Extended affine Lie algebras and Lie tori

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Abstract. In this announcement we describe the structure of an extended affine Lie algebra in terms of its centreless core, which is a Lie torus with trivial centre. The paper is an extended version of the two papers [**28**] and [**27**].

0. Introduction. Extended affine Lie algebras are a class of complex Lie algebras that includes finite-dimensional simple Lie algebras, affine Lie algebras and toroidal Lie algebras. They are closely related to Saito's elliptic Lie algebras ([**31**]). Originally proposed by the physicists Høegh-Krohn and B. Torrésani [**20**] under the name irreducible quasi-simple Lie algebras, extended affine Lie algebras have been put on a sound mathematical footing in the AMS-memoirs [**2**] by Allison, Azam, Berman, Gao and Pianzola. In particular, one can find there a detailed study of the root systems appearing in extended affine Lie algebras. The structure and representation theory of various classes of these Lie algebras has since been investigated in many papers, see section 4 for a (probably incomplete) survey. In this note we will describe the structure of extended affine Lie algebras in general.

Referring the reader to the main body of this note for precise definitions, we will only give a rough sketch of the relevant structures in this introduction. Two important properties of an extended affine Lie algebra are the existence of an invariant nondegenerate form and a finite-dimensional self-centralizing ad-diagonalizable subalgebra H. Thus E has a root space decomposition $E = \oplus E_\xi$ and a root system R, consisting of those $\xi \in H^*$ with $E_\xi \neq 0$. The form on E gives rise to a partition $R = R^0 \cup R^{\rm an}$ into isotropic roots R^0 and anisotropic roots $R^{\rm an}$, generalizing the decomposition into imaginary and real roots in the affine case. Let E_c be the ideal generated by $\{E_\xi : \xi \in R^{\rm an}\}$, called the core of E. One assumes that E can be recovered from its core E_c in the sense that the kernel of the natural representation $E \to \mathrm{Der} E_c : x \mapsto \mathrm{ad}\, x|E_c$ lies in E_c. The core E_c may have a non-trivial centre, and it turns out to be easier to describe its central quotient $L = E_c/Z(E_c)$, where $Z(E_c)$ denotes the centre of E_c. The situation can thus be summarized by the following diagram

$$\begin{tikzcd} E_c \arrow[r] \arrow[d] & E \\ L & \end{tikzcd} \tag{0.1}$$

familiar from the affine case where E_c is the derived algebra and L a loop algebra. In general, the Lie algebras L appearing in (0.1) can be characterized without any reference to extended affine Lie algebras: they are Lie tori as defined in Yoshii's recent preprints [**36**] and [**32**]. Moreover, it is shown there that all centreless Lie tori appear as the "bottom algebra" in a diagram (0.1). The

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canonical approach to untangling the structure of an extended affine Lie algebra E is therefore to describe (I) the centreless Lie tori L and (II) how to get from L to E.

Regarding (I), an important property of a Lie torus L is that L is graded by a finite irreducible root system $\varDelta$. Although one knows the structure of root-graded Lie algebras in general (Allison-Benkart-Gao [**3**], Berman-Moody [**14**], Benkart-Zelmanov [**10**] and Neher [**25**]), it is not obvious which of them are in fact Lie tori. As of now, the precise structure of a centreless Lie torus L has been worked out for the case of a reduced $\varDelta$ and in a special case for $\varDelta = \mathrm{BC}_1$ (see section 4). Concerning (II), one knows how to go from L to E in case $\varDelta = \mathrm{A}_l$, $l \geq 2$ ([**12**], [**13**]). We mention that there are also some constructions known, which associate to any centreless Lie torus an extended affine Lie algebra, but which do in general not yield all extended affine Lie algebras with a given centreless core ([**2**, Ch. III] and Azam [**8**]). In the same vein, Allison-Berman-Pianzola [**5**] describe how one can obtain new extended affine Lie algebras through affinization of an extended affine Lie algebra.

In this note we announce a solution of (II) in general (Thm. 14 and Thm. 16). Our construction, given in section 13, describes all extended affine Lie algebras with a given centreless core. It resembles the construction of affine Lie algebras and gives a new interpretation to certain subalgebras appearing in the previously known solution for the case $\varDelta = \mathrm{A}$. They are described here as subalgebras of skew centroidal derivations. The proof that our construction gives all extended affine Lie algebras relies on several new results for a centreless Lie torus L: L is finitely generated as Lie algebra and the dimension of its graded components are uniformly bounded (Thm. 5); the centroid $\mathrm{Cent}(L)$ of L is always a Laurent polynomial ring, and if $\varDelta$ is not of type A then L is a free $\mathrm{Cent}(L)$-module of finite rank (Thm. 7); the derivation algebra of L splits the semidirect sum of the ideal of inner derivations and the subalgebra of centroidal derivations (Thm. 9).

While the work on Lie tori can be done for Lie algebras over fields of characteristic 0, one has up to now only considered complex extended affine Lie algebras since one of their defining axioms is a topological (discreteness) condition. To remedy this discrepancy, we are proposing here a new definition of an extended affine Lie algebra over an arbitrary field F of characteristic 0. Roughly speaking, we are allowing more possibilities for the subalgebra $H \subset E$. In case $F = \mathbb{C}$ the algebras satisfying the old axiom system are recovered as the discrete extended affine Lie algebras in our sense (Thm. 16).

