An introduction to extended affine Lie algebras

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Like Kac-Moody Lie algebras, extended affine Lie algebras are a generalization of affine Lie algebras and of simple finite-dimensional complex Lie algebras. But contrary to arbitrary Kac-Moody algebras, extended affine Lie algebras have concrete realizations. In this report we will give some rough ideas about their structure theory. The main examples of extended affine Lie algebras are toroidal Lie algebras.

0. A pivotal example: Untwisted affine Lie algebras. All algebras will be over a field F of characteristic 0. Let $\mathfrak{g}$ be a split simple Lie algebra $\mathfrak{g}$, let $L = \mathfrak{g} \otimes F[t^{\pm 1}]$ be the associated loop algebra, $K = L \oplus Fc$ its universal central extension, and put $E = K \oplus Fd$ (semidirect product), where d is the degree derivation of K sending $x \otimes t^n$ to $n(x \otimes t^n)$ and annihilating Fc. We thus have a diagram of 3 related Lie algebras (see below), where E or sometimes even K is referred to as an untwisted affine Kac-Moody Lie algebra. In the theory of extended affine Lie algebras, this diagram is generalized by replacing the loop algebra L by a centreless Lie torus, the algebra K by a central extension of L, which is also a Lie torus, and E by an extended affine Lie algebra, abbreviated EALA:

$$\begin{array}{lcl} K & \hookrightarrow & E \\ \downarrow & & \\ L & & \end{array} \qquad \rightsquigarrow \qquad \begin{array}{lcl} \text{Lie torus } K & \hookrightarrow & \text{EALA } E \\ \downarrow & & \\ \text{centreless} & & \\ \text{Lie torus } L & & \end{array}$$

1. Extended affine Lie algebras. Due to space limitations we will not give the precise definitions. Rather the reader is referred to [1] for a definition of an EALA over $\mathbb{C}$ and to [14] for a definition over a field F of characteristic 0. Closely related Lie algebras are considered in [7] and [8]. While these definitions do not agree in general, not even over $F = \mathbb{C}$, the following most important features (EA1)-(EA3) of an extended affine Lie algebra E are present in all approaches.

(EA1): E has a nondegenerate invariant symmetric bilinear form $(.|.)$.

(EA2): E contains a nontrivial finite-dimensional self-centralizing and ad-diagonalizable subalgebra H.

To prepare the axiom (EA3) note that by (EA2) the algebra E has a root space decomposition $E = \bigoplus_{\xi \in H^*} E_\xi$ with $E_0 = H$, where, as usual, $E_\xi = \{e \in E : [h,e] = \xi(h)e \text{ for all } h \in H\}$. The invariance of $(.|.)$ implies that $(E_\xi | E_\zeta) = 0$ for $\xi + \zeta \neq 0$. It follows that $(.|.)$ restricted to $H \times H$ is nondegenerate. Hence, every $\xi \in H^*$ is represented by a unique $t_\xi \in H$ via $(t_\xi | h) = \xi(h)$ for all $h \in H$. The subalgebra E_c of E, generated by $\{E_\xi : (t_\xi | t_\xi) \neq 0\}$ is called the *core of* E.

(EA3): For $\xi \in R^{\mathrm{an}}$ and $x_\xi \in E_\xi$, the endomorphism $\operatorname{ad} x_\xi \in \operatorname{End}_F E$ is locally nilpotent.

Depending on the authors, several other axioms are added to (EA1)–(EA3). They mostly concern the nature of the root system R of an EALA.

2. Lie tori. As indicated in the diagram above, the loop algebra in the construction of an untwisted affine Lie algebra is replaced by a Lie torus. The essential properties of a Lie torus L are the following: L has two compatible gradings, one by the abelian group $\mathbb{Z}^n$ and one by the root lattice of a finite irreducible root system Δ. With respect to the second grading, the only non-zero homogeneous spaces of L have degrees in $\Delta \cup \{0\}$. In addition, one requires the existence of enough $\mathfrak{s}l_2$-triples. The precise definition is given in [13], see [16] for a different approach.

