Locally finite root systems

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Abstract

We develop the basic theory of root systems R in a real vector space X which are defined in analogy to the usual finite root systems, except that finiteness is replaced by *local finiteness*: The intersection of R with every finite-dimensional subspace of X is finite. The main topics are Weyl groups, parabolic subsets and positive systems, weights, and gradings.

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Introduction

This papers deals with root systems R in a real vector space X which are defined in analogy to the usual finite root systems à la Bourbaki [**12**, VI], except that finiteness is replaced by *local finiteness*: The intersection of R with every finite-dimensional subspace of X is finite.

Our aim is to develop the basic theory of these locally finite root systems. The main topics of our work are Weyl groups, parabolic subsets and positive systems, weights, and gradings. The reader will find that much, but not all, of the well-known theory of finite root systems does generalize to this setting, although often different proofs are needed. But there are also completely new phenomena, unfamiliar from the theory of finite root systems. Most important among these is that a locally finite root system R does in general not have a root basis, i.e., a vector space basis $B \subset R$ of X such that every root in R is an integer linear combination of B with coefficients of the same sign. Thus, by necessity, our work presents a "basis-free" approach to root systems. An important new tool is the concept of quotients of root systems by full subsystems. When working with quotients, the usual requirement that $0 \notin R$ proves to be cumbersome, so our root systems always contain 0. This is also useful when considering root gradings of Lie algebras, and fits in well with the axioms for extended affine root systems in [**1**, Ch. II]. It also occurs naturally in the axiomatizations of root systems given by Winter [**75**] and Cuenca [**19**].

Throughout, we have attempted to develop the categorical aspect of root systems which, we feel, has hitherto been neglected. Thus we define the category **RS** whose objects are locally finite root systems, and whose *morphisms* are linear maps of the underlying vector spaces mapping roots to roots. Morphisms of this type were studied for example by Doković and Thǎńg [**25**]. A more restricted class of morphisms, called *embeddings* and defined by the condition that f preserve Cartan numbers, leads to the subcategory **RSE** of **RS** whose morphisms are embeddings. Many natural constructions, for example the coroot system, the Weyl group and the group of weights, turn out to be functors defined on this category.

Let us stress once more that a locally finite root system is infinite if and only if it spans an infinite-dimensional space. Hence, locally finite root systems are not the same as the root systems appearing in the theory of Kac-Moody algebras. The axiomatic approach to these types of root systems has been pioneered by Moody and his collaborators [**45**, **48**, **46**]. Further generalizations are given in papers by Bardy [**4**], Bliss [**6**], and Hée [**30**]. Roughly speaking, the intersection of locally finite root systems and the root systems of Kac-Moody algebras consists of the direct sums of finite roots systems and their countably infinite analogues, see Kac [**35**, 7.11] or Moody-Pianzola [**47**, 5.8]. Similarly, the infinite root systems considered here are not the same as the extended affine root systems which appear in the theory of extended affine Lie algebras [**1**, Ch. II] and elliptic Lie algebras [**66**, **67**]. The extended affine root systems which are also locally finite root systems, are exactly the finite root systems. Since extended affine root systems map onto finite root systems, one is led to speculate that there should be a theory of "extended affine locally finite root systems", encompassing both the theory of extended affine root systems and of locally finite root systems.

The motivation for our study comes from the applications we have in mind. Notably, this paper provides some of the combinatorial theory needed for our study of Steinberg groups associated to Jordan pairs [**42**]. It also gives justification for some results of the second-named author announced in [**57**] and already used in some papers [**58**, **59**, **60**]. Not surprisingly, locally finite root systems have also appeared in the study of infinite-dimensional Lie algebras. For example, countable locally finite root systems are the root systems of the infinite rank affine algebras (Kac [**35**, 7.11]). Semisimple L^*-algebras, certain types of Lie algebras on Hilbert spaces, have a root space decomposition (in the Hilbert space sense) indexed by a locally finite root system (Schue [**68**, **69**]), and the classification of these root systems can be used to classify L^*-algebras [**59**, §4]. Lie algebras graded by infinite locally finite root systems are described in [**60**] (and in [**29**] for Lie superalgebras). A special class of this type of Lie algebras are the semisimple locally finite split Lie algebras recently studied by Stumme [**71**], Neeb-Stumme [**54**] and Neeb [**51**, **52**]. Dimitrov-Penkov have studied these Lie algebras and their representations from the point of view of direct limits of finite-dimensional reductive Lie algebras [**23**]. Groups associated to the classes of Lie algebras mentioned above have also been studied. Often, these are groups of operators on Hilbert or Banach spaces, analogues of the classical groups in finite dimension, see for example de la Harpe [**20**], Neeb [**50**, **53**], Natarajan, Rodríguez-Carrington and Wolf [**49**], Neretin [**61**], Ol'shanskii [**62**], Pickrell [**63**] and Segal [**70**].

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We now give a summary of the contents of this work. Unless specified otherwise, the term "root system" will always mean a locally finite root system.

A certain amount of the theory can be done in much greater generality than just for root systems in real vector spaces. Therefore, the first two sections are devoted to investigating the category $\mathbf{SV}_k$ of sets R in vector spaces X over some field k which satisfy $0 \in R$ and $X = \operatorname{span}(R)$,although the reader might be well-advised to start with §3 and return to sections 1 and 2 only when necessary. In §1 we introduce the concepts of full subsets, tight subspaces and tight intersections which allow us to define a good notion of quotients and to prove the standard First and Second Isomorphism Theorems in $\mathbf{SV}_k$ (1.7 and 1.9). In the following section we introduce local finiteness. As this property is not inherited by arbitrary quotients, we are led to consider a more stringent quantitative finiteness condition, called *strong boundedness* which is crucial in proving the existence of A-bases for R (2.11), for A a subring of k. Here A-bases are k-free subsets B of R such that every element of R is an A-linear combination of B.

The theory of root systems proper starts in §3. We introduce the usual concepts known from the theory of finite root systems as well as the categories **RS** and **RSE** mentioned above, and show that the locally finite root systems are precisely the direct limits in **RSE** of the finite root systems. We also prove the usual decomposition of a root system into a direct sum of irreducible components, based on the concept of connectedness. In §4 we prove that the vector space X spanned by a root system R carries so-called invariant inner products, defined by the condition that all reflections are orthogonal. There even exist normalized invariant inner products for which all isomorphisms are isometric. A discussion of the coroot system follows.

In §5 we study the Weyl group of a root system R, i.e., the group generated by all reflections. These Weyl groups are locally finite in the sense that any finite subset generates a finite subgroup. However, one of the major results for finite root systems fails: The Weyl group of an uncountable irreducible root system is not a Coxeter group (9.9). As a substitute, we provide a presentation which uses the reflections in all, instead of merely the simple roots. This is of course well-known for finite root systems (Carter [**17**]). Besides the usual Weyl group $W(R)$ we introduce a whole chain of Weyl groups $W(R, \mathbf{c})$, defined as generated by reflections in an orthogonal system of cardinality less than $\mathbf{c}$ where $\mathbf{c}$ is an infinite cardinal. We also define the *big Weyl group* $\overline{W}(R)$ as the closure of $W(R)$ in the finite topology. It turns out (9.6) that $\overline{W}(R)$ is the group generated by all reflections in orthogonal systems of arbitrary size. This is one of the results of §9, devoted to a detailed study of the Weyl groups and automorphism groups of the infinite irreducible root systems. Another is the determination of the outer automorphism groups (9.5) and of the normal subgroup structure of $W(R)$ (9.8).

Two types of bases are considered in §6. First, specializing the concept of A-bases of §2 to $A = \mathbb{Z}$ leads to so-called integral bases of root systems. We show that integral bases not only exist, a result also proven by Stumme with different methods in [**71**, Th. IV.6], but more generally integral bases always extend from a full subsystem, i.e., the intersection of R with a subspace, to the whole root system. This is an application of strong boundedness of root systems, proven in 6.2. The second type of bases are root bases in the sense mentioned earlier. We show in 6.7 and 6.9 that an irreducible root system admits a root basis if and only if it is countable.

The following §7 is the first of two sections devoted to weights. Besides the group $\mathcal{Q}(R)$ of radicial weights (also known as the root lattice) and the full group of weights $\mathcal{P}(R)$, we introduce new weight groups $\mathcal{P}_{\rm fin}(R)$, $\mathcal{P}_{\rm bd}(R)$ and $\mathcal{P}_{\rm cof}(R)$, called *finite, bounded* and *cofinite* weights. For R finite, $\mathcal{P}_{\rm bd}(R) = \mathcal{P}_{\rm fin}(R) = \mathcal{P}(R)$ and $\mathcal{P}_{\rm cof}(R) = \mathcal{Q}(R)$, but not so in general. The groups $\mathcal{Q}(R) \subset \mathcal{P}_{\rm fin}(R) \subset \mathcal{P}_{\rm bd}(R)$ are free abelian and the quotient $\mathcal{P}_{\rm fin}(R)/\mathcal{Q}(R)$ is a torsion group. Also, $\mathcal{P}_{\rm cof}(R) \subset \mathcal{P}(R)$ are the $\mathbb{Z}$-duals of the groups of finite and radicial weights of the coroot system $R^\vee$, and their quotient is the Pontrjagin dual of $\mathcal{P}_{\rm fin}(R^\vee)/\mathcal{Q}(R^\vee)$ (7.5). We give two presentations for the abelian group $\mathcal{Q}(R)$ and apply them to the description of gradings which in §17 leads to an easy classification of 3-graded root systems [**57**]. We also introduce *basic weights* which generalize the fundamental weights familiar from the theory of finite root systems but make sense even when R has no root basis.

In §8, we classify locally finite root systems, using simplifications of methods due to Kaplansky and Kibler [**37**, **38**] and to Neeb and Stumme [**54**]. There are no surprises: These root systems are either finite or the infinite, possibly uncountable, analogues of the classical root systems of type A, B, C, D and BC. In each case, we also work out the various weight groups introduced in §7.

The sections 10 – 16 deal with various aspects of positivity. Many properties of the theory of parabolic subsets and positive systems can be developed in the broader framework of symmetric sets in real vector spaces, which we do in §10. The following §11 is concerned with properties of parabolic subsets specific to root systems. Notably, we prove presentations of both the root lattice (11.12) and the Weyl group (11.13, 11.17), based on the unipotent part of a parabolic subset, which seem to be new even in the finite case.

In §12, the closed and full subsystems of the infinite irreducible root systems are investigated. We associate combinatorial invariants to a closed subsystem which determine it uniquely (12.5). The main results are the infinite analogue of the Borel-de Siebenthal theorem describing the maximal closed subsystems (12.13), and the classification of the full subsystems modulo the operation of the big Weyl group (12.17). A similar method is used in §13 to classify parabolic subsets of the infinite irreducible root systems (13.11). This provides a new unified approach to earlier work of Dimitrov-Penkov [**23**]. These results are specialized in §14 to positive systems. For finite root systems, positive systems are just the "positive roots" with respect to a root basis and there is a one-to-one correspondence between root bases and positive systems. The corresponding result for locally finite root systems is no longer true: Positive systems always exist while root bases may not. Nevertheless, the notion of simple root with respect to a positive system P is still meaningful and is closely tied to the extremal rays of the convex cone $\mathbb{R}_+[P]$ generated by P. This leads to a geometric characterization of those positive systems which are determined by a root basis: they are exactly those positive systems P for which $\mathbb{R}_+[P]$ is spanned by its extremal rays (14.4).

In §15 we introduce, for a parabolic subset P, the cone $D(P)$ of linear forms which are positive on $P^\vee$. When R is finite and P is a positive system, $D(P)$ is the closure of the Weyl chamber defined by P. Let us note here that the usual definition of Weyl chamber may yield the empty set in case of an infinite root system. We then introduce facets and develop many of their basic properties, familiar from the finite case. Section 16 introduces dominant and fundamental weights relative to a parabolic subset P, the latter being defined as the basic weights contained in $D(P)$. A detailed analysis of the fundamental weights of the irreducible infinite root systems follows. As a consequence, we show that the fundamental weights are in one-to-one correspondence with the extremal rays of $D(P)$ (16.9), that they generate a weak-$*$-dense subcone of $D(P)$, (16.11), and that every dominant weight is a weak-$*$-convergent linear combination of fundamental weights with nonnegative integer coefficients (16.18). While our approach to these results provides very detailed information, it does use the classification, and a classification-free proof would of course be desirable.

The last two sections are devoted to gradings of root systems, starting with the most general situation of a root system graded by an abelian group A, and progressing to $\mathbb{Z}$-gradings and finally special types of $\mathbb{Z}$-gradings, called 3- and 5-gradings. From the detailed description of weights obtained earlier, we derive easily the classification of 3-gradings. The final §18 is concerned with a more detailed theory of 3-graded root systems, and introduces in particular so-called *elementary configurations*. These allow us to give concise formulations of the presentations of the root lattice and the Weyl group of a 3-graded root system in terms of the 1-part, specializing 11.12 and 11.17. Elementary configurations provide the combinatorial framework for dealing with certain families of tripotents in Jordan triple systems [**56**] or idempotents in Jordan pairs [**60**, **55**].

By the very definition of locally finite root systems, it is not surprising that we often prove results by making use of the corresponding results for finite root systems. The reader is expected to be reasonably familiar with the basic reference [**12**, VI, §1]. For convenience, appendix A provides a summary of those results in [**12**] which are relevant for our work. In appendix B we prove a number of facts on a class of convex cones which appear naturally in our context as the cones spanned by parabolic subsets of irreducible infinite root systems.

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§1. The category of sets in vector spaces

1.1. *Basic concepts.* Let k be a field. We introduce the category $\mathbf{SV}_k$ of sets in k-vector spaces as follows and refer to [**43**] for notions of category theory. The *objects* of $\mathbf{SV}_k$ are the pairs (R, X) where X is a k-vector space, and $R \subset X$ is a subset which spans X and contains the zero vector. To have a typographical distinction between the elements of R and those of X, the former will usually be denoted by Greek letters $\alpha, \beta, \ldots$, and the latter by $x, y, z, \ldots$.

The *morphisms* $f\colon (R, X) \to (S, Y)$ are the k-linear maps $f\colon X \to Y$ such that $f(R) \subset S$. Hence f is an isomorphism in $\mathbf{SV}_k$ if and only if f is a vector space isomorphism mapping R onto S. Clearly, the pair $0 = (\{0\}, \{0\})$ is a zero object of $\mathbf{SV}_k$.

There are two forgetful functors $\mathcal{S}$ and $\mathcal{V}$ from $\mathbf{SV}_k$ to the category $\mathbf{Set}_*$ of pointed sets and the category $\mathbf{Vec}_k$ of k-vector spaces, respectively, given by $\mathcal{S}(R, X) = R$ and $\mathcal{V}(R, X) = X$ on objects, and $\mathcal{S}(f) = f\big|R$ and $\mathcal{V}(f) = f$ on morphisms, respectively. Here the base point of the pointed set R is defined to be the null vector. We will use the notation

$$R^\times := R \setminus \{0\}$$

for the set of non-zero elements of R. Thus $R = \{0\} \mathbin{\dot\cup} R^\times$.

Clearly $\mathcal{V}$ is faithful and so is $\mathcal{S}$ because, due to the requirement that R span X, a linear map on X is uniquely determined by its restriction to R. It is easy to see that $\mathcal{V}$ has a right adjoint which assigns to any vector space X the pair $(X, X) \in \mathbf{SV}_k$. Also, $\mathcal{S}$ has a left adjoint $\mathcal{L}$, which assigns to any $S \in \mathbf{Set}_*$ the following object. Denote by 0 the base point of S and let, as above, $S^\times = S \setminus \{0\}$. Then $\mathcal{L}(S)$ is the pair $(\{0\} \cup \{\varepsilon_s : s \in S^\times\}, k^{(S^\times)})$, i.e., the free k-vector space on $S^\times$ and its canonical basis $\{\varepsilon_s : s \in S^\times\}$ together with the null vector 0. For a morphism $f\colon S \to T$ of pointed sets, the induced morphism $\mathcal{L}(f)$ maps ε_s to $\varepsilon_{f(s)}$. The adjunction condition

$$\mathbf{SV}_k(\mathcal{L}(S), (R, X)) \cong \mathbf{Set}_*(S, \mathcal{S}(R, X)) = \mathbf{Set}_*(S, R)$$

is clear from the universal property of the free vector space on a set. As a consequence, $\mathcal{S}$ commutes with limits and $\mathcal{V}$ commutes with colimits. This can also be seen in the following lemmas and propositions.

We next investigate some further basic properties of the category $\mathbf{SV}_k$.

1.2. Lemma. *Let $f\colon (R, X) \to (S, Y)$ be a morphism of $\mathbf{SV}_k$.*

(a) *f is a monomorphism $\iff$ $\mathcal{S}(f)$ is a monomorphism, i.e., $f\big|R\colon R \to S$ is injective.*

(b) *f is an epimorphism $\iff$ $\mathcal{V}(f)$ is an epimorphism, i.e., $f\colon X \to Y$ is surjective.*

(c) *$\mathbf{SV}_k$ admits finite direct products and arbitrary coproducts, given by*

$$\prod_{i=1}^{n}(R_i, X_i) = \big(\prod_{i=1}^{n} R_i, \prod_{i=1}^{n} X_i\big), \qquad \coprod_{i\in I}(R_i, X_i) = \big(\bigcup_{i\in I} R_i, \bigoplus_{i\in I} X_i\big).$$

Proof. (a) Let f be a monomorphism, i.e., left cancelable, and let $\alpha, \beta \in R$ with $f(\alpha) = f(\beta)$. Let $g, h\colon (\{0,1\}, k) = \mathcal{L}(\{0,1\}) \to (R, X)$ be defined by $g(1) = \alpha$ and $h(1) = \beta$. Then $f \circ g = f \circ h$ implies $g = h$ and hence $\alpha = \beta$. Thus $\mathcal{S}(f)$ is injective. The reverse implication follows from the fact that $\mathcal{S}$ is faithful.

(b) Let f be an epimorphism, i.e., cancelable on the right, but suppose $f\colon X \to Y$ is not surjective. Then $Y' = f(X) \subsetneqq Y$. Let $Z = Y/Y'$, $g\colon Y \to Z$ the canonical map, and $h = 0\colon Y \to Z$. Then $(Z, Z) \in \mathbf{SV}_k$, and $g \circ f = h \circ f = 0$ but $g \neq h$, contradiction. Again the reverse implication follows from faithfulness of $\mathcal{V}$.

(c) The proof consists of a straightforward verification. Note that $0 \in R_i$ and finiteness of the product is essential for $\prod_1^n R_i$ to span $\prod_1^n X_i$. Also, the union of the R_i in the second formula is understood as the union of the canonical images of the R_i under the inclusion maps $X_i \to \bigoplus_{j\in I} X_j$.

1.3. *Spans and cores, full subsets and tight subspaces.* Let $(R, X) \in \mathbf{SV}_k$. For a subset $S \subset R$ we denote by $\operatorname{span}(S)$ the linear span of S, and we define the *rank* of S by

$$\operatorname{rank}(S) = \dim(\operatorname{span}(S)).$$

For a vector subspace $V \subset X$, the *core* of V is

$$\operatorname{core}(V) = R \cap V.$$

The following rules are easily established:

$$\operatorname{core}(\operatorname{span}(S)) \supset S, \tag{1}$$
$$\operatorname{span}(\operatorname{core}(V)) \subset V, \tag{2}$$
$$\operatorname{span}(\operatorname{core}(\operatorname{span}(S))) = \operatorname{span}(S), \tag{3}$$
$$\operatorname{core}(\operatorname{span}(\operatorname{core}(V))) = \operatorname{core}(V). \tag{4}$$

A subset F of R is called *full* if $F = \operatorname{core}(\operatorname{span}(F))$, equivalently, because of (4), if $F = \operatorname{core}(V)$ for some subspace V. Dually, a subspace U of X is called *tight* if $U = \operatorname{span}(\operatorname{core}(U))$, equivalently, by (3), if $U = \operatorname{span}(S)$ for some subset S of R. The assignments $F \mapsto \operatorname{span}(F)$ and $U \mapsto \operatorname{core}(U)$ are inverse bijections between the set of full subsets of R and the set of tight subspaces of X. Also, for any subset S of R, $\operatorname{core}(\operatorname{span}(S))$ is the smallest full subset containing S. Dually, for any subspace V, $\operatorname{span}(\operatorname{core}(V))$ is the largest tight subspace contained in V. Note the *transitivity of fullness:* F' full in F and F full in R implies F' full in R. This is immediate from the definitions.

It is easy to see that the intersection of two full subsets is again full, and the sum of two tight subspaces is again tight. But the union of two full subsets is in general not full, nor is the intersection of two tight subspaces tight, see 1.8.

1.4. *Exactness.* For a monomorphism $f\colon (R, X) \to (S, Y)$ of $\mathbf{SV}_k$, the map $\mathcal{V}(f)\colon X \to Y$ of vector spaces is in general very far from being injective. Dually, the induced map $\mathcal{S}(f) = f\big|R\colon R \to S$ of an epimorphism need not be surjective.

For example, let k be a field of characteristic zero and let $(R, X) = \mathcal{L}(\mathbb{N})$, so X is the free vector space with basis ε_n, $n \geqslant 0$, and R consists of these basis vectors together with 0. Define $f\colon X \to k$ by $f(\varepsilon_n) = n$. Then $f\colon (R, X) \to (k, k)$ is a monomorphism and an epimorphism but of course not an isomorphism.

Stricter classes of mono- and epimorphisms are defined by means of exactness conditions as follows. A sequence of two morphisms

$$(E): \quad (S, Y) \xrightarrow{f} (R, X) \xrightarrow{g} (T, Z)$$

in $\mathbf{SV}_k$ is called *exact* if the sequences in $\mathbf{Set}_*$ and $\mathbf{Vec}_k$ obtained from it by applying the functors $\mathcal{S}$ and $\mathcal{V}$ are exact. Sequences of more than two morphisms are exact if every two-term subsequence is exact. The exactness of (E) can be expressed as follows:

$$(E) \text{ is exact } \iff \operatorname{Ker} \mathcal{V}(g) = \operatorname{span}(f(S)) \text{ and } f(S) = \operatorname{core}(\operatorname{Ker} \mathcal{V}(g)). \tag{1}$$

Indeed, the sequence $Y \to X \to Z$ of vector spaces is exact if and only if $\operatorname{Ker} \mathcal{V}(g) = \operatorname{Im} \mathcal{V}(f) = f(Y) = f(\operatorname{span}(S)) = \operatorname{span}(f(S))$, by linearity of f, and the sequence $S \to R \to T$ of pointed sets is exact if and only if $f(S) = \operatorname{Ker} \mathcal{S}(g) = \{\alpha \in R : g(\alpha) = 0\} = \operatorname{core}(\operatorname{Ker} \mathcal{V}(g))$. — We now consider some special cases.

(a) A sequence $0 \longrightarrow (S, Y) \xrightarrow{f} (R, X)$ is exact if and only if the linear map $f\colon Y \to X$ is injective. In particular, f is then a monomorphism by 1.2(a). We call such monomorphisms *exact monomorphisms*. Isomorphism classes of exact monomorphisms can be naturally identified with the inclusions $i\colon (S, \operatorname{span}(S)) \subset (R, X)$ where S is a subset of R.

(b) A sequence $(R, X) \xrightarrow{g} (T, Z) \longrightarrow 0$ is exact if and only if $g(R) = T$. Since Z is spanned by T, 1.2(b) shows that g is then an epimorphism, called an *exact epimorphism*. Isomorphism classes of exact epimorphisms can be naturally identified with the canonical maps $p = \operatorname{can}\colon (R, X) \to (\operatorname{can}(R), X/V)$ where V is any vector subspace of X.

(c) A sequence $0 \longrightarrow (S, Y) \xrightarrow{f} (R, X) \longrightarrow 0$ is exact if and only if f is an isomorphism.

(d) A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow (S, Y) \xrightarrow{f} (R, X) \xrightarrow{g} (T, Z) \longrightarrow 0\,. \tag{2}$$

After the identifications of (a) and (b), (2) becomes

$$0 \longrightarrow (R', X') \xrightarrow{i} (R, X) \xrightarrow{p} (R/R', X/X') \longrightarrow 0 \tag{3}$$

where now $R' \subset R$ and $X' \subset X$ are a subset and a vector subspace, respectively, such that

$$X' = \operatorname{span}(R') \quad \text{and} \quad R' = \operatorname{core}(X'). \tag{4}$$

Here $R/R' = \operatorname{can}(R)$ denotes the canonical image of R in X/X'.

1.5. *Quotients by full subsets and tight subspaces.* From 1.4.4 it is clear that in an exact sequence 1.4.3, R' is full and X' is tight. Conversely, any full subset R' of R gives rise to a short exact sequence 1.4.3 by setting $X' = \operatorname{span}(R')$, and so does any tight subspace X' by setting $R' = \operatorname{core}(X')$. We then call

$$(R, X)/(R', X') := (R/R', X/X') \tag{1}$$

the *quotient* of (R, X) by the full subset R' (or the tight subspace X'). Since R spans X, we have

$$\operatorname{rank}(R/R') = \dim(X/X'),$$

also called the *corank of R' in R*.

A *finite quotient* is by definition a quotient by a finite-dimensional tight subspace X', equivalently, by a full subset R' of finite rank.

For $\alpha \in R$, the *coset of* $\alpha \bmod R'$ is the set $R \cap (\alpha + X')$, i.e., the fiber through α of $\mathcal{S}(p)$. The coset of an element $\alpha' \in R'$ is $R \cap (\alpha' + X') = R \cap X' = \operatorname{core}(X') = R'$. Clearly R is the disjoint union of its cosets $\bmod R'$ so the number of cosets is the cardinality of R/R'. Note, however, that unlike the cosets of a subgroup in a group, the cosets $\bmod R'$ may have different cardinalities. For example, in the root system $R = \mathrm{B}_2 = \{0\} \cup \{\pm\varepsilon_1, \pm\varepsilon_2\} \cup \{\pm\varepsilon_1 \pm \varepsilon_2\} \subset \mathbb{R}^2$ (see 8.1), the full subset $R' = \{0\} \cup \{\pm(\varepsilon_1 + \varepsilon_2)\}$ has five cosets, two of cardinality 1, two of cardinality 2 and one of cardinality 3.

1.6. Lemma. *Let $(R, X) = \coprod (R_i, X_i) = (\bigcup R_i, \bigoplus X_i)$ be the coproduct of a family (R_i, X_i) in $\mathbf{SV}_k$ as in* 1.2.

(a) *The tight subspaces of X are precisely the subspaces $X' = \bigoplus X_i'$ where the X_i' are tight subspaces of X_i'.*

(b) *The full subsets of R are precisely the subsets $R' = \bigcup R_i'$ where the R_i' are full subsets of R_i.*

(c) *Quotients commute with coproducts: If $X' \subset X$ is tight with core R' then, with the above notations,*

$$(R/R', X/X') \;\cong\; \coprod_{i \in I} (R_i/R_i', X_i/X_i').$$

Proof. (a) X' is tight if and only if X' is the span of a subset of R. Since R is the union of the $R_i \subset X_i$, the assertion follows.

(b) R' is full if and only if it is the core of $\operatorname{span}(R')$ which is a tight subspace. Now our claim follows from (a).

(c) This is immediate from (a) and (b).

We now prove the First Isomorphism Theorem in the category $\mathbf{SV}_k$. The canonical map $p\colon X \to X/X'$ of a quotient of (X, R) as in 1.5.1 will often be denoted by a bar.

1.7. Proposition (First Isomorphism Theorem). *Let $(\bar{R}, \bar{X}) = (R/R', X/X')$ be a quotient of (R, X).*

(a) *For any subset S of R, $p(\operatorname{span}(S)) = \operatorname{span}(p(S))$, and for any subspace $V \supset X'$ of X,*

$$p(\mathrm{core}(V)) = \mathrm{core}(p(V)). \tag{1}$$

(b) *Let $Y \supset X'$ be a tight subspace. Then $\bar{Y}$ is tight in $\bar{X}$, and the assignment $Y \mapsto \bar{Y}$ is a bijection between the set of tight subspaces of X containing X', and the set of all tight subspaces of X/X', with inverse map $U \mapsto p^{-1}(U)$, for a tight subspace $U \subset \bar{X}$.*

(c) *Let $S \supset R'$ be a full subset. Then $\bar{S}$ is full in $\bar{R}$, and the assignment $S \mapsto \bar{S}$ is a bijection from the set of full subsets $S \supset R'$ of R to the set of full subsets of $\bar{R}$.*

(d) *Let $Y \supset X'$ be tight with* $\mathrm{core}(Y) = S$. *Then the canonical vector space isomorphism $X/Y \xrightarrow{\cong} \bar{X}/\bar{Y}$ is also an isomorphism*

$$(R/S, X/Y) \xrightarrow{\cong} (\bar{R}/\bar{S}, \bar{X}/\bar{Y}) = \Big(\frac{R/R'}{S/R'}, \frac{X/X'}{Y/X'}\Big) \tag{2}$$

in the category $\mathbf{SV}_k$.

Proof. (a) The first statement is clear from linearity of p. Now let $V \supset X'$. Then $p(\mathrm{core}(V)) = p(R \cap V) \subset p(R) \cap p(V) = \bar{R} \cap p(V) = \mathrm{core}(p(V))$. Conversely, if $\beta \in \mathrm{core}(p(V))$ then $\beta = \bar{\alpha}$ for some $\alpha \in R$ and also $\beta = \bar{v}$ for some $v \in V$. Hence $\alpha - v \in \mathrm{Ker}(p) = X' \subset V$, showing $\alpha \in R \cap V = \mathrm{core}(V)$ and hence $\beta = \bar{\alpha} \in p(\mathrm{core}(V))$.

(b) Let $Y = \mathrm{span}(\mathrm{core}(Y)) \supset X'$ be a tight subspace. Since p commutes with spans and cores by (a), it follows that $p(Y) = p(\mathrm{span}(\mathrm{core}(Y))) = \mathrm{span}(\mathrm{core}(p(Y)))$, so that $p(Y)$ is tight. Conversely, let $U \subset \bar{X}$ be tight. Then $U = p(Y)$ for $Y := p^{-1}(U)$, so it suffices to show that Y is tight. By tightness of U and (a), $p(Y) = \mathrm{span}(\mathrm{core}(p(Y))) = p(\mathrm{span}(\mathrm{core}(Y)))$. It follows that $Y \subset \mathrm{span}(\mathrm{core}(Y)) + X'$. But $X' = \mathrm{span}(R')$ is contained in Y, hence $R' = \mathrm{core}(X') \subset \mathrm{core}(Y)$ and therefore $X' \subset \mathrm{span}(\mathrm{core}(Y))$, showing that $Y = \mathrm{span}(\mathrm{core}(Y))$ is tight.

(c) By (1) applied to $V = \mathrm{span}(S) \supset X'$ and linearity of p, we see $p(S) = p(\mathrm{core}(\mathrm{span}(S))) = \mathrm{core}(\mathrm{span}(p(S)))$, so $p(S)$ is full. Conversely, let $F \subset \bar{R}$ be full with linear span U, and let $V = p^{-1}(U) \supset X'$. Then $S := \mathrm{core}(V) \supset R'$ is full, and $p(S) = p(\mathrm{core}(V)) = \mathrm{core}(p(V))$ (by (1)) $= \mathrm{core}(U) = \mathrm{core}(\mathrm{span}(F)) = F$, by fullness of F.

(d) By (a) and (b), $\bar{Y}$ is tight in $\bar{X}$ with core $\bar{S}$. Hence the quotient on the right hand side of (2) makes sense. From the First Isomorphism Theorem in the category of vector spaces, the canonical map $f\colon X/Y \to \bar{X}/\bar{Y}$, $x + Y \mapsto \bar{x} + \bar{Y}$, is a vector space isomorphism. Hence it suffices to show that $f(R/S) = \bar{R}/\bar{S}$. This is evident from the fact that the canonical maps $R \to R/S$, $\bar{R} \to \bar{R}/\bar{S}$ and $R \to \bar{R}$ are surjective.

1.8. *Tight intersections.* Let $(R, X) \in \mathbf{SV}_k$ and let S and R' be full subsets of R with linear spans $Y = \mathrm{span}(S)$ and $X' = \mathrm{span}(R')$, respectively. The intersection of (S, Y) and (R', X') in the categorical sense, i.e., the pullback of the inclusions $(S, Y) \xrightarrow{j} (R, X) \xleftarrow{i} (R', X')$ exists in $\mathbf{SV}_k$, and is easily seen to be

$$(S, Y) \cap (R', X') = (S \cap R', \mathrm{span}(S \cap R')). \tag{1}$$

Note that, by fullness of S and R',

$$S \cap R' = R \cap Y \cap X' = \operatorname{core}(Y \cap X') = S \cap X' = R' \cap Y, \tag{2}$$

so $S \cap R'$ is again full in R and also in S and R', and

$$Y' := \operatorname{span}(S \cap R') = \operatorname{span}(\operatorname{core}(Y \cap X')) \subset Y \cap X' \tag{3}$$

is the largest tight subspace of $Y \cap X'$. But the subspace $Y \cap X'$ is in general not tight, reflecting the fact that the functor $\mathcal{V}$ does not commute with all projective limits (cf. 1.1). We say S and R' *intersect tightly* if $Y \cap X'$ *is* tight, i.e., if equality holds in (3).

For example, in the root system $R = \mathrm{B}_3 = \{0\} \cup \{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_3\} \cup \{\pm\varepsilon_1 \pm \varepsilon_2, \pm\varepsilon_1 \pm \varepsilon_3, \pm\varepsilon_2 \pm \varepsilon_3\} \subset \mathbb{R}^3$, the full subsets $S = \{0\} \cup \{\pm(\varepsilon_1 - \varepsilon_2)\} \cup \{\pm\varepsilon_3\}$ and $R' = \{0\} \cup \{\pm\varepsilon_1\} \cup \{\pm(\varepsilon_2 - \varepsilon_3)\}$ do not intersect tightly, since $S \cap R' = \{0\}$ while $\operatorname{span}(S) \cap \operatorname{span}(R')$ is the line $\mathbb{R}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3)$. On the other hand, S and $R'' = \{0\} \cup \{\pm(\varepsilon_1 - \varepsilon_2)\} \cup \{\pm\varepsilon_2\}$ do intersect tightly.

Returning to the general situation, we have an exact sequence of vector spaces

$$0 \longrightarrow (Y \cap X')/Y' \longrightarrow Y/Y' \xrightarrow{\kappa} X/X' \longrightarrow X/(Y + X') \longrightarrow 0 \tag{4}$$

where κ: $Y/Y' \to X/X'$ is induced from the inclusion j: $Y \subset X$. Note the following equivalent characterizations of tight intersection:

(i) S and R' intersect tightly,

(ii) κ is injective,

(iii) any subset of Y which is linearly independent modulo Y' remains so modulo X'.

Indeed, the equivalence of (i) and (ii) is clear from (4), and (iii) is simply a reformulation of (ii).

We now state the Second Isomorphism Theorem in the category $\mathbf{SV}_k$.

1.9. Proposition (Second Isomorphism Theorem). *Let $(R, X) \in \mathbf{SV}_k$ and let S and R' be full subsets of R with linear spans $Y = \operatorname{span}(S)$ and $X' = \operatorname{span}(R')$. Then the following conditions are equivalent:*

(i) *S and R' intersect tightly, and S meets every coset of $R \bmod R'$,*

(ii) *the canonical homomorphism κ of 1.8.4 is an isomorphism*

$$(S, Y)\big/\big((S, Y) \cap (R', X')\big) \;\cong\; (R, X)\big/(R', X').$$

Proof. We use the notations introduced in 1.8 and also set $S' := S \cap R'$, so that $Y' = \operatorname{span}(S')$.

(i) $\Longrightarrow$ (ii): By tightness of $Y \cap X'$ and (ii) of 1.8, κ: $Y/Y' \to X/X'$ is injective. Since S meets every coset of $R \bmod R'$, we have $R \subset S + X'$ and hence $X = \operatorname{span}(R) = \operatorname{span}(S) + X' = Y + X'$, so 1.8.4 shows that κ is a vector space isomorphism. It remains to show $\kappa(S/S') = R/R'$. Let p: $(R, X) \to (R/R', X/X')$ and q: $(S, Y) \to (S/S', Y/Y')$ be the canonical maps. Then the diagram

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d, "q"'] & X \arrow[d, "p"] \\ Y/Y' \arrow[r, "\kappa"'] & X/X' \end{tikzcd}$$

is commutative. Since q: $S \to S/S'$ is surjective and S meets every coset of $R \bmod R'$, we have $\kappa(S/S') = p(S) = p(R) = R/R'$.

(ii) $\Longrightarrow$ (i): Since κ is a vector space isomorphism $Y/Y' \xrightarrow{\cong} X/X'$, 1.8.4 shows $(Y \cap X')/Y' = 0$ or $Y' = Y \cap X'$, so S and R' intersect tightly. Also, $\kappa(S/S') = R/R'$ means that for every $\alpha \in R$ there exists $\beta \in S$ with $p(\beta) = \kappa(q(\beta)) = p(\alpha)$, that is, $\beta \equiv \alpha \bmod X'$, so β is in the coset of $\alpha \bmod R'$.

We next investigate equalizers and coequalizers in the category $\mathbf{SV}_k$. Note that, due to the existence of a zero element, the notions of kernel and cokernel of a morphism f in $\mathbf{SV}_k$, i.e., equalizer and coequalizer of the pair of morphisms $(f, 0)$, are well defined.