Some indications are provided for the proof of each announced result; details will appear elsewhere. The author thanks Bruce Allison and Yoji Yoshii for having provided him with their preprints [**7**], [**36**] and [**32**].

1. Notations and terminology. All vector spaces and algebras considered in this note will be defined over a field F of characteristic 0, except when indicated otherwise. For a subset R of a vector space V, $\mathrm{span}_{\mathbb{Q}}(R)$ denotes the rational span of R. For an abelian group G and a subset $R \subset G$ we denote by $\langle R \rangle$ the subgroup generated by R. We note that if G is free, e.g. if G is the underlying group of a vector space, then so is $\langle R \rangle$.

Root systems will always contain 0. This has some notational advantages and follows the conventions in [**2**]. We will call $\varDelta$ a finite root system if $\varDelta^\times := \varDelta \setminus \{0\}$ is a root system in the sense of [**16**, Ch.VI, §1.1]. In particular, $\varDelta$ need not be reduced. For $\alpha, \beta \in \varDelta$ we denote by $\langle \alpha, \beta^\vee \rangle$ the Cartan integer of α, β (thus $\langle \alpha, \beta^\vee \rangle = n(\alpha, \beta)$ in the notation of [**16**]) and by $\mathcal{Q}(\varDelta) = \langle \varDelta \rangle$ the root lattice of $\varDelta$. We denote by $\varDelta_{\mathrm{ind}} = \{0\} \cup \{\alpha \in \varDelta^\times : \alpha/2 \notin \varDelta\}$ the subsystem of indivisible roots of $\varDelta$.

2. Lie tori. Let $\varDelta$ be a finite irreducible root system and let $\varLambda$ be a free abelian group of finite rank. A *Lie torus of type* $(\varDelta, \varLambda)$ is a Lie algebra L satisfying the following axioms:

(LT1) L has a $(\mathcal{Q}(\Delta) \oplus \Lambda)$-grading of the form

$$L = \bigoplus_{\alpha \in \mathcal{Q}(\Delta),\, \lambda \in \Lambda} L_\alpha^\lambda\,, \quad [L_\alpha^\lambda,\, L_\beta^\mu] \subset L_{\alpha+\beta}^{\lambda+\mu}, \quad \text{satisfying } L_\alpha^\lambda = 0 \text{ if } \alpha \notin \Delta. \tag{2.1}$$

(LT2) For $\alpha \in \Delta^\times$ and $\lambda \in \Lambda$ we have
  (i) $\dim L_\alpha^\lambda \le 1$, with $\dim L_\alpha^0 = 1$ if $\alpha \in \Delta_{\mathrm{ind}}$,
  (ii) if $\dim L_\alpha^\lambda = 1$ then there exists $(e_\alpha^\lambda, f_\alpha^\lambda) \in L_\alpha^\lambda \times L_{-\alpha}^{-\lambda}$ such that $h_\alpha^\lambda = [e_\alpha^\lambda, f_\alpha^\lambda] \in L_0^0$ acts on $x_\beta^\mu \in L_\beta^\mu$ $(\beta \in \Delta, \mu \in \Lambda)$ by

$$[h_\alpha^\lambda\,,\, x_\beta^\mu] = \langle \beta, \alpha^\vee \rangle x_\beta^\mu.$$

(LT3) For $\lambda \in \Lambda$ we have $L_0^\lambda = \sum_{\alpha \in \Delta^\times,\, \mu \in \Lambda} [L_\alpha^\mu\,,\, L_{-\alpha}^{\lambda-\mu}]$.

(LT4) $\Lambda = \langle \{\lambda \in \Lambda : L_\alpha^\lambda \neq 0 \text{ for some } \alpha \in \Delta\} \rangle$.

The rank of Λ is called the *nullity* of L. If (Δ, Λ) is not important or clear from the context, we will simply call L a *Lie torus*. Similarly, a Lie torus of type Δ and nullity n is a Lie torus of type (Δ, Λ) for some Λ of rank n.

Examples of Lie tori will be given in section 4 below. It will emerge that Lie tori can be constructed using certain Λ-graded, not necessarily associative algebras, like Jordan, alternative or structurable algebras, which have been called Jordan tori, alternative tori or structurable tori respectively. This, together with the fact that toroidal Lie algebras are one of the main examples of Lie tori, is the justification for the name "Lie torus".

It is natural to consider Lie tori for more general groups Λ and with less restrictive conditions as (LT2i), see [**34**], [**35**] and [**36**] for some work in this direction. However, the results stated below require the axioms above.