Due to the efforts of many people, one now has a complete and precise classification of centreless Lie tori: [10] for $\Delta = \mathrm{A}_l$, $l \geq 3$, and $\Delta = \mathrm{D}_l$, $l \geq 4$ and $\Delta = \mathrm{E}_l$, $l = 6, 7, 8$; [11] for $\Delta = \mathrm{A}_2$; [15] for $\Delta = \mathrm{A}_1$; [3] for Δ reduced, but not simply-laced; [6] and [5] for $\Delta = \mathrm{BC}_1$; [12] for $\Delta = \mathrm{BC}_2$; and finally [4] for $\Delta = \mathrm{BC}_l, l \geq 3$. For example, if $\mathfrak{g}$ is a split simple finite-dimensional Lie algebra of type Δ the associated multiloop algebra $L = \mathfrak{g} \otimes F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is always a centreless Lie torus. If Δ is of type D or E, this covers all possibilities. However, already for Δ of type A, more general algebras do occur. Many of them are related to quantum tori. It is amazing that in order to classify Lie tori one needs all important classes of nonassociative algebras, namely alternative algebras for $\Delta = \mathrm{A}_2$, Jordan algebras for $\Delta = \mathrm{A}_1$, structurable algebras for $\Delta = \mathrm{BC}_1, \mathrm{BC}_2$, and a mixture of these for the other types.

The first step in understanding the structure of extended affine Lie algebras is the following.

3. Proposition. ([1, 3, 14, 17]) *Let E be an extended affine Lie algebra. Then its core E_c is a Lie torus, and $E_c/Z(E_c)$ is a centreless Lie torus.*

The proposition above begs the question: Does every centreless Lie torus L arise from an extended affine Lie algebra E? If yes, how can one reconstruct E from L? These questions will be answered in Th. 4 below. As can already be seen from the papers [10] and [11] there are in general infinitely many extended affine Lie algebras associated to a given centreless Lie torus.

4. Construction. Let L be a centreless Lie torus. As explained above, L is $\mathbb{Z}^n$-graded, say $L = \bigoplus_{\lambda \in \mathbb{Z}^n} L^\lambda$. Let ∂_i, $1 \leq i \leq n$ be the ith degree derivation, $\partial_i x = \lambda_i x$ for $x \in L^\lambda$, $\lambda = (\lambda_1, \dots, \lambda_n)$ and let $\mathcal{D} = \mathrm{span}_F\{\partial_1, \dots, \partial_n\}$. Also, let $C = \{\chi \in \mathrm{End}_F L : [\chi, \mathrm{ad}\, x] = 0 \text{ for all } x \in L\}$ be the centroid of L. It is shown in [9] that $C = \bigoplus_{\xi \in \Xi} C^\xi$ is graded by a subgroup Ξ of $\mathbb{Z}^n$. We note that $C\mathcal{D}$ is a subalgebra of derivations, the so-called *skew-centroidal derivations* of L. It is also known that L has an essentially unique invariant nondegenerate symmetric bilinear form $(.|.)$ [17]. We denote by $\mathrm{SCDer}L$ the subalgebra of $C\mathcal{D}$ consisting of the skew-symmetric derivations in $C\mathcal{D}$. It is a Ξ-graded algebra.