1.10. Proposition. (a) *The category* $\mathbf{SV}_k$ *admits equalizers: If* f, g: $(R, X) \to (S, Y)$ *are morphisms then an equalizer of* f *and* g *is the inclusion* $(R', X') \subset (R, X)$ *where* $R' = \{\alpha \in R : f(\alpha) = g(\alpha)\}$ *and* $X' = \operatorname{span}(R')$.

(b) *For a subset* R' *of* R *with linear span* X' *the following conditions are equivalent:*

(i) R' *is full,*
(ii) *every morphism* h: $(T, Z) \to (R, X)$ *with* $h(Z) \subset X'$ *factors via* (R', X'),
(iii) (R', X') *is the kernel of a morphism with domain* (R, X),
(iv) (R', X') *is the equalizer of a double arrow with domain* (R, X).

Proof. (a) Clearly $(R', X') \in \mathbf{SV}_k$ and the inclusion $(R', X') \subset (R, X)$ is a monomorphism. Let h: $(T, Z) \to (R, X)$ be a morphism with $f \circ h = g \circ h$. Then $f(h(\alpha)) = g(h(\alpha))$ for all $\alpha \in T$, whence $h(T) \subset R'$. Since T spans Z and h is linear, we have $h(Z) \subset X'$, so h factors via (R', X').

(b) (i) $\Longleftrightarrow$ (ii): Let R' be full. For $\beta \in T$ we have $h(\beta) \in R \cap X' = R'$ so h factors via (R', X'). To prove the converse, let $\alpha \in R \cap X'$ and consider the morphism h: $(\{0, 1\}, k) \to (R, X)$ given by $h(1) = \alpha$. Then $h(k) = k \cdot \alpha \subset X'$, so h factors via (R', X') and we conclude $h(1) = \alpha \in R'$.

(i) $\Longrightarrow$ (iii): Let p: $(R, X) \to (R/R', X/X')$ be the quotient of (R, X) by R' as in 1.5.1. Then by (a), the kernel of p is $\{\alpha \in R : p(\alpha) = 0\} = R \cap X' = R'$ together with its linear span X'.

(iii) $\Longrightarrow$ (iv): Obvious.

(iv) $\Longrightarrow$ (i): This follows from the description of the equalizer in (a).

1.11. Proposition. (a) *The category* $\mathbf{SV}_k$ *admits coequalizers: If* f, g: $(S, Y) \to (R, X)$ *are morphisms then a coequalizer of* f *and* g *is* p: $(R, X) \to (R'', X'')$ *where* $X'' = X/(f - g)(Y)$, p: $X \to X''$ *is the canonical projection and* $R'' = p(R)$.

(b) *For a morphism* p: $(R, X) \to (R'', X'')$ *the following conditions are equivalent:*

(i) $p(R) = R''$, *and the kernel* $\operatorname{Ker} \mathcal{V}(p) \subset X$ *of the linear map* p *is spanned by its intersection with* $R - R = \{\alpha - \beta : \alpha, \beta \in R\}$,
(ii) $p(R) = R''$, *and whenever* h: $(R, X) \to (T, Z)$ *is a morphism such that* $\mathcal{S}(h)$: $R \to T$ *factors via* $\mathcal{S}(p)$ *in* $\mathbf{Set}_*$, *then* h *factors via* p *in* $\mathbf{SV}_k$,
(iii) p *is the coequalizer of a pair of morphisms with codomain* (R, X).

Proof. (a) Let h: $(R, X) \to (T, Z)$ be a morphism with the property that $h \circ f = h \circ g$. We must show that $h = h' \circ p$ factors via p. Clearly, there is a unique linear map $h'\colon X'' \to Z$ with this property, and $h'(R'') \subset T$ follows readily from the definition of R''.

(b) (i) $\Longrightarrow$ (ii): That $\mathcal{S}(h)$ factors via $\mathcal{S}(p)$ means that $p(\alpha) = p(\beta)$ implies $h(\alpha) = h(\beta)$, for all $\alpha, \beta \in R$. Hence $\alpha - \beta \in \operatorname{Ker} \mathcal{V}(p)$ implies $\alpha - \beta \in \operatorname{Ker} \mathcal{V}(h)$. Since by assumption $\operatorname{Ker} \mathcal{V}(p)$ is spanned by all these differences, it follows that $\operatorname{Ker} \mathcal{V}(p) \subset \operatorname{Ker} \mathcal{V}(h)$, so there exists a unique linear map $h'\colon X'' \to Z$ such that $h = h' \circ p$ in $\mathbf{SV}_k$.

(ii) $\Longrightarrow$ (i): Let $V \subset X$ be the linear span of all $\alpha - \beta$, where $\alpha, \beta \in R$ and $p(\alpha) = p(\beta)$. Define $Z = X/V$, $h = \text{can}\colon X \to Z$, and $T = h(R)$. Then $p(\alpha) = p(\beta)$ implies $h(\alpha - \beta) = 0$ or $h(\alpha) = h(\beta)$, so $\mathcal{S}(h)$ factors via $\mathcal{S}(p)$. By assumption, this implies that $h = h' \circ p$ factors via p in $\mathbf{SV}_k$. Hence also $\mathcal{V}(h) = \mathcal{V}(h') \circ \mathcal{V}(p)$, and thus $\operatorname{Ker} \mathcal{V}(p) \subset \operatorname{Ker} \mathcal{V}(h) = V$, as required.

(i) $\Longrightarrow$ (iii): Let $\{\alpha_i - \beta_i : i \in I\} \subset R - R$ be a spanning set of $\operatorname{Ker} \mathcal{V}(p)$ where I is a suitable index set. Let $Y = k^{(I)}$ be the free vector space with basis $(\varepsilon_i)_{i \in I}$ and let $S = \{0\} \cup \{\varepsilon_i : i \in I\}$. Define morphisms $f, g\colon (S, Y) \to (R, X)$ by $f(\varepsilon_i) = \alpha_i$ and $g(\varepsilon_i) = \beta_i$. Then (a) shows that p is the coequalizer of f and g.

(iii) $\Longrightarrow$ (i): Let p be the coequalizer of $f, g\colon (S, Y) \to (R, X)$. By (a), the kernel of $\mathcal{V}(p)$ is $(f - g)(Y)$, and since Y is spanned by S, the kernel of p is spanned by $\{f(\gamma) - g(\gamma) : \gamma \in S\} \subset R - R$. Also by (a), we have $R'' = p(R)$.

1.12. Corollary. *The category $\mathbf{SV}_k$ has all finite limits and all colimits.*

This follows from 1.2(c), 1.10(a) and 1.11(a) and standard results in category theory.

While by Prop. 1.10(b) every equalizer in $\mathbf{SV}_k$ is a kernel, the dual statement is not true. Rather, there is the following characterization of cokernels:

1.13. Corollary. *A morphism $p\colon (R, X) \to (R'', X'')$ is the cokernel of some $f\colon (S, Y) \to (R, X)$ if and only if $p(R) = R''$ and $\operatorname{Ker} \mathcal{V}(p)$ is tight.*

This follows from 1.11 by specializing $g = 0$.

1.14. Corollary. *A sequence as in* 1.4.2 *is exact if and only if f is the kernel of g and g is the cokernel of f.*

§2. Finiteness conditions and bases

2.1. *Local finiteness.* We keep the notations introduced in §1. An object (R, X) of $\mathbf{SV}_k$ is called *locally finite* if it satisfies the following equivalent conditions:

(i) every finite-dimensional subspace V of X has finite $\operatorname{core}(V) = R \cap V$,

(ii) every finite-ranked subset F of R is finite.

To see the equivalence, apply (ii) to $\operatorname{core}(V)$ and (i) to $\operatorname{span}(F)$, respectively. We also note that it suffices to have (i) for tight subspaces only, since $\operatorname{core}(V) = \operatorname{core}(V')$ where $V' = \operatorname{span}(\operatorname{core}(V)) \subset V$, by 1.3.4. Similarly, it suffices to require (ii) for full subsets.

Obviously, if (R, X) is locally finite and $S \subset R$ is any subset containing 0, then $(S, \operatorname{span}(S))$ is locally finite. From 1.2(c) it follows easily that finite direct products and arbitrary coproducts of locally finite sets are again locally finite. Also, finite quotients (cf. 1.5) of a locally finite (R, X) are again locally finite. Indeed, let $(\bar{R}, \bar{X}) = (R/R', X/X')$ where X' is finite-dimensional. By 1.7(b), a finite-dimensional tight subspace of $\bar{X}$ is of the form $\bar{V}$ where $V \supset X'$ is tight. Since $\dim(V) = \dim(X') + \dim(\bar{V}) < \infty$, we have $\operatorname{core}(V)$ finite, and hence so is $\operatorname{core}(\bar{V})$ by 1.7.1. However, local finiteness is not inherited by arbitrary quotients, as Example 2.3 below shows.

Let $\mathbf{c}$ be an infinite cardinal, and denote by $|M|$ the cardinality of a set M. If (R, X) is locally finite then for any full subset $S \subset R$ of infinite rank,

$$|S| < \mathbf{c} \quad \Longleftrightarrow \quad \operatorname{rank}(S) < \mathbf{c}. \tag{1}$$

Indeed, let $B \subset S$ be a vector space basis of $Y = \operatorname{span}(S)$. Then $\dim(Y) = |B| \leqslant |S|$ proves the implication from left to right. Conversely, let $\mathbf{2}^{(B)}$ denote the set of finite subsets of B. Then S is the union of the finite sets $\operatorname{core}(\operatorname{span}(F))$, $F \in \mathbf{2}^{(B)}$, and hence $|S| \leqslant \aleph_0 \cdot |\mathbf{2}^{(B)}| = \aleph_0 \cdot |B| = |B|$, by standard facts of cardinal arithmetic, see for example [**18**].

2.2. *Boundedness and strong boundedness.* We now introduce finiteness conditions which not only require the core of any finite-dimensional subspace V of X to be finite, but actually bound its cardinality by a function of the dimension of V. First we define the admissible bounding functions. A function $b\colon \mathbb{N} \to \mathbb{N}$ is called a *bound* if it is superadditive, i.e., $b(m+n) \geqslant b(m) + b(n)$, and satisfies $b(1) \geqslant 1$. This last requirement merely serves to avoid trivial cases. It is easy to see that $b(0) = 0$, and that b is increasing. Also $b(n) \geqslant nb(1) \geqslant n$, and $b_0(n) = n$ is the smallest bound. Other examples are functions of type $b(n) = c(a^n - 1)$ for integers $c \geqslant 1$ and $a \geqslant 2$. Now we say (R, X) is *bounded by* b, or b-*bounded* for short, if

$$\big|\operatorname{core}(V)^{\times}\big| \;\leqslant\; b\big(\dim(V)\big), \tag{1}$$

for every finite-dimensional subspace V of X. Since b is increasing, it suffices to have (1) for tight subspaces only. An equivalent condition is

$$|F^{\times}| \leqslant b\big(\operatorname{rank}(F)\big), \tag{2}$$

for every finite subset of R. Indeed, if (2) holds and V is a finite-dimensional subspace of X, then $|F^\times| \leqslant b(\dim(V))$ for every finite subset F of $\operatorname{core}(V) = V \cap R$, which implies (1). The other implication is obvious. It is clear that a bounded (R, X) is locally finite.

Finite quotients of a b-bounded (R, X) are in general no longer bounded by b, and arbitrary quotients need not even be locally finite, see 2.3. We therefore define (R, X) to be *strongly bounded by* b if ((R, X) itself and) every finite quotient of (R, X) (as in 1.5) is bounded by b. Then strong b-boundedness descends to all finite quotients. This follows from the First Isomorphism Theorem by a similar argument as the local finiteness of finite quotients in 2.1. We will show in Theorem 2.6 that in fact all quotients inherit strong b-boundedness.

2.3. *Example.* Let k be a field of characteristic zero, let $X = k^{(\mathbb{N})}$ with basis ε_i, $i \in \mathbb{N}$, and let $R^\times = \{\varepsilon_i : i \geqslant 1\} \cup \{\varepsilon_j + j\varepsilon_0 : j \geqslant 1\}$. Then (R, X) is bounded by $b(n) = 2n$. Indeed, if $F \subset R$ is finite then

$$F^\times = \{\varepsilon_i : i \in I\} \cup \{\varepsilon_j + j\varepsilon_0 : j \in J\},$$

for suitable finite subsets I, J of $\mathbb{N}_+$. It follows that

$$\operatorname{span}(F) = \left\{ \begin{array}{ll} \Big(\bigoplus_{i \in I} k \cdot \varepsilon_i\Big) \oplus \Big(\bigoplus_{j \in J} k \cdot (\varepsilon_j + j\varepsilon_0)\Big) & \text{if } I \cap J = \emptyset \\ k \cdot \varepsilon_0 \oplus \bigoplus_{i \in I \cup J} k \cdot \varepsilon_i & \text{if } I \cap J \neq \emptyset \end{array} \right\},$$

with dimension

$$\operatorname{rank}(F) = \left\{ \begin{array}{ll} |I| + |J| & \text{if } I \cap J = \emptyset \\ 1 + |I \cup J| & \text{if } I \cap J \neq \emptyset \end{array} \right\} \geqslant \max(|I|, |J|) \geqslant \frac{1}{2}(|I| + |J|).$$

Hence $|F^\times| \leqslant |I| + |J| \leqslant 2 \operatorname{rank}(F)$, proving our assertion. On the other hand, there exists no bound b such that all finite quotients of (R, X) are b-bounded. Indeed, let $X_n = \operatorname{span}\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $R_n = R \cap X_n$. Then $X/X_n \cong k \cdot \bar\varepsilon_0 \bigoplus_{i>n} k \cdot \bar\varepsilon_i$ and $R/R_n \cong \{0\} \cup \{\bar\varepsilon_i : i > n\} \cup \{\bar\varepsilon_0, 2\bar\varepsilon_0, \ldots, n\bar\varepsilon_0\}$. Letting $Y_n = k \cdot \varepsilon_0 + X_n$, we have $\operatorname{core}(Y_n)^\times = \{\varepsilon_1, \ldots, \varepsilon_n\} \cup \{\varepsilon_1 + \varepsilon_0, \ldots, \varepsilon_n + n\varepsilon_0\}$. Thus $\dim(Y_n/X_n) = 1$ but $|\operatorname{core}(Y_n/X_n)^\times| = n$. Also, for $R' = \{0\} \cup \{\varepsilon_i : i \geqslant 1\} = \bigcup_{n \geqslant 1} R_n$, with linear span $X' = \bigoplus_{i \geqslant 1} k \cdot \varepsilon_i = \bigcup_{n \geqslant 1} X_n$, we have $X/X' \cong k$ one-dimensional but $R/R' \cong \mathbb{N} \subset k$ infinite, showing that quotients do not inherit local finiteness.

2.4. Lemma. (a) *If (R, X) is (strongly) bounded by b and $Y \subset X$ is a tight subspace with core S, then (S, Y) is again (strongly) bounded by b.*

(b) *If (R_i, X_i) ($i \in I$) are (strongly) bounded by b then so is their coproduct (R, X) (cf.* 1.2*).*

Proof. (a) This is obvious from the definitions.

(b) Since coproducts commute with quotients by 1.6, it suffices to prove the statement about boundedness. Thus let $V \subset X = \bigoplus_{i \in I} X_i$ be a tight subspace. By 1.6, $V = \bigoplus_{i \in I} V_i$ where $V_i = V \cap X_i$. Hence if V is finite-dimensional, we have $V_j \neq 0$ only for j in a finite subset J of I. Therefore

$$\operatorname{core}(V)^\times = \dot{\bigcup_{j\in J}} \big(\operatorname{core}(V_j)^\times\big) \quad \text{(disjoint union)}.$$

Since all (R_i, X_i) are bounded by b, it follows from superadditivity of b that

$$\begin{aligned} |\operatorname{core}(V)^\times| &\leqslant \sum_{j\in J} \big(|\operatorname{core}(V_j)^\times|\big) \;\leqslant\; \sum_{j\in J} b(\dim(V_j)) \\ &\leqslant\; b\big(\sum_{j\in J}\dim(V_j)\big) \;=\; b(\dim(V)). \end{aligned}$$

2.5. Lemma. *Let $(R, X) \in \mathbf{SV}_k$, let $R' \subset R$ be a full subset with linear span X', and let $\mathbf{c}$ be an infinite cardinal. Then any subset E of R of cardinality $|E| < \mathbf{c}$ is contained in a full subset S of R which intersects R' tightly (see* 1.8*) and has* $\operatorname{rank}(S) < \mathbf{c}$.

Proof. After replacing X by $\operatorname{span}(E) + X'$ and R by its intersection with this subspace, it is no restriction to assume that X is spanned by $E \cup R'$. Choose a subset B of E representing a vector space basis of X/X', let $X'' = \operatorname{span}(B)$ so that $X = X'' \oplus X'$, and let $\pi\colon X \to X'$ be the projection along X''. Since X' is spanned by R', there exists, for every $\alpha \in E$, a finite subset T_α of R' such that $\pi(\alpha) \in \operatorname{span}(T_\alpha)$. Let $T = \bigcup_{\alpha\in E} T_\alpha \subset R'$ and let $Y' := \operatorname{span}(T)$. Then we have $\pi(E) \subset Y'$. Moreover, $\dim Y' \leqslant \sum_{\alpha\in E} |T_\alpha| < \mathbf{c}$ since each T_α is finite and $|E| < \mathbf{c}$. Let $Y := X'' \oplus Y'$. Then $S = \operatorname{core}(Y)$ has the asserted properties. Indeed, S is full, being the core of a subspace. By construction, $E \subset X'' \oplus \pi(E) \subset X'' \oplus Y' = Y$ whence $E \subset R \cap Y = \operatorname{core}(Y) = S$. To show that S and R' intersect tightly, first note that $Y = \operatorname{span}(S)$ is tight, being the sum of the two tight subspaces $X'' = \operatorname{span}(B)$ and $Y' = \operatorname{span}(T)$. Hence we must show that $Y \cap X'$ is spanned by $S \cap R'$. From $Y = X'' \oplus Y'$ and $X = X'' \oplus X'$ as well as $Y' \subset X'$ it is clear that $Y \cap X' = Y'$. Now $Y' = \operatorname{span}(T)$ by definition, $T \subset R'$ by construction and clearly $T \subset \operatorname{core}(Y') \subset \operatorname{core}(Y) = S$. Finally, $\operatorname{rank}(S) = \dim(Y) = |B| + \dim Y' < \mathbf{c} + \mathbf{c} = \mathbf{c}$, since $\mathbf{c}$ is an infinite cardinal. This completes the proof.

2.6. Theorem. *If (R, X) is strongly bounded by b then so are all quotients $(\bar R, \bar X) = (R/R', X/X')$.*

Proof. We need to show boundedness of all quotients of $(\bar R, \bar X)$ by finite-dimensional tight subspaces U of $\bar X$. In view of the First Isomorphism Theorem 1.7, such a quotient is isomorphic to the quotient of (R, X) by the tight subspace $p^{-1}(U) \supset X'$. Therefore, after replacing X' by $p^{-1}(U)$, it suffices to show that all quotients $(\bar R, \bar X)$ of (R, X) are bounded by b.

Thus let now $V \subset \bar X$ be a tight finite-dimensional subspace. After replacing X by the tight subspace $p^{-1}(V) \supset X'$ and R by the core of this subspace, we may even assume that $\bar X$ is finite-dimensional, and only have to show that $|\bar R^\times| \leqslant b(\dim(\bar X))$. Consider a finite subset of $\bar R$ which we may assume of the form $\bar E$ where E is a finite subset of R. By Lemma 2.5, applied in case $\mathbf{c} = \aleph_0$, there exists a finite-ranked full subset $S \subset R$ containing E and intersecting R' tightly. We let $Y = \operatorname{span}(S)$, $Y' = Y \cap X'$, and $S' = S \cap R' = \operatorname{core}(Y')$. Then $Y' \subset Y$ are finite-dimensional tight subspaces of X. Since $\kappa\colon Y/Y' \to X/X'$ is injective by (ii) of 1.8, we have

$$\dim(Y/Y') \leqslant \dim(X/X') = \dim(\bar{X}).$$

As (R,X) is strongly bounded by b, the finite quotient $(R/S', X/Y')$ is bounded by b. From monotonicity of b it now follows that

$$|(S/S')^\times| = |\operatorname{core}(Y/Y')^\times| \leqslant b(\dim(Y/Y')) \leqslant b(\dim(\bar{X})).$$

Moreover, $\bar{S} = \kappa(S)$ so we also have $|\bar{E}^\times| \leqslant |\bar{S}^\times| \leqslant b(\dim(\bar{X}))$. As $\bar{E}$ was an arbitrary finite subset of $\bar{R}$, we conclude $|\bar{R}^\times| \leqslant b(\dim(\bar{X}))$, as desired.

2.7. *A-Bases and the extension property.* For the remainder of this section, we fix a subring A of the base field k. Let $(R,X) \in \mathbf{SV}_k$. A subset B of R is called an *A-basis of R* if

(i) B is k-free, and
(ii) every element of R is an A-linear combination of B.

Suppose (R,X) admits an A-basis B. Since R spans X, it is clear that B is in particular a vector space basis of X. Denoting by $A[R]$ the A-submodule of X generated by R, we see that

$$A[R] = \bigoplus_{\beta \in B} A \cdot \beta \tag{1}$$

is a free A-module with basis B. Also, the canonical homomorphism $A[R] \otimes_A k \to X$ is an isomorphism of k-vector spaces since it maps the k-basis $\{\beta \otimes 1 : \beta \in B\}$ of $A[R] \otimes_A k$ bijectively onto the k-basis B of X.

It turns out that a stronger condition than mere existence of A-bases is more useful. We say (R,X) has the *extension property for A* or the *A-extension property* if for every pair $S' \subset S$ of full subsets of R, with spans $Y' \subset Y$, every A-basis of (S',Y') extends to an A-basis of (S,Y). Also, (R,X) is said to have the *finite A-extension property* if this holds for all full subsets $S' \subset S$ of finite rank. As long as the ring A remains fixed, we will usually omit it when speaking of the extension properties.

The extension property is equivalent to the *existence of adapted bases* in the following sense: for all $(S',Y') \subset (S,Y)$ as above, there exist A-bases B' of (S',Y') and B of (S,Y) such that $B' \subset B$. Indeed, the extension property applied to $S' = 0$, $B' = \emptyset$ implies the existence of bases, so in particular S' has a basis which, again by the extension property, can be extended to a basis of S. Conversely, suppose adapted bases exist and let B_1' be a basis of S'. We can then choose adapted bases $B' \subset B$ of $S' \subset S$. Then $B_1 := (B \setminus B') \cup B_1'$ is a basis of S extending B_1'. An analogous statement holds for the finite extension property.

Finally, (R,X) is said to be *A-exact* if for every full subset R' with span X', the sequence

$$0 \longrightarrow A[R'] \xrightarrow{\ i\ } A[R] \xrightarrow{\ p\ } A[R/R'] \longrightarrow 0 \tag{2}$$

is an exact sequence of A-modules. Here i and p are induced from the inclusion $(R',X') \subset (R,X)$ and the canonical map $(R,X) \to (R/R', X/X')$. Hence it is clear that i is injective and p is surjective, so exactness of (2) is equivalent to the intersection condition

$$A[R'] = A[R] \cap X'. \tag{3}$$

2.8. Lemma. *Let $R' \subset R$ be full and suppose that* 2.7.3 *holds. Let B' be an A-basis of R', let C be an A-basis of R/R', and let $\Gamma \subset R$ be a set of representatives of C. Then $B = B' \cup \Gamma$ is an A-basis of R.*

Proof. B is k-free: If $\sum_{\beta\in B} a_\beta\beta = 0$, then all a_γ, $\gamma \in \Gamma$, vanish since $\bar{\Gamma} = C$ is in particular a k-basis of X/X'. But then all a_β, for $\beta \in B'$, also vanish, by k-linear independence of B'. It remains to show that $R \subset A[B]$. For $\alpha \in R$ there exist $a_\gamma \in A$ ($\gamma \in \Gamma$), such that $\bar{\alpha} = \sum_{\gamma\in\Gamma} a_\gamma\bar{\gamma}$, whence $\alpha - \sum_{\gamma\in\Gamma} a_\gamma\gamma \in A[R] \cap X' = A[R']$, by 2.7.3. Thus by 2.7.1 applied to R' and B' it follows that α is an A-linear combination of B.

We now give criteria for the (finite) extension property. A *subquotient* of (R, X) is defined as a full $(T, Z) \subset (\bar{R}, \bar{X})$ of some quotient $(\bar{R}, \bar{X}) = (R/R', X/X')$. By 1.7, the subquotients are precisely the $(R''/R', X''/X')$ where $R'' \supset R'$ is full with span X''. By a *finite subquotient* we mean one for which R'' has finite rank.

2.9. Proposition. *For $(R, X) \in \mathbf{SV}_k$, the following conditions are equivalent:*

(i) *(R, X) has the (finite) A-extension property,*

(ii) *(R, X) is A-exact, and every (finite) subquotient of (R, X) has an A-basis.*

Proof. (i) $\Longrightarrow$ (ii): We first show (R, X) is A-exact. Since the extension property is stronger than the finite extension property, it suffices to prove that the latter implies A-exactness. Thus let $R' \subset X'$ be full with linear span X'. We must verify 2.7.3. The inclusion from left to right is trivial. For the converse, let $x' = \sum_{i=1}^n a_i\alpha_i \in A[R] \cap X'$, where $a_i \in A$ and $\alpha_i \in R$. By Lemma 2.5, there exists a full finite-ranked subset S of R containing $E = \{\alpha_1, \dots, \alpha_n\}$ and intersecting R' tightly. We let $Y = \operatorname{span}(S)$, $S' = S \cap R'$ and $Y' = Y \cap X'$. Then $Y' = \operatorname{span}(S')$ by tightness of Y', and $x' \in A[S] \cap Y'$ because $E \subset S$. By the finite extension property, there exist A-bases B' of S' and $B \supset B'$ of S. Writing $x' = \sum_{\beta\in B} a_\beta\beta$ and keeping in mind that B' is a k-basis of Y', it follows that $a_\beta = 0$ for $\beta \in B \setminus B'$. Hence $x' \in A[B'] = A[S'] \subset A[R']$, as desired.

Next, consider a (finite) subquotient $(T, Z) = (R''/R', X''/X')$ of (R, X). By the (finite) extension property, there exist A-bases $B' \subset B''$ of $R' \subset R''$. Then it is easy to see that $\operatorname{can}(B \setminus B')$ is an A-basis of (T, Z).

(ii) $\Longrightarrow$ (i): Let $S' \subset S$ be full (finite-ranked) subsets with spans $Y' \subset Y$, and let $B' \subset S'$ be an A-basis. By assumption, $(S/S', Y/Y')$ has an A-basis. Now Lemma 2.8 shows that B' extends to an A-basis of (S, Y).

2.10. Proposition. (a) *A-exactness descends to all quotients: If (R, X) is A-exact then so is every quotient of (R, X).*

(b) *The A-extension property descends to all quotients.*

(c) *If all quotients of (R, X) are locally finite, then the finite A-extension property for (R, X) descends to all quotients.*

Proof. (a) Let $(\bar{R}, \bar{X}) = (R/R', X/X')$. By 1.7, a full subset of $\bar{R}$ is of the form $\bar{S}$ where $S \subset R$ is full and contains R'. We let $Y = \operatorname{span}(S)$ and then must show that $A[\bar{R}] \cap \bar{Y} \subset A[\bar{S}]$. Thus let $\bar{x} \in A[\bar{R}] \cap \bar{Y}$. Then, because of $X' \subset Y$, we have $x \in A[R] \cap Y$, and this equals $A[S]$, by 2.7.3, applied to (S, Y) instead of (R', X'). Hence $\bar{x} \in A[\bar{S}]$, as asserted.

(b) We use the criterion given in Prop. 2.9. By (a), A-exactness descends to $(\bar{R}, \bar{X})$. Furthermore, by the First Isomorphism Theorem, a subquotient of $(\bar{R}, \bar{X})$ is of the form $\bar{R}_1/\bar{R}_0 \cong R_1/R_0$, for full $R_1 \supset R_0 \supset R'$. Since R_1/R_0 has an A-basis by 2.9, so does $\bar{R}_1/\bar{R}_0$.

(c) We again use the criterion of 2.9, and in view of (a) only must show that all finite subquotients of $\bar{R}$ have an A-basis. Thus consider a subquotient $\bar{R}_1/\bar{R}_0$ with $\operatorname{rank}(\bar{R}_1) < \infty$. Since R/R_0 is by assumption locally finite and $\operatorname{rank}(R_1/R_0) = \operatorname{rank}(\bar{R}_1/\bar{R}_0) \leqslant \operatorname{rank}(\bar{R}_1) < \infty$, we have R_1/R_0 finite. Let $E \subset R_1$ be a set of representatives of R_1/R_0. By Lemma 2.5, there exists a finite-ranked full $S_1 \subset R_1$ intersecting R_0 tightly. By the finite extension property of R and 2.9, $S_1/S_1 \cap R_0$ has an A-basis. Since $S_1/S_1 \cap R_0 \cong R_1/R_0$ by the Second Isomorphism Theorem 1.9, $\bar{R}_1/\bar{R}_0 \cong R_1/R_0$ has an A-basis.

2.11. THEOREM. *Let A be a subring of the base field k. If $(R, X) \in \mathbf{SV}_k$ has the finite A-extension property and all quotients of (R, X) are locally finite then it has the A-extension property.*

Proof. By 2.9 and 2.10(a), it only remains to show that all subquotients R''/R' of R have an A-basis. Since the assumptions on R clearly pass to full subsets, we can assume $R'' = R$. By (c) of Prop. 2.10, R/R' has the finite extension property and by the First Isomorphism Theorem 1.7, all quotients of R/R' are isomorphic to quotients of R and are therefore locally finite. Thus, we may even replace R/R' by R and then merely have to show that R itself has an A-basis. Consider the set $\mathfrak{M}$ of all pairs (S, B) where S is a full subset of R, and $B \subset S$ is an A-basis of S. Note that $\mathfrak{M}$ is not empty since $(\{0\}, \emptyset) \in \mathfrak{M}$. Define a partial order on $\mathfrak{M}$ by $(S_1, B_1) \leqslant (S_2, B_2)$ if and only if $S_1 \subset S_2$ and $B_1 \subset B_2$. Then it is easy to see that $\mathfrak{M}$ is inductively ordered. By Zorn's Lemma, $\mathfrak{M}$ contains a maximal element (R_0, B_0), and we must show $R_0 = R$. Assume, for a contradiction, that $R_0 \neq R$. Then there exists $\alpha \in R \setminus R_0$, and even $\alpha \notin X_0 := \operatorname{span}(R_0)$, by fullness of R_0. Hence X_0 is a hyperplane in $X_1 := X_0 \oplus \mathbb{R}\alpha$, and $R_1 = \operatorname{core}(X_1)$ is a full subset of R, with linear span X_1. Since by assumption all quotients of (R, X) are locally finite, this is in particular so for $(R, X)/(R_0, X_0)$. Hence R_1/R_0 is finite, being a subset of the line $X_1/X_0 \subset X/X_0$. Let $E \subset R_1$ be a set of representatives of R_1/R_0. By Lemma 2.5, applied to $(R_0, X_0) \subset (R_1, X_1)$, there exists a finite-ranked (and therefore even finite, by local finiteness of R) full subset S_1 of R_1 containing E and intersecting R_0 tightly. We let $Y_1 = \operatorname{span}(S_1)$ and $Y_0 = Y_1 \cap X_0 = \operatorname{span}(S_0)$, where $S_0 := S_1 \cap R_0$. Then by the Second Isomorphism Theorem 1.9, $(S_1/S_0, Y_1/Y_0) \cong (R_1/R_0, X_1/X_0)$. Since (R, X) has the finite extension property, Proposition 2.9(ii) shows that the finite subquotient S_1/S_0 has an A-basis. Hence also R_1/R_0 has an A-basis, which consists of a single element, say $\{\bar{\gamma}\}$, since $\operatorname{rank}(R_1/R_0) = 1$. From A-exactness of R and Lemma 2.8, it follows that $B_1 := B_0 \cup \{\gamma\}$ is an A-basis of R_1. Hence $(R_0, B_0) < (R_1, B_1)$, contradicting maximality of (R_0, B_0) and completing the proof.

The assumption on the local finiteness of all quotients is, by Theorem 2.6, in particular satisfied as soon as (R, X) is strongly bounded. We explicitly formulate this important special case and some of its consequences (see 2.7) in the following corollary.

2.12. Corollary. *If $(R, X) \in \mathbf{SV}_k$ has the finite extension property for a subring A of k and is strongly bounded, then it has the extension property for A. In particular, every full $R' \subset R$ has an A-basis, every A-basis of R' extends to an A-basis of R, the sequence* 2.7.2 *is exact, and $A[R'] = A[R] \cap \operatorname{span}(R')$.*

§3. Locally finite root systems

3.1. *Reflections.* Let X be a vector space over a field k of characteristic $\neq 2$. An element $s \in \mathrm{GL}(X)$ is called a *reflection* if $s^2 = \mathrm{Id}$ and its fixed point set is a hyperplane. Picking a nonzero element α in the (-1)-eigenspace of s we have

$$s(x) = s_{\alpha,l}(x) := x - \langle x, l \rangle \alpha, \tag{1}$$

where l is the unique linear form on X with $\operatorname{Ker} l = \operatorname{Ker}(\mathrm{Id} - s)$ and $\langle \alpha, l \rangle = 2$. Here $\langle\ ,\ \rangle$ denotes the canonical pairing between X and its dual X^*. Conversely, given a linear form l on X and a vector $\alpha \in X$ satisfying $\langle \alpha, l \rangle = 2$, the right hand side of (1) defines a reflection.

For the following lemma see also [**12**, VI, §1, Lemma 1]. We use the notations and terminology of §1 and §2.

3.2. Lemma (Uniqueness of reflections). *Let the base field k have characteristic zero, let $(R, X) \in \mathbf{SV}_k$ be locally finite, and let $\alpha \in R^\times$. Then there exists at most one reflection s of X such that $s(\alpha) = -\alpha$ and $s(R) = R$.*

Proof. Let $s = s_{\alpha,l}$ and $s' = s_{\alpha,l'}$ be reflections with the stated properties. Then $t = ss'$ is given by $t(x) = x + \langle x, d \rangle \alpha$ where $d = l' - l$, and clearly $t(\alpha) = \alpha$. Assuming $d \neq 0$, we can find $\beta \in R$ such that $\langle \beta, d \rangle \neq 0$, because R spans X. Then the vectors $t^n(\beta) = \beta + n\langle \beta, d \rangle \alpha$ $(n \in \mathbb{N})$ form an infinite set in $R \cap (k\alpha + k\beta)$, contradicting local finiteness of R.

3.3. Definition. We define locally finite root systems in analogy to Bourbaki's definition [**12**, VI, §1, Def. 1]. The base field k is now taken to be the real numbers. A pair $(R, X) \in \mathbf{SV}_{\mathbb{R}}$ is called a *locally finite root system* if it satisfies the following conditions:

(i) R is locally finite,

(ii) for every $\alpha \in R^\times = R \setminus \{0\}$ there exists $\alpha^\vee$ in the dual X^* of X such that $\langle \alpha, \alpha^\vee \rangle = 2$ and the reflection $s_\alpha := s_{\alpha,\alpha^\vee}$ maps R into itself,

(iii) $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in R^\times$.

By Lemma 3.2, the reflection s_α in the root α is uniquely determined. Hence $\alpha^\vee$ is uniquely determined as well so that condition (iii) makes sense, and ${}^\vee\colon R^\times \to X^*$ is a well-defined map. We extend this map to all of R by defining

$$0^\vee := 0 \quad \text{and} \quad s_0 := \mathrm{Id}. \tag{1}$$

Then $s_\alpha(R) = R$ for all $\alpha \in R$. As usual, we call $\alpha^\vee$ the *coroot* determined by α. For all $\alpha \in R$ the reflection s_α is explicitly given by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha. \tag{2}$$

Henceforth, the unqualified term "root system" will always mean a locally finite root system.

Let us repeat here that, according to the definitions of 1.1, always $0 \in R$ and R spans X. Traditionally, root systems do not contain 0. On the other hand, the requirement $0 \in R$ is a natural one, for instance when considering morphisms and quotients, or Lie algebras graded by root systems. It is also part of the axioms for extended affine root systems [**1**, Ch. 2]. Moreover, root systems "with 0 added" occur naturally in the axiomatization of root systems given by Winter [**75**] and Cuenca Mira [**19**].

To distinguish the non-zero elements of R, we will call "roots" the elements of $R^\times$. Root systems in the classical sense are precisely the sets $R^\times \subset X$, where (R, X) is a locally finite root system in the above sense with R finite (equivalently, $\operatorname{rank}(R) = \dim(X)$ finite).

3.4. *Subsystems and full subsystems.* A subset $S \subset R$ is called a *subsystem* if $0 \in S$ and $s_\alpha(S) \subset S$ for all $\alpha \in S$. Then clearly S is itself a root system in the subspace $Y = \operatorname{span}(S)$ spanned by S. The reflection of Y and the coroot in Y^* determined by a root $\alpha \in S$ are the restrictions $s_\alpha \big| Y$ and $\alpha^\vee \big| Y$, respectively.

