$ is graded-simple if and only if $(A, \pi, ^-)$ is graded-simple. In this case, $H_2(A, A_0, \pi, ^-)$ is graded isomorphic to one of the following:

(I) $H_2(A, A_0, \pi, ^-)$ for a graded-simple associative unital A;

(II) $\operatorname{Mat}_2(B)$ for a graded-simple associative unital B with graded automorphism $^-$, where $\overline{(b_{ij})} = (\overline{b_{ij}})$ for $(b_{ij}) \in \operatorname{Mat}_2(B)$ and $P(x)y = x\overline{y}x$;

(III) $\operatorname{Mat}_2(B)$ for a graded-simple associative unital B with graded involution ι, where $\overline{(b_{ij})} = (b_{ij}^\iota)$ for $(b_{ij}) \in \operatorname{Mat}_2(B)$ and $P(x)y = x\overline{y}^t x$;

(IV) polarized $H_2(B, B_0, \pi) \oplus H_2(B, B_0, \pi)$ for a graded-simple B with graded involution π;

(V) polarized $\operatorname{Mat}_2(B) \oplus \operatorname{Mat}_2(B)$ for a graded-simple associative unital B and $P(x)y = xyx$.

The examples (IV) and (V) are special cases of polarized Jordan triple systems. Recall that a Jordan triple system T is called *polarized* if there exist submodules $T^\pm$ such that $T = T^+ \oplus T^-$ and for $\sigma = \pm$ we have $P(T^\sigma)T^\sigma = 0 = \{T^\sigma, T^\sigma, T^{-\sigma}\}$ and $P(T^\sigma)T^{-\sigma} \subseteq T^\sigma$. In this case, $V = (T^+, T^-)$ is a Jordan pair. Conversely, to any Jordan pair $V = (V^+, V^-)$ we can associate a polarized Jordan triple system $T(V) = V^+ \oplus V^-$ with quadratic map P defined by $P(x)y = Q(x^+)y^- \oplus Q(x^-)y^+$ for $x = x^+ \oplus x^-$ and $y = y^+ \oplus y^-$. In particular, for any Jordan triple system T the pair (T, T) is a Jordan pair and hence it has an associated polarized Jordan triple system which we denote $T \oplus T$. It is clear that if T is a Λ-triangulated Jordan triple system then so is $T \oplus T$.

Example 3.2. *Λ-triangulated ample Clifford systems $AC(q, S, D_0)$.* This example is a subtriple of a full Clifford system which we will define first. It is given in terms of

(i) a Λ-graded unital commutative associative k-algebra $D = \bigoplus_{\lambda\in\Lambda} D^\lambda$ endowed with an involution $^-$ of degree 0,

(ii) a Λ-graded D-module $M = \bigoplus_{\lambda\in\Lambda} M^\lambda$,

(iii) a Λ-graded D-quadratic form $q : M \to D$, hence $q(M^\lambda) \subset D^{2\lambda}$ and $q(M^\lambda, M^\mu) \subset M^{\lambda+\mu}$,

(iv) a hermitian isometry $S : M \to M$ of q of order 2 and degree 0, i.e., $S(dx) = \overline{d}S(x)$ for $d \in D$, $q\big(S(x)\big) = \overline{q(x)}$, $S^2 = Id$ and $S(M^\lambda) = M^\lambda$, and

(v) $u \in M^0$ with $q(u) = 1$ and $S(u) = u$.

Given these data, we define

$$V := De_1 \oplus M \oplus De_2,$$

where $De_1 \oplus De_2$ is a free Λ-graded D-module with basis (e_1, e_2) of degree 0. Then V is a Jordan triple system, called a *full Clifford system* and denoted by $FC(q, S)$, with respect to the product

$$P(c_1e_1 \oplus m \oplus c_2e_2)\,(b_1e_1 \oplus n \oplus b_2e_2) = d_1e_1 \oplus p \oplus d_2e_2, \quad \text{where}$$

$$\begin{aligned} d_i &= c_i^2\,\overline{b_i} + c_i\,q\big(m, S(n)\big) + \overline{b_j}\,q(m) \\ p &= [c_1\overline{b_1} + c_2\overline{b_2} + q\big(m, S(n)\big)]m + [c_1c_2 - q(m)]S(n). \end{aligned}$$

It is easily seen that $FC(q, S)$ is Λ-triangulated by $(u; e_1, e_2)$.