3. Some properties of Lie tori. Let L be a Lie torus of type (Δ, Λ). Then L has a Λ-grading

$$L = \bigoplus_{\lambda \in \Lambda} L^\lambda\,, \quad L^\lambda := \bigoplus_{\alpha \in \Delta} L_\alpha^\lambda \tag{3.1}$$

as well as a $\mathcal{Q}(\Delta)$-grading

$$L = \bigoplus_{\alpha \in \Delta} L_\alpha\,, \quad L_\alpha := \bigoplus_{\lambda \in \Lambda} L_\alpha^\lambda\,. \tag{3.2}$$

The subalgebra $\mathfrak{g}$ of L^0 generated by $\{L_\alpha^0 : \alpha \in \Delta^\times\}$ is a finite-dimensional split simple Lie algebra of type Δ_{ind} with splitting Cartan subalgebra

$$\mathfrak{h} = \sum_{\alpha \in \Delta^\times} [L_\alpha^0\,,\, L_{-\alpha}^0]\,. \tag{3.3}$$

With respect to $\mathfrak{g}$, $\mathfrak{h}$ and the decomposition (3.2), L is a Δ-graded Lie algebra, see e.g. [**3**, Def. 1.2 and Def. 1.12]. It is then easily seen that our definition of a Lie torus is equivalent to the one given in [**36**] and [**32**]. It thus follows from [**32**, Thm. 2.2 and Thm. 7.1] that L has a non-zero invariant (necessarily) symmetric bilinear form $(\ |\ )$, which is Λ-graded in the sense that $(L^\lambda | L^\mu) = 0$ if $\lambda + \mu \neq 0$. Moreover, any such form is unique up to a non-zero scalar, and is nondegenerate if L is *centreless*, i.e., the centre $Z(L)$ vanishes.

Let $C \subset Z(L) = \bigoplus_{\lambda\in\Lambda} (Z(L) \cap L^\lambda)$ be a Λ-graded subspace of $Z(L)$. Then L/C is canonically a Lie torus of type (Δ, Λ). In particular, $L/Z(L)$ is a centreless Lie torus. Conversely, the universal central extension of a Lie torus (more generally, any Λ-cover of L in the sense of [**26**, 1.15]) is again a Lie torus.

4. Examples. (a) Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra of type Δ, and let $F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be the ring of Laurent polynomials in n variables. Then $\mathfrak{g} \otimes F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is a centreless Lie torus of type Δ and nullity n. Hence, by section 3, its universal central extension, i.e., the associated *toroidal Lie algebra* [**24**], is also a Lie torus of type Δ and nullity n. Conversely, by [**12**, Thm. 1.37], every Lie torus of type $\Delta = \mathrm{D}_l$, $l \geq 4$, or $\Delta = \mathrm{E}_l$, $l = 6, 7, 8$ and nullity n is a central extension of $\mathfrak{g} \otimes F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

(b) The special case $n = 1$ and $F = \mathbb{C}$ of example (a) is worth pointing out. Then the *loop algebra* $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$ and its universal central extension $\hat{L}(\mathfrak{g})$ ([**22**, 7.2]) are Lie tori of nullity 1. More generally, it follows from the proof of [**4**, Thm. 1.19] that the complex Lie tori of nullity 1 are precisely the derived affine Lie algebras and their central quotients.

(c) Let $\mathbf{q} = (q_{ij}) \in \mathrm{M}_n(F)$ be a $(n \times n)$-matrix over F satisfying $q_{ii} = 1 = q_{ij} q_{ji}$ for $1 \leq i, j \leq n$, and let $F_{\mathbf{q}}$ be the associated *quantum torus*, which, by definition, is the unital associative algebra with $2n$ generators $t_1^{\pm 1}, \dots, t_n^{\pm 1}$ and defining relations $t_i t_i^{-1} = 1 = t_i^{-1} t_i$ and $t_i t_j = q_{ij} t_j t_i$ for $1 \leq i, j \leq n$. Denote by $[F_{\mathbf{q}}, F_{\mathbf{q}}]$ the span of all commutators $[a, b] = ab - ba$ with $a, b \in F_{\mathbf{q}}$. Then $\mathfrak{sl}_{l+1}(F_{\mathbf{q}}) = \{x \in \mathrm{M}_{l+1}(F_{\mathbf{q}}) : \mathrm{tr}(x) \in [F_{\mathbf{q}}, F_{\mathbf{q}}]\}$ is a Lie torus of type A_l, $l \geq 1$, and nullity n. Of course, if all $q_{ij} = 1$ then $F_{\mathbf{q}} = F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ and $\mathfrak{sl}_{l+1}(F_{\mathbf{q}}) = \mathfrak{sl}_{l+1}(F) \otimes F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is an example considered in (a). It is shown in [**12**, Thm. 2.65] that every Lie torus of type A_l, $l \geq 3$, and nullity n is a central extension of $\mathfrak{sl}_{l+1}(F_{\mathbf{q}})$ for some quantum torus $F_{\mathbf{q}}$.

(d) Lie tori of type A_2 are classified in [**12**] and [**13**]. The centreless Lie tori of type A_1 are precisely the Tits-Kantor-Koecher algebras of the so-called Jordan tori, classified in [**33**]. A description of the centreless Lie tori is given in [**6**] for Δ of type $\mathrm{B}_l, \mathrm{C}_l, \mathrm{F}_4, \mathrm{G}_2$ and, under additional assumptions, in [**7**] for $\Delta = \mathrm{BC}_1$. Our results described below are for the most part independent of these classifications, see the discussion in section 7.