In particular, every full subset S of R (as defined in 1.3) is a subsystem, naturally called a *full subsystem*. Indeed, if α and β are in S then, by 3.3.2, $s_\alpha\beta \in R \cap (\mathbb{R}\alpha + \mathbb{R}\beta) \subset R \cap \operatorname{span}(S) = \operatorname{core}(\operatorname{span}(S)) = S$, since S is full. As a consequence:

$$\textit{Locally finite root systems are bounded by the function } b(n) = 4n^2. \tag{1}$$

Indeed, let V be a tight subspace of dimension n of X. Then $F = \operatorname{core}(V)$ is a finite root system of rank n. From the classification of finite root systems [**12**] it follows by a case-by-case verification that $|F^\times| \leqslant 4n^2$ in case F is irreducible. This estimate holds in the reducible case as well, because of the well-known decomposition of F into irreducible components and Lemma 2.4(b).

For $\alpha, \beta \in R$ the set $R \cap (\mathbb{R}\alpha + \mathbb{R}\beta)$ is a root system of rank at most two. The possible relations between two roots α and β of R are therefore the same as in the finite case which are reviewed in A.2. Thus, the *Cartan numbers* $\langle \alpha, \beta^\vee \rangle$ can only take the values $0, \pm 1, \pm 2, \pm 3, \pm 4$. We also note that for any $\alpha \in R^\times$, there are the following possibilities for the roots contained in the line spanned by α:

$$R^\times \cap \mathbb{R}\alpha = \left\{ \begin{array}{l} \{\pm\alpha\} \\ \{\pm\alpha/2, \pm\alpha\} \\ \{\pm\alpha, \pm 2\alpha\} \end{array} \right\}. \tag{2}$$

As usual, a root system is called *reduced* if the first alternative in (2) holds for all $\alpha \in R^\times$. The relation between irreducible reduced and non-reduced root systems is the same as in the finite case, see 8.5 and A.7, A.8. Finally, a root α is said to be *divisible* or *indivisible* according to whether $\alpha/2$ is a root or not. The union of $\{0\}$ and the set of indivisible roots is denoted R_{ind}. It is obvious that (R_{ind}, X) is a subsystem of (R, X).

3.5. *Orthogonality.* For any subset $T \subset R$ we define

$$T^\perp := \bigcap_{\alpha \in T} \operatorname{Ker} \alpha^\vee. \tag{1}$$

Then

$$\operatorname{span}(T) \cap T^{\perp} = \{0\}. \tag{2}$$

Indeed, let $x \in \operatorname{span}(T) \cap T^{\perp}$. Since x is a finite linear combination of elements of T, there exists a finite subsystem $S \subset T$ such that $x \in \operatorname{span}(S)$. In particular, $\langle x, \alpha^{\vee} \rangle = 0$ for all $\alpha \in S$, and this implies $x = 0$ since it is known that the coroots of the finite root system S span the full dual of the vector space $\operatorname{span}(S)$ in which S lives [**12**, VI, §1.1, Prop. 2]. In case $T = R$, we see from $\operatorname{span}(R) = X$ that

$$R^{\perp} = \{0\}. \tag{3}$$

Hence, denoting by $X^{\vee} \subset X^*$ the $\mathbb{R}$-linear span of $\{\alpha^{\vee} : \alpha \in R\}$, the canonical pairing $X \times X^{\vee} \to \mathbb{R}$ is nondegenerate.

For $\alpha, \beta \in R$ we define *orthogonality* by

$$\alpha \perp \beta \quad \iff \quad \alpha \in \beta^{\perp}. \tag{4}$$

Here $\beta^{\perp}$ is short for $\{\beta\}^{\perp}$ in the sense of (1). The relation $\alpha \perp \beta$ is symmetric, as follows from well-known facts on finite root systems by considering $R \cap (\mathbb{R}\alpha + \mathbb{R}\beta)$, see A.2. For subsets $S, T \subset R$ we use the notation $S \perp T$ to mean $\alpha \perp \beta$ for all $\alpha \in S$ and $\beta \in T$.

3.6. *Morphisms, embeddings and the categories* **RS** *and* **RSE**. We denote by **RS** the full subcategory of $\mathbf{SV}_{\mathbb{R}}$ whose objects are root systems. Thus a morphism $f\colon (R, X) \to (S, Y)$ in **RS** is merely a linear map $f\colon X \to Y$ with $f(R) \subset S$. Note that $f(R)$ need not be a subsystem, even when $f\colon X \to Y$ is a vector space isomorphism. For example, let $R = \mathrm{A}_1 \oplus \mathrm{A}_1 = \{0, \pm\alpha_1, \pm\alpha_2\}$ and let $S = \mathrm{A}_2 = \{0, \pm\beta_1, \pm\beta_2, \pm(\beta_1 + \beta_2)\}$ where $\langle \beta_1, \beta_2^{\vee} \rangle = -1 = \langle \beta_2, \beta_1^{\vee} \rangle$. Let f be the vector space isomorphism given by $f(\alpha_i) = \beta_i$, $i = 1, 2$. Then f is a morphism of **RS** but $f(R)$ is not a subsystem of S. Nevertheless, morphisms between root systems in this sense are of interest; in particular, we note that morphisms between finite root systems with the additional property that $f(R) = S$ (i.e., exact epimorphisms in the sense of 1.4(b)) were classified by Doković and Thăńg [**25**].

A morphism $f\colon (R, X) \to (S, Y)$ of **RS** is called an *embedding of root systems* if $f\colon X \to Y$ is injective and $f(R)$ is a subsystem of S. We denote by **RSE** the (non-full) subcategory of **RS** whose objects are root systems and whose morphisms are embeddings of root systems.

Clearly, an *isomorphism* $f\colon (R, X) \to (S, Y)$ in the category **RS** is just a vector space isomorphism $f\colon X \to Y$ such that $f(R) = S$. In particular, an isomorphism in **RS** is an embedding, so the isomorphisms in **RS** and in **RSE** are the same.

3.7. Lemma. *For a morphism $f\colon (R, X) \to (S, Y)$ of* **RS**, *the following conditions are equivalent:*

(i) *f is an embedding,*
(ii) *$\langle f(\beta), f(\alpha)^{\vee} \rangle = \langle \beta, \alpha^{\vee} \rangle$ for all $\alpha, \beta \in R$,*
(iii) *$\langle f(x), f(\alpha)^{\vee} \rangle = \langle x, \alpha^{\vee} \rangle$ for all $x \in X$, $\alpha \in R$,*
(iv) *$f(s_\alpha(\beta)) = s_{f(\alpha)}(f(\beta))$ for all $\alpha, \beta \in R$,*
(v) *$f(s_\alpha(x)) = s_{f(\alpha)}(f(x))$ for all $x \in X$, $\alpha \in R$.*

Proof. The equivalence of (ii) – (v) is straightforward from 3.3.2 and the fact that R spans X. Suppose that these conditions hold. Then (iv) shows that $f(R)$ is a subsystem of S. Moreover, by (iii), any x in the kernel of f lies in $R^\perp$ which is $\{0\}$ by 3.5.3, so f is an embedding. Conversely, let this be the case and let $\alpha \in R^\times$. Since $f(R)$ is a subsystem, $s_{f(\alpha)}(f(\beta)) = f(\beta - \langle f(\beta), f(\alpha)^\vee\rangle\alpha) \in f(R)$ for every $\beta \in R$. Hence, defining $s\colon X \to X$ by $s(x) = x - \langle f(x), f(\alpha)^\vee\rangle\alpha$, we have $f(s(\beta)) = s_{f(\alpha)}(f(\beta)) \in f(R)$ and therefore $s(\beta) \in R$, by injectivity of f. One checks that $s(\alpha) = -\alpha$ and $s(x) = x$ for every $x \in X$ satisfying $\langle f(x), f(\alpha)^\vee\rangle = 0$ which is a subspace of codimension 1. Now Lemma 3.2 says that $s = s_\alpha$, which implies (iv).

Remark. We will see in Cor. 7.7 that any map $f\colon R \to S$ satisfying (ii) can be extended to an embedding $(R, X) \to (S, Y)$.

3.8. Definition. A morphism $f\colon (S, Y) \to (R, X)$ between root systems is called a *full embedding* if it satisfies the following equivalent conditions:

(i) f is an embedding and $f(S)$ is a full subsystem of R,

(ii) $S = f^{-1}(R)$ is the full pre-image of R under the linear map $f\colon Y \to X$.

We prove the equivalence of these conditions. Suppose that (i) holds. Then $S \subset f^{-1}(R)$ is clear. For the reverse inclusion, let $y \in f^{-1}(R)$, so $f(y) = \alpha \in R$. Then $\alpha \in R \cap f(Y) = f(S)$ since $f(S)$ is full in R, say, $\alpha = f(\beta)$ for some $\beta \in S$. As f is injective, we conclude $y = \beta \in S$.

Conversely, let $S = f^{-1}(R)$. Then in particular, $f^{-1}(0) = \mathrm{Ker}(f) \subset S$, whence $\mathrm{Ker}(f) = 0$ by local finiteness of S. Moreover, $f(S) = f(f^{-1}(R)) = R \cap f(Y)$ is a full subsystem of R, showing (i).

From the characterization (ii) above it is immediate that the composition of full embeddings is again a full embedding. Thus we have a (again not full) subcategory **RSF** of **RSE**, whose objects are root systems and whose morphisms are full embeddings.

3.9. *Automorphisms and the Weyl group.* We denote by $\mathrm{Aut}(R) \subset \mathrm{GL}(X)$ the *automorphism group* of a root system $R \subset X$. By 3.6, $f \in \mathrm{GL}(X)$ is an automorphism if and only if $f(R) = R$. Automorphisms are in particular embeddings and thus satisfies the equivalent conditions of Lemma 3.7. From the definition of a root system it is clear that each reflection $s_\alpha \in \mathrm{Aut}(R)$, so 3.7 yields, after replacing x by $s_\alpha(x)$, the formulas

$$\langle x, (s_\alpha(\beta))^\vee\rangle = \langle s_\alpha(x), \beta^\vee\rangle, \tag{1}$$

$$s_{s_\alpha(\beta)} = s_\alpha s_\beta s_\alpha. \tag{2}$$

By working out the right hand side of (1) with 3.3.2, we obtain the equivalent formula

$$(s_\alpha(\beta))^\vee = \beta^\vee - \langle \alpha, \beta^\vee\rangle\alpha^\vee. \tag{3}$$

Note in particular that

$$\alpha \perp \beta \quad \Longrightarrow \quad s_\alpha s_\beta = s_\beta s_\alpha. \tag{4}$$

Indeed, $\langle \beta, \alpha^\vee\rangle = 0$ implies $s_\alpha(\beta) = \beta$ by 3.3.2 and therefore $s_\beta = s_\alpha s_\beta s_\alpha$ by (2).

We say a transformation $f \in \mathrm{GL}(X)$ is *finitary* or *of finite type* if its fixed point set
$$X^f := \{x \in X : f(x) = x\}$$
has finite codimension. The finitary transformations form a normal subgroup $\mathrm{GL}_{\mathrm{fin}}(X)$ of $\mathrm{GL}(X)$, and thus
$$\mathrm{Aut}_{\mathrm{fin}}(R) := \mathrm{Aut}(R) \cap \mathrm{GL}_{\mathrm{fin}}(X)$$
is a normal subgroup of $\mathrm{Aut}(R)$. Since $X^{s_\alpha} = \operatorname{Ker}\alpha^\vee$ is a hyperplane, every reflection s_α is of finite type. We denote by $W = W(R) \subset \mathrm{Aut}_{\mathrm{fin}}(R)$ the group generated by all s_α, $\alpha \in R^\times$ and call it the *Weyl group* of R. From 3.7(v) we see that $W(R)$ is a normal subgroup of $\mathrm{Aut}(R)$.

3.10. Lemma. *The category* **RS** *admits arbitrary coproducts, given by*
$$(R, X) = \coprod_{i\in I}(R_i, X_i) = (\bigcup_{i\in I} R_i, \bigoplus_{i\in I} X_i),$$
for a family $(R_i, X_i)_{i\in I}$ of root systems.

Proof. By 1.2(c) and 2.1, (R, X) is locally finite. We extend each $\alpha_i^\vee$ (where $\alpha_i \in R_i$) to a linear form on X by $\langle X_j, \alpha_i^\vee\rangle = 0$ for $i \neq j$. Then it is easily seen that R is a root system in X and that (R, X) is the coproduct of the (R_i, X_i) in the category **RS**.

By abuse of notation, we also write $R = \bigoplus_{i\in I} R_i$ and call R the *direct sum* of the R_i. After identifying R_i with a subset of R, each R_i is a full subsystem of R, and
$$R_i \perp R_j \quad \text{for } i \neq j. \tag{1}$$
Note, however, that (R, X) is *not* the coproduct of the (R_i, X_i) in the category **RSE**! Indeed, the required universal property fails: If $f_i\colon (R_i, X_i) \to (S, Y)$ are embeddings then the induced morphism $f\colon (R, X) \to (S, Y)$ is in general not an embedding of root systems. In fact, it is easily seen that even the coproduct of the simplest root system $\mathrm{A}_1 = \{0, \pm\alpha\}$ with itself does not exist in **RSE**.

A subsystem S of a root system R is said to be a *direct summand* if there exists a second subsystem S' of R such that $R = S \oplus S'$.

3.11. Lemma. *A subsystem S of a root system (R, X) is a direct summand if and only if S is full and $(R \setminus S) \perp S$. In this case, R is the direct sum of S and $R \cap S^\perp$.*

Proof. That the conditions on S are necessary is clear from the definition of a direct summand in 3.10. Conversely, suppose they are satisfied and let $Y = \mathrm{span}(S)$, so $S = R \cap Y$ by fullness of S. Also, let $Z = \mathrm{span}(R \setminus S)$. Then $(R \setminus S) \perp S$ implies $Y \cap Z = \{0\}$ by 3.5.2. Furthermore, $X = \mathrm{span}(R) = \mathrm{span}(S) + \mathrm{span}(R \setminus S) = Y + Z$, and clearly $T := R \cap Z = \{0\} \cup (R \setminus S)$, showing that R is the direct sum of S and T.

3.12. *Irreducibility and connectedness.* A nonzero root system is called *irreducible* if it is not isomorphic to a direct sum of two nonzero root systems. We will show that any root system R decomposes uniquely into a direct sum of irreducible root systems. For this purpose, we introduce the notion of connectedness.

Let A be a subset of a root system R with $0 \in A$. Two roots α and β of $A^\times = A \setminus \{0\}$ are said to be *connected in* A if there exist finitely many roots $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$, $\alpha_i \in A^\times$, such that $\alpha_{i-1} \not\perp \alpha_i$, for $i = 1, \ldots, n$. We then call $\alpha_0, \ldots, \alpha_n$ a *chain connecting* α *and* β *in* A. Connectedness is an equivalence relation on the set $A^\times$. A *connected component of* A is defined as the union of $\{0\}$ with an equivalence class of $A^\times$. Naturally, A is called *connected* if there is only one connected component. In particular this applies to $A = R$.

One can always achieve $n \leqslant 2$ in a chain connecting α and β in $R^\times$. Indeed, let $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ be a connecting chain of minimal length and suppose $n > 2$. Possibly after replacing α_1 by $-\alpha_1$ we may assume $\langle \alpha_1, \alpha_2^\vee \rangle > 0$. Then $\alpha_1 - \alpha_2 \in R$ by A.3. Since $\alpha_i \perp \alpha_j$ for $|i - j| > 1$ by minimality, we obtain $\alpha \not\perp (\alpha_1 - \alpha_2) \not\perp \alpha_3$ and so $\alpha = \alpha_0, \alpha_1 - \alpha_2, \alpha_3, \ldots, \alpha_n = \beta$ is a connecting chain of smaller length, contradiction. Note that the same argument applies to any closed subsystem, as defined in 10.2.

3.13. Proposition (Decomposition into irreducible components). *A root system is irreducible if and only if it is connected. Every root system is the direct sum of its connected components.*

Proof. We first note that a connected root system is irreducible. Indeed, if $R = \bigoplus_{i \in I} R_i$ is a direct sum of nonzero root systems R_i, then $R_i \perp R_j$ for $i \neq j$ (by 3.10.1) shows that no $\alpha \in R_i^\times$ can be connected to any $\beta \in R_j^\times$. That, conversely, an irreducible root system is connected, is a consequence of the decomposition into connected components which we show next. Let $\mathfrak{C}$ be the set of connected components of a root system R. From the definition of connectedness it is clear that $S \perp T$ for different $S, T \in \mathfrak{C}$. Moreover, each connected component $S \in \mathfrak{C}$ is a subsystem of R. Indeed, let $\alpha, \beta \in S$ and suppose $\gamma := s_\alpha(\beta) \notin S$. Since $0 \in S$, we must have $\gamma \neq 0$ and then also $\beta \neq 0$. Then γ is in a connected component different from S and hence is orthogonal to both α and β. This implies $\gamma = s_\alpha(\gamma) = s_\alpha^2(\beta) = \beta$ and hence $\beta \perp \beta$, which is impossible. Thus S is a connected, hence irreducible, subsystem of R. Furthermore, X is the direct sum of the subspaces $\operatorname{span}(S)$, $S \in \mathfrak{C}$. Indeed, $X = \operatorname{span}(R)$ and $R = \bigcup \mathfrak{C}$ imply that X is the sum of the subspaces $\operatorname{span}(S)$, $S \in \mathfrak{C}$. To show that the sum is direct, let $S_1, \ldots, S_n \in \mathfrak{C}$ be pairwise different, and suppose that $\sum_1^n x_i = 0$ for $x_i \in \operatorname{span}(S_i)$. By orthogonality of the S_i we then have, for all $\alpha \in S_j$, that

$$0 = \Big\langle \sum_1^n x_i, \alpha^\vee \Big\rangle = \langle x_j, \alpha^\vee \rangle.$$

This shows $x_j \in \operatorname{span}(S_j) \cap S_j^\perp = \{0\}$ by 3.5.2. Thus R is the direct sum of its connected components as a root system.

In the sequel, the terminologies "irreducible component" and "connected component" will be used interchangeably.

3.14. PROPOSITION (Direct limits of root systems). *The category* **RSE** *admits all direct limits (i.e., filtered colimits)* $\varinjlim(R_\lambda, X_\lambda)$. *If the* (R_λ, X_λ) *are irreducible so is their limit.*

Proof. Let Λ be a directed index set, and let $((R_\lambda, X_\lambda), f_{\mu\lambda})$ be a directed system in **RSE** indexed by Λ, i.e., a family $(R_\lambda, X_\lambda)_{\lambda\in\Lambda}$ of root systems together with root system embeddings $f_{\mu\lambda}$: $(R_\lambda, X_\lambda) \to (R_\mu, X_\mu)$ for all $\lambda \preccurlyeq \mu$, satisfying $f_{\lambda\lambda} = \mathrm{Id}$ and $f_{\nu\lambda} = f_{\nu\mu} \circ f_{\mu\lambda}$ for $\lambda \preccurlyeq \mu \preccurlyeq \nu$. In particular, $(X_\lambda)_{\lambda\in\Lambda}$ is then a directed system of real vector spaces and hence has a direct limit $X = \varinjlim X_\lambda$, namely the quotient of the disjoint union of the X_λ by the equivalence relation $x \sim y \iff x \in X_\lambda$, $y \in X_\mu$ and $f_{\nu\lambda}(x) = f_{\nu\mu}(y)$ for some $\nu \succcurlyeq \lambda$ and $\nu \succcurlyeq \mu$. We denote as usual by f_λ: $X_\lambda \to X$ the canonical maps. Since the maps $f_{\mu\lambda}$ are injective, so are the f_λ [**10**, III, §7.6, Remarque 1]. We therefore identify the X_λ and the R_λ with their images in X. It is then straightforward to show that the union R of the R_λ satisfies all the axioms of a locally finite root system in X, with the exception of local finiteness. The latter can be seen as follows. Suppose F is a finite subset of R. Since Λ is directed, there exists an index λ_0 such that $F \subset R_{\lambda_0}$. By 3.4.1, R_{λ_0} is bounded by the function $b(n) = 4n^2$. Hence $|F^\times| \leqslant b(\mathrm{rank}(F))$, showing that R is also bounded by b; in particular, it is locally finite. Finally, the R_λ are subsystems of R and the universal property of (R, X) is easily checked.

Now suppose that the (R_λ, X_λ) are irreducible, and let $\alpha, \beta \in R^\times$. Then there exists an index λ_0 such that $\alpha, \beta \in R_{\lambda_0}$. By irreducibility and 3.13, there exists a chain $\alpha = \alpha_0 \not\perp \alpha_1 \not\perp \cdots \not\perp \alpha_n = \beta$ in R_{λ_0} connecting α and β, and since R_{λ_0} is a subset of R, this is also a chain connecting α and β in R, showing R is connected and hence irreducible.

3.15. COROLLARY. (a) *The locally finite root systems are precisely the direct limits of the finite root systems.*

(b) *The irreducible locally finite root systems are precisely the direct limits of the irreducible finite root systems.*

Proof. (a) By 3.14, a direct limit of finite root systems is a (locally finite) root system. Conversely, it follows from local finiteness that in any locally finite root system (R, X), the finite subsystems (and even the full finite subsystems) form a directed system with respect to inclusion, whose direct limit is canonically isomorphic to R.

(b) Again by 3.14, a direct limit of finite irreducible root systems is irreducible. Conversely, let (R, X) be irreducible. It suffices to show that the finite irreducible subsystems form a directed system with respect to inclusion. For this, it suffices to have any finite subset of $R^\times$ contained in a finite irreducible subsystem. Thus let $F = \{\alpha_1, \ldots, \alpha_n\} \subset R^\times$ be finite. By irreducibility of R, there exist chains connecting α_1 to α_2, α_2 to α_3, and so on. Then the union of these chains is a finite connected subset C of R contained in the irreducible finite full subsystem $R \cap \mathrm{span}(C)$ of R.

As a corollary of this proof we note

3.16. COROLLARY. *Any finite subset of an irreducible root system R is contained in a finite full irreducible subsystem of R.*

§4. Invariant inner products and the coroot system

4.1. *Invariant bilinear forms.* Let (R, X) be a root system. A bilinear form $B\colon X \times X \to \mathbb{R}$ is called *invariant* if it is invariant under the Weyl group, i.e., if $B(wx, wy) = B(x, y)$ for all $w \in W(R)$ and $x, y \in X$. As $W(R)$ is generated by the reflections s_α, $\alpha \in R^\times$, which have period two, invariance of B is equivalent to

$$B(s_\alpha x, y) = B(x, s_\alpha y), \tag{1}$$

for all $\alpha \in R^\times$ and $x, y \in X$. Expanding both sides with 3.3.2, one finds that (1) is equivalent to $\langle x, \alpha^\vee\rangle B(\alpha, y) = \langle y, \alpha^\vee\rangle B(x, \alpha)$. By specializing $x = \alpha$ and $y = \alpha$ and using the fact that R spans X, it follows easily that B is invariant if and only if it is symmetric and satisfies

$$2B(x, \alpha) = B(\alpha, \alpha)\langle x, \alpha^\vee\rangle \tag{2}$$

for all $x \in X$ and $\alpha \in R^\times$. From (2) it is clear that $\alpha \perp \beta$ (in the sense of 3.5) implies $B(\alpha, \beta) = 0$. If $B(\alpha, \alpha) \neq 0$ then (2) shows

$$\langle \beta, \alpha^\vee\rangle = \frac{2B(\beta, \alpha)}{B(\alpha, \alpha)}, \tag{3}$$

and hence s_α is by 3.3.2 the orthogonal reflection in the hyperplane orthogonal to α. This is in particular so if B is a positive definite invariant bilinear form, also called an *invariant inner product*.

We denote by $\mathcal{I}(R)$ the set of invariant bilinear forms on X, which is obviously a real vector space. In fact, $\mathcal{I}$ is a contravariant functor on the category **RSE** of root systems and embeddings, since for an embedding $f\colon (S, Y) \to (R, X)$ and an invariant bilinear form on X, the bilinear form $\mathcal{I}(f)(B) := B'$, defined by

$$B'(x, y) := B(f(x), f(y)) \quad (x, y \in Y) \tag{4}$$

is an invariant bilinear form on Y. This follows immediately from 3.7(iii) and (2). We note that B' is an invariant inner product along with B, since embeddings are injective.

If $(R, X) = \coprod(R_i, X_i)$ is a direct sum of root systems as in 3.10 then $R_i \perp R_j$ for $i \neq j$ and therefore $B(X_i, X_j) = 0$ for $i \neq j$, because the R_i span X_i. Conversely, if B_i are invariant bilinear forms on X_i then the orthogonal sum of the B_i yields an invariant bilinear form B on X. Hence the functor $\mathcal{I}$ converts direct sums to direct products:

$$\mathcal{I}(\bigoplus R_i) \cong \prod \mathcal{I}(R_i). \tag{5}$$

In particular, this applies to the decomposition of a root system into irreducible components (3.13).

4.2. THEOREM. (a) *Every locally finite root system (R, X) admits an invariant inner product. If (R, X) is irreducible, the space $\mathcal{I}(R)$ of invariant bilinear forms on X is one-dimensional.*

(b) *Conversely, let $(R, X) \in \mathbf{SV}_{\mathbb{R}}$ and suppose there exists an inner product $(\ |\ )$ on X such that $s_\alpha(R) \subset R$ for all $\alpha \in R^\times$ where $s_\alpha(x) = x - \dfrac{2(x|\alpha)}{(\alpha|\alpha)}\alpha$ is the orthogonal reflection in α with respect to $(\ |\ )$, and such that the integrality condition $\dfrac{2(\beta|\alpha)}{(\alpha|\alpha)} \in \mathbb{Z}$ holds for all $\alpha, \beta \in R^\times$. Then (R, X) is a locally finite root system and $(\ |\ )$ is an invariant inner product.*

Proof. (a) By 3.13 and the remarks at the end of 4.1, we may assume R irreducible. By 3.15(b), R is the direct limit (in the category **RSE**) of finite irreducible root systems (R_λ, X_λ), $\lambda \in \varLambda$. We may assume that the directed set $\varLambda$ has a smallest element λ_0. Indeed, if this is not the case, choose some $\lambda_0 \in \varLambda$ and replace $\varLambda$ by the cofinal subset $\{\lambda \in \varLambda : \lambda \succcurlyeq \lambda_0\}$. Since morphisms in **RSE** are in particular injective linear maps, we may identify the R_λ with subsystems of R and R with their union, and similarly for X_λ and X. It is known (A.1) that finite irreducible root systems admit invariant inner products which are unique up to a positive factor. Fix an invariant inner product B_{λ_0} on X_{λ_0} and let B_λ be the unique extension of B_{λ_0} to an invariant inner product on X_λ. Then we have

$$B_\lambda \big| X_\lambda \cap X_\mu = B_\mu \big| X_\lambda \cap X_\mu$$

for all λ, μ. Indeed, since $\varLambda$ is directed, there exists $\nu \succcurlyeq \lambda, \mu$, and since B_ν is the unique extension of B_{λ_0} to X_ν, we have $B_\lambda = B_\nu \big| X_\lambda$ and $B_\mu = B_\nu \big| X_\mu$, whence our assertion. Now it is an easy matter to show that there exists a unique inner product B on X whose restriction to each X_λ equals B_λ and hence satisfies 4.1.2 for all $x \in X_\lambda$, $\alpha \in R_\lambda$. Since any $x \in X$ and $\alpha \in R$ is contained in some X_λ, we see that B satisfies 4.1.2 for all $x \in X$ and $\alpha \in R^\times$, and hence is an invariant inner product.

Next, let B' be any invariant bilinear form on X. By A.1, there exist $c_\lambda \in \mathbb{R}$ such that $B' \big| X_\lambda = c_\lambda B_\lambda$. By restricting further to X_{λ_0}, we see that $c_\lambda = c_{\lambda_0}$ for all λ. Hence $B' = c_{\lambda_0} B$, showing that $\mathcal{I}(R)$ is one-dimensional.

(b) By the definition of a root system in 3.3, it only remains to show local finiteness of R. Suppose α and $c\alpha$ are in $R^\times$ for some $c > 0$. Then it follows easily from the integrality condition that $c \in \{1/2, 1, 2\}$, and hence the intersection of $R^\times$ with any half-line $\mathbb{R}_+ \cdot \alpha$ has at most two elements. Now let $\alpha, \beta \in R^\times$ and assume that β is not a positive multiple of α. Then the angle φ between α and β is at least $\pi/6$. Indeed, $\cos\varphi = (\alpha|\beta)/\|\alpha\|\,\|\beta\|$, and the integrality condition implies

$$4\cos^2\varphi = \frac{2(\alpha|\beta)}{(\alpha|\alpha)} \cdot \frac{2(\beta|\alpha)}{(\beta|\beta)} \in \mathbb{Z}.$$

From this, one sees easily that $\cos\varphi \in \{-1, \pm\sqrt{3}/2, \pm\sqrt{2}/2, \pm 1/2, 0\}$ and hence $\cos\varphi \leqslant \sqrt{3}/2$ or $\varphi \geqslant \pi/6$, see also A.2. We now prove local finiteness and consider a finite-dimensional tight subspace V of X, with $F := \operatorname{core}(V) = V \cap R$. Since $|F^\times \cap \mathbb{R}_+ \cdot \alpha| \leqslant 2$, it suffices to show that the image C of $F^\times$ in the unit sphere S of V (under the map $\alpha \mapsto \alpha/\|\alpha\|$) is finite. Now $\angle(\alpha, \beta)$ is just the distance of the normalized points $\alpha/\|\alpha\|$ and $\beta/\|\beta\|$ in the standard metric of S. Since S is compact and the distance between two different points of C is at least $\pi/6$, the assertion follows. Finally, s_α is the orthogonal reflection in the hyperplane orthogonal to α. Hence we have $\langle x, \alpha^\vee \rangle = 2(x|\alpha)/(\alpha|\alpha)$ so 4.1.2 shows that $(\ |\ )$ is indeed an invariant inner product.

4.3. COROLLARY. (a) *Orthogonality as defined in* 3.5 *is equivalent to orthogonality with respect to any invariant inner product.*

(b) *Two roots α, β are linearly dependent if and only if $s_\alpha = s_\beta$.*

(c) *The definition of a root system in* [**57**] *is equivalent to the definition given in* 3.3.

Proof. (a) and (b) are clear from the formulas in 4.1.3, while (c) follows from part (b) of the theorem and the definitions in [**57**].

4.4. PROPOSITION. (a) *Let (R, X) be a root system and let $\alpha, \beta \in R^\times$ belong to the same connected component, with a connecting chain $\alpha = \alpha_0 \not\perp \alpha_1 \not\perp \ldots \not\perp \alpha_n = \beta$. Then*

$$c_{\alpha\beta} := \prod_{i=1}^{n} \frac{\langle \alpha_{i-1}, \alpha_i^\vee \rangle}{\langle \alpha_i, \alpha_{i-1}^\vee \rangle} \tag{1}$$

is independent of the choice of the connecting chain. If $(\ |\ )$ is an invariant inner product on X then

$$c_{\alpha\beta} = \frac{(\alpha|\alpha)}{(\beta|\beta)}. \tag{2}$$

If $f\colon (S, Y) \to (R, X)$ is an embedding and $\alpha, \beta \in S^\times$ belong to the same connected component then so do $f(\alpha), f(\beta) \in R^\times$, and we have

$$c_{\alpha\beta} = c_{f(\alpha)f(\beta)}. \tag{3}$$

(b) *Let (R, X) be irreducible. Then there are the following possibilities for the set of values of the function $(\alpha, \beta) \mapsto c_{\alpha\beta}$ on $R^\times \times R^\times$:*

(i) $\{1\}$,
(ii) $\{\frac{1}{2}, 1, 2\}$,
(iii) $\{\frac{1}{3}, 1, 3\}$,
(iv) $\{\frac{1}{4}, 1, 4\}$,
(v) $\{\frac{1}{4}, \frac{1}{2}, 1, 2, 4\}$.

In case (iii), *$R \cong \mathrm{G}_2$, in case* (iv), *$R \cong \mathrm{BC}_1$, and in case* (v), *R is not reduced and of rank $\geqslant 2$. If $(\ |\ )$ is an invariant inner product on X then the function $\alpha \mapsto (\alpha|\alpha)$ on $R^\times$ has at most three values.*

Proof. (a) Let $(\ |\ )$ be any invariant inner product. Then by 4.1.3,

$$\begin{aligned}\prod_{i=1}^{n} \frac{\langle \alpha_{i-1}, \alpha_i^\vee \rangle}{\langle \alpha_i, \alpha_{i-1}^\vee \rangle} &= \prod_{i=1}^{n} \frac{2(\alpha_{i-1}|\alpha_i)}{(\alpha_i|\alpha_i)} \cdot \frac{(\alpha_{i-1}|\alpha_{i-1})}{2(\alpha_i|\alpha_{i-1})} \\ &= \prod_{i=1}^{n} \frac{(\alpha_{i-1}|\alpha_{i-1})}{(\alpha_i|\alpha_i)} = \frac{(\alpha_0|\alpha_0)}{(\alpha_n|\alpha_n)} = \frac{(\alpha|\alpha)}{(\beta|\beta)},\end{aligned}$$

showing that $c_{\alpha\beta}$ is independent of the choice of connecting chain as well as (2). The remaining statements are immediate from 3.7(ii) and (1).

(b) By Corollary 3.16, any finite set of roots in R is contained in a finite irreducible subsystem S of R, so all remaining statements follow from well-known results on finite root systems (see A.6).

In more detail, since $(\alpha|\alpha)$ takes at most three different values on any finite irreducible $S^\times$, the same holds for $R^\times$. Suppose there are three different root lengths. Then R contains a finite irreducible subsystem S with the same property. Hence $S \cong \mathrm{BC}_n$ for some $n \geqslant 2$ (see 8.1.5 for a description of BC_n), and we are in case (v). Similarly, assuming that $(\alpha|\alpha)$ takes two different values, there exists a finite irreducible subsystem S with this property and therefore we have one of the cases (ii), (iii), or (iv).

In case (iii), there exist roots $\alpha, \beta \in S$ such that $(\alpha|\alpha) = 3(\beta|\beta)$. Hence α and β generate a subsystem of type G_2. If R has rank $\geqslant$3, we also could choose S of rank $\geqslant$3, contradicting the well-known fact that G_2 does not embed in any irreducible finite root system of rank > 2 (see below). Thus $R \cong \mathrm{G}_2$. Finally, in case (iv), it follows from similar arguments that $R = \{0, \pm\alpha, \pm 2\alpha\}$ is of type BC_1.

The non-embeddability of G_2 is of course immediate from the classification of finite root systems but can also be proven directly, using only the easier classification of root systems of rank two.

4.5. LEMMA. *Let R be a root system and $S \subset R$ be a subsystem which is isomorphic to the root system* G_2. *Then S is a direct summand of R.*

Proof. Let $V = \mathrm{span}(S)$. Then V is a two-dimensional subspace of X, and from the classification of root systems of rank two [**12**, Planche X], it follows that $R \cap V = S$, so S is full. To show that S is a direct summand, we use Lemma 3.11, and thus have to show that any $\gamma \in R \setminus S$ is perpendicular to S. Assume, for a contradiction, that there exists $\gamma \in R \setminus S$ but $\gamma \not\perp S$, and let $(\ |\ )$ be an invariant inner product. Let $u := \gamma/\|\gamma\|$ and denote by v the orthogonal projection of u onto V, which then satisfies $0 < \|v\| < 1$. The twelve nonzero roots of $S \cong \mathrm{G}_2$ divide the Euclidean plane V into twelve 30° sectors, the Weyl chambers of S. By a suitable choice of coordinates, we may identify V with $\mathbb{C}$ and assume that $v = x + iy$ lies in the 30° sector bounded by 1 and $\zeta = \exp(\pi i/6) = (1/2)(\sqrt{3} + i)$. Then

$$x > 0, \quad y \geqslant 0, \quad 0 < \|v\|^2 = x^2 + y^2 < 1, \quad \frac{y}{x} \leqslant \tan 30^\circ = \frac{1}{\sqrt{3}}. \tag{1}$$

The nonzero roots of S, normalized to length 1, are now the twelfth roots of unity ζ^k, $k = 1, \ldots, 12$. From A.2 and the fact that $\gamma \notin S$ we know that the possible cosines of the angles between γ and a root $\alpha \in S$ are $0, \pm 1/2, \pm\sqrt{2}/2, \pm\sqrt{3}/2$. Hence we have

$$x = \cos\angle(\gamma, 1) \in M := \{\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\}, \quad y = \cos\angle(\gamma, \zeta^3) \in \{0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\}. \tag{2}$$

It is easily checked that (1) and (2) imply $y = 0$. Now $\cos\angle(\gamma, \zeta) = (v|\zeta) = x\sqrt{3}/2 \in \{\sqrt{3}/4, \sqrt{6}/4, 3/4\}$ is not one of the admissible values for the cosine between γ and an element of S, contradiction.