But in general we need not take the full Peirce spaces De_i in order to get a Λ-triangulated Jordan triple system. Indeed, let us define a *Clifford-ample subspace of* $(D, {}^-, q)$ as above as a Λ-graded k-submodule $D_0 = \bigoplus_{\lambda\in\Lambda}(D_0 \cap D^\lambda)$ such that $D_0 = \overline{D_0}$, $1 \in D_0$ and $D_0q(M) \subseteq D_0$. Then

$$AC(q, S, D_0) := D_0\, e_1 \oplus M \oplus D_0\, e_2,$$

also denoted $AC(q, M, S, D, {}^-, D_0)$ if more precision is helpful, is a Λ-graded subsystem of the full Clifford system $FC(q, S)$ which is triangulated by $(u; e_1, e_2)$. It is called a *Λ-triangulated ample Clifford system.*

One can prove that $AC(q, S, D_0)$ is graded-simple if and only if q is graded-nondegenerate (in the obvious sense) and $(D, {}^-)$ is graded-simple. In this case, either $D = F$ is a graded-division algebra or D is graded isomorphic to $F \boxplus F$ for a graded-division algebra F with ${}^-$ the exchange automorphism. In the latter case $AC(q, S, D_0) = AC(q, S, F_0) \oplus AC(q, S, F_0)$ is polarized with a Clifford ample subspace $F_0 \subseteq F$.

It is an important fact that one can find the above two examples of Λ-triangulated Jordan triple systems inside any faithfully Λ-triangulated Jordan triple system. More precisely, let $J = J_1 \oplus M \oplus J_2$ be faithfully Λ-triangulated by $(u; e_1, e_2)$, put $C_0 = L(J_1)$ and let C be the subalgebra of $\mathrm{End}_k(M)$ generated by C_0. Then C is naturally Λ-graded, $c \mapsto \overline{c} = P(e) \circ c \circ P(e)$ is an automorphism of C of degree 0 and $L(x_1) \cdots L(x_n) \mapsto (L(x_1) \cdots L(x_n))^\pi = L(x_n) \cdots L(x_1)$ induces a (well-defined) involution of C of degree 0. One can prove that the Λ-graded subsystem

$$J_h = J_1 \oplus Cu \oplus J_2$$

is Λ-triangulated by $(u; e_1, e_2)$ and graded isomorphic to $H_2(A, A_0, \pi, ^-)$ under the map

$$x_1 \oplus cu \oplus y_2 \mapsto \begin{pmatrix} L(x_1) & c \\ c^\pi & L(y_2^*) \end{pmatrix}$$

for $A = C|_{Cu}$, $A_0 = C_0|_{Cu}$. Moreover, J has a Λ-graded subsystem

$$J_q = K_1 \oplus N \oplus K_2$$

for appropriately defined submodules $K_i \subset J_i$ and $N \subset M$, which is Λ-triangulated by $(u; e_1, e_2)$ and graded isomorphic to $AC(q, S, D_0)$ under the map

$$x_1 \oplus n \oplus x_2 \mapsto L(x_1) \oplus n \oplus L(x_2^*)$$

where $D_0 = L(K_1)$, D is the subalgebra of $\mathrm{End}_k(N)$ generated by D_0, $q(n) = L(P(n)e_2)$ and $S(n) = P(e)n$. (Roughly speaking, J_q is the biggest graded subsystem of J where the identity $(x_1 - x_1^*) \cdot N \equiv 0$ holds). Moreover, the two isomorphisms above map the triangle $(u; e_1, e_2)$ of J onto the standard triangle of $H_2(A, A_0, \pi, ^-)$ or $AC(q, S, D_0)$, respectively. We note that for $\Lambda = \{0\}$ these two partial coordinatization theorems were proven in [7].