5. Theorem. *Let L be a Lie torus of type (Δ, Λ).*

(a) *L is finitely generated as Lie algebra, and has uniformly bounded dimension with respect to the $(\mathcal{Q}(\Delta) \oplus \Lambda)$-grading (2.1), i.e., there exists a $M \in \mathbb{N}$ such that $\dim_F L_\alpha^\lambda \leq M$ for all $\alpha \in \Delta$ and $\lambda \in \Lambda$.*

(b) *The Lie algebra $\mathrm{Der}_F L$ of F-linear derivations of L is $(\mathcal{Q}(\Delta) \oplus \Lambda)$-graded:*

$$\mathrm{Der}_F L = \bigoplus_{\alpha\in\Delta,\, \lambda\in\Lambda} (\mathrm{Der}_F L)_\alpha^\lambda, \tag{5.1}$$

where $(\mathrm{Der}_F L)_\alpha^\lambda$ consists of those derivations mapping L_β^μ to $L_{\alpha+\beta}^{\lambda+\mu}$. Moreover, $\mathrm{Der}_F L$ has uniformly bounded dimension with respect to the $(\mathcal{Q}(\Delta) \oplus \Lambda)$-grading (5.1).

In (a) the point is of course to prove $\dim_F L_0^\lambda \leq M$ for all $\lambda \in \Lambda$. This can be done by using that for each $\alpha \in \Delta^\times$ the subset $\{\lambda \in \Lambda : L_\alpha^\lambda \neq 0\} \subset \Lambda$ is a reflection subspace of Λ, hence finitely generated as reflection space ([**29**, 2.2]). It then follows that $\mathrm{Der}_F L$ is $(\mathcal{Q}(\Delta) \oplus \Lambda)$-graded and has uniformly bounded dimension. This in turn implies that in case $Z(L) = 0$ also L has uniformly bounded dimension. The generalization to an arbitrary L uses the duality between $H_2(L)$ and outer skew derivations, well-known in the finite-dimensional setting, but which can be extended to the case of graded Lie algebras with finite-dimensional grading subspaces.

6. Let L be a Lie torus of type $(\varDelta, \varLambda)$. Recall that the *centroid* of L, denoted $\mathrm{Cent}(L)$, is the set of all $\chi \in \mathrm{End}_F L$ satisfying $[\chi, \mathrm{ad}\, x] = 0$ for all $x \in L$. Since L is perfect, $\mathrm{Cent}(L)$ is a unital associative commutative algebra, and one can thus consider L as a module or as a Lie algebra over $\mathrm{Cent}(L)$. Since L is $\varDelta$-graded, a $\chi \in \mathrm{Cent}(L)$ leaves every root space L_α invariant. Moreover, χ is uniquely determined by $\chi|L_\alpha$ for a short root α. It follows that $\mathrm{Cent}(L)$ is $\varLambda$-graded,

$$\mathrm{Cent}(L) = \bigoplus_{\lambda \in \varLambda} \mathrm{Cent}(L)^\lambda\,, \quad \text{with } \dim_F \mathrm{Cent}(L)^\lambda \leq 1,$$

where $\mathrm{Cent}(L)^\lambda$ consists of endomorphisms of degree λ with respect to the $\varLambda$-grading (3.1) of L. We put $\varGamma = \{\lambda \in \varLambda : \mathrm{Cent}(L)^\lambda \neq 0\}$. The following result justifies to call $\varGamma$ the *centroid grading group.*

7. Theorem. *Let L be a centreless Lie torus of type $(\varDelta, \varLambda)$.*

(a) *$\varGamma$ is a subgroup of $\varLambda$, and* $\mathrm{Cent}(L)$ *is isomorphic to the group ring $F[\varGamma]$, hence to a Laurent polynomial ring in several variables.*

(b) *L is a free* $\mathrm{Cent}(L)$*-module. If $\varDelta \neq \mathrm{A}_l$, then L has finite rank as* $\mathrm{Cent}(L)$*-module.*

Part (a) follows from the fact that L is simple as $\varLambda$-graded Lie algebra. Since every graded module over a group ring is free, see e.g. [**29**, 2.8], it suffices in (b) to prove that L is finitely generated as a $\mathrm{Cent}(L)$-module. To do so, I use partial coordinatization theorems for various types of $\varDelta$, which permit to prove (b) without knowing the precise structure of L. In more detail, if $\varDelta = \mathrm{BC}_1$ and L is not A_1-graded, L is the Kantor Lie algebra of a structurable torus $(A, ^-)$ that is not a Jordan torus ([**7**, 5.6]). By [**1**, Prop. 8], the centroid of any Kantor Lie algebra is isomorphic to the centre of $(A, ^-)$, so that it suffices to prove that $(A, ^-)$ is finitely generated over its centre, which was done in [**7**, Thm. 8.8]. For $\varDelta = \mathrm{BC}_l$, $l \geq 2$, one can proceed in an analogous manner: L is the Kantor Lie algebra of a certain structurable algebra $(A, ^-)$ (for general BC_2-graded Lie algebras this has already been established in [**3**, Thm. 6.19(i)]), and one then shows that for a Lie torus this structurable algebra is finitely generated over its centre. For $\varDelta = \mathrm{B}_l, \mathrm{C}_l, \mathrm{F}_4$ or G_2 and $F = \mathbb{C}$, finite generation of the $\mathrm{Cent}(L)$-module L can be inferred from the classification theorems in [**7**, §4 and §5]. One can however prove partial coordinatization theorems in general, sufficient to establish (b) without doing the detailed analysis of [**6**]. For $\varDelta = \mathrm{D}_l$, E_6, E_7 or E_8, finite generation is obvious from the easily established structure theorems for these types (see Example 4.a) and the fact that the centroid of $\mathfrak{g} \otimes F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is $F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