4.6. *Short and long roots, and the normalized invariant inner product.* Let (R, X) be a root system. A root $\alpha \in R^\times$ is called *short* (*long*) if $c_{\alpha\beta} \leqslant 1$ ($c_{\alpha\beta} \geqslant 1$) for all roots $\beta \neq 0$ in the connected component of R containing α. In view of 4.4.2, this is equivalent to $(\alpha|\alpha) \leqslant (\beta|\beta)$ or $(\alpha|\alpha) \geqslant (\beta|\beta)$ where $(\ |\ )$ is any invariant inner product, explaining the terminology. Clearly the set of short roots and also the set of long roots of each connected component of R is not empty. Of course, these sets may be identical, and there also may be roots which are neither short nor long, namely in case R has a non-reduced component of rank $\geqslant 2$. Note that divisible roots are automatically long.

By Theorem 4.2(a) and the remarks at the end of 4.1, there exists a unique invariant inner product $(\ |\ )$ on X with the property that $(\alpha|\alpha) = 2$ for all short roots. We call this the *normalized invariant inner product.* Then by 4.4, $(\alpha|\alpha)$ is even for all $\alpha \in R$, and hence $(\alpha|\beta) = (1/2)(\alpha|\alpha)\langle\beta, \alpha^\vee\rangle \in \mathbb{Z}$, for all $\alpha, \beta \in R^\times$. This makes

$$\Phi(x) := (1/2)(x|x) \tag{1}$$

an integer-valued quadratic form on the root lattice $\mathfrak{Q}(R)$, the abelian group generated by R (see 6.1), whose associated bilinear form is

$$\Phi(x + y) - \Phi(x) - \Phi(y) = (x|y). \tag{2}$$

Following standard practice, we call a root system (R, X) *simply laced* if $c_{\alpha\beta} = 1$ for all $\alpha, \beta \in R^\times$ which lie in the same connected component of R; equivalently, if all roots are short (long).

A root system which is not simply laced is said to be *multiply laced.* More precisely, an irreducible root system R will be called *m-laced* if $m := \max\{c_{\alpha\beta} : \alpha, \beta \in R^\times\}$. Thus an irreducible R is simply laced if and only if it is 1-laced, and by Prop. 4.4(b), the possible values of m are $1, 2, 3, 4$, with $m = 3$ if and only if $R \cong \mathrm{G}_2$, and $m = 4$ if and only if R is not reduced.

4.7. Corollary. *An isomorphism $f\colon (R, X) \to (S, Y)$ between root systems maps short (long) roots to short (long) roots, and is isometric with respect to the normalized invariant inner products.*

Proof. By 3.7(ii), f preserves (non-)orthogonality, hence connected components, and by 4.4.1 it satisfies $c_{f(\alpha),f(\beta)} = c_{\alpha\beta}$ for all α, β in the same connected component. Therefore f maps short roots to short roots and long roots to long roots. Now the isometric property of f is clear from the formula 4.1.4.

4.8. Lemma. *Let (R, X) be a root system, and let*

$$R^\vee := \{\alpha^\vee : \alpha \in R\}, \qquad X^\vee := \operatorname{span}(R^\vee) \subset X^*. \tag{1}$$

Also, let $(\ |\ )$ be an invariant inner product, and define $\flat\colon X \to X^*$ by $\langle x, y^\flat\rangle = (x|y)$ for all $x, y \in X$. Then

$$\alpha^\vee = \frac{2\alpha^\flat}{(\alpha|\alpha)} \tag{2}$$

for all $\alpha \in R^\times$, and $\flat\colon X \to X^\vee$ is a vector space isomorphism.

Proof. Since $(\ |\ )$ is nondegenerate, the map $\flat$ is injective, and from 4.1.2 we see that

$$\langle x, \alpha^\vee \rangle = \frac{2(x|\alpha)}{(\alpha|\alpha)}$$

for all $x \in X$. This is equivalent to formula (2). As X and $X^\vee$ are spanned by R and $R^\vee$, respectively, it follows that $\flat\colon X \to X^\vee$ is a vector space isomorphism.

4.9. Theorem (The coroot system). (a) *There is a covariant functor* $\mathcal{C}$ *from the category* **RSE** *of root systems and embeddings (see* 3.6*) to itself given on objects by* $\mathcal{C}(R, X) = (R^\vee, X^\vee)$ *as in* 4.8.1*, and on embeddings* $f\colon (S, Y) \to (R, X)$ *by* $\mathcal{C}(f) = f^\vee$*, where* $f^\vee$ *is the unique linear map* $f^\vee\colon Y^\vee \to X^\vee$ *satisfying*

$$f^\vee(\beta^\vee) = f(\beta)^\vee \tag{1}$$

for all $\beta \in S$*. We call* $\mathcal{C}(R, X) = (R^\vee, X^\vee)$ *the* coroot system *of* (R, X)*. If* f *is a full embedding then so is* $f^\vee$*.*

(b) *The map* $\iota_{(R,X)}\colon (R, X) \to (R^{\vee\vee}, X^{\vee\vee})$*, given by*

$$\langle \xi, \iota_{(R,X)}(x) \rangle = \langle x, \xi \rangle, \tag{2}$$

for all $x \in X$*,* $\xi \in X^\vee$*, is an isomorphism of root systems. It induces a natural isomorphism* $\iota\colon \mathrm{Id}_{\mathbf{RSE}} \to \mathcal{C} \circ \mathcal{C}$ *on the category* **RSE**.

(c) *The functor* $\mathcal{C}$ *commutes with direct sums and direct limits and preserves connected components. In particular,* R *is irreducible if and only if* $R^\vee$ *is so.*

In view of (b), we will usually identify $R^{\vee\vee}$ and R. Sometimes the coroot system is referred to as the "dual root system". However, unlike the dual of a vector space, the functor $\mathcal{C}$ is a *covariant*, and not a contravariant functor on the category **RSE**.

Proof. (a) Let $(\ |\ )$ be an invariant inner product on X, and consider the vector space isomorphism $\flat\colon X \to X^\vee$ as in Lemma 4.8. We denote by $(\ |\ )'$ the inner product on $X^\vee$ for which $\flat$ is an isometry, and show that $R^\vee$ is a root system by verifying the conditions of Theorem 4.2(b) for this inner product. Thus let $s_{\alpha^\vee}$ be the orthogonal reflection in $\alpha^\vee$ with respect to $(\ |\ )'$. By definition, $(\alpha^\flat|\beta^\flat)' = (\alpha|\beta)$. Hence a simple computation with 4.8.2 shows that

$$\langle \beta^\vee, \alpha^{\vee\vee} \rangle = \frac{2(\beta^\vee|\alpha^\vee)}{(\alpha^\vee|\alpha^\vee)} = \frac{2(\alpha|\beta)}{(\beta|\beta)} = \langle \alpha, \beta^\vee \rangle \in \mathbb{Z}, \tag{3}$$

so the integrality condition of 4.2(b) is satisfied. For later use, note that (3), together with the fact that $X^\vee$ is spanned by $R^\vee$, implies

$$\alpha^{\vee\vee} = j(\alpha)\big|X^\vee, \tag{4}$$

where $j\colon X \to X^{**}$ is the canonical map. Next,

$$s_{\alpha^\vee}(\beta^\vee) = \beta^\vee - \frac{2(\beta^\vee|\alpha^\vee)}{(\alpha^\vee|\alpha^\vee)}\alpha^\vee = \beta^\vee - \langle \alpha, \beta^\vee \rangle \alpha^\vee = (s_\alpha(\beta))^\vee \tag{5}$$

by 3.9.3, so $s_{\alpha^\vee}(R^\vee) = R^\vee$, as required.

Now let $f\colon (S,Y) \to (R,X)$ be an embedding of root systems. Clearly, there is at most one linear map $f^\vee$ satisfying (1) since $S^\vee$ spans $Y^\vee$. To prove existence, let $(x|y)_Y := (f(x)|f(y))$ be the invariant inner product induced on Y as in 4.1.4, and $\flat\colon Y \to Y^\vee$ the vector space isomorphism induced by $(\ |\ )_Y$. Define $f^\vee$ by commutativity of the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\flat"'] & X \arrow[d, "\flat"] \\ Y^\vee \arrow[r, "f^\vee"'] & X^\vee \end{tikzcd}$$

Then by 4.8.2, for any $\beta \in S^\times$,

$$f^\vee(\beta^\vee) = f^\vee\Big(\frac{2\beta^\flat}{(\beta|\beta)_Y}\Big) = \frac{2f(\beta)^\flat}{(f(\beta)|f(\beta))} = f(\beta)^\vee,$$

which proves the existence of the linear map $f^\vee$ satisfying (1). For $f^\vee$ to be an embedding, we need to check, by condition (ii) of 3.7, that $\langle \beta^\vee, \alpha^{\vee\vee}\rangle = \langle f^\vee(\beta^\vee), f^\vee(\alpha^\vee)^\vee\rangle$, which follows easily from (3), (1) and the fact that f is an embedding. Now it is clear that $\mathcal{C}$ is a covariant functor from **RSE** to itself.

Suppose that f is a full embedding (3.8), so $f(S)$ is a full subsystem of R, and let $\alpha^\vee \in R^\vee \cap \operatorname{span}(f^\vee(S^\vee)) = R^\vee \cap \operatorname{span}(f(S)^\vee) = R^\vee \cap f(Y)^\flat$. By 4.8.2, $\alpha^\flat \in f(Y)^\flat$ and hence $\alpha \in f(Y)$. Since $f(S)$ is full in R, we have $\alpha = f(\beta) \in f(S)$, and therefore by (1), $\alpha^\vee = f^\vee(\beta^\vee) \in f^\vee(S^\vee)$, so $f^\vee(S^\vee)$ is full in $R^\vee$, showing that $f^\vee$ is again a full embedding.

(b) We first show that $\iota_{(R,X)}\colon (R,X) \to (R^{\vee\vee}, X^{\vee\vee})$ is an isomorphism of root systems. Formula (2) says that

$$\iota_{(R,X)}(x) = j(x)\big|X^\vee.$$

Hence (4) implies

$$\iota_{(R,X)}(\alpha) = \alpha^{\vee\vee}, \tag{6}$$

for $\alpha \in R$. Thus $\iota_{(R,X)}$ maps R onto $R^{\vee\vee}$. For $\iota_{(R,X)}$ to be an isomorphism of root systems, it therefore suffices, by 3.6, that $\iota_{(R,X)}$ be a vector space isomorphism. Surjectivity is clear since X and $X^{\vee\vee}$ are spanned by R and $R^{\vee\vee}$, respectively. If $\iota_{(R,X)}(x) = 0$ then $\langle x, R^\vee\rangle = \{0\}$ by (2), and hence $x \in R^\perp = 0$ by 3.5.3.

It remains to show naturality of ι. This means that the diagram

$$\begin{tikzcd} (S,Y) \arrow[r, "\iota_{(S,Y)}"] \arrow[d, "f"'] & (S^{\vee\vee}, Y^{\vee\vee}) \arrow[d, "f^{\vee\vee}"] \\ (R,X) \arrow[r, "\iota_{(R,X)}"'] & (R^{\vee\vee}, X^{\vee\vee}) \end{tikzcd}$$

is commutative, for any embedding f. Thus let $y \in Y$. Then $\iota_{(R,X)}(f(y))$ and $f^{\vee\vee}(\iota_{(S,Y)}(y))$ are elements of $X^{\vee\vee} \subset X^{\vee *}$; so to prove them equal we evaluate on an element $\xi \in X^\vee$. Since Y and $X^\vee$ are spanned by S and $R^\vee$, respectively, we may assume $y = \beta \in S$ and $\xi = \alpha^\vee \in R^\vee$. Then $\langle \alpha^\vee, \iota_{(R,X)}(f(\beta))\rangle = \langle f(\beta), \alpha^\vee\rangle$ by (2), while

$$\begin{aligned}\langle \alpha^\vee, f^{\vee\vee}(\iota_{(S,Y)}(\beta))\rangle &= \langle \alpha^\vee, f^{\vee\vee}(\beta^{\vee\vee})\rangle \quad \text{(by (6))} = \langle \alpha^\vee, (f^\vee(\beta^\vee))^\vee\rangle \\ &= \langle \alpha^\vee, f(\beta)^{\vee\vee}\rangle \quad \text{(by (1))} = \langle f(\beta), \alpha^\vee\rangle \quad \text{(by (3))}.\end{aligned}$$

Thus ι is a natural transformation.

(c) This follows easily from the definitions.

4.10. Corollary. *Let (R, X) be a root system and $S \subset R$ a subsystem, spanning the subspace Y. Then the coroot system $S^\vee$ and its linear span may be canonically identified with $\{\beta^\vee : \beta \in S\} \subset R^\vee$ and its linear span in $X^\vee$.*

Proof. This follows by applying Th. 4.9(a) to the inclusion $i\colon (S, Y) \to (R, X)$.

4.11. Corollary. *Let (R, X) be a root system and $(R^\vee, X^\vee)$ its coroot system.*

(a) *The relative root lengths of R and $R^\vee$ are related by*

$$c_{\alpha^\vee\beta^\vee} = c_{\alpha\beta}^{-1} = c_{\beta\alpha}. \tag{1}$$

In particular, $\alpha \in R$ is short if and only if $\alpha^\vee \in R^\vee$ is long, and vice versa.

(b) *The assignment $g \mapsto g^\vee$ is an isomorphism $\mathrm{Aut}(R) \cong \mathrm{Aut}(R^\vee)$ mapping $\mathrm{Aut}_{\mathrm{fin}}(R)$ to $\mathrm{Aut}_{\mathrm{fin}}(R^\vee)$. It satisfies*

$$(s_\alpha)^\vee = s_{\alpha^\vee} \tag{2}$$

and hence maps $W(R)$ to $W(R^\vee)$, and

$$g^\vee(\alpha^\vee) = \alpha^\vee \circ g^{-1} \tag{3}$$

for every $g \in \mathrm{Aut}(R)$.

Proof. (a) Formula (1) follows easily from 4.9.3 and the definition of $c_{\alpha\beta}$ in 4.4.1.

(b) The first statement is clear from (a) of Theorem 4.9. Also, 4.9.5 and 4.9.1 imply (2). Thus ${}^\vee\colon \mathrm{Aut}(R) \to \mathrm{Aut}(R^\vee)$ is an isomorphism mapping $W(R)$ onto $W(R^\vee)$. Let $(\ |\ )$ be the normalized invariant inner product and $\flat\colon X \to X^\vee$ the induced vector space isomorphism. By 4.7, every automorphism g of R is then an isometry. It follows easily that $\flat \circ g = g^\vee \circ \flat$ which implies (3) in view of 4.8.2. Since $\flat$ is a vector space isomorphism, $g^\vee$ is of finite type if and only if g is so.

4.12. Corollary. *Let (R, X) be simply laced, and let $\flat\colon X \to X^\vee$ be the vector space isomorphism induced by the normalized invariant inner product $(\ |\ )$. Then $\flat\colon (R, X) \to (R^\vee, X^\vee)$ is an isomorphism of root systems.*

Proof. This is immediate from $(\alpha|\alpha) = 2$ for all $\alpha \in R^\times$, and from formula 4.8.2.

4.13. Corollary. *Let (R, X) be irreducible and m-laced, with $m = m(R) \in \{1, 2, 3, 4\}$ as in 4.6. Also let $(\ |\ )$ and $(\ |\ )^\vee$ be the normalized invariant inner products on X and $X^\vee$, respectively, and let $\flat\colon X \to X^\vee$ and $\flat^\vee\colon X^\vee \to X$ be the induced vector space isomorphisms of 4.8, where we identify X and $(X^\vee)^\vee$ by 4.9.2. Then $m(R) = m(R^\vee)$, and*

$$\flat^\vee \circ \flat = m\,\mathrm{Id}_X, \qquad \flat \circ \flat^\vee = m\,\mathrm{Id}_{X^\vee}. \tag{1}$$

In particular, if $m = 4$, then $\frac{1}{2}\flat\colon (R, X) \to (R^\vee, X^\vee)$ is an isomorphism of root systems.

Proof. $m(R) = m(R^\vee)$ follows readily from 4.11.1. Let $(\ |\ )'$ be the scalar product on $X^\vee$ for which $\flat$ is an isometry, so $(x^\flat|y^\flat)' = (x|y)$ for all $x, y \in X$. Formula 4.9.5 implies $s_{\alpha^\vee}(\beta^\flat) = (s_\alpha\beta)^\flat$ whence $(s_{\alpha^\vee}(\beta^\flat)|s_{\alpha^\vee}(\gamma^\flat))' = (\,(s_\alpha\beta)^\flat|(s_\alpha\gamma)^\flat\,)' = (s_\alpha\beta|s_\alpha\gamma) = (\beta|\gamma) = (\beta^\flat|\gamma^\flat)'$ for all $\alpha, \beta, \gamma \in R^\times$. Thus $(\ |\ )'$ is an invariant inner product on $X^\vee$, and hence by Th. 4.2(a), $(\ |\ )'$ and $(\ |\ )^\vee$ differ by a scalar factor, say, $(\ |\ )^\vee = \lambda(\ |\ )'$. To determine λ, let α and β be, respectively, a long and a short root of R. Then $m = c_{\alpha\beta} = (\alpha|\alpha)/(\beta|\beta) = (\alpha|\alpha)/2$, so $\alpha^\flat = m\alpha^\vee$ by 4.8.2. By Cor. 4.11(a), $\alpha^\vee$ is a short root of $R^\vee$ whence $(\alpha^\vee|\alpha^\vee)^\vee = 2$. It follows that $2m = (\alpha|\alpha) = (\alpha^\flat|\alpha^\flat)' = m^2(\alpha^\vee|\alpha^\vee)' = \lambda^{-1}m^2(\alpha^\vee|\alpha^\vee)^\vee = 2\lambda^{-1}m^2$, and therefore $\lambda = m$. Now we have $(x^\flat|y^\flat)^\vee = m(x|y)$, which is equivalent to the first formula of (1). The second formula follows by interchanging the roles of R and $R^\vee$ and using the canonical isomorphism $R^{\vee\vee} \cong R$ of Th. 4.9(a).

Now suppose $m = 4$. By Prop. 4.4(b), R is not reduced, and a root α is short if and only if $2\alpha \in R$ is long. Let $f := \frac{1}{2}\flat$. It suffices to show that $f(R) \subset R^\vee$. From 4.8.2 it follows easily that $f(\alpha) = (2\alpha)^\vee \in R^\vee$ when α is short, and $f(\beta) = (\beta/2)^\vee \in R^\vee$ when $\beta = 2\alpha$ is long. If $\gamma \in R^\times$ is neither long nor short then by Prop. 4.4(b), Case (v), we have $c_{\gamma\alpha} = (\gamma|\gamma)/(\alpha|\alpha) = 2$ or $(\gamma|\gamma) = 4$, and hence $f(\gamma) = \gamma^\vee \in R^\vee$.

Remark. For $m \neq 1, 4$, R and $R^\vee$ need not be isomorphic, and even when they are (for example in case $R = \mathrm{F}_4$ or G_2), an isomorphism between R and $R^\vee$ is not a multiple of $\flat$.

4.14. *Root systems over arbitrary fields of characteristic zero.* For applications of root systems in the theory of Lie algebras, it is useful to have a more general definition of root systems.

Let k be a field of characteristic zero. We define *locally finite root systems over* k by replacing real vector spaces by k-vector spaces in the definition 3.3. The following remarks on descent and base fields extensions show how to reduce the study of root systems over k to that of root systems over $\mathbb{R}$.

First let $R \subset X$ be a root system over k and denote by $X_\mathbb{Q}$ the rational span of R. Then $R \subset X_\mathbb{Q}$ is locally finite and $\alpha^\vee(X_\mathbb{Q}) \subset \mathbb{Q}$, from which it easily follows that $(R, X_\mathbb{Q})$ is a root system over $\mathbb{Q}$.

Suppose again that $R \subset X$ is a root system over k and let K be an extension field of k. We identify X with a subset of $X_K = X \otimes_k K$. Then for any subset $F \subset R$, finite or not, we have

$$R \cap \operatorname{span}_\mathbb{Q}(F) = R \cap \operatorname{span}_k(F) = R \cap \operatorname{span}_K(F), \tag{1}$$

where $\operatorname{span}_L(F)$ denotes the span of F over $L = \mathbb{Q}, k$ or K. It suffices to prove $R \cap \operatorname{span}_K(F) \subset R \cap \operatorname{span}_\mathbb{Q}(F)$. Any $\beta \in R \cap \operatorname{span}_K(F)$ can be written in the form $\beta = \sum_{i=1}^n x_i\alpha_i$ where $x_i \in K$ and $\alpha_i \in F$. We can assume that the α_i are linearly independent over K. The x_i are then a solution of the system of linear equations $\langle\beta, \alpha_j^\vee\rangle = \sum_{i=1}^n x_i\langle\alpha_i, \alpha_j^\vee\rangle$ with integral coefficients. Hence the x_i lie in $\mathbb{Q}$ as soon as we know that the matrix $(\langle\alpha_i, \alpha_j^\vee\rangle)$ is invertible. By assumption, $S := R \cap \operatorname{span}_k(F)$ is a finite root system in $\operatorname{span}_k(F)$. Therefore, by [**12**, VI, §1.1 Prop. 3], $\operatorname{span}_k(F)$ carries a nondegenerate symmetric bilinear form B invariant under the reflections s_α and with $B(\alpha, \alpha) \neq 0$ for all $\alpha \in S$. The arguments in 4.1 work for root systems over k, in particular we have the formula 4.1.3 from which it easily follows that the matrix $(\langle \alpha_i, \alpha_j^\vee \rangle)$ is invertible. Thus (1) holds, and hence $R \subset X_K$ is locally finite. By taking the canonical K-extensions of the linear forms $\alpha^\vee \in X^*$ one sees that (R, X_K) is a root system over K.

Many results proven here for root systems over $\mathbb{R}$, for example Theorem 8.4, are in fact true for root systems over k. These generalizations will be left to the reader.

§5. Weyl groups

5.1. *The finite topology.* Let X be an arbitrary set. We equip X with the discrete topology and the symmetric group $\mathrm{Sym}(X)$ of all bijections of X with the finite topology [**24**, 2.4]. Thus a basis of neighborhoods of Id_X consists of the sets $\{g \in \mathrm{Sym}(X) : g\big|F = \mathrm{Id}_F\}$, where F runs over the finite subsets of X. This is just the topology of pointwise convergence. Hence, a net $(g_\lambda)_{\lambda\in\Lambda}$ in $\mathrm{Sym}(X)$, where Λ is a directed index set, converges to g if and only if $\lim g_\lambda(x) = g(x)$ for all $x \in X$. Since X is discrete, this means in turn that there exists λ_x such that for all $\lambda \succcurlyeq \lambda_x$ we have $g_\lambda(x) = g(x)$, i.e., the $g_\lambda(x)$ become eventually constant. It is well known and easy to see that with this topology, $\mathrm{Sym}(X)$ is a Hausdorff topological group, which is discrete if and only if X is finite.

Now let X be a vector space. Then $\mathrm{GL}(X)$, being the automorphism group of a relational structure, is a closed subgroup of $\mathrm{Sym}(X)$ [**24**, 2.4.10], and $\mathrm{GL}(X)$ (with the induced topology) is discrete if and only if X is finite-dimensional [**33**, II, §3].

Suppose $X = \bigoplus_{i\in I} X_i$ is a direct sum of vector spaces. We identify the product $\prod_{i\in I} \mathrm{GL}(X_i)$ with the subgroup of $\mathrm{GL}(X)$ leaving each X_i invariant. It is easily seen that the topology induced from $\mathrm{GL}(X)$ on $\prod_{i\in I} \mathrm{GL}(X_i)$ coincides with the product topology of the topological spaces $\mathrm{GL}(X_i)$ with the finite topology. Moreover, the description of limits given above, shows that $\prod_{i\in I} \mathrm{GL}(X_i)$ is a closed subgroup of $\mathrm{GL}(X)$.

Let $(G_i)_{i\in I}$ be a family of groups where each G_i is a subgroup of $\mathrm{GL}(X_i)$. We denote by $\bigoplus_{i\in I} G_i$ the restricted direct product, that is, the subgroup of the full direct product $\prod_{i\in I} G_i$ consisting of all elements having only finitely many components different from the identity. Then

$$\overline{\bigoplus_{i\in I} G_i} = \prod_{i\in I} \overline{G_i}, \tag{1}$$

where the closure on the left is taken in $\mathrm{GL}(X)$, equivalently, in $\prod_{i\in I} \mathrm{GL}(X_i)$, while $\overline{G_i}$ is calculated in $\mathrm{GL}(X_i)$. To prove (1), recall that the projection maps π_j onto the factors $\mathrm{GL}(X_j)$ are continuous, and hence $\pi_j(\overline{\bigoplus_{i\in I} G_i}) \subset \overline{\pi_j(\bigoplus_{i\in I} G_i)} = \overline{G_j}$ which proves the inclusion "$\subset$" in (1). To prove the reverse inclusion, we first note that $\overline{G_i} \subset \overline{\bigoplus_{i\in I} G_i}$. Hence also $\bigoplus_{i\in I} \overline{G_i}$, being the subgroup generated by the $\overline{G_i}$, is contained in $\overline{\bigoplus_{i\in I} G_i}$. Now suppose $g = (g^{(i)})_{i\in I} \in \prod_{i\in I} \overline{G_i}$, let Λ be the set of finite subsets of I, directed by inclusion, and define a net $(g_F)_{F\in\Lambda}$ in $\bigoplus_{i\in I} \overline{G_i}$ by

$$g_F^{(i)} = \left\{ \begin{matrix} g^{(i)} & \text{if } i \in F \\ 1 & \text{if } i \notin F \end{matrix} \right\}.$$

Then we have $g = \lim g_F$. Indeed, let $x = (x_i)_{i\in I} \in X = \bigoplus_{i\in I} X_i$ and, say, $x_i \neq 0$ if and only if $i \in E$. Then E is a finite subset of I, and $g_F(x) = g(x)$ for all $F \supset E$, proving our assertion. Since $g_F \in \bigoplus_{i\in I} \overline{G_i} \subset \overline{\bigoplus_{i\in I} G_i}$, it follows that $g \in \overline{\bigoplus_{i\in I} G_i}$. Thus "$\supset$" holds in (1).

5.2. $\mathrm{Aut}(R)$ *as a topological group and the big Weyl group.* Let (R, X) be a root system. It is easy to see that $\mathrm{Aut}(R)$ is closed in $\mathrm{GL}(X)$. We always consider $\mathrm{Aut}(R)$ as a topological group with the topology induced from $\mathrm{GL}(X)$. The closure $\overline{W}(R)$ of the Weyl group $W(R)$ will be called the *big Weyl group* of R. We also introduce the following two outer automorphism groups:

$$\mathrm{Out}_{\mathrm{fin}}(R) := \mathrm{Aut}_{\mathrm{fin}}(R)/W(R), \qquad \mathrm{Out}(R) := \mathrm{Aut}(R)/\overline{W}(R). \tag{1}$$

If R is finite we clearly have $\mathrm{Aut}_{\mathrm{fin}}(R) = \mathrm{Aut}(R)$, $\overline{W}(R) = W(R)$ and hence $\mathrm{Out}_{\mathrm{fin}}(R) = \mathrm{Out}(R)$.

These Weyl groups behave as follows with respect to direct sums:

$$W(R) \cong \bigoplus_{i \in I} W(R_i), \tag{2}$$

$$\overline{W}(R) \cong \prod_{i \in I} \overline{W}(R_i). \tag{3}$$

Indeed, (2) is immediate from the definitions, while (3) follows from (2) and 5.1.1.

As a special case of Cor. 4.7 we note that an invariant bilinear form as in 4.1 is also invariant under the big Weyl group $\overline{W}(R)$. Regarding the behavior of the big Weyl group with respect to the coroot system, the map $g \mapsto g^\vee$, $g \in \mathrm{GL}(X)$ as in Cor. 4.11(b), induces a topological isomorphism ${}^\vee\colon \mathrm{Aut}(R) \to \mathrm{Aut}(R^\vee)$ and hence maps $\overline{W}(R)$ isomorphically onto $\overline{W}(R^\vee)$.

5.3. Lemma (generalized reflections). *Let (R, X) be a root system and let $\Omega \subset R^\times$ be an orthogonal subset, i.e., $\alpha \perp \beta$ for all $\alpha \neq \beta$ in Ω. Let Λ be the set of finite subsets of Ω, and define $s_F = \prod_{\alpha \in F} s_\alpha$ for all $F \in \Lambda$. Then the net $(s_F)_{F \in \Lambda}$ in $W(R)$ converges to an element $s_\Omega \in \overline{W}(R)$, called the* generalized reflection in *Ω. Explicitly, it is given by*

$$s_\Omega(x) = x - \sum_{\alpha \in \Omega} \langle x, \alpha^\vee \rangle \alpha, \tag{1}$$

the sum on the right having only finitely many nonzero terms for every $x \in X$. The generalized reflection s_Ω satisfies $s_\Omega^2 = \mathrm{Id}$, with $(+1)$-eigenspace $\Omega^\perp$ and (-1)-eigenspace $\mathrm{span}(\Omega)$.

Proof. Note first that the order of factors in s_F is immaterial by 3.9.4. We claim that $\Omega_x := \{\alpha \in \Omega : \langle x, \alpha^\vee \rangle \neq 0\}$ is finite, for all $x \in X$. Since R spans X, this will follow from

$$\beta \in R^\times \quad \Longrightarrow \quad \mathrm{Card}\{\alpha \in \Omega : \beta \not\perp \alpha\} \leqslant 4, \tag{2}$$

To prove (2), let $\alpha_1, \ldots, \alpha_n$ be pairwise orthogonal roots with $\beta \not\perp \alpha_i$ for $i = 1, \ldots, n$. Possibly after replacing α_i by its negative we may assume $\langle \beta, \alpha_i^\vee \rangle < 0$. Let Y be the linear span of $\alpha_0 := \beta, \alpha_1, \ldots, \alpha_n$ and consider the full finite subsystem $S = R \cap Y$. Let $(\ |\ )$ be a $W(S)$-invariant inner product on Y. Putting $u_j := \alpha_j / \|\alpha_j\|$, we then have $(u_0|u_i) = \cos \angle(\alpha_0, \alpha_i) \leqslant -1/2$ for $i = 1, \ldots, n$, by A.2. Hence, for all non-negative $x_j \in \mathbb{R}$,

$$0 \leqslant \Big\| \sum_{j=0}^{n} x_j u_j \Big\|^2 = x_0^2 + 2x_0 \sum_{i=1}^{n} x_i (u_0|u_i) + \sum_{i=1}^{n} x_i^2 \leqslant \sum_{j=0}^{n} x_j^2 - x_0 \sum_{i=1}^{n} x_i.$$

Specializing $x_0 = n/2$ and $x_i = 1$ for $i \geqslant 1$ yields $0 \leqslant (n^2/4) + n - (n/2) \cdot n = n - n^2/4$ or $n \leqslant 4$, as asserted.

Returning to an element $x \in X$, we now have, for all finite $F \subset \Omega$ with $F \supset \Omega_x$, that

$$s_F(x) = \Big(\prod_{\alpha \in \Omega_x} s_\alpha\Big)\Big(\prod_{\beta \in F \setminus \Omega_x} s_\beta\Big)(x) = \Big(\prod_{\alpha \in \Omega_x} s_\alpha\Big)(x) = s_{\Omega_x}(x),$$

since $\langle x, \beta^\vee \rangle = 0$ and hence $s_\beta(x) = x$ for $\beta \notin \Omega_x$. In view of 5.2, this proves the existence of s_Ω. Formula (1) is clear from 3.3.2 and orthogonality of Ω. Since all $s_F^2 = \mathrm{Id}$, this is true for s_Ω as well. The assertions concerning the eigenspaces of s_Ω follow easily from (1).

Remark. The proof of (2) shows that from any point in the graph of a Cartan matrix whose associated bilinear form is positive semidefinite, can issue at most 4 branches. On the other hand, the configuration of roots realizing the extended Dynkin diagram of the root system D_4 shows that $n = 4$ does actually occur (cf. [**47**, 3.5]).

5.4. *More Weyl groups and automorphism groups.* Let (R, X) be an infinite locally finite root system with $\dim(X) = \mathbf{d}$, and let $\mathbf{c}$ be an infinite cardinal. We define $W(R, \mathbf{c})$ to be the subgroup of $\overline{W}(R)$ generated by all s_Ω where $\Omega \subset R^\times$ is an orthogonal system of cardinality $< \mathbf{c}$. It is easily seen that the groups $W(R, \mathbf{c})$ form an ascending chain of normal subgroups of $\mathrm{Aut}(R)$, all contained in $\overline{W}(R)$, with smallest member $W(R) = W(R, \aleph_0)$. Since the cardinality of an orthogonal system is at most $\mathbf{d}$, this chain becomes stationary at $\mathbf{d}^+$, the cardinal successor of $\mathbf{d}$. In fact, $W(R, \mathbf{d}^+) = \overline{W}(R)$ will be shown in 9.6 as a consequence of the classification.

Similar definitions can be made for automorphism groups. Let $\mathrm{GL}(X, \mathbf{c})$ be the set of $f \in \mathrm{GL}(X)$ whose fixed point set X^f has codimension $< \mathbf{c}$, and put $\mathrm{Aut}(R, \mathbf{c}) := \mathrm{Aut}(R) \cap \mathrm{GL}(X, \mathbf{c})$. Since $\mathrm{GL}(X, \mathbf{c})$ is a normal subgroup of $\mathrm{GL}(X)$ [**65**], the groups $\mathrm{Aut}(R, \mathbf{c})$ are normal subgroups of $\mathrm{Aut}(R)$. Clearly, $\mathrm{Aut}(R, \aleph_0) = \mathrm{Aut}_{\mathrm{fin}}(R)$ and $\mathrm{Aut}(R, \mathbf{d}^+) = \mathrm{Aut}(R)$. For a generalized reflection s_Ω the codimension of its fixed point set equals the dimension of the (-1)-eigenspace, and by 5.3 this is equal to $|\Omega|$, so that $W(R, \mathbf{c}) \subset \mathrm{Aut}(R, \mathbf{c})$. The outer automorphism groups defined in 5.2.1 are then part of the series $\mathrm{Out}(R, \mathbf{c}) := \mathrm{Aut}(R, \mathbf{c})/W(R, \mathbf{c})$ of outer automorphism groups, with $\mathrm{Out}(R, \aleph_0) = \mathrm{Out}_{\mathrm{fin}}(R)$ and $\mathrm{Out}(R, \mathbf{d}^+) = \mathrm{Out}(R)$. Examples will be calculated in 9.5.

The groups $W(R, \mathbf{c})$ behave in the expected way when passing to the coroot system. Indeed, the map $\alpha \mapsto \alpha^\vee$ sends orthogonal systems to orthogonal systems of the same cardinality, and thus $W(R, \mathbf{c})^\vee = W(R^\vee, \mathbf{c})$. Similar statements hold for the automorphism groups $\mathrm{Aut}(R, \mathbf{c})$.

5.5. Proposition. *Let (R, X) be a root system with Weyl group $W = W(R)$, and let $\mathfrak{C}$ be the set of connected components of R as in* 3.12. *For any subset $\mathfrak{S} \subset \mathfrak{C}$ let $X_{\mathfrak{S}} = \sum_{C \in \mathfrak{S}} \mathrm{span}(C)$. Then the map $\mathfrak{S} \mapsto X_{\mathfrak{S}}$ is a lattice isomorphism between the power set of $\mathfrak{C}$ and the lattice of W-submodules of X. In particular, W acts* *completely reducibly on X, every W-submodule has a unique complement, and W acts irreducibly if and only if R is irreducible. The same statements hold for the big Weyl group $\overline{W}(R)$ in place of $W(R)$, and hence also for all $W(R,\mathbf{c})$.*

Proof. Let $C \in \mathfrak{C}$ and $\alpha \in R^\times$, $\beta \in C$. Then $s_\alpha\beta \in C$ if $\alpha \in C$ since C is a subsystem, while $\alpha \perp \beta$ and hence again $s_\alpha\beta = \beta \in C$ in case $\alpha \in R\backslash C$. This shows that $\operatorname{span}(C)$ is a W-submodule of X, and therefore so is $X_{\mathfrak{S}}$. Since by 3.13 X is the direct sum of the subspaces $\operatorname{span}(C)$, $C \in \mathfrak{C}$, it follows easily that the map $\mathfrak{S} \mapsto X_{\mathfrak{S}}$ is injective and satisfies $X_{\mathfrak{S}\cap\mathfrak{T}} = X_{\mathfrak{S}} \cap X_{\mathfrak{T}}$ and $X_{\mathfrak{S}\cup\mathfrak{T}} = X_{\mathfrak{S}} + X_{\mathfrak{T}}$. It remains to show that every W-submodule Y of X is of the form $X_{\mathfrak{S}}$. We define $S := R \cap Y$ and $T = R \setminus Y$ and claim that $Y \subset T^\perp$. Indeed, let $y \in Y$ and $\beta \in T$. Since Y is a W-submodule, $y - s_\beta y = \langle y, \beta^\vee\rangle\beta \in Y$ which implies $\langle y, \beta^\vee\rangle = 0$ since $\beta \notin Y$. In particular, $R = S \cup T$ is an orthogonal decomposition so that $S = \bigcup \mathfrak{S}$ is the union of a set $\mathfrak{S} \subset \mathfrak{C}$ of connected components of R. Clearly $X_{\mathfrak{S}} = \operatorname{span}(S) \subset Y$. As $X = \operatorname{span}(S) \oplus \operatorname{span}(T)$ we conclude $Y = \operatorname{span}(S) \oplus \big(Y \cap \operatorname{span}(T)\big)$, and $Y \cap \operatorname{span}(T) \subset T^\perp \cap S^\perp = R^\perp = 0$, as desired.