A question that arises naturally is the following: *Let J be faithfully Λ-triangulated by $(u; e_1, e_2)$. When is J_h or J_q the whole J?*

(i) If $M = Cu$, then $J = J_h$ and thus J is graded isomorphic to a hermitian matrix system,

(ii) If u is C-faithful and $(x_1 - x_1^*) \cdot m = 0$ for all $x_1 \in J_1$ and $m \in M$, then $J = J_q$ and thus J is graded isomorphic to an ample Clifford system.

One can show that (ii) holds whenever C is commutative and $^-$-simple. In fact, (i) or (ii) above holds if $(C, \pi, ^-)$ is graded-simple.

Proposition 3.3. *Let J be a graded-simple Λ-triangulated Jordan triple system satisfying one of the following conditions*

(a) *every $m \in M$ is a linear combination of invertible homogeneous elements, or*

(b) Λ *is torsion-free.*

Then $(C, \pi, ^-)$ *is graded-simple. In this case,* u *is* C*-faithful and* $M = Cu$ *or* C *is commutative.*

All together, we have the following result:

Theorem 3.4. *A graded-simple* Λ*-triangulated Jordan triple system satisfying* (a) *or* (b) *of* Prop. 3.3 *is graded isomorphic to one of the following: a non polarized Jordan triple system*

(I) $H_2(A, A_0, \pi, ^-)$ *for a graded-simple* A *with graded involution* π *and automorphism* $^-$*;*

(II) $\mathrm{Mat}_2(B)$ *with* $P(x)y = x\overline{y}x$ *for a graded-simple associative unital* B *with graded automorphism* $^-$ *and* $\overline{(y_{ij})} = (\overline{y_{ij}})$ *for* $(y_{ij}) \in \mathrm{Mat}_2(B)$*;*

(III) $\mathrm{Mat}_2(B)$ *with* $P(x)y = x\overline{y}^t x$ *for a graded-simple associative unital* B *with graded involution* ι *and* $\overline{(y_{ij})} = (y_{ij}^\iota)$ *for* $(y_{ij}) \in \mathrm{Mat}_2(B)$*;*

(IV) $AC(q, S, F_0)$ *for a graded-nondegenerate* q *over a graded-division* F *with Clifford-ample subspace* F_0*;*

or a polarized Jordan triple system

(V) $H_2(B, B_0, \pi) \oplus H_2(B, B_0, \pi)$ *for a graded-simple* B *with graded involution* π*;*

(VI) $\mathrm{Mat}_2(B) \oplus \mathrm{Mat}_2(B)$ *for a graded-simple associative unital* B *with* $P(x)y = xyx$*;*

(VII) $AC(q, S, F_0) \oplus AC(q, S, F_0)$ *for* $AC(q, S, F_0)$ *as in* (IV).

Conversely, all Jordan triple systems in (I)-(VII) *are graded-simple* Λ*-triangulated.*

Since $\Lambda = \{0\}$ is a special case of our assumption (b), the theorem above generalizes [7, Prop. 4.4].

4 Graded-simple triangulated Jordan pairs and algebras

We consider Jordan algebras and Jordan pairs over arbitrary rings of scalars. In order to apply our results, we will view Jordan algebras as Jordan triple systems with identity elements. Thus, to a Jordan algebra J we associate the Jordan triple system $T(J)$ defined on the k-module J with Jordan triple product $P_x y = U_x y$. The element $1_J \in J$ satisfies $P(1_J) = \mathrm{Id}$. Conversely, every Jordan triple system T containing an element $1 \in T$ with $P(1) = \mathrm{Id}$ is a Jordan algebra with unit element 1 and multiplication $U_x y = P_x y$. A Λ-graded Jordan algebra J is called *Λ-triangulated* by $(u; e_1, e_2)$ if $e_i = e_i^2 \in J^0$, $i = 1, 2$, are supplementary orthogonal idempotents and $u \in J_1(e_1)^0 \cap J_1(e_2)^0$ with $u^2 = 1$ and $u^3 = u$. Thus, with our definition of a triangle in a Jordan algebra, J is Λ-triangulated by $(u; e_1, e_2)$ iff $T(J)$ is Λ-triangulated by $(u; e_1, e_2)$.