Remarks. (a) Let $L = \mathfrak{sl}_{l+1}(F_{\mathbf{q}})$ as in Example 4(c). In this case, $\mathrm{Cent}(L) = Z(F_{\mathbf{q}})\mathrm{Id}$, where $Z(F_{\mathbf{q}})$ denotes the centre of $F_{\mathbf{q}}$, and L has finite rank over $\mathrm{Cent}(L)$ if and only if $F_{\mathbf{q}}$ has finite rank over $Z(F_{\mathbf{q}})$, equivalently $[\varLambda : \varGamma] < \infty$. Using the description of $Z(F_{\mathbf{q}})$ given in [**12**, 2.44], it is easy to construct examples for which $\mathrm{rank}(\varGamma)$ takes on every value between 0 and n. In particular, L is in general not a finitely generated $\mathrm{Cent}(L)$-module.

(b) The theorem above together with the results of [**13**] and [**33**] implies that any centerless Lie torus over an algebraically closed field, which does not have finite rank over its centroid, is isomorphic to an example $\mathfrak{sl}_{l+1}(F_{\mathbf{q}})$.

(c) Let L be a centreless Lie torus. Then $\mathrm{Cent}(L)$ is an integral domain, acting without torsion on L. Let K be the quotient field of $\mathrm{Cent}(L)$, and let

$$\tilde{L} = L \otimes_{\mathrm{Cent}(L)} K \tag{7.1}$$

be its *central closure*, where in this tensor product L is considered as Lie algebra over $\mathrm{Cent}(L)$. Then L imbeds into $\tilde{L}$ and is a $\mathrm{Cent}(L)$-form of $\tilde{L}$. If L has finite rank over $\mathrm{Cent}(L)$, $\tilde{L}$ is a simple finite-dimensional Lie algebra over K.

8. Centroidal derivations. Let L be a centreless Lie torus of type (Δ, Λ), nullity n and centroidal grading group Γ. Recall the Λ-grading (3.1) of L. Any $\theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, F)$ induces a so-called *degree derivation* ∂_θ of L, defined by $\partial_\theta(x^\lambda) = \theta(\lambda)x^\lambda$ for $x^\lambda \in L^\lambda$. We put

$$\mathcal{D} = \{\partial_\theta : \theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, F)\}$$

and note that $\theta \mapsto \partial_\theta$ is an isomorphism from $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, F)$ onto $\mathcal{D}$, hence $\mathcal{D} \cong F^n$. Moreover, $\mathcal{D}$ induces the Λ-grading of L, i.e.,

$$L^\lambda = \{x \in L : \partial_\theta(x) = \theta(\lambda)x \text{ for all } \partial_\theta \in \mathcal{D}\}. \tag{8.1}$$

If $\chi \in \mathrm{Cent}(L)$ then $\chi\partial \in \mathrm{Der}_F L$ for any $\partial \in \mathrm{Der}_F L$. It follows that

$$\mathrm{CDer}_F L = \mathrm{Cent}(L)\mathcal{D} = \bigoplus_{\mu \in \Gamma} \mathrm{Cent}(L)^\mu \mathcal{D}$$

is a Γ-graded subalgebra of $\mathrm{Der}_F L$, called the algebra of *centroidal derivations of* L. Let $m = \mathrm{rank}(\Gamma)$, so $0 \le m \le n$, and identify $\mathrm{Cent}(L) = F[t_1^{\pm 1}, \dots, t_m^{\pm 1}] = \bigoplus_{\mu \in \Gamma} t^\mu \mathcal{D}$ where $t^\mu = t_1^{\mu_1} \cdots t_m^{\mu_m}$ for $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{Z}^m$. Then $\mathrm{CDer}_F L = \bigoplus_{\mu \in \mathbb{Z}_m} t^\mu \mathcal{D}$ and the product of $\mathrm{CDer} L$ is given by

$$[t^\mu \partial_\theta \,,\, t^\nu \partial_\psi] = t^{\mu+\nu}(\theta(\nu)\partial_\psi - \psi(\mu)\partial_\theta)\,. \tag{8.2}$$

Thus $\mathrm{CDer}_F L$ is a generalized Witt algebra in the sense of [**29**, 1.9], a generalization of the generalized Witt algebras studied in [**17**]. One easily obtains from (8.2) that $\mathcal{D}$ is an ad-diagonalizable subalgebra of $\mathrm{CDer}_F L$ with weight spaces $t^\mu \mathcal{D}$.