The assertion concerning the big Weyl group follows from the simple observation that any W-submodule Y of X is stable under $\overline{W}(R)$. Indeed, let $\bar{w} = \lim w_\lambda \in \overline{W}(R)$ where $(w_\lambda)_{\lambda\in\Lambda}$ is a net in W, and let $y \in Y$. Then $w_\lambda(y) \in Y$ for all $\lambda \in \Lambda$. Since $\bar{w}(y) = w_\lambda(y)$ for all sufficiently large λ, we have $\bar{w}(y) \in Y$. Finally, the analogous statements for the $W(R,\mathbf{c})$ are clear from the fact that they are all sandwiched between $W(R)$ and $\overline{W}(R)$.

5.6. COROLLARY. *Let R be irreducible and let $\alpha, \beta \in R^\times$. Then there exists $w \in W(R)$ such that $\langle w\alpha, \beta^\vee\rangle > 0$. If α and β have the same length with respect to an invariant inner product then even $w\alpha = \beta$ for some $w \in W(R)$ holds.*

This can be shown in the same way as [**12**, VI, §1.3, Prop. 11]

5.7. THEOREM (functoriality of Weyl groups). (a) *Let $f\colon (R', X') \to (R, X)$ be an embedding of root systems and let $X'' := X/f(X')$ be the cokernel of f, with $p\colon X \to X''$ the canonical map. For every $w \in W(R',\mathbf{c})$ there exists a unique $\tilde{w} \in W(R,\mathbf{c})$ making the diagram*

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "f"] \arrow[d, "w"'] & X \arrow[r, "p"] \arrow[d, "\tilde{w}"] & X'' \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & X' \arrow[r, "f"'] & X \arrow[r, "p"'] & X'' \arrow[r] & 0 \end{tikzcd} \tag{1}$$

commutative. The map $w \mapsto \tilde{w}$ is a group monomorphism $W(f,\mathbf{c})\colon W(R',\mathbf{c}) \to W(R,\mathbf{c})$ which satisfies

$$\widetilde{s_\Omega} = s_{f(\Omega)} \tag{2}$$

for all orthogonal systems Ω of R'. In this way, the Weyl groups $W(R,\mathbf{c})$ become covariant functors $W_{\mathbf{c}} = W(-,\mathbf{c})$ from the category **RSE** *of root systems and embeddings to the category of groups. These functors commute with direct limits.*

(b) *Let $W(R',\mathbf{c})$ act trivially on X'' and via the homomorphism $W(f,\mathbf{c})$ on X. Then the sequence*

$$0 \longrightarrow X' \xrightarrow{f} X \xrightarrow{p} X'' \longrightarrow 0 \tag{3}$$

of $W(R',\mathbf{c})$-modules is exact, and the fixed point set of $W(R',\mathbf{c})$ on X is

$$\{x \in X : \tilde{w}(x) = x \text{ for all } w \in W(R',\mathbf{c})\} = f(R')^{\perp}. \tag{4}$$

The sequence (3) *splits (as a sequence of $W(R',\mathbf{c})$-modules) if and only if*

$$X = f(X') \oplus f(R')^{\perp}. \tag{5}$$

In this case, $f(R')^{\perp}$ is the unique $W(R',\mathbf{c})$-submodule of X complementary to $f(X')$.

(c) *If R' is finite or $f(R')$ is a direct summand of R then* (3) *splits.*

Proof. (a) Let $(\ |\ )$ be an invariant inner product on X. For any $w \in \mathrm{GL}(X')$ let us call $\tilde{w} \in \mathrm{GL}(X)$ an *extension* of w if $\tilde{w}$ leaves $(\ |\ )$ invariant, and makes (1) commutative. Observe that if w_1, w_2 have extensions $\tilde{w}_1, \tilde{w}_2$ then $\tilde{w}_1\tilde{w}_2$ is an extension of w_1w_2, and $\tilde{w}^{-1}$ is an extension of w^{-1}. Also, the fact that f is injective ensures that $\tilde{w} = \mathrm{Id}$ implies $w = \mathrm{Id}$. Hence, (a) will follow once we show that extensions are unique, and that the generators of $W(R',\mathbf{c})$ have extensions which belong to $W(R,\mathbf{c})$.

For uniqueness, suppose that $w \in \mathrm{GL}(X')$ has two extensions $\tilde{v}$ and $\tilde{w}$. Then $u = \tilde{v}^{-1}\tilde{w}$ is an extension of $\mathrm{Id}_{X'}$, and thus acts like the identity on $Y := f(X')$. Now let $x \in X$ be arbitrary. As u induces the identity on $X'' = X/Y$, we have $u(x) \equiv x \bmod Y$, so $x - u(x) \in Y$. Now for any $y \in Y$, $(y|x) = (u(y)|u(x)) = (y|u(x))$ or $(y|x - u(x)) = 0$. Since $(\ |\ )$ is nondegenerate on Y, this proves $u(x) = x$, as asserted.

Next, let Ω be an orthogonal system in R' with $|\Omega| < \mathbf{c}$. We claim that $\widetilde{s_\Omega} = s_{f(\Omega)}$ is the extension of s_Ω. Indeed, by 3.7(ii), $f(\Omega)$ is an orthogonal system in R, and obviously $|f(\Omega)| < \mathbf{c}$, so $s_{f(\Omega)} \in W(R,\mathbf{c})$. As remarked in 5.2, any element of $W(R,\mathbf{c})$ leaves $(\ |\ )$ invariant. Thus it remains to show that $s_{f(\Omega)}$ makes (1) commutative. By 5.3.1 and 3.7(iii), we have

$$s_{f(\Omega)}(f(x')) = f(x') - \sum_{\alpha\in\Omega}\langle f(x'), f(\alpha)^{\vee}\rangle f(\alpha) = f\big(x' - \sum_{\alpha\in\Omega}\langle x', \alpha^{\vee}\rangle\alpha\big) = f(s_\Omega(x'))$$

for all $x' \in X'$ which shows that the left hand square of (1) commutes. Again by 5.3.1,

$$s_{f(\Omega)}(x) = x - \sum_{\alpha\in\Omega}\langle x, f(\alpha)^{\vee}\rangle f(\alpha) \equiv x \bmod f(X'), \tag{6}$$

whence also the right hand square of (1) commutes. The statement concerning direct limits follows easily from the definitions.

(b) From (1) it is clear that (3) is (with the indicated actions) an exact sequence of $W(R',\mathbf{c})$-modules. Formula (4) is a consequence of (6). Now the remaining statements follow easily.

(c) The case of a direct summand is clear. If R' is finite, let $e_1, \ldots, e_n$ be an orthonormal basis of $f(X')$. Then the map $x \mapsto \sum_{i=1}^{n}(x|e_i)e_i$ is the orthogonal projection of X onto $f(X')$, so we have (5).

Remarks. (a) For an infinite R', the sequence (3) need not be split, even when f is a full embedding. For example, let I be an infinite set, let $(R', X') = (\dot{\mathrm{A}}_I, \dot{X})$ and consider the embedding $(R', X') \hookrightarrow (R, X) = \mathrm{B}_I$ (notations of 8.1). Here X'' is one-dimensional. The Weyl group of R' is the finitary symmetric group acting by permutation of the standard basis ε_i on X (see 9.5.2), and hence has fixed point set $\{0\}$ on X.

(b) Functoriality of the big Weyl group $\overline{W}(R)$ with respect to embeddings will be shown in 9.7 as a consequence of the equality $\overline{W}(R) = W(R, \mathbf{d}^+)$ where $\mathbf{d} = \mathrm{rank}(R)$. It would be desirable to have a direct proof of this fact.

5.8. Corollary. *Let (R, X) be a root system and $S \subset R$ a subsystem, with linear span $Y = \mathrm{span}(S)$. Also let $W_{S,\mathbf{c}} \subset W(R, \mathbf{c})$ be the subgroup generated by all generalized reflections s_Ω, where $\Omega \subset S^\times$ and $|\Omega| < \mathbf{c}$. Then the restriction map $W_{S,\mathbf{c}} \to W(S, \mathbf{c})$, $w \mapsto w\big|Y$, is an isomorphism.*

This follows easily from 5.7 applied to the embedding $(S, Y) \hookrightarrow (R, X)$. We will frequently *identify* the groups $W_{S,\mathbf{c}}$ and $W(S, \mathbf{c})$. In case $\mathbf{c} = \aleph_0$, we will use the simpler notation W_S instead of $W_{S,\aleph_0}$.

Remarks. (a) If $W(R)$ is a Coxeter group one knows that W_S, being a so-called reflection subgroup, is again a Coxeter group [**22**, **27**]. In this case the corollary is obvious. However, in general $W(R)$ is not a Coxeter group, see 9.9.

(b) The subgroups W_S, for S a full subsystem of R, are called *parabolic subgroups* of $W(R)$. An explanation for this terminology will be given in 15.8, Remark (b). It is easy to see that in the case of finite Weyl groups our concept of parabolic subgroups coincides with the usual one, as for example defined in [**32**, 1.10], [**17**, 2.5], [**36**, 5.1].

Recall from [**39**] that a group is called *locally finite* if every finite subset generates a finite subgroup.

5.9. Corollary. *The Weyl group of a locally finite root system is locally finite.*

Proof. Since $W(R)$ is generated by the reflections s_α, $\alpha \in R^\times$, it suffices to show that, for every finite subset F of $R^\times$, the subgroup G generated by $\{s_\alpha : \alpha \in F\}$ is finite. By local finiteness of R, F is contained in the finite subsystem $S = R \cap \mathrm{span}(F)$ and hence $G \subset W_S \cong W(S)$ which is finite. (An alternative proof would be to use 3.15 and 5.7(a).)

5.10. Corollary. *Let (R, X) be a root system, and let $w \in W(R)$.*

(a) *Then w is the product of reflections in roots contained in $(X^w)^\perp$, the orthogonal complement of the fixed point set X^w of w with respect to any invariant inner product.*

(b) *Let $l_T(w)$ be the length of w with respect to the generating set $T = \{s_\alpha : \alpha \in R\}$ of $W(R)$, cf.* [**12**, IV, §1.1, Déf. 1]. *Then*

$$l_T(w) = \mathrm{codim}\, X^w, \tag{1}$$

and this is also the length of w with respect to $\{s_\alpha : \alpha \in R'\}$ for any subsystem R' of R such that $w \in W_{R'}$.

(c) *Let $w = s_{\alpha_1} \cdots s_{\alpha_n}$ for $\alpha_i \in R$. Then $l_T(w) = n$ if and only if the $\alpha_1, \dots, \alpha_n$ are linearly independent.*

Proof. (a) Choose an invariant inner product on X. We may assume $w \in W_S$ for a finite subsystem (S, Y) of (R, X). Hence by 5.7(c), applied to the inclusion $(S, Y) \hookrightarrow (R, X)$, we have the orthogonal w-invariant decomposition $X = Y \oplus Y^\perp$ as in 5.7.5, and $Y = Y_w \oplus Y^w$ where Y^w denotes the fixed point set of w in Y and Y_w its orthogonal complement in Y. Since w has no non-zero fixed point in Y_w and acts like the identity on $Y^\perp$, it follows that $X^w = Y^w \oplus Y^\perp$ and therefore $(X^w)^\perp = Y_w$. By [**12**, V, §3.3, Prop. 2], w is a product of reflections in roots contained in Y_w.

(b) For any decomposition $w = s_{\alpha_1} \cdots s_{\alpha_n}$ let $V = \operatorname{span}\{\alpha_1, \dots, \alpha_n\}$. Then $V^\perp \subset X^w$, and $V \oplus V^\perp = X$ because V is finite-dimensional. Hence $V + X^w = X$, so $\operatorname{codim} X^w \leqslant \dim V \leqslant n$, and then also $\operatorname{codim} X^w \leqslant l_T(w)$.

To prove the inequality $l_T(w) \leqslant \operatorname{codim} X^w$, we use the setting and notation of the proof of (a): $w \in W_S$ for a finite subsystem (S, Y) of (R, X). By a result of Carter [**16**, Lemma 2], w is a product of $\dim Y_w = \operatorname{codim} X^w$ reflections s_β, $\beta \in Y_w$. Therefore $l_T(w) \leqslant \operatorname{codim} X^w$, so by what we have already shown, $l_T(w) = \operatorname{codim} X^w$. Moreover, $l_T(w)$ is also the length of $w \in W_S$ with respect to $\{s_\alpha : \alpha \in S\}$, from which the second part of (b) easily follows.

(c) Because of (b) it is sufficient to prove this for a finite root system where it was shown by Carter [**16**, Lemma 3].

We will see in 9.6 that this corollary is no longer true for the Weyl groups $W(R, \mathbf{c})$, $\mathbf{c} > \aleph_0$.

5.11. Corollary. *Let R_1 and R_2 be full subsystems of R. Then $R_1 \cap R_2$ is again full, and the corresponding parabolic subgroups satisfy*

$$W(R_1 \cap R_2) = W(R_1) \cap W(R_2). \tag{1}$$

Proof. That $R_1 \cap R_2$ is again full is obvious from the definitions, cf. 1.8.2. The inclusion "$\subset$" in (1) being obvious, let, conversely, $w \in W(R_1) \cap W(R_2)$. There exist finite full subsystems F_i of R_i such that $w \in W(F_1) \cap W(F_2)$. Put $Y_i = \operatorname{span}(F_i)$. With respect to an invariant inner product we then have $X^w \supset Y_1^\perp + Y_2^\perp = (Y_1 \cap Y_2)^\perp$ where the last equality follows from [**8**, §1.6, Cor. 2 of Prop. 4]. Since $Y_1 \cap Y_2$ is finite-dimensional, we obtain $(X^w)^\perp \subset (Y_1 \cap Y_2)^{\perp\perp} = Y_1 \cap Y_2$. Hence, by 5.10, w is a product of roots in $Y_1 \cap Y_2$. But $R \cap Y_1 \cap Y_2 = F_1 \cap F_2 \subset R_1 \cap R_2$ and so $w \in W(F_1 \cap F_2) \subset W(R_1 \cap R_2)$.

5.12. Theorem. *The Weyl group $W(R)$ of a root system (R, X) is presented by generators $\{g_\alpha : \alpha \in R^\times\}$ and relations*

$$g_\alpha = g_\beta \quad \text{for } \alpha \text{ and } \beta \text{ linearly dependent,} \tag{1}$$

$$g_\alpha g_\beta g_\alpha = g_{s_\alpha \beta} \quad \text{for all } \alpha, \beta \in R^\times. \tag{2}$$

Proof. Let Γ be the group presented by generators g_α, $\alpha \in R^\times$, and the relations (1) and (2). By Corollary 4.3(b) and by 3.9.2, the generators s_α of $W(R)$ satisfy these relations. Hence there is a surjective homomorphism $\varphi \colon \Gamma \to W(R)$

mapping g_α to s_α, and it remains to show that φ is injective as well. Thus let $g = g_{\alpha_1} \cdots g_{\alpha_n} \in \operatorname{Ker}\varphi$ and consider the subspace $Y \subset X$ spanned by $\alpha_1, \dots, \alpha_n$. Then $R' = R \cap Y$ is a finite root system in Y containing $\alpha_1, \dots, \alpha_n$. Let Γ' be the group defined analogously to Γ, with generators g'_α and relations (1) and (2), for $\alpha, \beta \in R'^\times$. Then we have homomorphisms $\psi\colon \Gamma' \to \Gamma$ and $\varphi'\colon \Gamma' \to W(R')$, sending g'_α to g_α and $s_\alpha|Y$, respectively, for all $\alpha \in R'^\times$. By Prop. 5.8, we identify $W(R')$ with the subgroup of $W(R)$ generated by $\{s_\alpha : \alpha \in R'^\times\}$. Then the following diagram is commutative:

$$\begin{tikzcd} \Gamma' \arrow[r, "\psi"] \arrow[d, "\varphi'"'] & \Gamma \arrow[d, "\varphi"] \\ W(R') \arrow[r, "\text{incl}"'] & W(R) \end{tikzcd}$$

Hence if φ' is injective, then the restriction of φ to the image of ψ is injective. As $g = \psi(g'_{\alpha_1} \cdots g'_{\alpha_n})$ belongs to that image, $g = 1$ will follow. Thus to prove injectivity of φ, we may replace R by R', in other words, we may assume R finite.

Let, then, R be a finite root system, and let B be a root basis of R. By [**12**, VI, §1.5, Th. 2(vii)], $W(R)$ is presented by generators $\{s_\alpha : \alpha \in B\}$ and relations

$$(s_\alpha s_\beta)^{m_{\alpha\beta}} = 1, \tag{3}$$

where $m_{\alpha\beta}$ is the order of $s_\alpha s_\beta$ in $W(R)$ (cf. A.2). Thus to construct a homomorphism from $W(R)$ to Γ mapping $s_\alpha \mapsto g_\alpha$, we must verify (3) in Γ, with s_α replaced by g_α. For $\alpha = \beta$, (3) just says $s_\alpha^2 = 1$. In Γ, we have $g_\alpha = g_{-\alpha} = g_{s_\alpha \alpha} = g_\alpha^3$ and therefore also $g_\alpha^2 = 1$. Next, let $\alpha, \beta \in B$ be different. Then α and β are linearly independent and hence span a plane $P = \mathbb{R}\alpha \oplus \mathbb{R}\beta$. With the restriction of an invariant inner product, P is Euclidean, and we equip P with the orientation determined by the ordered basis α, β. Let ϱ_t be the rotation of P with matrix $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ relative to a positively oriented orthonormal basis, and let $\vartheta \in \,]0, 2\pi[$ be the unique angle such that $\varrho_\vartheta(\alpha) = c\beta$ is a positive multiple of β. Since s_α is the orthogonal reflection of P in the line $P \cap \alpha^\perp$ and similarly for s_β, an elementary computation shows that $r := (s_\alpha s_\beta)|P = \varrho_{-2\vartheta}$. Also, $s_\alpha s_\beta$ acts as the identity on $P^\perp$, and since $X = P \oplus P^\perp$, the order of r is $m_{\alpha\beta}$.

From (2) we obtain

$$g_{r^n(\alpha)} = (g_\alpha g_\beta)^n g_\alpha (g_\beta g_\alpha)^n = (g_\alpha g_\beta)^{2n} g_\alpha, \tag{4}$$

for all $n \in \mathbb{N}$. If $m_{\alpha\beta} = 2n$ is even then $r^n = -\mathrm{Id}$. Hence (1) and (4) yield $g_\alpha = g_{-\alpha} = (g_\alpha g_\beta)^{2n} g_\alpha$ and therefore $(g_\alpha g_\beta)^{2n} = 1$, as required. If $m_{\alpha\beta} = 2n+1$ is odd, we have $\varrho_{-\vartheta}^{2n+1} = -\mathrm{Id}$ and hence $-c\beta = \varrho_{-\vartheta}^{2n+1}(c\beta) = \varrho_{-2\vartheta}^n \varrho_{-\vartheta}(c\beta) = r^n(\alpha)$. Thus again by (1) and (4) we see that $g_{-c\beta} = g_\beta = (g_\alpha g_\beta)^{2n} g_\alpha$ or $(g_\alpha g_\beta)^{2n+1} = 1$.

5.13. Remark. For a finite reduced root system, this presentation of the Weyl group is proven in [**17**, Theorem 2.4.3] with a different (perhaps more complicated) proof.

The result is in fact true for any Coxeter system (W, S): Consider the geometric realization of W in $E = \bigoplus_{s \in S} \mathbb{R} e_s$ as in [**12**, V, §4], and let $R = \{w(e_s) : s \in S,\ w \in$

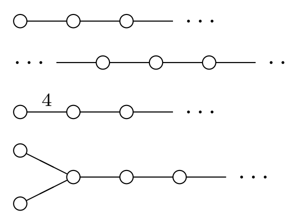
$W\}$ be the "root system" of (W, S), see [**21**]. Then W is presented by generators $\{g_\alpha : \alpha \in R\}$ and the relations 5.12.1 and 5.12.2 above.

Indeed, for any $\alpha = w(e_s) \in R$ one has a well-defined reflection $s_\alpha = wsw^{-1}$ satisfying $s_\alpha(x) = x - 2B(x, \alpha)\alpha$ for all $x \in E$, where B is the bilinear form associated to (W, S). Since R is reduced, the s_α satisfy 5.12.1. Because W leaves B invariant, we also have $ws_\alpha w^{-1} = s_{w(\alpha)}$ and hence in particular 5.12.2. Thus, with the above notations, there is a homomorphism from Γ to W mapping g_α to s_α, and the proof above shows that it is an isomorphism.

Concerning Coxeter groups we follow the terminology of [**12**, IV, §1].

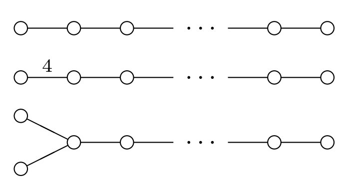
5.14. Proposition. *Let (W, S) be an irreducible Coxeter system with an infinite but locally finite W. Then W and S are countable and the Coxeter graph of (W, S) is isomorphic to exactly one of the following graphs:*

($A_{+\infty}$)
(A_∞)
(B_∞)
(D_∞)



Proof. Our hypotheses imply that S is infinite. We will use the following observation. Let S' be a finite connected subset of S with at least 9 elements, and let W' be the subgroup of W generated by S'. Then (W', S') is a finite irreducible Coxeter system, and so the classification of these groups in [**12**, VI, §4, Th. 1] implies that the Coxeter graph of (W', S'), or S' for short, is one of the following:

(A_l)
(B_l)
(D_l)



Let us first assume that S contains a finite subset S_0 whose graph is (B_l) or (D_l). Then any finite connected subgraph S' of S containing S_0 is of the same type as S_0, i.e., of type (B_n) or (D_n) for a suitable n. Since S is connected, every $s \in S$ lies in such a subgraph. Moreover, for a given n there is exactly one subgraph of type (B_n) respectively (D_n) in S. This implies that S is countable and of type (B_∞) or (D_∞).

We can now assume that all finite connected subgraphs of S are of type (A_n). If there exists an $s \in S_0$ which is only connected to one other element of S, the argument used above proves that S is countable and of type ($A_{+\infty}$). Otherwise, it follows that S is countable of type (A_∞). In all cases S is countable and hence so is W.

Remark. It follows from this result and 5.9 that the Weyl group of an uncountable irreducible root system cannot be the group of an *irreducible* Coxeter system. In fact, we will show later in 9.9 that it is not a Coxeter group at all.

§6. Integral bases, root bases and Dynkin diagrams

6.1. Definition. Let $(R, X) \in \mathbf{SV}_{\mathbb{R}}$. We specialize the situation of 2.7 to the case $k = \mathbb{R}$ and $A = \mathbb{Z}$. A $\mathbb{Z}$-basis of (R, X) as defined in 2.7 will also be called an *integral basis* of (R, X). In agreement with established notation for root systems, we denote by

$$\mathcal{Q}(R) := \mathbb{Z}[R]$$

the additive subgroup of X generated by R.

As an example, let R be an extended affine root system in $V = V^0 \oplus \dot{V}$ (notation of [**1**, II]). Then $S \cup \dot{R} \subset R \subset \mathbb{Z}[R] = \mathbb{Z}[S] \oplus \mathbb{Z}[\dot{R}]$ where both (S, V^0) and $(\dot{R}, \dot{V})$ have integral bases, and hence so does R.

A subset B of R is called a *root basis of* R if

(i) B is $\mathbb{R}$-free, and
(ii) every element of R is a $\mathbb{Z}$-linear combination of B with coefficients of the same sign.

This is motivated by the situation for finite root systems [**12**, VI, No. 1.5], where root bases in this sense are simply called bases. In particular, finite root systems and, as will be shown later, countable root systems, always admit root bases. Other examples are the root systems of Kac-Moody algebras.

Just as for integral bases, we say (R, X) has the *(finite) extension property for root bases* if for every pair $S' \subset S$ of (finite-ranked) full subsets of R, every root basis of S' extends to a root basis of S. Again, it is easy to see that this is equivalent to the existence of adapted bases: For all $S' \subset S$ as above, there exist root bases B' of S' and B of S such that $B' \subset B$.

As in 2.7 one shows that the (finite) extension property is equivalent to the *existence of adapted root bases*: for all $S' \subset S$ as above, there exist root bases B' of S' and B of S such that $B' \subset B$.

We list some easily proven properties of root bases for a general $(R, X) \in \mathbf{SV}_{\mathbb{R}}$, not necessarily a root system.

(a) Any root basis of R is in particular an integral basis.

(b) Every subset B' of a root basis B of R is a root basis of the full subset $R' = R \cap X'$ where $X' = \operatorname{span}(B')$, and $p(B \setminus B')$ is a root basis of R/R' where $p\colon X \to X/X'$ is the canonical projection.

(c) Suppose $(R, X) = \coprod_{i \in I} (R_i, X_i)$ is the coproduct of (R_i, X_i) as in 1.2(c). If B is a root basis of R then each $B_i = B \cap R_i$ is a root basis of R_i and, conversely, if B_i are root bases of R_i then $B = \bigcup_{i \in I} B_i$ is a root basis of R.

(d) Suppose $B_0 \subset B_1 \subset \cdots \subset B_n \subset \cdots$ is an increasing chain of subsets of R where each B_n is a root basis of $R_n = R \cap \operatorname{span}(B_n)$. Then $B = \bigcup_{n \in \mathbb{N}} B_n$ is a root basis of $R \cap \operatorname{span}(B)$.

We continue with properties of root bases of root systems.

(e) A root basis B of a root system R which is connected in the sense of 3.12 is called *irreducible* . Then B is irreducible if and only if R is irreducible.

(f) If $B = \bigcup B_i$ is an orthogonal decomposition, i.e., $B_i \perp B_j$ for $i \neq j$, of a root basis B of R, then R is the direct sum of the root systems R_i spanned by B_i.

6.2. Lemma. *Let (R, X) be a root system.*

(a) *(R, X) has the finite extension property for root bases and for integral bases.*

(b) *(R, X) is strongly bounded by the function*

$$s(n) = 2(7^n - 1). \tag{1}$$

Proof. (a) Let $F' \subset F$ be full subsystems of finite rank of R. Then F is a finite root system and F' is a full subsystem of F. By A.12, every root basis of F' extends to a root basis of F. By 6.1, adapted root bases exist for finite full subsets $F' \subset F$ of R. Since root bases are in particular integral bases, the same holds for integral bases and so, again by 6.1, R has the finite extension property for integral bases.

(b) Let $(\bar{R}, \bar{X}) = (R/F', X/V')$ be a finite quotient, so F' is full of finite rank (hence finite by local finiteness), and $V' = \operatorname{span}(F')$ is finite-dimensional. By 2.2.1, we must show that $|\operatorname{core}(U)^\times| \leqslant s(\dim(U))$, for every finite-dimensional tight subspace $U \subset \bar{X}$. By 1.7, $U = \bar{V}$ where $V = p^{-1}(U) \supset V'$ is again tight, and $\dim(V) = \dim(V') + \dim(U) < \infty$. Hence $F := \operatorname{core}(V)$ is a finite root system in V, with $F' = \operatorname{core}(V')$ as a full subsystem. By (a) there exist root bases B' of F' and B of F such that $B' \subset B$. Let $B \setminus B' = \{\beta_1, \ldots, \beta_n\} \subset B = \{\beta_1, \ldots, \beta_l\}$. Then $\bar{\beta}_1, \ldots, \bar{\beta}_n$ is a vector space basis of U so $n = \dim(U)$. Every $\alpha \in F$ has the form $\alpha = \sum_{i=1}^{l} n_i \beta_i$ where the n_i are integers of the same sign. From the classification of finite root systems it is known that $|n_i| \leqslant 6$. Hence every element of $\bar{F}$ is a linear combination of $\bar{\beta}_1, \ldots, \bar{\beta}_n$ with integer coefficients n_i of the same sign and satisfying $|n_i| \leqslant 6$. It follows that $\bar{F}$ has at most $2(7^n - 1)$ nonzero elements. Since $\operatorname{core}(U) = \bar{F}$ by 1.7.1, the assertion follows.

6.3. *The category $\overline{\mathbf{RS}}$.* The category $\mathbf{RS}$ of root systems and morphisms (cf. 3.6) is not closed under taking quotients with respect to full subsets. We therefore introduce the full subcategory $\overline{\mathbf{RS}}$ of $\mathbf{SV}_{\mathbb{R}}$ whose objects are quotients $(R, X) = (R_1/R_0, X_1/X_0)$ of root systems by full subsystems. Note that

$$\mathbf{RS} \subset \overline{\mathbf{RS}}$$

as a full subcategory since $(R, X) \cong (R, X)/0$. From the First Isomorphism Theorem 1.7 it follows that the category $\overline{\mathbf{RS}}$ is closed under taking full subsets and forming quotients by full subsets.

6.4. Theorem. *Every $(R, X) \in \overline{\mathbf{RS}}$ is strongly bounded by the function s of 6.2.1 and has the extension property for integral bases. Hence, if R' is a full subset of R then R, R' and R/R' have integral bases, every integral basis of R' extends to an integral basis of R, and the sequence*

$$0 \longrightarrow \mathcal{Q}(R') \longrightarrow \mathcal{Q}(R) \longrightarrow \mathcal{Q}(R/R') \longrightarrow 0 \tag{1}$$

is a split exact sequence of free abelian groups.

Proof. Let $(R, X) = (R_1/R_0, X_1/X_0)$ be written as the quotient of a root system (R_1, X_1) by a full subsystem (R_0, X_0). By Lemma 6.2, (R_1, X_1) has the finite extension property for integral bases and is strongly bounded by the function s. Hence (R_1, X_1) has the extension property by Cor. 2.12. Now Prop. 2.10(b) and Theorem 2.6 show that (R, X) has the extension property for integral bases and is strongly bounded by s as well. The remaining statements follow from Cor. 2.12 applied to (R, X).

6.5. Corollary ([**71**, Th. VI.6]). *Every root system admits an integral basis.*

6.6. Lemma. *Every $(R, X) \in \overline{\mathbf{RS}}$ has the finite extension property for root bases.*

Proof. By 6.1, it suffices to show that all finite-ranked full subsets $F' \subset F$ of R admit adapted root bases. Since the category $\overline{\mathbf{RS}}$ is closed under taking full subsets, we may replace R by F and thus assume $F = R$. By Theorem 6.4, R is strongly bounded, hence locally finite, so R and F' are finite.

Write $(R, X) = (R_1, X_1)/(R_0, X_0)$ as a quotient of root systems, and let $E \subset R_1$ be a set of representatives of R. By Lemma 2.5 there exists a finite full subsystem $S \supset E$ of R_1 which intersects R_0 tightly. Let $Y = \mathrm{span}(S)$, $S_0 = S \cap R_0$ and $Y_0 = Y \cap X_0 = \mathrm{span}(S_0)$ (by tightness). Then the Second Isomorphism Theorem 1.9 yields an isomorphism κ: $(S, Y)/(S_0, Y_0) \cong (R, X)$ which we treat as an identification. By the First Isomorphism Theorem 1.7, $F' = S'/S_0$ where S' is a full subsystem of S with $S_0 \subset S' \subset S$. Since S has the finite extension property for root bases by 6.2(a), there exist adapted root bases $B_0 \subset B' \subset B$ for $S_0 \subset S' \subset S$. By 6.1(b), $p(B' \setminus B_0) \subset p(B \setminus B_0)$ are the required adapted root bases of $F' \subset F = R$.

While integral bases exist under fairly general assumptions, this is not the case for root bases. Indeed, we will show below that every countable root system has a root basis. Therefore, in view of 6.9(a) below, an irreducible root system has a root basis if and only if it is countable.

6.7. Proposition. *Let $(R, X) \in \overline{\mathbf{RS}}$ with R countable, and let R' be a finite full subset. Then every root basis B' of R' extends to a root basis of R, and R/R' has root bases. In particular, every countable R has a root basis.*

Proof. We choose an enumeration $R = \{\alpha_n : n \in \mathbb{N}\}$ of R such that $R' = \{\alpha_n : 0 \leqslant n \leqslant k\}$ for some $k \in \mathbb{N}$. For $n \geqslant k$ we define $R_n = R \cap \mathrm{span}\{\alpha_m : m \leqslant n\}$. Then

$$R' = R_k \subset R_{k+1} \subset \cdots \subset \bigcup_n R_n = R,$$

where each R_n is a finite full subset of R, and hence also of R_{n+1}. By 6.6, R' has a root basis B', and every root basis B_n of R_n extends to a root basis B_{n+1} of R_{n+1}. By induction we therefore obtain a chain $B' \subset B_{k+1} \subset \cdots \subset B_n \subset \cdots$ of root bases B_n of R_n. Then $B = \bigcup_n B_n$ is a root basis of R by 6.1(c), and hence R/R' has a root basis by 6.1(b). The last part follows from the first by taking $R' = 0$.

6.8. *Dynkin diagrams.* Let B be a root basis of a root system R. The *Dynkin diagram* $\mathrm{Dyn}(B)$ of B is defined as the graph with a vertex $\overset{\beta}{\circ}$ for every $\beta \in B$ such that $2\beta \notin R$, a "double vertex" $\overset{\beta}{\circledcirc}$ if β is a *multipliable root* in the sense that $2\beta \in R$, and $k = \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle$ edges between α and β, with a $>$ sign going from the longer to the shorter root. The Dynkin diagram in this sense of a reduced root systems is of course just the usual one. The Dynkin diagram of the nonreduced root system BC_n with root basis $\beta_1 = \varepsilon_1, \beta_2 = \varepsilon_2 - \varepsilon_1, \dots, \beta_n = \varepsilon_n - \varepsilon_{n-1}$ is

$$\overset{\beta_1}{\circledcirc} \Leftarrow \overset{\beta_2}{\circ} — \cdots — \overset{\beta_n}{\circ}.$$

From $\mathrm{Dyn}(B)$ we can read off the *Cartan matrix* $(\langle \alpha, \beta^\vee \rangle)_{\alpha,\beta \in B}$, and also whether a root $\beta \in B$ is multipliable or not.

6.9. Theorem. *Let (R, X) be a root system admitting a root basis B and Dynkin diagram $\Delta = \mathrm{Dyn}(B)$, let $W(R)$ be the Weyl group of R, and put $S = \{s_\alpha : \alpha \in B\} \subset W(R)$.*

(a) *$(W(R), S)$ is a Coxeter system. If R is irreducible then $(W(R), S)$ is irreducible and hence $W(R)$ and R are at most countable.*

(b) *For every root $\alpha \in R^\times$ there exists $w \in W(R)$ such that $w(\alpha) \in B$ or that $w(\alpha/2) \in B$.*

(c) *Let (R', X') be a second root system admitting a root basis B' with Dynkin diagram $\Delta' = \mathrm{Dyn}(B')$. Assume further that $f\colon \Delta' \to \Delta$ is a morphism of Dynkin diagrams, i.e., f preserves double vertices and satisfies*

$$\langle f(\alpha), f(\beta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle \tag{1}$$

for all $\alpha, \beta \in B'$. Then f extends to an embedding $f\colon (R', X') \to (R, X)$ of root systems (cf. 3.6*) with the property that $f(R')$ is a full subsystem of R. In particular, if $f(B') = B$ then f is an isomorphism of root systems.*

We call two root bases B' and B *isomorphic* if their Dynkin diagrams are isomorphic. Then (c) implies that this is equivalent to the existence of a root system isomorphism mapping B' onto B. Let us point out that it is not always true that two root bases of R are conjugate under $\mathrm{Aut}(R)$, see 6.11.

Proof. (a) For an arbitrary $\alpha \in R$ there exists a finite subset Ψ of B such that $\alpha \in R_\Psi = R \cap \mathrm{span}(\Psi)$. Let $S_\Psi = \{s_\beta : \beta \in \Psi\}$. Then $(W(R_\Psi), S_\Psi)$ is a Coxeter system, in particular s_α is a product of reflections in S_Ψ. This shows that $W(R)$ is generated by $S = \{s_\alpha : \alpha \in B\}$. For $(W(R), S)$ to be a Coxeter system it is sufficient to verify the exchange condition [**12**, IV, §1.6, Th. 1]. Let, then, $w \in W(R)$ and $s \in S$ satisfy $l(sw) < l(w)$ where l is the length function on $W(R)$ with respect to the generating set S. Suppose further that $w = s_1 s_2 \cdots s_q$, $s_i \in S$, and $sw = t_1 t_2 \cdots t_p$, $t_i \in S$, are reduced decompositions. Let $S' = \{s, s_1, \dots, s_q, t_1, \dots, t_p\}$ and let $\Psi \subset B$ such that $S' = S_\Psi$. Then the elements $w, s \in W(R_\Psi)$ have $l'(sw) < l'(w)$ where now l' is the length function for the Coxeter system $(W(R_\Psi), S_\Psi)$. Since the exchange condition holds for this Coxeter system, it follows for $(W(R), S)$.

Suppose R is irreducible and infinite. By 6.1(e) so is B, and therefore $(W(R), S)$ is an irreducible Coxeter system. By 5.9, $W(R)$ is locally finite. Hence $W(R)$ and S are countable by 5.14. Since B is in bijection with S via $\alpha \mapsto s_\alpha$ and $R \subset \bigoplus_{\delta \in B} \mathbb{Z}\delta$, it follows that R is countable.

(b) This is true for finite reduced root systems by A.10, and with trivial modifications also in the non-reduced case. The general case then follows by applying 3.16 and 5.8.