As a second ingredient of our construction, let $D = \bigoplus_{\xi \in \Xi} D^\xi$ be a graded subalgebra of SCDerL with the property that D^0 induces the $\mathbb{Z}^n$-grading of L, i.e., the L^λ are the joint eigenspaces of D^0. The graded dual $D^{\mathrm{gr}*}$ is canonically a D-module. Associated to D and the bilinear form $(.|.)$ is a 2-cocycle $\sigma : L \times L \to D^{\mathrm{gr}*}$, given by $\sigma(x, y)(d) = (dx|y)$ for $x, y \in L$ and $d \in D$. Thus, we have a central extension $K = L \oplus D^{\mathrm{gr}*}$ with product $[x \oplus \phi\,,\, y \oplus \psi]_K = [x, y]_L \oplus \sigma(x, y)$, where $x, y \in L$ and $\phi, \psi \in D^{\mathrm{gr}*}$. We can also form the semidirect product $E = K \oplus D$. While this will be an extended affine Lie algebra, it is not general enough. Rather, one can twist the product on E by a special 2-cocycle $\tau : D \times D \to D^{\mathrm{gr}*}$. We denote the corresponding Lie algebra by $E(L, D, \tau)$.

4. Theorem ([14]) *The Lie algebra $E(L, D, \tau)$ is an extended affine Lie algebra. Conversely, every extended affine Lie algebra E arises in this way with $L = E_c/Z(E_c)$ and appropriate choices of D and τ.*

An important point in the proof of this theorem is

5. Theorem ([13]) *Let L be a centreless Lie torus of type $\Delta \neq \mathrm{A}$. Then L is finitely generated as a module over its centroid.*

As an immediate corollary of this result, it follows from [2] that over an algebraically closed field of characteristic 0 every centreless Lie torus of type $\Delta \neq \mathrm{A}$ is a so-called multiloop algebra.

References

[1] B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola, *Extended affine Lie algebras and their root systems*, Mem. Amer. Math. Soc., vol. **126** (1997), no. 603.

[2] B. Allison, S. Berman, J. Faulkner and A. Pianzola, *Realization of graded-simple algebras as loop algebras*, to appear in Forum Math.

[3] B. Allison and Y. Gao, *The root system and the core of an extended affine Lie algebra*, Selecta Math. (N.S.), **7** (2001), 149–212.

[4] B. Allison, G. Benkart and Y. Gao, *Extended affine Lie algebras of type* $\mathrm{BC}_l, l \geq 3$, preprint.

[5] B. Allison, J. Faulkner and Y. Yoshii, *Structurable tori*, in preparation.

[6] B. Allison and Y. Yoshii, *Structurable tori and extended affine Lie algebras of type* BC_1, J. Pure Appl. Algebra **184** (2003), 105–138.

[7] S. Azam, *Generalized reductive Lie algebras: connections with extended affine Lie algebras and Lie tori*, Canad. J. Math. **58** (2006), 225–248.

[8] S. Azam, V. Khalili and M. Yousofzadeh, *Extended affine root systems of type BC*, J. Lie Theory **15** (2005), 145–181.

[9] G. Benkart and E. Neher, *The centroid of extended affine and root graded Lie algebras*, J. Pure Appl. Algebra, **205** (2006), 117–145.

[10] S. Berman, Y. Gao and Y. Krylyuk, *Quantum tori and the structure of elliptic quasi-simple Lie algebras*, J. Funct. Anal., **135**, (1996), 339–389.

[11] S. Berman, Y. Gao, Y. Krylyuk and E. Neher, *The alternative torus and the structure of elliptic quasi-simple Lie algebras of type* A_2, Trans. Amer. Math. Soc., **347**, (1995), 4315–4363.

[12] J. Faulkner, *Lie tori of type* BC_2 *and structurable quasi-tori*, preprint 2005.

[13] E. Neher, *Lie tori*, C. R. Math. Acad. Sci. Soc. R. Can. **26** (2004), 84–89.

[14] E. Neher, *Extended affine Lie algebras*, C. R. Math. Acad. Sci. Soc. R. Can. **26** (2004), 90–96.

[15] Y. Yoshii, *Coordinate algebras of extended affine Lie algebras of type* A_1, J. Algebra **234** (2000), 128–168.

[16] Y. Yoshii, *Root systems extended by an abelian group and their Lie algebras*, J. Lie Theory **14** (2004), 371–394.

[17] Y. Yoshii, *Lie tori − a simple characterization of extended affine Lie algebras*, to appear in Publ. RIMS.