Let $\mathrm{SCDer}_F L$ be the subalgebra of $\mathrm{CDer}_F L$ consisting of skew derivations with respect to a nondegenerate invariant Λ-graded form $(\ |\ )$ on L, cf. section 3. Then

$$\mathrm{SCDer}_F L = \bigoplus_{\mu \in \Gamma} (\mathrm{SCDer}_F L)^\mu = \bigoplus_{\mu \in \Gamma} t^\mu \{\partial_\theta \in \mathcal{D} : \theta(\mu) = 0\}$$

is Γ-graded with 0-component $\mathcal{D}$. It follows from (8.2) that $\mathrm{SCDer}_F L$ is the semidirect product of the subalgebra $\mathcal{D}$ and the ideal

$$\mathcal{D}' = \bigoplus_{0 \ne \mu \in \Gamma} (\mathrm{SCDer}_F L)^\mu.$$

9. Theorem. *Let L be a centreless Lie torus. Denote by* $\mathrm{IDer} L$ *the ideal of inner derivations of L. Then*

$$\mathrm{Der}_F L = \mathrm{IDer} L \rtimes \mathrm{CDer}_F L \quad \textit{(semidirect product).} \tag{9.1}$$

In case L has finite rank as $\mathrm{Cent}(L)$-module, this result can be proven by using that its central closure $\tilde{L}$, see (7.1), is a finite-dimensional simple Lie K-algebra and hence all K-linear derivations are inner. In the remaining case, where L is a $\mathrm{Cent}(L)$-module of infinite rank, Δ is of type A by Thm. 7. Then the result follows from [**12**, 2.17, 2.53], [**13**, Thm. 1.40] and [**29**, Thm. 4.11].

For Δ of type B or D the splitting (9.1) has also been proven in [**18**, Cor. 4.9 and Cor. 4.10] using different methods. We note also that the decomposition (9.1) is not the one proven in [**9**, Thm. 3.12] for arbitrary Δ-graded Lie algebras: the subalgebra $\mathrm{Der}_*(\mathfrak{a}, \mathcal{S})$ of [**9**] contains $\mathrm{CDer}_F L$ but has in general a non-zero intersection with $\mathrm{IDer} L$.

We will now turn to extended affine Lie algebras. Their definition requires some notation, which we shall establish in the next section.

10. A preliminary setting. Let E be a Lie algebra satisfying the following two axioms (EA1) and (EA2):

(EA1) E has a nondegenerate invariant symmetric bilinear form $(\ |\ )$.

(EA2) E contains a nontrivial finite-dimensional self-centralizing and ad-diagonalizable subalgebra H.

By (EA2) E has a root space decomposition

$$E = \bigoplus_{\xi \in H^*} E_\xi\,, \qquad E_0 = H,$$

where, as usual, $E_\xi = \{e \in E : [h,e] = \xi(h)e \text{ for all } h \in H\}$. The invariance of $(\ |\ )$ implies that $(E_\xi|E_\zeta) = 0$ for $\xi + \zeta \neq 0$. It follows that $(\ |\ )$ restricted to $H \times H$ is nondegenerate. We can therefore transfer the restricted form $(\ |\ )|H \times H$ to a nondegenerate symmetric bilinear form on H^* by setting $(\xi|\zeta) = (t_\xi|t_\zeta)$ where $t_\xi \in H$ is defined by $(t_\xi|h) = \xi(h)$ for all $h \in H$. We define

$$\begin{aligned} R &= \{\xi \in H^* : E_\xi \neq 0\} \quad \text{(root system of } E),\\ R^0 &= \{\xi \in R : (\xi|\xi) = 0\} \quad \text{(isotropic roots)},\\ R^{\mathrm{an}} &= \{\xi \in R : (\xi|\xi) \neq 0\} \quad \text{(anisotropic roots)}. \end{aligned}$$

The subalgebra E_c of E, generated by $\{E_\xi : (\xi|\xi) \neq 0\}$ is called the *core of E*. It is in fact an ideal if E is an extended affine Lie algebra as defined below.

11. Definition. An *extended affine Lie algebra of nullity n*, or extended affine Lie algebra for short, is a Lie algebra E satisfying (EA1), (EA2) of section 10 and, in addition, the following axioms (EA3) – (EA6) (see section 10 for unexplained notation):

(EA3) For $\xi \in R^{\mathrm{an}}$ and $x_\xi \in E_\xi$, $\mathrm{ad}\, x_\xi \in \mathrm{End}_F L$ is locally nilpotent.

(EA4) R^{an} is *irreducible*, i.e., $R^{\mathrm{an}} = R_1 \cup R_2$ and $(R_1|R_2) = 0$ implies $R_1 = \emptyset$ or $R_2 = \emptyset$.

(EA5) E is *tame* in the sense that $\{e \in E : [e, E_c] = 0\} \subset E_c$.

(EA6) $\Lambda := \langle R^0 \rangle \subset V$ is a free abelian group of rank n.

It is appropriate to immediately point out that this definition for $F = \mathbb{C}$ is more general than the usual definition of an extended affine Lie algebra ([**2**] or section 15 below). The relation between the two definitions is discussed in sections 15 and 16. As we will see there is a close connection between extended affine Lie algebras and Lie tori. The following proposition is the first step in this direction. It can be proven using the techniques of [**2**, Ch. I] and Thm. 5.