(c) If $f(\alpha) = f(\beta)$ for $\alpha, \beta \in B'$ we obtain $\langle \alpha, \beta^\vee \rangle = 2 = \langle \beta, \alpha^\vee \rangle$ and therefore $\alpha = \beta$, by the list of possible Cartan numbers of two roots. Since B' is in particular a basis of X' as a vector space, f extends to an injective linear map on $f\colon X' \to X$. Then $\langle f(x'), f(\beta)^\vee \rangle = \langle x', \beta^\vee \rangle$ holds for all $x' \in X'$ and therefore

$$f \circ s_\beta = s_{f(\beta)} \circ f \quad \text{for all } \beta \in B'. \tag{2}$$

By (a), the s_β ($\beta \in B'$) generate $W(R')$, so for every $w \in W(R')$ there exists $\tilde{w} \in W(R)$ satisfying $fw = \tilde{w}f$, and therefore also $\tilde{w}^{-1}f = f'w^{-1}$. By (b), an arbitrary $\alpha \in R'$ can be written in the form $\alpha = w(c\beta)$ for some $\beta \in B'$ and $c \in \{1, 2\}$. Let us first assume $c = 1$. Then $f(\alpha) = fw(\beta) = \tilde{w}f(\beta) \in W(R)B \subset R$. If $c = 2$ then β is a double vertex. Hence so is $f(\beta)$ and thus $2f(\beta) \in R$. It follows again that $f(\alpha) = fw(2\beta) = \tilde{w}f(2\beta) \in R$, so we have $f(R') \subset R$. Finally, using (2), $fs_\alpha = fs_{w(c\beta)} = fws_{c\beta}w^{-1} = \tilde{w}fs_{c\beta}w^{-1} = \tilde{w}s_{f(c\beta)}fw^{-1} = \tilde{w}s_{f(c\beta)}\tilde{w}^{-1}f = s_{\tilde{w}f(c\beta)}f = s_{f(\alpha)}f$ which by 3.7(v) proves that f is an embedding. The fact that $f(R')$ is a full subsystem of R follows easily from 6.1(b).

6.10. Corollary. *Let B be a root basis of a root system R, let H be the stabilizer of B in* $\mathrm{Aut}(R)$ *and let* $\mathrm{Aut}(\varDelta)$ *be the automorphism group of the Dynkin diagram* $\varDelta = \mathrm{Dyn}(B)$ *of B. Then the restriction map* $\mathrm{res}\colon H \to \mathrm{Aut}(\varDelta)$ *is an isomorphism of topological groups, where H has the topology induced from* $\mathrm{Aut}(R)$ *and* $\mathrm{Aut}(\varDelta)$ *has the finite topology, i.e., the topology induced from $\varDelta^\varDelta$ where $\varDelta$ is discrete, cf.* 5.1.

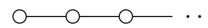
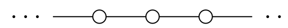
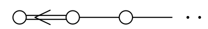
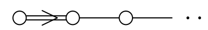
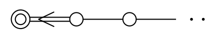
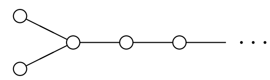
Proof. From Theorem 6.9(c) it is clear that res is an isomorphism of groups, and continuity in both directions is easily checked. The details are left to the reader.

6.11. *Classification of Dynkin diagrams.* Let B be a root basis of a root system R with Dynkin diagram $\mathrm{Dyn}(B)$. Because of the results listed in 6.1, it suffices to classify Dynkin diagrams of irreducible root bases. The finite case being well-known, we restrict our attention to the case of a countable irreducible B. Their classification is described in [**35**, Exercise 4.14]. In our setting, it is an easy consequence of 6.9(a) and 5.14, taking into account the two possibilities for the root lengths in the case B_∞. The result is listed in the table at the end of this section. We use the notations introduced in 8.1 and let

$$B_0 = \{\varepsilon_{i+1} - \varepsilon_i : i \in \mathbb{N}\}$$

where $\mathbb{N}$ denotes the non-negative integers. Note that the root systems $\dot{\mathrm{A}}_\mathbb{N}$ and $\dot{\mathrm{A}}_\mathbb{Z}$ are isomorphic (choose a bijection between $\mathbb{N}$ and $\mathbb{Z}$) but admit non-isomorphic root bases (A_∞) and ($\mathrm{A}_{+\infty}$). This shows that two root bases of an infinite root system R are in general not conjugate under $\mathrm{Aut}(R)$. Also, it is evident from the table that

$$\mathrm{Aut}(\mathrm{Dyn}(B)) = \left\{ \begin{array}{ll} (\mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{Z} & \text{in case } (\mathrm{A}_\infty) \\ \mathbb{Z}/2\mathbb{Z} & \text{in case } (\mathrm{D}_\infty) \\ \{1\} & \text{otherwise} \end{array} \right\}.$$

Type	B	$\mathrm{Dyn}(B)$	R
$(\mathrm{A}_{+\infty})$	B_0		$\dot{\mathrm{A}}_{\mathbb{N}}$
(A_{∞})	$\{\varepsilon_{i+1} - \varepsilon_i : i \in \mathbb{Z}\}$		$\dot{\mathrm{A}}_{\mathbb{Z}}$
(B_{∞})	$\{\varepsilon_0\} \cup B_0$		$\mathrm{B}_{\mathbb{N}}$
(C_{∞})	$\{2\varepsilon_0\} \cup B_0$		$\mathrm{C}_{\mathbb{N}}$
(BC_{∞})	$\{\varepsilon_0\} \cup B_0$		$\mathrm{BC}_{\mathbb{N}}$
(D_{∞})	$\{\varepsilon_0 + \varepsilon_1\} \cup B_0$		$\mathrm{D}_{\mathbb{N}}$

§7. Weights and coweights

7.1. Definition. With any root system (R, X), we associate the following abelian groups:

(a) The group $\mathcal{Q}(R) = \mathbb{Z}[R]$ as in 6.1, also called the group of *radicial weights* or the *root lattice*.

(b) The group

$$\mathcal{P}^\vee(R) := \{q \in X^* : \langle R, q\rangle \subset \mathbb{Z}\}, \tag{1}$$

called the group of *coweights* of R.

(c) The group

$$\mathcal{Q}^\vee(R) := \mathcal{Q}(R^\vee) = \mathbb{Z}[R^\vee] \tag{2}$$

of *radicial coweights*, i.e., the group of radicial weights of the coroot system $R^\vee$.

(d) The group

$$\mathcal{P}(R) := \mathcal{P}^\vee(R^\vee) \tag{3}$$

of coweights of $R^\vee$, called the group of *weights* of R. According to (1), the elements of $\mathcal{P}(R)$ are the linear forms $p \in (X^\vee)^*$ with the property that

$$\langle R^\vee, p\rangle \subset \mathbb{Z}. \tag{4}$$

Clearly, the assignment $R \mapsto \mathcal{Q}(R)$ is a covariant functor from the category **RS** of root systems and morphisms to the category **Ab** of abelian groups. Similarly, $\mathcal{P}^\vee$ is a contravariant functor from **RS** to **Ab**: Any morphism $f\colon (S, Y) \to (R, X)$ of root systems induces a homomorphism $\mathcal{P}^\vee(f)\colon \mathcal{P}^\vee(R) \to \mathcal{P}^\vee(S)$ by $\mathcal{P}^\vee(f)(q) = q \circ f$. It should also be noted that the functors $\mathcal{Q}$ and $\mathcal{P}^\vee$ make sense not only for root systems but for an arbitrary $(R, X) \in \mathbf{SV}_\mathbb{R}$.

In the remaining two cases, $\mathcal{Q}^\vee = \mathcal{Q} \circ \mathcal{C}$ and $\mathcal{P} = \mathcal{P}^\vee \circ \mathcal{C}$ are obtained by composing the functors $\mathcal{Q}$ and $\mathcal{P}^\vee$ with the coroot system functor $\mathcal{C}$ of Th. 4.9. As $\mathcal{C}$ is a covariant functor from **RSE** (root systems with embeddings as morphisms) to itself, it follows that $\mathcal{Q}^\vee\colon \mathbf{RSE} \to \mathbf{Ab}$ is covariant and $\mathcal{P}\colon \mathbf{RSE} \to \mathbf{Ab}$ is contravariant. In more detail, the map $\mathcal{P}(f)\colon \mathcal{P}(R) \to \mathcal{P}(S)$ induced from an embedding $f\colon (S, Y) \to (R, X)$ of root systems is given by

$$\mathcal{P}(f)(p) = p \circ f^\vee\colon Y^\vee \to X^\vee \to \mathbb{R}. \tag{5}$$

From $\mathcal{P} = \mathcal{P}^\vee \circ \mathcal{C}$ and the natural isomorphism $\mathcal{C} \circ \mathcal{C} \cong \mathrm{Id}$ of 4.9(b) it follows that there is a natural isomorphism

$$\mathcal{P}^\vee \cong \mathcal{P} \circ \mathcal{C}. \tag{6}$$

This explains the terminology "coweights" (which is somewhat unfortunate as $\mathcal{P}^\vee$ is a *contra*variant functor).

7.2. Proposition (lifting weights). *Let $f\colon (S,Y) \to (R,X)$ be a full embedding (cf. 3.8) of root systems. Then the homomorphisms $\mathcal{P}^\vee(f)\colon \mathcal{P}^\vee(R) \to \mathcal{P}^\vee(S)$ and $\mathcal{P}(f)\colon \mathcal{P}(R) \to \mathcal{P}(S)$ are surjective. In particular, if $R' \subset R$ is a full subsystem then every (co)weight of R' can be lifted to a (co)weight of R.*

Proof. By Th. 4.9(a), $f^\vee$ is a full embedding along with f. Hence it suffices to prove the statement concerning $\mathcal{P}^\vee(f)$, the other then follows by $\mathcal{P} = \mathcal{P}^\vee \circ \mathcal{C}$. We may also identify (S,Y) with its image $(R',X') = (f(S), f(Y))$ in (R,X). By 6.4, $\mathcal{Q}(R')$ is free abelian and a direct summand in the free abelian group $\mathcal{Q}(R)$, and there exists an integral basis B' of R' which extends to an integral basis B of R. Then B' and B are bases of the free abelian groups $\mathcal{Q}(R')$ and $\mathcal{Q}(R)$, and also vector space bases of X' and X (2.7). Hence any coweight $q \in \mathcal{P}^\vee(R')$ (which is uniquely determined by its values on B') extends to a coweight $\tilde{q}$ of R, for instance by defining $\tilde{q}(B \setminus B') = 0$.

7.3. *More weight groups.* We keep the notations of 7.1. From the inclusions $\mathcal{Q}^\vee(R) = \mathcal{Q}(R^\vee) \subset X^\vee \subset X^*$ and the fact that $\langle R, R^\vee\rangle \subset \mathbb{Z}$ by the integrality condition (iii) in the definition of a root system (3.3) we obtain the inclusion

$$\mathcal{Q}^\vee(R) \subset \mathcal{P}^\vee(R) \tag{1}$$

which, however, is not functorial in R, and so does not make $\mathcal{Q}^\vee$ a subfunctor of $\mathcal{P}^\vee$. By $\mathcal{Q}^\vee = \mathcal{Q} \circ \mathcal{C}$ and treating 7.1.6 as an identification, we see that also

$$\mathcal{Q}(R) \subset \mathcal{P}(R), \tag{2}$$

where we now identify an element $x \in X$ with the linear form $j(x) \in (X^\vee)^*$ given by $\langle \xi, j(x)\rangle = \langle x, \xi\rangle$, for all $\xi \in X^\vee$. We define the following subgroups of $\mathcal{P}(R)$:

$$\mathcal{P}_{\rm fin}(R) = \{x \in X : \langle x, R^\vee\rangle \subset \mathbb{Z}\} = X \cap \mathcal{P}(R) \qquad (\textit{finite weights}), \tag{3}$$

$$\mathcal{P}_{\rm cof}(R) = \{p \in (X^\vee)^* : \langle \mathcal{P}_{\rm fin}(R^\vee), p\rangle \subset \mathbb{Z}\} \qquad (\textit{cofinite weights}), \tag{4}$$

$$\mathcal{P}_{\rm bd}(R) = \{p \in (X^\vee)^* : \langle R^\vee, p\rangle \text{ is bounded}\} \qquad (\textit{bounded weights}). \tag{5}$$

From $\langle R, R^\vee\rangle \subset \mathbb{Z}$ it follows that $\mathcal{Q}(R) \subset \mathcal{P}_{\rm fin}(R)$. Hence also $\mathcal{Q}(R^\vee) \subset \mathcal{P}_{\rm fin}(R^\vee)$ which implies $\mathcal{P}_{\rm cof}(R) \subset \mathcal{P}(R)$. We introduce the quotient groups

$$\Theta(R) = \mathcal{P}_{\rm fin}(R)/\mathcal{Q}(R), \qquad \Theta^*(R) = \mathcal{P}(R)/\mathcal{P}_{\rm cof}(R),$$

and summarize the relations between these groups in the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{Q}(R) \arrow[r] \arrow[d, "i'"'] & \mathcal{P}_{\rm fin}(R) \arrow[r] \arrow[d, "i"] & \Theta(R) \arrow[r] \arrow[d, "i''"] & 0 \\ 0 \arrow[r] & \mathcal{P}_{\rm cof}(R) \arrow[r] & \mathcal{P}(R) \arrow[r] & \Theta^*(R) \arrow[r] & 0 \end{tikzcd} \tag{6}$$

Here i' and i are injective, being the restrictions of $j\colon X \to (X^\vee)^*$ to the respective subgroups, and i'' is the unique homomorphism making the diagram commutative. In general, i'' is neither injective nor surjective, see the remark at the end of 8.7.

For R finite, $(X^\vee)^*$ is canonically identified with X and therefore $\mathcal{P}_{\rm fin}(R) = \mathcal{P}(R)$. It is also known that $\mathcal{Q}(R)$ is a free abelian group of rank equal to the rank of R, with $\mathcal{P}(R)$ canonically isomorphic to $\mathrm{Hom}(\mathcal{Q}(R^\vee), \mathbb{Z})$. Hence $\mathcal{P}_{\rm cof}(R) \cong \mathrm{Hom}(\mathcal{P}(R^\vee), \mathbb{Z}) \cong \mathcal{Q}(R)$. Also, $\Theta(R)$ is finite and i'' is an isomorphism [**12**, VI, §1, No. 9]. We will generalize these results to the infinite case below.

The various weight groups behave as follows with respect to direct sums: If $(R, X) = \coprod(R_i, X_i)$ then

$$\mathcal{Q}(R) = \bigoplus \mathcal{Q}(R_i), \qquad \mathcal{P}_{\rm fin}(R) = \bigoplus \mathcal{P}_{\rm fin}(R_i), \qquad \Theta(R) = \bigoplus \Theta(R_i), \quad (7)$$

$$\mathcal{P}_{\rm cof}(R) = \prod \mathcal{P}_{\rm cof}(R_i), \qquad \mathcal{P}(R) = \prod \mathcal{P}(R_i), \qquad \Theta^*(R) = \prod \Theta^*(R_i). \quad (8)$$

This follows easily from the definitions.

In contrast to the weight groups $\mathcal{P}(R)$, the groups $\mathcal{P}_{\rm fin}(R)$ of finite weights do not depend functorially on R with respect to embeddings $f\colon (S, Y) \to (R, X)$. Let $p = x \in X \cap \mathcal{P}(R) = \mathcal{P}_{\rm fin}(R)$ be a finite weight of R. Then $\mathcal{P}(f)(x) \in \mathcal{P}_{\rm fin}(S)$ if and only if there exists an $y \in Y$ such that $\langle x, f(\alpha)^\vee\rangle = \langle y, \alpha^\vee\rangle = \langle f(y), f(\alpha)^\vee\rangle$ (by 3.7(iii)), for all $\alpha \in S$, equivalently, if we can write $x = f(y) + z \in f(Y) \oplus f(S)^\perp$. In general, this is not the case. For an example, let $R = \mathrm{B}_\mathbb{N}$ and $S = \dot{\mathrm{A}}_\mathbb{N}$ as in 8.1, with f the inclusion $\dot{X} \subset X$. Then $S^\perp = 0$ because $x = \sum x_i\varepsilon_i \perp S$ means that $(x|\varepsilon_i - \varepsilon_j) = x_i - x_j = 0$ for all $i \neq j$, so all components of x are equal. Since only finitely many components of x are nonzero, this implies $x = 0$. Now for instance $\varepsilon_0 \in R \subset \mathcal{Q}(R) = \mathcal{P}_{\rm fin}(R)$ (by 8.7) but $\mathrm{res}(\varepsilon_0) \notin \mathcal{P}_{\rm fin}(S)$ because $\varepsilon_0 \notin Y \oplus S^\perp = Y$.

Let us finally note that finite weights are bounded:

$$\mathcal{P}_{\rm fin}(P) \subset \mathcal{P}_{\rm bd}(R). \quad (9)$$

Indeed, for all $\beta \in R$ and $x = \sum_{\alpha\in R} c_\alpha \alpha \in X$, we have

$$|\langle x, \beta^\vee\rangle| \leqslant \sum_{\alpha\in R} |c_\alpha| \cdot |\langle \alpha, \beta^\vee\rangle| \leqslant 4 \sum_{\alpha\in R} |c_\alpha|,$$

independent of β, by A.2.

7.4. *Weights and automorphisms.* The automorphism group of R acts on $X^\vee$ via the isomorphism $g \mapsto g^\vee$ as in 4.11(b) and therefore also on the various weight groups by

$$\langle \alpha^\vee, g(p)\rangle = \langle (g^{-1})^\vee(\alpha^\vee), p\rangle = \langle (g^{-1}(\alpha))^\vee, p\rangle, \quad (1)$$

for all $\alpha \in R$, $p \in \mathcal{P}(R)$. In particular, for $g = s_\beta$ this yields by 4.9.5 the formula

$$s_\beta(p) = p - \langle \beta^\vee, p\rangle\beta, \quad (2)$$

which implies

$$p - w(p) \in \mathcal{Q}(R), \quad (3)$$

for all $w \in W(R)$. Hence,

$W(R)$ acts trivially on the groups $\mathcal{P}(R)/\mathcal{Q}(R)$, $\Theta(R)$ and $\Theta^*(R)$. (4)

For the action of an element $\bar{w}$ of the big Weyl group $\overline{W}(R)$ on $\mathcal{P}(R)$ we still have

$$p - \bar{w}(p) \in \mathcal{P}_{\mathrm{cof}}(R). \tag{5}$$

Indeed, let $\bar{w} = \lim w_\lambda$ be the limit of a net $(w_\lambda)_{\lambda\in\Lambda}$ in $W(R)$. Then $w_\lambda(p) - p \in \mathcal{Q}(R)$ for all $\lambda \in \Lambda$ by (3). Let $y \in \mathcal{P}_{\mathrm{fin}}(R^\vee) \subset X^\vee$ so that $\langle \mathcal{Q}(R), y\rangle \subset \mathbb{Z}$. It follows that $\langle y - (w_\lambda^{-1})^\vee(y), p\rangle = \langle y, p - w_\lambda(p)\rangle \in \mathbb{Z}$ for all $\lambda \in \Lambda$. Since the map $g \mapsto g^\vee$ from $\mathrm{Aut}(R)$ to $\mathrm{Aut}(R^\vee)$ is a topological isomorphism by 5.2, we have $(\bar{w}^{-1})^\vee(y) = \lim(w_\lambda^{-1})^\vee(y)$, so there exists λ_0 such that $(\bar{w}^{-1})^\vee(y) = (w_\lambda^{-1})^\vee(y)$ for all $\lambda \succ \lambda_0$. This implies

$$\begin{aligned}\langle y, p - \bar{w}(p)\rangle &= \langle y - (\bar{w}^{-1})^\vee(y), p\rangle \\ &= \langle y - (w_{\lambda_0}^{-1})^\vee(y), p\rangle = \langle y, p - w_{\lambda_0}(p)\rangle \in \mathbb{Z}.\end{aligned}$$

As $y \in \mathcal{P}_{\mathrm{fin}}(R^\vee)$ was arbitrary, we conclude $\langle \mathcal{P}_{\mathrm{fin}}(R^\vee), \bar{p} - w(p)\rangle \subset \mathbb{Z}$, i.e., $p - \bar{w}(p) \in \mathcal{P}_{\mathrm{cof}}(R)$. From (5) we see that

$$\overline{W}(R) \textit{ acts trivially on } \Theta^*(R). \tag{6}$$

7.5. Theorem. *Let (R, X) be a root system.*

(a) *The groups $\mathcal{P}_{\mathrm{bd}}(R)$, $\mathcal{P}_{\mathrm{fin}}(R)$ and $\mathcal{Q}(R)$ are free abelian groups and $\Theta(R)$ is a torsion group.*

(b) *The canonical homomorphisms $\mu\colon \mathcal{P}_{\mathrm{fin}}(R)\otimes_{\mathbb{Z}}\mathbb{R} \to X$ and $\nu\colon \mathcal{Q}(R)\otimes_{\mathbb{Z}}\mathbb{R} \to X$ are isomorphisms.*

(c) *There are isomorphisms*

$$\varrho'\colon \mathcal{P}_{\mathrm{cof}}(R) \xrightarrow{\cong} \mathrm{Hom}(\mathcal{P}_{\mathrm{fin}}(R^\vee), \mathbb{Z}), \tag{1}$$

$$\varrho\colon \mathcal{P}(R) \xrightarrow{\cong} \mathrm{Hom}(\mathcal{Q}(R^\vee), \mathbb{Z}), \tag{2}$$

$$\varrho''\colon \Theta^*(R) \xrightarrow{\cong} \mathrm{Hom}(\Theta(R^\vee), \mathbb{Q}/\mathbb{Z}), \tag{3}$$

given by $\varrho'(p') = p'\big|\mathcal{P}_{\mathrm{fin}}(R^\vee)$, $\varrho(p) = p\big|\mathcal{Q}(R^\vee)$, and $\varrho''(p'')([l]) = \langle p, l\rangle + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, for $p' \in \mathcal{P}_{\mathrm{cof}}(R)$, $p \in \mathcal{P}(R)$, $p'' = p + \mathcal{P}_{\mathrm{cof}}(R) \in \Theta^*(R)$, and $[l] = l + \mathcal{Q}(R^\vee) \in \Theta(R^\vee)$.

Proof. (a) Since $R^\vee$ spans $X^\vee$, the map $p \mapsto (\langle \beta^\vee, p\rangle)_{\beta\in R^\times}$ is an injective homomorphism of $\mathcal{P}_{\mathrm{bd}}(R)$ into the group of all integer-valued bounded functions on the set $R^\times$. By a theorem of Specker and Nöbeling [**5**, Cor. 1.2], such a group is free abelian. Since a subgroup of a free abelian group is again free abelian, it follows that $\mathcal{P}_{\mathrm{bd}}(R)$ and its subgroups $\mathcal{P}_{\mathrm{fin}}(R)$ and $\mathcal{Q}(R)$ are free abelian as well. (Note that by 6.5, we even know that $\mathcal{Q}(R)$ admits bases contained in R.)

Let $x \in \mathcal{P}_{\mathrm{fin}}(R)$, and choose a finite subsystem $S \subset R$ such that $x \in \mathrm{span}(S)$. Then, identifying $S^\vee$ with a subset of $R^\vee$ as in 4.10, we have $\langle x, S^\vee\rangle \subset \mathbb{Z}$ whence x is a weight of S. Since $\mathcal{P}(S)/\mathcal{Q}(S)$ is finite, it follows that $nx \in \mathcal{Q}(S) \subset \mathcal{Q}(R)$ for some $n \in \mathbb{N}$. Thus $\Theta(R)$ is a torsion group.

(b) By 6.5, R admits integral bases so the natural map $\nu\colon \mathcal{Q}(R) \otimes_{\mathbb{Z}} \mathbb{R} \to X$ is an isomorphism by 2.7. As $\mathcal{P}_{\mathrm{fin}}(R) \supset \mathcal{Q}(R)$ also spans X, it is clear that μ is surjective. Tensoring the exact sequence

$$0 \longrightarrow \mathcal{Q}(R) \longrightarrow \mathcal{P}_{\mathrm{fin}}(R) \longrightarrow \Theta(R) \longrightarrow 0$$

with $\mathbb{R}$ and taking into account that $\Theta(R) \otimes_{\mathbb{Z}} \mathbb{R} = 0$ because $\Theta(R)$ is a torsion group, we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{Q}(R) \otimes_{\mathbb{Z}} \mathbb{R} \arrow[r] \arrow[d, "\nu"'] & \mathcal{P}_{\mathrm{fin}}(R) \otimes_{\mathbb{Z}} \mathbb{R} \arrow[d, "\mu"] \\
X \arrow[r, "\mathrm{Id}"'] & X
\end{tikzcd}$$

where the top map is surjective because tensoring is right exact. Since ν is an isomorphism so must be μ.

(c) Let $\mathcal{Q}(R^\vee) = G$, $\mathcal{P}_{\mathrm{fin}}(R^\vee) = F$, and $\Theta(R^\vee) = T$ for shorter notation. Then the short exact sequence

$$0 \longrightarrow G \longrightarrow F \longrightarrow T \longrightarrow 0$$

yields the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(T,\mathbb{Z}) \longrightarrow \mathrm{Hom}(F,\mathbb{Z}) \longrightarrow \mathrm{Hom}(G,\mathbb{Z}) \longrightarrow & \\ \longrightarrow \mathrm{Ext}^1(T,\mathbb{Z}) \longrightarrow \mathrm{Ext}^1(F,\mathbb{Z}) \longrightarrow \cdots & \end{aligned} \tag{4}$$

By (a), applied to the coroot system $R^\vee$, we have F free and T a torsion group. Hence $\mathrm{Hom}(T,\mathbb{Z}) = \mathrm{Ext}^1(F,\mathbb{Z}) = 0$ and $\mathrm{Ext}^1(T,\mathbb{Z}) \cong \mathrm{Hom}(T,\mathbb{Q}/\mathbb{Z})$, the Pontrjagin dual [**74**, Ex. 3.3.3]. Thus (4) gives the bottom row of the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{P}_{\mathrm{cof}}(R) \arrow[r] \arrow[d, "\varrho'"'] & \mathcal{P}(R) \arrow[r] \arrow[d, "\varrho"] & \Theta^*(R) \arrow[r] \arrow[d, "\varrho''"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(F,\mathbb{Z}) \arrow[r] & \mathrm{Hom}(G,\mathbb{Z}) \arrow[r] & \mathrm{Hom}(T,\mathbb{Q}/\mathbb{Z}) \arrow[r] & 0
\end{tikzcd} \tag{5}$$

Commutativity of (5) is easily checked. Since $G = \mathcal{Q}(R^\vee)$ spans $X^\vee$, it is clear that ϱ and therefore also ϱ' are injective. To see that ϱ is also surjective, let $\varphi\colon G \to \mathbb{Z}$ be linear. Then φ induces an $\mathbb{R}$-linear map $\tilde{\varphi}\colon \mathcal{Q}(R^\vee) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$, which by (b) yields a $p\colon X^\vee \to \mathbb{R}$ such that $p \circ \nu = \tilde{\varphi}$. It follows that $\varphi = \varrho(p)$. Surjectivity of ϱ' follows in the same way. Finally, it is easy to see by chasing the diagram (5) that ϱ'' is an isomorphism as well.

7.6. Proposition. *The group $\mathcal{Q}(R)$ of radicial weights is isomorphic to the abelian group presented by generators $[\alpha]$, $\alpha \in R$, and relations $[\alpha+\beta] = [\alpha] + [\beta]$ for all $\alpha, \beta \in R$ such that also $\alpha + \beta \in R$.*

Proof. Let A be the abelian group with the presentation given above. There is a canonical epimorphism $\psi\colon A \to \mathcal{Q}(R)$ mapping $[\alpha]$ to α, so it suffices to show that ψ is injective. First note that $[0] = 0$ in A since $[0] = [0+0] = [0] + [0]$. This implies $0 = [\alpha - \alpha] = [\alpha] + [-\alpha]$ or $[-\alpha] = -[\alpha]$ for all $\alpha \in R$. Suppose $x = \sum_{i=1}^n [\alpha_i] \in \mathrm{Ker}\,\psi$, and let $S \subset R$ be a finite subsystem containing $\alpha_1, \ldots, \alpha_n$. By A.15, there exists a homomorphism $\varphi\colon \mathcal{Q}(S) \to A$ such that $\varphi(\alpha) = [\alpha]$, for all $\alpha \in S$. Since $\psi \circ \varphi$ is the inclusion $\mathcal{Q}(S) \hookrightarrow \mathcal{Q}(R)$, it follows that $0 = \psi(x) = \psi(\varphi(\sum_{i=1}^n \alpha_i)) = \sum_{i=1}^n \alpha_i$, and therefore $x = \varphi(\sum_{i=1}^n \alpha_i) = 0$.

7.7. Corollary. *Let (R, X) and (S, Y) be root systems and let $f\colon R \to S$ be a map satisfying $\langle f(\alpha), f(\beta)^\vee\rangle = \langle \alpha, \beta^\vee\rangle$ for all $\alpha, \beta \in R$. Then f extends uniquely to an embedding $f\colon (R, X) \to (S, Y)$.*

Proof. Uniqueness of f is clear since R spans X. For existence, it suffices, because of 3.7 and the isomorphism $\mathcal{Q}(R) \otimes_{\mathbb{Z}} \mathbb{R} \cong X$ of 7.5(b), to show that f extends to a homomorphism $f\colon \mathcal{Q}(R) \to \mathcal{Q}(S)$ of abelian groups. For this we use the presentation 7.6. Thus let α, β and $\gamma := \alpha + \beta \in R$. Then for all $\delta \in R$ we have $\langle f(\alpha) + f(\beta), f(\delta)^\vee\rangle = \langle \alpha + \beta, \delta^\vee\rangle = \langle \gamma, \delta^\vee\rangle = \langle f(\gamma), f(\delta)^\vee\rangle$, so $z := f(\alpha) + f(\beta) - f(\gamma)$ is orthogonal to $\operatorname{span} f(R)$ with respect to an invariant inner product. As $z \in \operatorname{span} f(R)$, it follows that $z = 0$, so f preserves the defining relations of $\mathcal{Q}(R)$.

7.8. Corollary. *$\mathcal{Q}(R)$ is also presented by generators $\hat\alpha$, $\alpha \in R$, and relations $\widehat{2\alpha} = 2\hat\alpha$ for all $\alpha \in R$ such that also $2\alpha \in R$, and $\hat\beta - \langle \beta, \alpha^\vee\rangle\hat\alpha = \widehat{s_\alpha\beta}$ for all $\alpha \in R^\times$, $\beta \in R$.*

Proof. Let B be the group with the indicated generators and relations. Clearly, there is an epimorphism from B to $\mathcal{Q}(R)$ sending $\hat\alpha$ to α. In the opposite direction, define $\varphi\colon R \to B$ by $\varphi(\alpha) = \hat\alpha$. It suffices to show that φ extends to a homomorphism from $\mathcal{Q}(R)$ to B. By 7.6, this is the case if and only if

$$\alpha, \beta, \alpha + \beta \in R \quad\Longrightarrow\quad \widehat{\alpha + \beta} = \hat\alpha + \hat\beta. \tag{1}$$

Thus let α, β and $\gamma := \alpha + \beta$ be in R, and first consider the case where α and β are linearly dependent. Note that $2 \cdot 0 = 0$ implies $2 \cdot \hat 0 = \hat 0$ and thus $\hat 0 = 0 \in B$. If α or β is zero then (1) is clear from $\hat 0 = 0$, whereas $\gamma = 0 \neq \alpha, \beta$ means $\beta = -\alpha \in R^\times$ and then $\hat\beta = \widehat{s_\alpha\alpha} = \hat\alpha - 2\hat\alpha = -\hat\alpha$. If none of α, β and γ is zero then either $\alpha = \beta$ (and then (1) is clear) or $\beta = -2\alpha$ (and then $\hat\alpha + \hat\beta = \hat\alpha + \widehat{-2\alpha} = \hat\alpha - \widehat{2\alpha} = \hat\alpha - 2\hat\alpha = -\hat\alpha = \widehat{-\alpha} = \hat\gamma$) or $\beta = -\alpha/2$ (and then $\hat\alpha + \hat\beta = \widehat{2\gamma} - \widehat{\alpha/2} = 2\hat\gamma - \hat\gamma = \hat\gamma$). We now assume that α and β are linearly independent. Then so are α, γ and also β, γ. If $\langle \alpha, \beta^\vee\rangle = -1$ then $\hat\alpha + \hat\beta = \widehat{s_\beta\alpha} = \widehat{\alpha + \beta}$. Hence we can assume that $\langle \alpha, \beta^\vee\rangle \neq -1$ and, by symmetry, that $\langle \beta, \alpha^\vee\rangle \neq -1$. But then $\langle \alpha, \beta^\vee\rangle \geqslant 0$ by A.2, whence $\langle \gamma, \alpha^\vee\rangle = 2 + \langle \beta, \alpha^\vee\rangle \geqslant 2$ and therefore $\langle \alpha, \gamma^\vee\rangle = 1$, again by A.2. Hence $s_\gamma\alpha = \alpha - \gamma = -\beta$ and therefore $-\hat\beta = \widehat{-\beta} = \widehat{s_\gamma\alpha} = \hat\alpha - \hat\gamma$, as desired.

A subsystem S of R is called *closed* if $\alpha, \beta \in S$ and $\alpha + \beta \in R$ imply $\alpha + \beta \in S$. Refer to §10, in particular to Lemma 10.4, for further properties of closed subsystems.

7.9. Lemma. *Let (R, X) be a root system, A an abelian group, and $h\colon \mathcal{Q}(R) \to A$ a homomorphism. Define*

$$R_a := R_a(h) := \{\alpha \in R : h(\alpha) = a\} \qquad (a \in A). \tag{1}$$

Then

$$R = \dot{\bigcup_{a \in A}} R_a \qquad (\textit{disjoint union}), \tag{2}$$

the sets R_a satisfy

$$\big(R_a + R_b\big) \cap R \subset R_{a+b}, \tag{3}$$

$$R_{-a} = -R_a, \tag{4}$$

and R_0 is a closed subsystem of R. Conversely, for any decomposition (2) *with the property* (3) *there exists a unique homomorphism $h\colon \mathcal{Q}(R) \to A$ such that $R_a = R_a(h)$. If A is a subgroup of a vector space Y then h extends to a linear map $f\colon X \to Y$ and $R_0(h)$ is a full subsystem.*

Proof. It is evident that the sets defined by (1) satisfy (2) and (3), and that $R_0 = R \cap \operatorname{Ker}(h)$ is closed. As to (4), $0+0=0$ and (3) implies $0 \in R_0$, and $\alpha \in R_a$, $-\alpha \in R_b$ yields $0 = \alpha + (-\alpha) \in R_0 = R_{a+b}$, whence $b = -a$.

Conversely, suppose that (2) and (3) hold. By 7.6, there exists a unique homomorphism $h\colon \mathcal{Q}(R) \to A$ such that $R_a = R \cap h^{-1}(a)$. If $A \subset Y$ where Y is a vector space, then h extends to a linear map $f\colon X \to Y$ by 7.5(b). Hence $R_0(h) = R_0(f) = R \cap \operatorname{Ker}(f)$ is full.

7.10. *Rank of linear forms, basic weights and coweights.* Let (R, X) be a root system. The *rank* of a linear form $f \in X^*$ (relative to R) is defined by

$$\operatorname{rank}(f) = \operatorname{rank}(R/R_0(f)) = \dim(X/\operatorname{span}(R_0(f))), \tag{1}$$

where

$$R_0(f) = R \cap \operatorname{Ker}(f) = \{\alpha \in R : \langle \alpha, f\rangle = 0\} \tag{2}$$

is as in 7.9.1. Thus $\operatorname{rank}(f)$ is a measure of the lack of tightness of $\operatorname{Ker}(f)$. In particular, $\operatorname{rank}(f) = 0$ if and only if $f = 0$, and $\operatorname{rank}(f) = 1$ if and only if $\operatorname{Ker}(f)$ is a tight hyperplane. Analogously, we define the rank of a linear form in $X^{\vee *}$ with respect to $R^\vee$.

A coweight q is called *indivisible* if it is so as an element of the abelian group $\mathcal{P}^\vee(R) \cong \operatorname{Hom}(\mathcal{Q}(R), \mathbb{Z})$. Since $\langle \mathcal{Q}(R), q\rangle = m\mathbb{Z}$ is a subgroup of $\mathbb{Z}$, q is indivisible if and only if $q\colon \mathcal{Q}(R) \to \mathbb{Z}$ is surjective, and every nonzero coweight is a positive integer multiple of an indivisible coweight. A coweight q is called *basic* if it is indivisible and has rank one. Conversely, we will show below in 7.12 that every rank one linear form is a multiple of a basic coweight. Basic weights are of course defined analogously. We denote the set of basic weights and coweights by

$$\mathcal{B}(R) \quad \text{and} \quad \mathcal{B}^\vee(R),$$

respectively. In the following sections, we will often formulate results for coweights, because of notational convenience, and leave the dual formulation for weights to the reader. The basic coweights of the infinite irreducible root systems will be determined in 8.12.