This close relation to Λ-triangulated Jordan triple systems also indicates how to get examples of Λ-triangulated Jordan algebras: We take a Jordan triple system which is Λ-triangulated by $(u; e_1, e_2)$ and require $P(e) = \mathrm{Id}$ for $e = e_1 + e_2$. Doing this for our two basic examples 3.1 and 3.2, yields the following examples of Λ-triangulated Jordan algebras.

(A) *Hermitian matrix algebra*: This is the Jordan triple system $H_2(A, A_0, \pi, {}^-)$ with ${}^- = \mathrm{Id}$, which we will write as $H_2(A, A_0, \pi)$. Note that this is a Jordan algebra with product $U(x)y = P(x)y = xyx$ and identity element $E = E_{11} + E_{22}$. If, for example, $A = B \boxplus B^{\mathrm{op}}$ with π the exchange involution, then $H_2(A, A_0, \pi)$ is graded isomorphic to $\mathrm{Mat}_2(B)$ where $\mathrm{Mat}_2(B)$ is the Jordan algebra with product $U_x y = xyx$.

(B) *Quadratic form Jordan algebra*: This is the ample Clifford system $AC(q, S, D, {}^-, D_0)$ with ${}^- = \mathrm{Id}$ and $S|_M = \mathrm{Id}$. Since then $P(e) = \mathrm{Id}$ we get indeed a Λ-triangulated Jordan algebra denoted $AC_{\mathrm{alg}}(q, D, D_0)$. Note that this Jordan algebra is defined on $D_0 e_1 \oplus M \oplus D_0 e_2$ and has product $U_x y = q(x, \tilde{y})x - q(x)\tilde{y}$ where $q(d_1 e_1 \oplus m \oplus d_2 e_2) = d_1 d_2 - q(m)$ and $(d_1 e_1 \oplus m \oplus d_2 e_2)\tilde{} = d_2 e_1 \oplus -m \oplus d_1 e_1$. (If $\frac{1}{2} \in k$ it is therefore a reduced spin factor in the sense of [6, II, §3.4].)

From the classification given in Th. 3.4 we get:

Theorem 4.1. *A graded-simple Λ-triangulated Jordan algebra satisfying*

- (a) *every $m \in M$ is a linear combination of invertible homogeneous elements of M, or*
- (b) *Λ is torsion-free,*

is graded isomorphic to one of the following Jordan algebras:

- (I) *$H_2(A, A_0, \pi)$ for a graded-simple A with graded involution π;*

(II) $\mathrm{Mat}_2(B)$ *for a graded-simple associative unital* B*;*

(III) $AC_{\mathrm{alg}}(q, F, F_0)$ *for a graded-nondegenerate* $q : M \to F$ *over a commutative graded-division algebra* F *and a Clifford-ample subspace* F_0*.*

Conversely, all Jordan algebras in (I)–(III) *are graded-simple* Λ*-triangulated.*

Note that for $\Lambda = \{0\}$ this theorem generalizes the well known Capacity Two Theorem for Jordan algebras.

With the above algebra classification at hand and taking into account that a Λ-triangulated Jordan pair can be viewed as a disguised Λ-triangulated Jordan algebra, we get the following classification of Λ-triangulated Jordan pairs:

Theorem 4.2. *A graded-simple* Λ*-triangulated Jordan pair satisfying*

(a) *every* $m \in M^\sigma$ *is a linear combination of invertible homogeneous elements of* M^σ*, or*

(b) Λ *is torsion-free,*

is graded isomorphic to a Jordan pair (J, J) *where*

(I) $J = H_2(A, A_0, \pi)$ *is the hermitian matrix algebra of a graded-simple* A *with graded involution* π*;*

(II) $J = \mathrm{Mat}_2(B)$ *for a graded-simple associative unital* B*;*

(III) $J = AC(q, \mathrm{Id}, F_0)$ *for a graded-nondegenerate* q *over a graded-division algebra* F *with Clifford-ample subspace* F_0*.*

Conversely, all Jordan pairs described above are graded-simple Λ*-triangulated.*

Note that a Jordan pair V satisfies assumption (a) if it is the Jordan pair associated to a graded-division (B_2, Λ)-graded Lie algebra for an arbitrary Λ (cf. §1).

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