12. Proposition. *Let E be an extended affine Lie algebra with root system R, and put $\Lambda = \langle R^0 \rangle$.*

(a) *There exists a finite irreducible root system Δ, an imbedding $\Delta_{\mathrm{ind}} \hookrightarrow R$ and a family $(\Lambda_\alpha : \alpha \in \Delta) \subset \Lambda$ such that*

$$V = \mathrm{span}_{\mathbb{Q}}(\Delta) \oplus \mathrm{span}_{\mathbb{Q}}(R^0) \quad \textit{and} \quad R = \bigcup_{\alpha \in \Delta} (\alpha \oplus \Lambda_\alpha).$$

The subspaces $(E_c)_\alpha^\lambda = E_c \cap E_{\alpha \oplus \lambda}$ give E_c the structure of a Lie torus of type (Δ, Λ).

(b) *The root spaces E_ξ of E have uniformly bounded finite dimension.*

We note that the family $(\Lambda_\alpha : \alpha \in \Delta^\times)$ is a "reduced root system of type Δ extended by Λ" in the terminology of [**36**]. Thm. 2.4 of this paper gives the structure of this family, generalizing [**2**, II, Thm. 2.37].

The proposition associates a centreless Lie torus to every extended affine Lie algebra E, namely $L = E_c/Z(E_c)$. We will now describe a construction which, conversely, associates an extended affine Lie algebra to any centreless Lie torus.

13. Construction. Let L be a centreless Lie torus of type (Δ, Λ) and nullity n, and let $(\ |\ )$ be a nondegenerate invariant Λ-graded symmetric bilinear form on L. We denote by Γ the centroid grading group (Thm. 7) and by $\mathrm{SCDer}_F L = \bigoplus_{\gamma\in\Gamma} (\mathrm{SCDer}_F L)^\gamma = \mathcal{D} \oplus \mathcal{D}'$ the skew centroidal derivations (section 8).

The second ingredient of our construction is a Γ-graded subalgebra of $\mathrm{SCDer}_F L$,

$$D = \bigoplus_{\gamma\in\Gamma} D_\gamma, \quad D_\gamma \subset (\mathrm{SCDer}_F L)^\gamma,$$

which has the property that D_0 induces the Λ-grading (3.1) of L, i.e., $L^\lambda = \{x \in L : \partial_\theta(x) = \theta(\lambda)x \text{ for all } \partial_\theta \in D_0\}$. Equivalently, the canonical evaluation map

$$\mathrm{ev}\colon \Lambda \to D_0^* : \lambda \mapsto \mathrm{ev}(\lambda), \quad \text{where } \big(\mathrm{ev}(\lambda)\big)(\partial_\theta) = \theta(\lambda), \tag{13.1}$$

is injective. Let $D^{\mathrm{gr}*} = \bigoplus_{\gamma\in\Gamma} D_\gamma^*$ be the graded dual space of D. Thus, $f \in D_\gamma^*$ is extended to a linear form on D by $f|D_\delta = 0$ for $\delta \neq \gamma$. We consider $D^{\mathrm{gr}*}$ as a Γ-graded vector space with γ-component $(D^{\mathrm{gr}*})_\gamma = D_{-\gamma}^*$. It is easily seen that

$$\sigma_D\colon L \times L \to D^{\mathrm{gr}*}, \quad \sigma_D(x, y)(d) = (dx|y)$$

is a 2-cocycle for L with values in the trivial L-module $D^{\mathrm{gr}*}$ which respects the gradings of L and $D^{\mathrm{gr}*}$.

The third ingredient of our construction is a 2-cocycle

$$\tau\colon D \times D \to D^{\mathrm{gr}*}$$

of D with values in $D^{\mathrm{gr}*}$, considered a D-module via the contragredient action $d \cdot f$, which is graded and invariant, i.e.,

$$\tau(D_\gamma, D_\delta) \subset (D^{\mathrm{gr}*})_{\gamma+\delta} = D_{-\gamma-\delta}^* \quad \text{and} \quad \tau(d_1, d_2)(d_3) = \tau(d_2, d_3)(d_1) \tag{13.2}$$

for $d_1, d_2, d_3 \in D$. Moreover, we suppose that

$$\tau(D_0, D) = 0. \tag{13.3}$$

Let $D' = \bigoplus_{0\neq\gamma\in\Gamma} D_\gamma$, an ideal of D. Because of condition (13.3), the map $\tau \mapsto \tau|D' \times D'$ is a bijection between the 2-cocycles of D satisfying (13.2) and (13.3) and the 2-cocycles of D' with values in the D'-module $D^{\mathrm{gr}*}$ satisfying (13.2).