Let B be an integral basis of R which exists by 6.4. We define the *dual coweights* q_β *of B* by

$$\langle \alpha, q_\beta\rangle = \delta_{\alpha\beta}, \tag{3}$$

for all $\alpha, \beta \in B$. Clearly the q_β are basic and, conversely, every basic coweight is of this form (whence the name). Indeed, let $B_0 \subset B = B_0 \cup \{\gamma\}$ be adapted integral bases for $R_0(q) \subset R$ which exist by 6.4. Then $\{\bar\gamma\}$ is an integral basis for $R/R_0(q)$, and $\langle \mathcal{Q}(R), q\rangle = \langle \mathbb{Z}\gamma, q\rangle = \mathbb{Z}\langle \gamma, q\rangle = \mathbb{Z}$ (by indivisibility) implies $\langle \gamma, q\rangle = \pm 1$. Thus possibly after replacing B by $-B$, we have $q = q_\gamma$. If R is finite, the same argument works for root bases in place of integral bases, due to 6.2(a). Applying this to the coroot system, we see:

$$\text{\textit{The basic weights of finite root systems are precisely the fundamental weights in the sense of} [\textbf{12}, VI, §1.10] \textit{with respect to some root basis.}} \tag{4}$$

Fundamental weights and coweights for infinite root systems will be considered in §16.

Let f be a linear form of rank one and suppose $(R, X) = \coprod(R_i, X_i)$ is a direct sum of root systems. Since $R_0(f)$ is a full subsystem spanning a hyperplane, 1.6(c) shows that f vanishes on all R_i with one exception. Conversely, each rank one linear form of R_i extends by zero to a rank one linear form of R. In particular, the basic coweights of R are given by

$$\mathcal{B}^\vee(R) \cong \dot{\bigcup_{i\in I}} \mathcal{B}^\vee(R_i), \tag{5}$$

and similarly for the basic weights.

7.11. Lemma. *Let $(\bar R, \bar X) = (R/R', X/X') \in \overline{\mathbf{RS}}$ be a quotient of a root system (R, X) by a full subsystem (R', X'), and suppose that $\bar R$ has rank one. If $\{\bar\gamma\}$ is a root basis of $\bar R$, then $\bar R = \{i\bar\gamma : i \in \mathbb{Z},\ -m \leqslant i \leqslant m\}$ for some $m \in \{1, \dots, 6\}$.*

Proof. By 6.4, $\bar R$ is finite. Choose a set $E \subset R$ of representatives of $\bar R$. By Lemma 2.5 and local finiteness of R, there exists a full finite subsystem F of R intersecting R' tightly, and by 1.9, $R/R' \cong F/F \cap R'$. Thus we may replace R by F and so assume R finite. After decomposing R into irreducible components, it follows from 1.6(c) and $\operatorname{rank}(\bar R) = 1$ that R' contains all irreducible components of R except one. Hence we may assume R irreducible. Now choose adapted root bases $B' \subset B = B' \cup \{\gamma\}$ for $R' \subset R$ (cf. Lemma 6.2(a)). Then $\{\bar\gamma\}$ is a root basis of $\bar R$ by 6.1(b). From the classification of finite root systems [**12**, VI], in particular, the list of coefficients of the highest root expressed as a linear combination of simple roots, as well as A.14, it follows that $\bar R$ has the form indicated.

Remark. Note that for $m = 1$ and $m = 2$ these quotients are (isomorphic to) the root systems A_1 and BC_1, but they are no longer root systems for $3 \leqslant m \leqslant 6$. From the classification it follows that all six possibilities for m do occur, but $3 \leqslant m \leqslant 6$ only when R is exceptional.

7.12. Proposition. (a) *Let f be a rank one linear form of a root system (R, X). Then the set of values $\langle R, f\rangle$ of f on R is of the form $\{-am, \dots, -a, 0, a, \dots, am\}$ for a unique positive real number a and integer m, $1 \leqslant m \leqslant 6$, and $a^{-1}f$ is a basic coweight.*

(b) *Let q be a basic coweight of R. Then $R_1(q) \neq \emptyset$, and $|\langle\alpha, q\rangle| \leqslant 6$, for all $\alpha \in R$. In particular, basic (co)weights are bounded.*

Proof. (a) Let $(\bar R, \bar X)$ be the quotient of (R, X) by $R_0(f)$. Then f induces a linear form $\bar f\colon \bar X \to \mathbb{R}$, and the assertion follows from the structure of $\bar R$ described in Lemma 7.11.

(b) Applying (a) to $f = q$, we have $a \in \mathbb{N}$, and $q = aq'$ where $q' = a^{-1}q$ is a basic coweight. Then $a = 1$ by indivisibility of q, and the remaining assertion is also clear from (a).

7.13. Proposition. *Let (R, X) be a root system and $R' \subset R$ a full subsystem, with $X' = \operatorname{span}(R')$.*

(a) *Any basic coweight q' of R' extends to a basic coweight q of R.*

(b) *Conversely, let q be a basic coweight of R and suppose that R' and $R_0(q)$ intersect tightly. Then the restriction $q' := q\big|X'$ is an integral multiple of a basic coweight of R'.*

Proof. (a) By 7.10, there exists an integral basis B' of R' such that $q' = q'_{\beta'}$ for some $\beta' \in B'$. By Theorem 6.4 we can extend B' to an integral basis B of R, and then it is clear that q' is the restriction of the basic weight $q_{\beta'}$ with respect to B.

(b) We have $\operatorname{Ker}(q') = X' \cap \operatorname{Ker}(q) = \operatorname{span}(R') \cap \operatorname{span}(R_0(q)) = \operatorname{span}(R' \cap R_0(q))$ (by tightness) $= \operatorname{span}(R'_0(q'))$. Thus either $q' = 0$ or $\operatorname{rank}(q') = 1$, and the claim follows from 7.12.

7.14. *Minuscule (co)weights and saturated sets.* A non-zero coweight q of a root system R is called *minuscule* if it does not vanish on any connected component of R and $\langle \alpha, q \rangle \in \{0, \pm 1\}$ for all $\alpha \in R$. Clearly, the automorphism group of R acts on the set of minuscule weights. Minuscule weights are of course defined analogously.

A subset $T \subset \mathcal{P}^\vee(R)$ is called *saturated* if for all $q \in T$ and all $\alpha \in R$, the coweight $q - t\alpha^\vee$ belongs to T, for all non-zero integers t between 0 and $\langle \alpha, q \rangle$. Since $s_\alpha(q) = q - \langle \alpha, q \rangle \alpha^\vee$, it is clear that a saturated subset of $\mathcal{P}^\vee(R)$ is invariant under the Weyl group.

7.15. Proposition. *Let R be an irreducible root system.*

(a) *A coweight q is minuscule if and only if the orbit $W(R) \cdot q$ is saturated (and hence the smallest saturated subset containing q).*

(b) *A minuscule coweight is basic.*

Proof. (a) Let q be minuscule. As remarked above, the orbit of q under the Weyl group consists of minuscule coweights. Hence for all $q' \in W(R) \cdot q$ and $\alpha \in R$, we have $\langle \alpha, q' \rangle \in \{0, \pm 1\}$ and therefore $q' - t\alpha = s_\alpha(q') \in W(R) \cdot q' = W(R) \cdot q$ for every nonzero t between 0 and $\langle \alpha, q' \rangle$.

Conversely, let $W(R) \cdot q$ be saturated, and suppose that there exists $\alpha \in R$ such that $|\langle \alpha, q \rangle| \geqslant 2$. Possibly after replacing α by its negative, we may assume that $\langle \alpha, q \rangle = n \geqslant 2$. Then $q - \alpha^\vee \in W(R) \cdot q$, say $q - \alpha^\vee = w(q)$. Since the Weyl group is locally finite by 5.9, there exists a finite subgroup $F \subset W(R)$ containing w and s_α. Choose an F-invariant inner product on X^*, and let $\|\cdot\|$ denote the Euclidean norm defined by this inner product. Then $\|q - n\alpha^\vee\| = \|s_\alpha(q)\| = \|q\| = \|w(q)\| = \|q - \alpha^\vee\|$, which contradicts elementary Euclidean geometry in the 2-dimensional subspace spanned by q and $\alpha^\vee$.

(b) Clearly, a minuscule coweight q is indivisible, so it remains to prove that it has rank one, i.e., that any two roots are congruent modulo $X_0 := \operatorname{span} R_0(q)$. By 7.9, we have a decomposition $R = R_1 \mathbin{\dot\cup} R_0 \mathbin{\dot\cup} R_{-1}$ with $R_i = R_i(q)$ and $R_{-1} = -R_1$. Thus it suffices to show that $\alpha - \beta \in X_0$ for all $\alpha, \beta \in R_1$. Since $(R_1 + R_1) \cap R = \emptyset$, we have $\alpha + \beta \notin R$ and thus $\langle \alpha, \beta^\vee \rangle \geqslant 0$ by A.3. If $\langle \alpha, \beta^\vee \rangle > 0$ then $\alpha - \beta$ is in R and then even in R_0. Otherwise, since R_1 is connected by 11.9, there exists a connecting chain $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ in R_1 with $\alpha_{i-1} \not\perp \alpha_i$. By the argument just used, we have $\alpha_{i-1} - \alpha_i \in R_0$ and thus $\alpha - \beta = (\alpha_0 - \alpha_1) + \cdots + (\alpha_{n-1} - \alpha_n) \in X_0$.

We can now prove a result on the set of closed subsystems of a root system R containing $R_0(q)$ where q is a basic coweight.

7.16. PROPOSITION. *Let q be a basic coweight of a root system (R, X), let $R_0 = R_0(q)$, and let $m = m(q)$ be the unique positive integer such that $q(R) = [-m, m] \cap \mathbb{Z}$ as in* 7.12. *For every integer $l \in [0, m]$ let*

$$R_{[l]} := R_{[l]}(q) := R \cap q^{-1}(l\mathbb{Z}) = \{\alpha \in R : q(\alpha) \in l\mathbb{Z}\}. \tag{1}$$

Then $l \mapsto R_{[l]}$ is a bijection between the set of integers in $[0, m]$ and the set of closed subsystems S of R with $R_0 \subset S$. This bijection satisfies $R_{[0]} = R_0$, $R_{[1]} = R$, and

$$R_{[k]} \subset R_{[l]} \quad \Longleftrightarrow \quad l \big| k. \tag{2}$$

Hence $R_{[l]}$ is a maximal closed proper subsystems of R if and only if either $m = 1$ and $l = 0$, or l is prime and $m \geqslant 2$.

Remark. If R is irreducible then the coweights with $m(q) = 1$ are precisely the minuscule coweights.

Proof. Clearly $R_{[l]}$ is a closed subsystem and we have $R_{[0]} = R_0$, $R_{[1]} = R$. Since every $l \in [0, m] \cap \mathbb{Z}$ occurs as a value of q on R by 7.12, it follows that $R_{[l]} = R_0$ if and only if $l = 0$. Therefore, it suffices to consider the case $l \in [1, m] \cap \mathbb{Z}$ on the one hand, and closed subsystems S of R properly containing R_0 on the other. Let S be such a subsystem. Then $\operatorname{span}(S) = X$, and hence q is a linear form of rank one for the root system (S, X). By 7.12, $q(S) = [-am', \ldots, -a, 0, a, \ldots, am']$ for some $a \in \mathbb{R}_{++}$ and $m' \in \mathbb{N}_+$. On the other hand, $q(S) \subset q(R) = [-m, m] \cap \mathbb{Z}$. Hence $a = l \in [1, m] \cap \mathbb{Z}$ (and of course $lm' \leqslant m$), so

$$q(S) = l \cdot \big([-m', m'] \cap \mathbb{Z}\big). \tag{3}$$

We claim that the assignment $S \mapsto \lambda(S) := l$ is inverse to the map $l \mapsto R_{[l]}$.

Indeed, given $l \in [1, m] \cap \mathbb{Z}$ it is clear from $q(R) = [-m, m] \cap \mathbb{Z}$ that $\lambda(R_{[l]}) = l$. Conversely, given a closed subsystem $S \supsetneqq R_0$ with associated $l = \lambda(S)$, we must show $S = R_{[l]}$. From (3) it follows that

$$q(\mathbb{Z}[S]) = \mathbb{Z}[q(S)] = l\mathbb{Z}. \tag{4}$$

We claim that

$$\mathbb{Z}[S] = \mathbb{Z}[R] \cap q^{-1}(l\mathbb{Z}). \tag{5}$$

Indeed, the inclusion from left to right in (5) follows from (4). Conversely, let $x \in \mathbb{Z}[R]$ and $q(x) \in l\mathbb{Z}$, say, $q(x) = nl$. We may identify R/R_0 with $[-m, m] \cap \mathbb{Z}$ and q with the canonical map $R \to R/R_0$. Then Th. 6.4 implies in particular that $\mathbb{Z}[R] \cap \operatorname{Ker}(q) = \mathbb{Z}[R_0]$. By (3) there exists $\alpha \in S$ with $q(\alpha) = l$. Hence $q(x - n\alpha) = 0$, so $y := x - n\alpha \in \mathbb{Z}[R] \cap \operatorname{Ker}(q) = \mathbb{Z}[R_0] \subset \mathbb{Z}[S]$. It follows that $x = y + n\alpha \in \mathbb{Z}[S]$ as well. Now S is a closed subsystem, so $S = R \cap \mathbb{Z}[S]$ by 10.4 (the reader can easily check that the straightforward proof of 10.4 is indeed independent of the result proven here). This then implies $S = R \cap q^{-1}(l\mathbb{Z})$ (by (5)) $= R_{[l]}$ (by (1)).

In (2), the implication from right to left is clear from the definitions. Conversely, $R_{[l]} \subset R_{[k]}$ implies $\mathbb{Z}[R_{[l]}] \subset \mathbb{Z}[R_{[k]}]$ and hence, by applying q and using (4), $l\mathbb{Z} \subset k\mathbb{Z}$, so $k\big|l$. The last statement is then immediate.

The following result was conjectured by M. Racine.

7.17. COROLLARY. *Let q be a basic coweight of the root system (R, X), and let $\alpha \in R_l = R_l(q)$, see* 7.9.1. *Then $R_l = (\alpha + \mathbb{N}[R_0]) \cap R$, where $\mathbb{N}[R_0]$ is the subsemigroup of $(X, +)$ generated by R_0.*

We remark that of course only the case $l \neq 0$ is of interest here.

Proof. It is easily seen, cf. 10.4, that $S_\alpha := R \cap \mathbb{Z}\big[\{\alpha\} \cup R_0\big]$ is a closed subsystem of R containing R_0 and α. Since $l \in q(S_\alpha) \subset l\mathbb{Z}$, it follows from Prop. 7.16 that $S_\alpha = R_{[l]}$. Hence $S_\alpha = R_{[l]} = S_\beta$ for any $\beta \in R_l$, proving $R_l \subset S_\alpha$. Clearly $R_l \cap \mathbb{Z}\big[\{\alpha\} \cup R_0\big] \subset \alpha + \mathbb{N}[R_0]$ because $R_0 = -R_0$. Therefore $R_l \subset (\alpha + \mathbb{N}[R_0])$. The other inclusion is obvious.

§8. Classification

8.1. *Classical root systems.* Let I be a non-empty set, let $X = \mathbb{R}^{(I)} = \bigoplus_{i\in I} \mathbb{R}\varepsilon_i$ be the free $\mathbb{R}$-vector space on the set I, and let

$$\dot{X} = \mathrm{Ker}(t) \subset X$$

be the kernel of the *trace form* t, defined as the linear form on X taking the value 1 on each ε_i. We define

$$\dot{\mathrm{A}}_I = \{\varepsilon_i - \varepsilon_j : i, j \in I\}, \tag{1}$$
$$\mathrm{D}_I = \dot{\mathrm{A}}_I \cup \{\pm(\varepsilon_i + \varepsilon_j) : i \neq j\}, \tag{2}$$
$$\mathrm{B}_I = \mathrm{D}_I \cup \{\pm\varepsilon_i : i \in I\}, \tag{3}$$
$$\mathrm{C}_I = \mathrm{D}_I \cup \{\pm 2\varepsilon_i : i \in I\} = \{\pm\varepsilon_i \pm \varepsilon_j : i, j \in I\}, \tag{4}$$
$$\mathrm{BC}_I = \mathrm{B}_I \cup \mathrm{C}_I = \{\pm\varepsilon_i : i \in I\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : i, j \in I\}. \tag{5}$$

Then $\dot{\mathrm{A}}_I$ is a locally finite root system in $\dot{X}$ and the others are locally finite root systems in X, with the exception of D_I for $|I| = 1$ where $\mathrm{D}_I = \{0\}$ does not span X. In all cases, an invariant inner product is given by $(\varepsilon_i|\varepsilon_j) = \delta_{ij}$. Indeed, with the definition $\langle\alpha, \beta^\vee\rangle = 2(\alpha|\beta)/(\beta|\beta)$, the proof becomes a straightforward verification which is left to the reader. For finite I, this is of course well known.

The rank of $\dot{\mathrm{A}}_I$ is $\mathrm{Card}(I) - 1$ while the rank in the other cases is $\mathrm{Card}(I)$. The notation $\dot{\mathrm{A}}$ (instead of A) serves to indicate this fact. For a finite I, say $|I| = n$, we will use the standard notation $\mathrm{B}_n = \mathrm{B}_I$, $\mathrm{C}_n = \mathrm{C}_I$, $\mathrm{D}_n = \mathrm{D}_I$ and $\mathrm{BC}_n = \mathrm{BC}_I$, while the usual notation A_n is linked to our notation by

$$\mathrm{A}_n = \dot{\mathrm{A}}_{\{0,1,\ldots,n\}} = \dot{\mathrm{A}}_{n+1}.$$

Also, our convention that $0 \in R$ accounts for the difference in the description of the irreducible root systems above and that given, e.g., in [**12**, Planches]. A root system R will be called *classical* if it is isomorphic to one of the root systems (1) – (5) for a suitable, possibly infinite, set I.

To describe the coroot systems of the classical root systems, we introduce the linear forms e_i on X defined by

$$\langle\varepsilon_i, e_j\rangle = \delta_{ij}. \tag{6}$$

We also denote the restriction of a linear form $f \in X^*$ to $\dot{X}$ by $\dot{f}$. Then it is easily verified that the coroots are given by

$(\varepsilon_i - \varepsilon_j)^\vee = \dot{e}_i - \dot{e}_j$ in case $\dot{\mathrm{A}}_I$, and
$(\varepsilon_i \pm \varepsilon_j)^\vee = e_i \pm e_j$ $(i \neq j)$, $\varepsilon_i^\vee = 2e_i$, $(2\varepsilon_i)^\vee = e_i$, in the other cases.

Hence the span of the coroots is

$$\operatorname{span}(\dot{\mathrm{A}}_I^\vee) = (\dot{X})^\vee = \operatorname{span}\{\dot{e}_i - \dot{e}_j : i, j \in I\}, \tag{7}$$

$$\operatorname{span}(R^\vee) = X^\vee = \bigoplus_{i\in I} \mathbb{R}e_i \quad \text{for } R \neq \dot{\mathrm{A}}_I. \tag{8}$$

To make the situation more symmetrical in the ε_i and e_j, we will identify $(\dot{X})^\vee$ with a subspace of $X^\vee$ as follows. Consider the *cotrace* $t^\vee$, i.e., the linear form $t^\vee$ on $X^\vee$ defined by $t^\vee(e_i) = 1$ for all $i \in I$, and let

$$(X^\vee)^{\boldsymbol{\cdot}} := \operatorname{Ker}(t^\vee). \tag{9}$$

Then it is easily seen that $(X^\vee)^{\boldsymbol{\cdot}}$ is spanned by the $e_i - e_j$, and the restriction map $f \mapsto \dot{f} = f\big|\dot{X}$ induces a vector space isomorphism

$$(X^\vee)^{\boldsymbol{\cdot}} \xrightarrow{\cong} (\dot{X})^\vee, \tag{10}$$

which obviously maps $e_i - e_j$ to $\dot{e}_i - \dot{e}_j$. We will treat (10) as an identification, and simply write

$$\dot{X}^\vee = \operatorname{Ker}(t^\vee) \subset X^\vee.$$

With these conventions, the coroot systems of the classical root systems are:

$$\dot{\mathrm{A}}_I^\vee = \{e_i - e_j : i, j \in I\}, \tag{11}$$

$$\mathrm{D}_I^\vee = \dot{\mathrm{A}}_I \cup \{\pm(e_i + e_j) : i \neq j\}, \tag{12}$$

$$\mathrm{B}_I^\vee = \mathrm{D}_I \cup \{\pm 2e_i : i \in I\} = \{\pm e_i \pm e_j : i, j \in I\}, \tag{13}$$

$$\mathrm{C}_I^\vee = \mathrm{D}_I \cup \{\pm e_i : i \in I\}, \tag{14}$$

$$\mathrm{BC}_I^\vee = \mathrm{B}_I^\vee \cup \mathrm{C}_I^\vee = \{\pm e_i : i \in I\} \cup \{\pm e_i \pm e_j : i, j \in I\}. \tag{15}$$

Clearly, $\mathrm{B}_I^\vee \cong \mathrm{C}_I$ and $\mathrm{C}_I^\vee \cong \mathrm{B}_I$, while the others are isomorphic to their coroot systems.

8.2. *Root systems of type* T *and locally of type* T. For infinite I it is easily checked that the five systems listed in 8.1.1 – 8.1.5 are pairwise not isomorphic. This is still true in the finite case except for the well-known isomorphisms

$$\dot{\mathrm{A}}_2 = \mathrm{A}_1 \cong \mathrm{B}_1 \cong \mathrm{C}_1, \quad \mathrm{B}_2 \cong \mathrm{C}_2, \quad \mathrm{D}_2 \cong \mathrm{A}_1 \oplus \mathrm{A}_1, \quad \dot{\mathrm{A}}_4 = \mathrm{A}_3 \cong \mathrm{D}_3. \tag{1}$$

Hence, for two classical root systems R and R' on index sets I and I' of cardinality $\geqslant 4$ to be isomorphic, it is necessary and sufficient that $\operatorname{Card} I = \operatorname{Card} I'$ and that they have the same *type* $\mathrm{T} \in \mathfrak{T}$, where

$$\mathfrak{T} := \{\dot{\mathrm{A}}, \mathrm{B}, \mathrm{C}, \mathrm{BC}, \mathrm{D}\}$$

is the set of possible types.

A root system R is said to be *of type* T if $R \cong \mathrm{T}_I$ for some set I and some type $\mathrm{T} \in \mathfrak{T}$. If $\operatorname{rank}(R) \geqslant 4$ then by the above remarks, R can be of type T for at most one type T, and the cardinality of the set I is uniquely determined. On the other hand, the classification of finite root systems shows that a finite irreducible root system of rank > 8 is of type T for some $\mathrm{T} \in \mathfrak{T}$.

An infinite root system R is called *locally of type* T if $R = \varinjlim R_\lambda$ is the direct limit of finite root systems R_λ of type T.

As noted in 8.1, we have $\mathrm{B}_I^\vee \cong \mathrm{C}_I$ and vice versa, while $\mathrm{T}_I^\vee \cong \mathrm{T}_I$ for the other types. Accordingly, we define an involutory map $\mathrm{T} \mapsto \mathrm{T}^\vee$ on $\mathfrak{T}$ by $\mathrm{B}^\vee := \mathrm{C}$, $\mathrm{C}^\vee := \mathrm{B}$, and $\mathrm{T}^\vee = \mathrm{T}$ for the other types. From 4.9(c) it follows easily that R locally of type T implies $R^\vee$ is locally of type $\mathrm{T}^\vee$.

8.3. Lemma. *An infinite irreducible root system has a well-defined local type, i.e., it is locally of type* T *for a unique* $\mathrm{T} \in \mathfrak{T}$.

Proof. Since R has infinite rank, we can and do fix a finite irreducible full subsystem R_0 with the following properties:

(i) $\mathrm{rank}(R_0) > 8$,
(ii) if R is multiply laced, then all root lengths occurring in R (of which there are at most three by 4.4) already occur in R_0,
(iii) if R is simply laced and contains a full subsystem of type D_4 then also R_0 contains such a subsystem.

Then by Cor. 3.15(b), $R = \varinjlim R_\lambda$ is the direct limit of its irreducible finite full subsystems $R_\lambda \supset R_0$ which, by 8.2, have unique types T_λ and T_0, respectively. Since R_0 is a full subsystem of R_λ, a root basis B_0 of R_0 extends to a root basis B_λ of R_λ (A.12). Hence the Dynkin diagram of B_0 (as defined in 6.8) is an induced subgraph of the Dynkin diagram of B_λ. Now a glance at the structure of Dynkin diagrams shows that, with the choices made above, we must have $\mathrm{T}_\lambda = \mathrm{T}_0$, so R is locally of type T_0. Assume that R is also locally of type T_1 for some $\mathrm{T}_1 \in \mathfrak{T}$. Then R contains a full finite subsystem R_1 of type T_1. Since R is the direct limit of its irreducible finite full subsystems, we can assume $R_0 \subset R_1$, so that the same argument as before shows $\mathrm{T}_0 = \mathrm{T}_1$.

8.4. Theorem. *Every irreducible locally finite root system R of infinite rank is isomorphic to one of the systems listed in* 8.1.1 – 8.1.5*, for a suitable infinite set I.*

Proof. In view of the preceding lemma, this is equivalent to showing:

If R is locally of type T *then it is of type* T.

We will do this for each type separately.

It is clear that a root system locally of type $\dot{\mathrm{A}}$ or D is simply laced, so all roots have the same length with respect to an invariant inner product $(\ |\ )$ which we assume to be the normalized one (see 4.6). Then the possible inner products of two roots are $\langle \alpha, \beta^\vee \rangle = (\alpha|\beta) = 0, \pm 1, \pm 2$, and the last case occurs only for $\alpha = \pm\beta$.

Case 1: R is locally of type $\dot{\mathrm{A}}$. Let us call a subset $C \subset R^\times$ a *collinear system* if $(\alpha|\beta) = 1 + \delta_{\alpha\beta}$ for all $\alpha, \beta \in C$. By considering the Gram matrix of C, it is easily seen that C is linearly independent. Clearly collinear systems exist, and they are inductively ordered by inclusion. By Zorn's Lemma, we thus may pick a maximal collinear system $C = \{\gamma_j : j \in J\}$. For $j \neq k$ we have $\gamma_j - \gamma_k = s_{\gamma_k}(\gamma_j) \in R$, so $S := C \cup (-C) \cup (C - C) \subset R$. It is easily checked that this is in fact a partition of S and that, letting 0 denote an element not in J and setting $I = \{0\} \dot\cup J$, we have an isomorphism $\dot{\mathrm{A}}_I \cong S$ by mapping $\varepsilon_j - \varepsilon_0 \mapsto \gamma_j$ for $j \in J$. Thus it remains to show that $S = R$.

Suppose to the contrary that $S \neq R$. Then there exists $\alpha \in R \setminus S$ and $\alpha \not\perp S$, else R would not be irreducible. There cannot exist j and k such that $(\alpha|\gamma_j) = 1 = -(\alpha|\gamma_k)$, because otherwise $(\alpha|\gamma_j - \gamma_k) = 2$ and therefore $\alpha = \gamma_j - \gamma_k \in S$. Thus, possibly after replacing α by its negative, we have $(\alpha|C) \subset \{0, 1\}$. For $i = 0, 1$, let $J_i = \{j \in J : (\alpha|\gamma_j) = i\}$. Then $J = J_0 \cup J_1$ and $J_1 \neq \emptyset$. We now distinguish the following two cases.

(a) $|J_1| = 1$, say $J_1 = \{1\}$. Then $\gamma_1 - \alpha = s_\alpha(\gamma_1) \in R$ and $(\gamma_1 - \alpha|\gamma_j) = 1$ for all $j \in J$, showing $C \cup \{\gamma_1 - \alpha\}$ collinear and contradicting maximality of C.

(b) $|J_1| \geqslant 2$, say $\{1, 2\} \subset J_1$. Note that we must have $J_0 \neq \emptyset$, else $C \cup \{\alpha\}$ would be collinear which is impossible by maximality of C. Let $3 \in J_0$. The vectors $\alpha, \gamma_1, \gamma_2, \gamma_3$ are linearly independent, as can easily be seen from their Gram determinant. Hence $V = \operatorname{span}\{\alpha, \gamma_1, \gamma_2, \gamma_3\}$ is 4-dimensional, and thus $R \cap V$ is a full irreducible subsystem of rank 4 of R which contains the following roots:

$$\begin{aligned}
&\pm\alpha,\\
&\pm\gamma_j,\ \gamma_j - \gamma_k, \quad (j, k \in \{1, 2, 3\},\ j \neq k),\\
&\pm(\gamma_i - \alpha) = \pm s_\alpha(\gamma_i), \quad (i = 1, 2),\\
&\pm(\gamma_i - \gamma_3 - \alpha) = \pm s_{\gamma_3}(\gamma_i - \alpha), \quad (i = 1, 2).
\end{aligned}$$

These are $2+12+4+4 = 22$ roots altogether, and hence $R \cap V$ cannot be isomorphic to a root system of type A_4 which has only $4\cdot 5 = 20$ roots. On the other hand, since R is locally of type $\dot{\mathrm{A}}$, there exists a full finite subsystem $F \cong \mathrm{A}_n$ of R containing $R \cap V$, and it is easy to see (and follows also from 12.3(b)) that a full irreducible subsystem of rank 4 of F is isomorphic to A_4, hence cannot contain more than 20 roots. This contradiction shows that also (b) is impossible, and completes the proof of Case 1.

Case 2: R is locally of type D. Let us call a subset $\varOmega = \{\varepsilon_i : i \in I\}$ of X an *orthosystem* if it is orthonormal with respect to $(\ |\ )$, and $\varepsilon_i \pm \varepsilon_j \in R$, for all $i, j \in I$, $i \neq j$. Since R is locally of type D, it contains orthosystems of arbitrarily large finite cardinality. Also, the set of orthosystems is inductively ordered by inclusion, so by Zorn's Lemma we may pick a maximal orthosystem $\varOmega = \{\varepsilon_i : i \in I\}$, and with $|I| > 8$. Then it is clear that $S := \{0\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : i \neq j,\ i, j \in I\} \subset R$ is a root system of type D, and it only remains to show that $S = R$.

Suppose to the contrary that $S \neq R$. Since R is irreducible, there exists a root $\alpha \in R\backslash S$ with $\alpha \not\perp S$, so we have $(\alpha|\beta) \in \{0, \pm 1\}$ for all $\beta \in S$. We now examine the inner products $(\alpha|\varepsilon_i)$. First, there must be at least one index, say $1 \in I$, such that $(\alpha|\varepsilon_1) \neq 0$ (otherwise α would be orthogonal to S), and even $(\alpha|\varepsilon_1) > 0$, possibly after replacing α by $-\alpha$. Next, choosing an index $j \neq i$, we have $\varepsilon_i = (1/2)(\beta + \gamma)$ where $\beta = \varepsilon_i + \varepsilon_j$ and $\gamma = \varepsilon_i - \varepsilon_j$ are in S. Hence

$$(\alpha|\varepsilon_i) = \frac{1}{2}\big((\alpha|\beta) + (\alpha|\gamma)\big) \in \{0, \pm\frac{1}{2}, \pm 1\}. \tag{1}$$

Suppose that $(\alpha|\varepsilon_k) = \pm 1/2$ for some $k \in I$. Then for all $i \neq k$, $\varepsilon_k + \varepsilon_i \in R$ and hence $(\alpha|\varepsilon_k + \varepsilon_i) = \pm(1/2) + (\alpha|\varepsilon_i) \in \{0, \pm 1\}$, which by (1) implies $|(\alpha|\varepsilon_i)| = 1/2$. On the other hand, Bessel's inequality yields $\sum_{j\in I}(\alpha|\varepsilon_j)^2 \leqslant (\alpha|\alpha) = 2$. Since I has more than 8 elements, this leads to a contradiction. Thus we now have $(\alpha|\varepsilon_i) \in \{0, \pm 1\}$ for all $i \in I$, and in particular $(\alpha|\varepsilon_1) = 1$. Furthermore, $(\alpha|\varepsilon_i) = 0$ for all $i \neq 1$, because if $(\alpha|\varepsilon_i) = c \in \{\pm 1\}$ for some $i \neq 1$, then $(\alpha|\varepsilon_1 + c\varepsilon_i) = 2$ and therefore $\alpha = \varepsilon_1 + c\varepsilon_i \in S$ which is not the case.

Now let 0 be an index not in I and put $\varepsilon_0 := \alpha - \varepsilon_1$. We claim that $\varOmega' = \varOmega \,\dot\cup\, \{\varepsilon_0\}$ is an orthosystem. This will contradict maximality of $\varOmega$ and complete the proof of Case 2.

Clearly $\varepsilon_0 \perp \varepsilon_i$ for all $i \in I$, and $(\varepsilon_0|\varepsilon_0) = (\alpha|\alpha) - 2(\alpha|\varepsilon_1) + (\varepsilon_1|\varepsilon_1) = 2-2+1 = 1$, so $\varOmega'$ is orthonormal. Also, $\varepsilon_0 + \varepsilon_1 = \alpha \in R$. It remains to show that $\varepsilon_0 - \varepsilon_1$ and

$\varepsilon_0 \pm \varepsilon_i$ are in R, for all $i \neq 1$. Pick an element $2 \neq 1$ in I. Then $w = s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_1 + \varepsilon_2} \in W(R)$, and $w(\varepsilon_0) = \varepsilon_0$ because ε_0 is perpendicular to ε_1 and ε_2, while $w(\varepsilon_1) = -\varepsilon_1$. It follows that $\varepsilon_0 - \varepsilon_1 = w(\alpha) \in R$. For $i \neq 1$ we have $\varepsilon_0 \pm \varepsilon_i = s_{\varepsilon_1 - \varepsilon_i}(\varepsilon_0 \pm \varepsilon_1) \in R$. This shows that Ω' is indeed an orthosystem, as desired.

Case 3: R is locally of type B*,* BC *or* C*.* If R is locally of type C then $R^\vee$ is locally of type B. Thus it suffices to deal with the first two possibilities, and it is clear that we are in the cases (ii) or (v) of Prop. 4.4. We normalize an invariant inner product by requiring the short roots to have length one. Let Σ be the set of short roots and $\Omega = \{\varepsilon_i : i \in I\} \subset \Sigma$ a subset such that $\Sigma = \Omega \,\dot\cup\, (-\Omega)$. Then Ω is orthonormal: By definition of Ω, two different $\varepsilon_i, \varepsilon_j \in \Omega$ are not multiples of each other, and they are short roots in some full finite subsystem $F \cong \mathrm{B}_n$ or $\cong \mathrm{BC}_n$, $n \geqslant 2$, where it is clear that two linearly independent short roots are orthogonal. In particular, then, Ω is linearly independent. Also, $\varepsilon_i \pm \varepsilon_j \in F \subset R$ and therefore $S := \{0\} \cup \Sigma \cup \{\pm\varepsilon_i \pm \varepsilon_j : i \neq j,\ i, j \in I\} \subset R$. Clearly, $S \cong \mathrm{B}_I$. If R is locally of type B then every long root of R is the sum of two orthogonal short roots, since this is so in the full finite irreducible subsystems of type B whose direct limit R is. This shows $R = S \cong \mathrm{B}_I$. If R is locally of type BC, a similar argument shows $R = S \cup 2\Sigma \cong \mathrm{BC}_I$.

The following description of non-reduced irreducible root systems is immediate from the classification above. It could also be proven without classification, by a reduction to the finite case (3.16 and A.7).

8.5. Corollary. *Let $R \subset X$ be a non-reduced irreducible root system. Let $(\ |\ )$ be the normalized invariant inner product as in* 4.6*, and let $R_i = \{\alpha \in R : (\alpha|\alpha) = 2i\}$. Also denote by R_{ind} the union of $\{0\}$ and the set of indivisible roots. Then*

(a) *$R_{\mathrm{ind}} = \{0\} \cup R_1 \cup R_2$ is an irreducible reduced root system in X,*

(b) *any two elements of R_1 are either proportional or orthogonal,*

(c) *$R = \{0\} \cup R_1 \cup R_2 \cup R_4$ and $R_4 = 2R_1$.*

8.6. *Notes.* Other proofs of the Classification Theorem 8.4 for reduced root systems were given by Kaplansky and Kibler [**37**, **38**], Neher [**57**, sect. 2], and by Neeb and Stumme [**54**].

The work of Kaplansky and Kibler is related to our root systems as follows. Let $R \subset X$ be an irreducible reduced root system of infinite rank. Then only the cases (i) and (ii) of Prop. 4.4 are possible. Using the normalized invariant inner product, it is immediately checked that R is, in the terminology of [**37**] and [**38**], an H-system and a J-system, respectively. By the results of [**37**, **38**], R is therefore isomorphic to $\dot{\mathrm{A}}_I$ or D_I in the simply-laced case, and to B_I or C_I in the doubly-laced case. Due to the fact that our root systems live in vector spaces over the reals which carry a positive definite invariant inner product, our proof is simpler than that of Kaplansky and Kibler who allow fields of positive characteristic.

The notion of local type is essentially due to Neeb and Stumme, and our Lemma 8.3 is equivalent to their [**54**, Prop. III.2]. However, we handle the classification of the types A and D in a different and much simpler way than [**54**]. Also, our proof avoids the machinery of grid bases used in [**57**].