Finally, for L, D and τ as above we define

$$E = E(L, D, \tau) = L \oplus D^{\mathrm{gr}*} \oplus D.$$

Then E is a Lie algebra with respect to the product ($x_i \in L, f_i \in D^{\mathrm{gr}*}, d_i \in D$)

$$\begin{aligned}[x_1 \oplus f_1 \oplus d_1\,,\, x_2 \oplus f_2 \oplus d_2] &= \big([x_1, x_2] + d_1(x_2) - d_2(x_1)\big)\\ &\oplus \big(\sigma_D(x_1, x_2) + d_1 \cdot f_2 - d_2 \cdot f_1 + \tau(d_1, d_2)\big) \oplus [d_1, d_2].\end{aligned}$$

E has a nondegenerate invariant form $(\ |\ )$ given by

$$(x_1 \oplus f_1 \oplus d_1 | x_2 \oplus f_2 \oplus d_2) = (x_1|x_2) + f_1(d_2) + f_2(d_1).$$

Let $H = \mathfrak{h} \oplus D_0^* \oplus D_0$ for $\mathfrak{h}$ as in (3.3). We identify $\Lambda = \mathrm{ev}(\Lambda) \subset D_0^*$ and view $\Lambda \subset H^*$ by letting $\lambda \in \Lambda$ act by 0 on $\mathfrak{h} \oplus D_0^*$. Similarly, any $\alpha \in \Delta \subset \mathfrak{h}^*$ gives rise to a linear form on H by putting $\alpha | D_0^* \oplus D_0 = 0$. With these identifications, H becomes a self-centralizing, ad-diagonalizable subalgebra of E whose root spaces are

$$E_{\alpha \oplus \lambda} = \begin{cases} L_\alpha^\lambda & ;\ \alpha \neq 0, \\ L_0^\lambda \oplus D_{-\lambda}^* \oplus D_\lambda & ;\ \alpha = 0. \end{cases}$$

It is then easy to verify part (a) of the following theorem.

14. Theorem. (a) *The algebra $E(L, D, \tau)$ constructed in section* 13 *is an extended affine Lie algebra of nullity n with respect to the form $(\ |\ )$ and the subalgebra H.*

(b) *Conversely, let E be an extended affine Lie algebra of nullity n and let $L = E_c/Z(E_c)$, which by Prop.* 12 *is a centreless Lie torus of nullity n. Then there exists a unique subalgebra $D \subset \mathrm{SCDer}_F L$ inducing the Λ-grading of L and a* 2*-cocycle $\tau \colon D \times D \to D^{\mathrm{gr}*}$ satisfying* (13.2) *and* (13.3) *such that $E \cong E(L, D, \tau)$.*

For the proof of part (b) we note that tameness of E and Prop. 12(b) imply that E can be described in terms of a Γ-graded subalgebra $D \subset \mathrm{Der}_F\, L$ with $D \cap \mathrm{IDer}_F L = 0$ and a 2-cocycle τ. Because of Thm. 9 one can take $D \subset \mathrm{SCDer}_F L$.

Remarks. (a) The construction of the Lie algebra $E(L, D, \tau)$ makes sense in the more general setting where L is just a Lie algebra with a nondegenerate invariant form and D is a subalgebra of skew-symmetric derivations of L. For finite-dimensional algebras the Lie algebra $E(L, D, 0)$, called a *double extension*, has been used to classify finite-dimensional solvable Lie algebras admitting a nondegenerate invariant form ([**21**], [**22**, Ex. 2.10, 2.11], [**23**]). The more general construction with a possibly non-zero τ appears in [**15**, §3]. In the setting of discrete extended affine Lie algebras of type $\Delta = \mathrm{A}_l$, $l \geq 2$, the construction $E(L, D, \tau)$ appears in [**12**] and [**13**].

(b) Since $\mathrm{SCDer}_F L$ induces the Λ-grading of L, $D = \mathrm{SCDer}_F L$ is the maximal choice for D. In this case the subalgebra $L \oplus (\mathrm{SCDer}_F L)^*$ of E is the universal central extension of L. As pointed out in [**12**, Remark 3.71(b)], there do indeed exist non-trivial 2-cocycles τ in this case, which first have appeared in the context of toroidal Lie algebras ([**30**], see also [**11**, (2.11), (2.12)]).

15. Definition. Let E be a Lie algebra over $F = \mathbb{C}$. We call E a *discrete extended affine Lie algebra* if E satisfies the axioms (EA1) – (EA5) and the following axiom

(DE) R is a discrete subset of H^*.

In view of [**5**, Lemma 3.62] a discrete extended affine Lie algebra as defined above is the same as a tame extended affine Lie algebra in the sense of [**2**]. We have included tameness in our definition, i.e., the axiom (EA5), since our results apply to tame extended affine Lie algebras only. Moreover, as an example in [**12**, §3] shows, there is little hope to get a precise description of extended affine Lie algebras that are not tame. Besides the mentioned example in [**12**], all of the known constructions yield tame extended affine Lie algebras.

16. Theorem. *Let* $F = \mathbb{C}$. (a) *Let* L *be a centreless Lie torus and let* D *be a graded subalgebra of* $\mathrm{SCDer}_{\mathbb{C}} L$ *such that the evaluation map* $\mathrm{ev}\colon \Lambda \to D_0^*$ *of* (13.1) *is not only injective but has also discrete image. Then, with* τ *as in the construction* 13*, the Lie algebra* $E(L, D, \tau)$ *is a discrete extended affine Lie algebra.*

(b) *Conversely, every discrete extended affine Lie algebra arises from the construction described in* (a).

We note that $\mathrm{ev}\colon \Lambda \to \mathcal{D}^*$ is always a discrete imbedding. Hence any complex Lie torus gives rise to a discrete extended affine Lie algebra ([**32**, Cor. 7.3]). A construction of discrete imbeddings in terms of the maximal choice $D = \mathcal{D}$ is given in [**19**, §2] (although not in this language).

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