8.7. *Description of weight groups.* We next describe the weight groups of the classical irreducible root systems listed in (1) – (5) of 8.1, with emphasis on the infinite case. Using the notations introduced there, we identify the dual X^* of $X = \bigoplus_{i\in I} \mathbb{R}\varepsilon_i$ with $\prod_{i\in I} \mathbb{R}e_i$, and the dual $X^{\vee *}$ of $X^\vee = \bigoplus_{i\in I} \mathbb{R}e_i$ with $\prod_{i\in I} \mathbb{R}\varepsilon_i$. Then the canonical map $j\colon X \to X^{\vee *}$ is just the inclusion. Now we consider the following abelian groups:

$$
\begin{aligned}
\Gamma &= \bigoplus_{i\in I} \mathbb{Z}\varepsilon_i \subset X, & \Gamma^\vee &= \bigoplus_{i\in I} \mathbb{Z}e_i \subset X^\vee,\\
\Gamma^* &= \prod_{i\in I} \mathbb{Z}e_i \subset X^*, & \Gamma^{\vee *} &= \prod_{i\in I} \mathbb{Z}\varepsilon_i \subset X^{\vee *},\\
\Gamma_0 &= \{x \in \Gamma : t(x) = 0\}, & \Gamma_0^\vee &= \{f \in \Gamma^\vee : t^\vee(f) = 0\},\\
\Gamma_0^* &= \{q \in \dot{X}^* : q(\Gamma_0) \subset \mathbb{Z}\}, & \Gamma_0^{\vee *} &= \{p \in (\dot{X}^\vee)^* : p(\Gamma_0^\vee) \subset \mathbb{Z}\},\\
\Gamma_2 &= \{x \in \Gamma : t(x) \in 2\mathbb{Z}\}, & \Gamma_2^\vee &= \{f \in \Gamma^\vee : t^\vee(f) \in 2\mathbb{Z}\},\\
\Gamma_2^* &= \{q \in X^* : q(\Gamma_2) \subset \mathbb{Z}\}, & \Gamma_2^{\vee *} &= \{p \in (X^\vee)^* : p(\Gamma_2^\vee) \subset \mathbb{Z}\}.
\end{aligned}
$$

Note that Γ^* and $\Gamma^{\vee *}$ can also be characterized as

$$
\Gamma^* = \{f \in X^* : f(\Gamma) \subset \mathbb{Z}\}, \qquad \Gamma^{\vee *} = \{f \in X^* : f(\Gamma) \subset \mathbb{Z}\}.
$$

Clearly, Γ and $\Gamma^\vee$ are free, with basis $(\varepsilon_i)_{i\in I}$ and $(e_i)_{i\in I}$, respectively. Likewise, Γ_0, Γ_2 and $\Gamma_0^\vee$, $\Gamma_2^\vee$ are free. Indeed, fix an element $0 \in I$. Then it is easily seen that

$$
\Gamma_0 = \bigoplus_{i\in I\setminus\{0\}} \mathbb{Z}(\varepsilon_i - \varepsilon_0), \qquad \Gamma_2 = \bigoplus_{i\in I} \mathbb{Z}(\varepsilon_i + \varepsilon_0), \tag{1}
$$

and analogous formulas hold for $\Gamma_0^\vee$ and $\Gamma_2^\vee$. These $\mathbb{Z}$-bases are vector space bases of $\dot{X}$, X, and $\dot{X}^\vee$, $X^\vee$, respectively. Hence there are natural isomorphisms

$$
\Gamma^* \cong \operatorname{Hom}(\Gamma, \mathbb{Z}) \cong \mathbb{Z}^I, \qquad \Gamma_n^* \cong \operatorname{Hom}(\Gamma_n, \mathbb{Z}) \quad (n = 0, 2), \tag{2}
$$

$$
\Gamma^{\vee *} \cong \operatorname{Hom}(\Gamma^\vee, \mathbb{Z}) \cong \mathbb{Z}^I, \qquad \Gamma_n^{\vee *} \cong \operatorname{Hom}(\Gamma_n^\vee, \mathbb{Z}) \quad (n = 0, 2), \tag{3}
$$

given by restricting a linear form on X, $\dot{X}$, $X^\vee$ or $\dot{X}^\vee$ to the respective subgroups Γ, Γ_n, $\Gamma^\vee$ or $\Gamma_n^\vee$. We also have

$$
\begin{aligned}
\Gamma &= \mathbb{Z}\cdot\varepsilon_0 + \Gamma_2, & \Gamma_2^* &= \mathbb{Z}\cdot\frac{t}{2} + \Gamma^*,\\
\Gamma^\vee &= \mathbb{Z}\cdot e_0 + \Gamma_2^\vee, & \Gamma_2^{\vee *} &= \mathbb{Z}\cdot\frac{t^\vee}{2} + \Gamma^{\vee *},
\end{aligned}
$$

and therefore, denoting by $\mathbb{Z}_n$ the cyclic group of order n,

$$
\Gamma/\Gamma_2 \cong \Gamma_2^*/\Gamma^* \cong \Gamma^\vee/\Gamma_2^\vee \cong \Gamma_2^{\vee *}/\Gamma^{\vee *} \cong \mathbb{Z}_2, \tag{4}
$$

the nontrivial element of the quotient being represented by ε_0, $t/2$, e_0 and $t^\vee/2$, respectively. Now the various weight and coweight groups for an infinite I are given in the following table. Here

$$\mathcal{P}^\vee_{\mathrm{fin}}(R) = \mathcal{P}_{\mathrm{fin}}(R^\vee), \qquad \Theta^\vee(R) = \Theta(R^\vee) = \mathcal{P}_{\mathrm{fin}}(R^\vee)/\mathcal{Q}^\vee(R),$$
$$\mathcal{P}^\vee_{\mathrm{cof}}(R) = \mathcal{P}_{\mathrm{cof}}(R^\vee), \qquad \Theta^{\vee *}(R) = \Theta^*(R^\vee).$$

R	$\dot{\mathrm{A}}_I$	B_I	C_I	BC_I	D_I
$\mathcal{Q}(R)$	Γ_0	Γ	Γ_2	Γ	Γ_2
$\mathcal{P}_{\mathrm{fin}}(R)$	Γ_0	Γ	Γ	Γ	Γ
$\Theta(R)$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
$\mathcal{P}_{\mathrm{cof}}(R)$	$\Gamma_0^{\vee *}$	$\Gamma^{\vee *}$	$\Gamma^{\vee *}$	$\Gamma^{\vee *}$	$\Gamma^{\vee *}$
$\mathcal{P}(R)$	$\Gamma_0^{\vee *}$	$\Gamma_2^{\vee *}$	$\Gamma^{\vee *}$	$\Gamma^{\vee *}$	$\Gamma_2^{\vee *}$
$\Theta^*(R)$	0	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$
$\mathcal{Q}^\vee(R)$	$\Gamma_0^\vee$	$\Gamma_2^\vee$	$\Gamma^\vee$	$\Gamma^\vee$	$\Gamma_2^\vee$
$\mathcal{P}^\vee_{\mathrm{fin}}(R)$	$\Gamma_0^\vee$	$\Gamma^\vee$	$\Gamma^\vee$	$\Gamma^\vee$	$\Gamma^\vee$
$\Theta^\vee(R)$	0	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$
$\mathcal{P}^\vee_{\mathrm{cof}}(R)$	Γ_0^*	Γ^*	Γ^*	Γ^*	Γ^*
$\mathcal{P}^\vee(R)$	Γ_0^*	Γ^*	Γ_2^*	Γ^*	Γ_2^*
$\Theta^{\vee *}(R)$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$

In each case, a coweight is bounded if and only if it is bounded on the bases $\{\varepsilon_i : i \in I\}$ and $\{\varepsilon_i \pm \varepsilon_0 : i \neq 0\}$ of Γ, Γ_0 and Γ_2, respectively, and similarly for weights.

We recall that, by definition, the groups $\Theta(R)$, $\Theta^*(R)$, $\Theta^\vee(R)$ and $\Theta^{\vee *}(R)$ are the quotients of the groups in the preceding two rows. Moreover, keeping in mind the various definitions of the weight groups, cf. 7.1 and 7.3, and the isomorphisms 7.5.1, 7.5.2 and 7.5.3, one sees that only $\mathcal{Q}(R)$ and $\mathcal{P}_{\mathrm{fin}}(R)$ have to be determined. The proofs are largely straightforward and left to the reader. We indicate the case $R = \mathrm{D}_I$; the other cases are similar (and simpler). Clearly $\mathrm{D}_I \subset \Gamma_2$ and therefore $\mathcal{Q}(\mathrm{D}_I) \subset \Gamma_2$. For the reverse inclusion, it suffices by the second formula of (1) to show that $2\varepsilon_0 \in \mathcal{Q}(\mathrm{D}_I)$. Choose an element $1 \in I \setminus \{0\}$. Then we have $2\varepsilon_0 = (\varepsilon_0 + \varepsilon_1) + (\varepsilon_0 - \varepsilon_1) \in \mathcal{Q}(\mathrm{D}_I)$.

It is easily seen that $\Gamma \subset \mathcal{P}_{\mathrm{fin}}(\mathrm{D}_I)$. Conversely let $x = \sum x_i\varepsilon_i \in \mathcal{P}_{\mathrm{fin}}(\mathrm{D}_I)$, and, say, $x_i = 0$ for $i \notin F$ where $F \subset I$ is finite. Choosing $k \in I \setminus F$ (which is always possible because I is infinite), we have, for all $j \in F$, that $\langle x, (\varepsilon_j - \varepsilon_k)^\vee\rangle = \langle x, e_j - e_k\rangle = \sum_{i\in F} x_i\langle \varepsilon_i, e_j - e_k\rangle\rangle = x_j \in \mathbb{Z}$, whence $x \in \Gamma$.

Thus $\mathcal{P}_{\mathrm{fin}}(\mathrm{D}_I) = \Gamma$ and $\Theta(\mathrm{D}_I) \cong \mathbb{Z}_2$ by (4). The remaining weight and coweight groups of D_I follow easily from (2), (3) and the description of $\mathrm{D}_I^\vee$ in 8.1.12.

Remark. We note that the homomorphism $i''\colon \Theta(R) \to \Theta^*(R)$ of 7.3.6 is zero for all the infinite irreducible root systems. Indeed, by the table, only the case $R = \mathrm{D}_I$ needs to be checked, and here it is given on the nontrivial element of $\Theta(\mathrm{D}_I)$ by $i''(\varepsilon_0 + \Gamma_2) = e_0 + \Gamma^{\vee *} = \Gamma^{\vee *}$.

For comparison purposes, we also list the weight groups of the finite classical root systems, see [**12**, VI, Planches] for details. Here v is the vector $\dfrac{1}{n+1}\displaystyle\sum_{i=1}^{n}(\varepsilon_i - \varepsilon_0)$. By finiteness, $\Gamma^{\vee *} = \Gamma$ and hence $\Gamma_2^{\vee *} = \mathbb{Z}\cdot(t^\vee/2) + \Gamma$, where $t^\vee = \sum_{i=1}^n \varepsilon_i$.

R	A_n	B_n	C_n	BC_n	D_n
$\mathcal{Q}(R)$	Γ_0	Γ	Γ_2	Γ	Γ_2
$\mathcal{P}(R)$	$\mathbb{Z}v + \Gamma_0$	$\Gamma_2^{\vee *}$	Γ	Γ	$\Gamma_2^{\vee *}$
$\Theta(R)$	$\mathbb{Z}_{n+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \times \mathbb{Z}_2$ if n is even $\mathbb{Z}_4$ if n is odd

As a consequence of these computations, we have the following improvements of 7.5(a):

8.8. Corollary. *Let R be a locally finite root system. Then $\Theta(R)$ is a direct sum and $\Theta^*(R)$ a direct product of finite abelian groups.*

Indeed, this is well-known in the finite case and holds by 8.7 for infinite irreducible R. The general case then follows from 7.3.7 and 7.3.8.

8.9. *Notation.* We now work out the basic weights and coweights for the classical root systems $R = \mathrm{T}_I$ of 8.1, where the index set may be finite or infinite. We keep the notations of 8.7 and also put

$$E := \{\varepsilon_i : i \in I\}.$$

For a subset $J \subset I$ we let $X_J \subset X$ and $X_J^\vee \subset X^\vee$ be the subspaces spanned by $\{\varepsilon_j : j \in J\}$ and $\{e_j : j \in J\}$, respectively. We also define linear forms $q_J \in X^*$ and $p_J \in X^{\vee *}$ by

$$\langle \varepsilon_i, q_J\rangle = \langle e_i, p_J\rangle = \chi_J(i) = \begin{Bmatrix} 1 & \text{if } i \in J \\ 0 & \text{otherwise} \end{Bmatrix}.$$

In particular, the trace and cotrace are

$$t = q_I, \qquad t^\vee = p_I.$$

If $\mathrm{T} \in \{\dot{\mathrm{A}}, \mathrm{B}, \mathrm{BC}, \mathrm{C}, \mathrm{D}\}$ is one of the types of root systems, we define

$$\mathrm{T}_J = \mathrm{T}_I \cap X_J.$$

Then T_J is a root system in X_J except when $\mathrm{T} = \dot{\mathrm{A}}$ where it is a root system in $\dot{X}_J = X_J \cap \dot{X}$. For example, $\dot{\mathrm{A}}_J = \{\varepsilon_i - \varepsilon_j : i, j \in J\}$, and $\mathrm{C}_J = \{\pm\varepsilon_i \pm \varepsilon_j,\ i, j \in J\}$. Clearly, T_J and $\mathrm{T}_{J'}$ are orthogonal for disjoint subsets J and J' of I.

Let $f \in X^*$ and define, for any $c \in \mathbb{R}$,

$$I_c := I_c(f) := \{i \in I : \langle \varepsilon_i, f\rangle = c\}.$$

Let $J \subset I$ be an arbitrary subset. The *sign change defined by* J is the linear transformation σ_J of X mapping ε_j to $-\varepsilon_j$ for $j \in J$ and fixing ε_i for $i \in I \setminus J$. It is immediate that the group $\mathbf{2}^I$ of sign changes acts by automorphisms of R unless R is of type $\dot{\mathrm{A}}$. (In fact, we will see in §9 that $\mathbf{2}^I \subset \overline{W}(R)$ if I is infinite and R is not of type $\dot{\mathrm{A}}$.) Hence it is no restriction to assume in these cases that f has non-negative values on E, possibly after replacing f with a suitable $f^\sigma := f \circ \sigma$, $\sigma \in \mathbf{2}^I$.

The restriction map $f \mapsto \dot{f} := f\big|\dot{X}$ induces an isomorphism between $X^*/\mathbb{R}\cdot t$ and the linear forms on $\dot{X}$. We now determine $R_0(f)$ (cf. 7.10.2) for the root systems $R = \mathrm{T}_I$, $\mathrm{T} \neq \dot{\mathrm{A}}$, and $R_0(\dot{f})$ in case $R = \dot{\mathrm{A}}_I$. This will quickly lead to a description of the basic and minuscule weights and coweights. Recall the definition of the rank of a linear form with respect to a root system from 7.10.

8.10. Lemma. *We use the notations introduced in* 8.9 *and consider an element* $f \in X^*$.

(a) *If* $R = \dot{\mathrm{A}}_I$, *we have*

$$R_0(\dot{f}) = \bigoplus_{c \in f(E)} \dot{\mathrm{A}}_{I_c(f)}, \tag{1}$$

$$\operatorname{rank}(\dot{f}) + 1 = \operatorname{Card}(f(E)). \tag{2}$$

(b) *If* $R = \mathrm{T}_I \in \{\mathrm{B}_I, \mathrm{BC}_I, \mathrm{C}_I, \mathrm{D}_I\}$ *and* f *has non-negative values on* E *then*

$$R_0(f) = \mathrm{T}_{I_0(f)} \oplus \bigoplus_{c \in f(E)\setminus\{0\}} \dot{\mathrm{A}}_{I_c(f)}, \tag{3}$$

$$\operatorname{rank}(f) = \left\{ \begin{array}{ll} \operatorname{Card}(f(E) \setminus \{0\}) + 1 & \text{if } R = \mathrm{D}_I \text{ and } |I_0(f)| = 1 \\ \operatorname{Card}(f(E) \setminus \{0\}) & \text{otherwise} \end{array} \right\}. \tag{4}$$

Proof. (a) Equation (1) follows easily from the definitions. Concerning (2), note that the $I_c = I_c(f)$ (for $c \in f(E)$) are disjoint non-empty subsets of I, and $X_J/\operatorname{span}(\dot{\mathrm{A}}_J)$ is one-dimensional, for any non-empty subset J of I.

(b) The inclusion from right to left in (3) is clear from the definitions. Conversely, let $\alpha \in R_0(f)$. If α is a multiple of some ε_i then $\langle \varepsilon_i, f\rangle = 0$ so $i \in I_0$ and $\alpha \in \mathrm{T}_{I_0}$. If $\alpha = \pm(\varepsilon_i + \varepsilon_j)$ for $i \neq j$ then $\langle \alpha, f\rangle = 0$ implies $\langle \varepsilon_i, f\rangle = \langle \varepsilon_j, f\rangle = 0$ since f is non-negative on E, so $\alpha \in \mathrm{T}_{I_0}$. Finally, if $\alpha = \varepsilon_i - \varepsilon_j$ for $i \neq j$ then $\langle \varepsilon_i, f\rangle = \langle \varepsilon_j, f\rangle = c$ whence $i, j \in I_c$ for some $c \geqslant 0$, and thus $\alpha \in \dot{\mathrm{A}}_{I_c}$. Since $\dot{\mathrm{A}}_{I_0} \subset \mathrm{T}_{I_0}$, the assertion follows. Now the formula for rank(f) follows as in case (a). The exceptional first case in (4) is due to the fact that D_J has rank zero if $|J| = 1$ but rank $|J|$ otherwise.

8.11. COROLLARY. *Let (R, X) be an irreducible root system and $f \in X^*$. Then* $\operatorname{rank}(f) \leqslant \operatorname{Card}(\mathbb{R})$, *and even* $\operatorname{rank}(f) \leqslant \aleph_0$ *in case f is a coweight.*

8.12. *Basic weights and coweights of classical root systems.* We now determine the basic and minuscule coweights of the classical root systems $R = \mathrm{T}_I$ listed in 8.1. The basic and minuscule weights are then the basic and minuscule coweights of the coroot system $R^\vee$. The results are listed in the following tables.

R	basic weights	basic coweights
$\dot{\mathrm{A}}_I$	$\dot{p}_J,\ \emptyset \neq J \subsetneqq I$	$\dot{q}_J,\ \emptyset \neq J \subsetneqq I$
B_I	$p^\sigma_J,\ \emptyset \neq J \subsetneqq I;\quad p^\sigma_I/2$	$q^\sigma_J,\ \emptyset \neq J \subsetneqq I$
C_I	$p^\sigma_J,\ \emptyset \neq J \subset I$	$q^\sigma_J,\ \emptyset \neq J \neq I;\quad q^\sigma_I/2$
BC_I	$p^\sigma_J,\ \emptyset \neq J \subset I$	$q^\sigma_J,\ \emptyset \neq J \subset I$
D_I	$p^\sigma_J,\ J \neq \emptyset,\ \lvert I \setminus J\rvert \geqslant 2;\quad p^\sigma_I/2$	$q^\sigma_J,\ J \neq \emptyset,\ \lvert I \setminus J\rvert \geqslant 2;\quad q^\sigma_I/2$

R	minuscule weights	minuscule coweights
$\dot{\mathrm{A}}_I$	all	all
B_I	$p^\sigma_I/2$	$q^\sigma_J,\ \lvert J\rvert = 1$
C_I	$p^\sigma_J,\ \lvert J\rvert = 1$	$q^\sigma_I/2$
BC_I	none	none
D_I	$p^\sigma_J,\ \lvert J\rvert = 1;\quad p^\sigma_I/2$	$q^\sigma_J,\ \lvert J\rvert = 1;\quad q^\sigma_I/2$

We use the notations of 8.9 and discuss the cases of Lemma 8.10.

(a) Let $R = \dot{\mathrm{A}}_I$. By 8.10.2, $\dot{f}$ has rank one if and only if f has exactly two values on E. Replacing f by $f + cq_I$ just amounts to shifting $f(E)$ by c and doesn't change $\dot{f}$. Thus we may assume that $f(E) = \{0, a\}$ for some $a > 0$, and then have $\dot{f} = a\dot{q}_J$ for $J = \{i \in I : \langle \varepsilon_i, f\rangle = a\}$, where $\emptyset \neq J \neq I$. The set of values of f on R is $\{-a, 0, a\}$. Hence the basic coweights of $\dot{\mathrm{A}}_I$ are precisely the linear forms $\dot{q}_J$ where $\emptyset \neq J \neq I$, and they are all minuscule.

(b) Let R and f be as in 8.10(b), in particular, f is non-negative on E. If $R \neq \mathrm{D}_I$, then f has rank one if and only if f has exactly one non-zero value, say a, on E, so $f = aq_J$ for a non-empty subset J of I. Now let $R = \mathrm{D}_I$. Then, as I has at least two elements, we have $\operatorname{rank}(f) \geqslant 2$ in the exceptional case of 8.10.4. Hence f has rank one relative to D_I if and only if $f = aq_J$ for some non-empty subset J of I with $\lvert I \setminus J\rvert \geqslant 2$.

The list of basic weights is obtained from the determination of the coroot systems $\mathrm{T}_I^\vee$ in 8.1 and the fact that passing from R to $R^\vee$ switches weights and coweights. The p_J are defined in 8.9, and the notation $\dot{p}_J$ and $\dot{q}_J$ indicates the restriction of the linear form p_J and q_J to the subspace $\dot{X}^\vee$ and $\dot{X}$, respectively. Also $\sigma \in \mathbf{2}^I$ denotes an arbitrary sign change, and $f^\sigma = f \circ \sigma$. We assume R irreducible and hence $|I| \geqslant 3$ for type D_I.

(c) The minuscule (co)weights are easily determined from the structure of R. As before, $|I| \geqslant 3$ for type D_I.

§9. More on Weyl groups and automorphism groups

9.1. *The group* $\mathrm{O}(\Gamma)$. In this section, we will study in more detail the Weyl groups and automorphism groups of the irreducible infinite root systems classified in 8.4. We keep the notations introduced in 8.1 and 8.7 but will assume I infinite (see, however, 9.5 for a discussion of automorphism groups including the finite case). Thus, $X = \mathbb{R}^{(I)}$ denotes the free vector space on an infinite set I, with basis $\{\varepsilon_i : i \in I\}$, and $\Gamma = \mathbb{Z}^{(I)} = \bigoplus_{i\in I} \mathbb{Z}\varepsilon_i$ the subgroup generated by this basis. We let $\mathrm{O}(X)$ be the orthogonal group of X with respect to the inner product given by $(\varepsilon_i|\varepsilon_j) = \delta_{ij}$. Also, let

$$\mathrm{O}(\Gamma) = \{f \in \mathrm{O}(X) : f(\Gamma) = \Gamma\} = \mathrm{Stab}_{\mathrm{O}(X)}(\Gamma),$$

the stabilizer of Γ in $\mathrm{O}(X)$.

Every permutation $\pi \in \mathrm{Sym}(I)$, the symmetric group of I, induces an orthogonal transformation of X, also denoted π and given by

$$\pi(\varepsilon_i) = \varepsilon_{\pi(i)}\,. \tag{1}$$

As in 8.9, we denote by $\mathbf{2}^I$ the group of all sign changes $\varepsilon_i \mapsto \sigma(i)\varepsilon_i$, $\sigma \in \{\pm1\}^I$. This notation is consistent with the interpretation of $\mathbf{2}^I$ as the power set of I, if we identify a subset J of I with the element σ_J of $\mathrm{O}(\Gamma)$ mapping ε_i to $(1-2\chi_J(i))\varepsilon_i$, i.e.,

$$\sigma_J(\varepsilon_i) = \left\{ \begin{array}{ll} -\varepsilon_i & \text{if } i \in J \\ \varepsilon_i & \text{if } i \notin J \end{array} \right\}. \tag{2}$$

Note that then $\sigma_J\sigma_K = \sigma_{J\cdot K}$ where $J\cdot K = (J\cup K)\setminus(J\cap K)$ denotes the symmetric difference of the subsets J and K of I. Clearly $\sigma_\emptyset = \mathrm{Id}$ while $\sigma_I = -\mathrm{Id}$.

If $f \in \mathrm{O}(\Gamma)$ then $f(\varepsilon_i)$ must be a finite integral linear combination of the ε_j of length one, hence of the form $\sigma(i)\varepsilon_{\pi(i)}$, where $\sigma\colon I \to \{\pm1\}$ and $\pi \in \mathrm{Sym}(I)$. It is easy to see that in this way

$$\mathrm{O}(\Gamma) \cong \mathrm{Sym}(I) \ltimes \mathbf{2}^I \tag{3}$$

(semidirect product), with $\mathrm{Sym}(I)$ acting on the right on $\mathbf{2}^I$ via $\sigma^\pi(i) = \sigma(\pi(i))$, and group multiplication $(\pi,\sigma)\cdot(\pi',\sigma') = (\pi\pi', \sigma^{\pi'}\sigma')$. Following a well-established terminology in the finite case, we call this group the *hyperoctahedral group on the set* I. We frequently treat (3) as an identification, and denote by $\mathrm{per}(f) = \pi \in \mathrm{Sym}(I)$ the *permutation part* of an element $f = (\pi,\sigma) \in \mathrm{O}(\Gamma)$. Thus per: $\mathrm{O}(\Gamma) \to \mathrm{Sym}(I)$ is surjective with kernel isomorphic to $\mathbf{2}^I$, and the sequence

$$1 \longrightarrow \mathbf{2}^I \longrightarrow \mathrm{O}(\Gamma) \xrightarrow{\mathrm{per}} \mathrm{Sym}(I) \longrightarrow 1$$

is exact and split.

Let $\mathbf{c}$ be an infinite cardinal, and recall from 5.4 the normal subgroup $\mathrm{GL}(X,\mathbf{c}) \subset \mathrm{GL}(X)$. We define

$$\mathrm{O}(\Gamma, \mathbf{c}) := \mathrm{O}(\Gamma) \cap \mathrm{GL}(X, \mathbf{c}),$$

in particular,

$$\mathrm{O}_{\mathrm{fin}}(\Gamma) := \mathrm{O}(\Gamma) \cap \mathrm{GL}(X, \aleph_0),$$

called the *finitary hyperoctahedral group.*

Next, let $\dot{X} \subset X$ and $\Gamma_0 = \Gamma \cap \dot{X}$ as in 8.7. Similarly as before, we define

$$\mathrm{O}(\Gamma_0) = \{g \in \mathrm{O}(\dot{X}) : g(\Gamma_0) = \Gamma_0\}, \quad \mathrm{O}(\Gamma_0, \mathbf{c}) = \mathrm{O}(\Gamma_0) \cap \mathrm{GL}(\dot{X}, \mathbf{c}).$$

Let $f = (\pi, \sigma) \in \mathrm{O}(\Gamma)$. It is easily seen that f stabilizes Γ_0 if and only if σ is constant, equal to 1 or to -1, whence

$$\mathrm{Stab}_{\mathrm{O}(\Gamma)}(\Gamma_0) = \mathrm{Sym}(I) \times \{\mathrm{Id}, -\mathrm{Id}\}. \tag{4}$$

The *support* of a permutation π of I is $\mathrm{supp}(\pi) = \{i \in I : \pi(i) \neq i\}$. We denote by $\mathrm{Sym}(I, \mathbf{c})$ the set of permutations π with $|\mathrm{supp}(\pi)| < \mathbf{c}$. We abbreviate $\mathfrak{S} := \mathfrak{S}_I := \mathrm{Sym}(I, \aleph_0)$, and call its elements *finitary permutations.* The support of an element $\sigma = \sigma_J \in \mathbf{2}^I$ is defined as $\mathrm{supp}(\sigma) = J$. We let $\mathbf{2}^{(I,\mathbf{c})}$ be the subgroup of $\mathbf{2}^I$ consisting of all σ with $|\mathrm{supp}(\sigma)| < \mathbf{c}$, and denote by $\mathbf{2}^{(I)} := \mathbf{2}^{(I,\aleph_0)}$ the group of finitary sign changes σ_F, $F \subset I$ finite.

9.2. *Signed cycle types.* Let $f = (\pi, \sigma) \in \mathrm{O}(\Gamma)$, and let $\mathfrak{Z}$ be the set of cycles of π, i.e., the set of orbits of the subgroup of $\mathrm{Sym}(I)$ generated by π. For every $K \in \mathfrak{Z}$ let $X_K = \bigoplus_{k\in K} \mathbb{R}\varepsilon_k$. Then $X = \bigoplus_{K\in\mathfrak{Z}} X_K$, and each subspace X_K is invariant under π and σ and hence under f. Let $f_K := f\big|X_K$, choose an element $k_0 \in K$ and let $e_i := f^i(\varepsilon_{k_0})$. Then the e_i, $i \in \mathbb{Z}$, span X_K and f_K acts via the shift $e_i \mapsto e_{i+1}$. There are two cases: If K is infinite, the e_i form a basis of X_K. If K is finite with n elements then $e_1, \ldots, e_n$ is a basis of X_K, and the matrix of f_K relative to this basis is

$$\begin{pmatrix} 0 & 0 & \ldots & \eta(f_K) \\ 1 & 0 & \ldots & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix}, \quad \text{where} \quad \eta(f_K) := \prod_{k\in K} \sigma(k) = (-1)^{n-1} \det f_K. \tag{1}$$

We say K is a *positive* or *negative cycle* of f according to whether $\eta(f_K)$ is $+1$ or -1. Let $\mathbf{a}_0$ be the number of infinite cycles and $\mathbf{a}_{\pm n}$ the number of positive or negative cycles of finite length n. The sequence $(\mathbf{a}_n)_{n\in\mathbb{Z}}$ of cardinal numbers is called the *signed cycle type of* f. It is easy to see that two elements of $\mathrm{O}(\Gamma)$ are conjugate if and only if they have the same signed cycle type, see also [**16**, p. 25] in the finite case. Moreover, any sequence $(\mathbf{a}_n)$ of cardinal numbers with $\dim X = |I| = \aleph_0 \mathbf{a}_0 + \sum_{n\geqslant 1} n(\mathbf{a}_{-n} + \mathbf{a}_n)$ occurs as the signed cycle type of some $f \in \mathrm{O}(\Gamma)$.

9.3. Proposition. (a) *With the above notations,*

$$\mathrm{O}(\Gamma, \mathbf{c}) = \mathrm{Sym}(I, \mathbf{c}) \ltimes \mathbf{2}^{(I,\mathbf{c})}. \tag{1}$$

(b) *Every $g \in \mathrm{O}(\Gamma_0, \mathbf{c})$ is of the form*

$$g(\varepsilon_i - \varepsilon_j) = \tau(\varepsilon_{\pi(i)} - \varepsilon_{\pi(j)}) \tag{2}$$

for a unique $\pi \in \mathrm{Sym}(I, \mathbf{c})$ *and a unique sign* $\tau \in \{\pm 1\}$*, with* $\tau = 1$ *for* $\mathbf{c} \leqslant |I|$*. The transformation* $f = (\pi, \tau\mathrm{Id}) \in \mathrm{O}(\Gamma, \mathbf{c})$ *is the unique extension of* g *as in* (2) *to a map in* $\mathrm{O}(\Gamma, \mathbf{c})$*. Hence,*

$$\mathrm{O}(\Gamma_0, \mathbf{c}) \cong \mathrm{Stab}_{\mathrm{O}(\Gamma,\mathbf{c})}(\Gamma_0) = \left\{ \begin{array}{ll} \mathrm{Sym}(I, \mathbf{c}) & \text{if } \mathbf{c} \leqslant |I| \\ \mathrm{Sym}(I) \times \{\pm \mathrm{Id}\} & \text{if } \mathbf{c} > |I| \end{array} \right\}. \tag{3}$$

Proof. (a) Let $f = \sigma \in \mathrm{O}(\Gamma)$, and decompose X into the subspaces X_K parameterized by the cycles of π as in 9.2. If K is an infinite cycle then f_K is the shift $e_i \mapsto e_{i+1}$ and therefore has no nonzero fixed points. If K is a finite cycle of f then 9.2.1 shows that f_K has fixed point set $\mathbb{R}(e_1 + \cdots + e_n)$ or $\{0\}$, depending on whether K is positive or negative. Hence

$$\begin{aligned} \big|\mathrm{supp}(\pi)\big| &= \aleph_0 \mathbf{a}_0 + \sum_{n \geqslant 2} n(\mathbf{a}_{-n} + \mathbf{a}_n), \\ \mathrm{codim}\, X^f &= \aleph_0 \mathbf{a}_0 + \sum_{n \geqslant 1} n\mathbf{a}_{-n} + \sum_{n \geqslant 2} (n-1)\mathbf{a}_n, \end{aligned}$$

in terms of the signed cycle type of f, from which we obtain the estimates

$$\mathrm{codim}\, X^f \leqslant \big|\mathrm{supp}(\pi)\big| + \mathbf{a}_{-1} \leqslant \big|\mathrm{supp}(\pi)\big| + \big|\mathrm{supp}(\sigma)\big|, \tag{4}$$

$$\big|\mathrm{supp}(\pi)\big| \leqslant 2 \cdot \mathrm{codim}\, X^f. \tag{5}$$

Since $\mathbf{c}$ is an infinite cardinal, $\mathbf{a} < \mathbf{c}$ and $\mathbf{b} < \mathbf{c}$ for cardinals $\mathbf{a}$ and $\mathbf{b}$ imply $\mathbf{a}+\mathbf{b} < \mathbf{c}$. Hence (4) shows that $\mathrm{Sym}(I, \mathbf{c}) \ltimes \mathbf{2}^{(I,\mathbf{c})} \subset \mathrm{O}(\Gamma, \mathbf{c})$. Conversely, let $f = \pi\sigma \in \mathrm{O}(\Gamma, \mathbf{c})$. Then (5) implies $\pi \in \mathrm{Sym}(I, \mathbf{c}) \subset \mathrm{O}(\Gamma, \mathbf{c})$, whence also $\sigma = \pi^{-1} f \in \mathrm{O}(\Gamma, \mathbf{c})$. Since $X/X^\sigma \cong X^{-\sigma}$, the (-1)-eigenspace of σ, which has basis $\{\varepsilon_i : i \in \mathrm{supp}(\sigma)\}$, it follows that $|\mathrm{supp}(\sigma)| < \mathbf{c}$, so $\sigma \in \mathbf{2}^{(I,\mathbf{c})}$.

(b) We pick an element $0 \in I$, let $I' := I \setminus \{0\}$ and consider the $\mathbb{Z}$-basis $\alpha_i = \varepsilon_0 - \varepsilon_i$ $(i \in I')$ of Γ_0. For $g \in \mathrm{O}(\Gamma_0)$ we have $g(\alpha_i) = \sum_{j \in I'} n_j \alpha_j$ where only finitely many of the integers n_j are different from zero. Since $(\alpha_j|\alpha_k) = 1 + \delta_{jk}$ and g is an orthogonal transformation, it follows that

$$2 = (\alpha_i|\alpha_i) = (g(\alpha_i)|g(\alpha_i)) = \sum_{j,k \in I'} (1 + \delta_{jk}) n_j n_k = \big(\sum_{j \in I'} n_j\big)^2 + \sum_{j \in I'} n_j^2.$$

This easily implies that either all n_j are zero except for one which has absolute value one, or exactly two of the n_j are non-zero, of absolute value one and of opposite sign. In the first case, $g(\alpha_i) = \pm\alpha_l$ while in the second, $g(\alpha_i) = \alpha_l - \alpha_m = \varepsilon_m - \varepsilon_l$. Hence in any case we see that

$$g(\alpha_i) = \varepsilon_{\varphi(i)} - \varepsilon_{\psi(i)}$$

with maps $\varphi, \psi \colon I' \to I$. We claim that either φ or ψ must be constant. Indeed, first note that

$$\operatorname{Card}\big(\{\varphi(i),\psi(i)\}\cap\{\varphi(j),\psi(j)\}\big)=1 \quad \text{for } i\neq j \text{ in } I'.$$

This is a consequence of $1=(\alpha_i|\alpha_j)=(\varepsilon_{\varphi(i)}-\varepsilon_{\psi(i)}|\varepsilon_{\varphi(j)}-\varepsilon_{\psi(j)})$ and orthonormality of the ε_i. We show next that there exists an element $i_0\in I$ such that

$$\bigcap_{i\in I'}\{\varphi(i),\psi(i)\}=\{i_0\}.$$

Indeed, we pick four different elements in I' which we denote by $\{1,2,3,4\}\subset I'$, and set $E_i=\{\varphi(i),\psi(i)\}$. Let $E_1=\{i_0,i_1\}$ and $E_2=\{i_0,i_2\}$. Assume $i_0\notin E_3$, so that necessarily $E_3=\{i_1,i_2\}$. Then the condition $\operatorname{Card}(E_4\cap E_i)=1$ for $i=1,2,3$ implies a contradiction.

Suppose that neither φ nor ψ are constant equal to i_0. Then there would exist $i\neq j$ in I' such that $\varphi(i)=i'\neq i_0$ and $\psi(j)=j'\neq i_0$. This would imply

$$1=(\alpha_i|\alpha_j)=(g(\alpha_i)|g(\alpha_j))=(\varepsilon_{i'}-\varepsilon_{i_0}|\varepsilon_{i_0}-\varepsilon_{j'})=-\delta_{i'j'}-1,$$

contradiction. Thus either φ or ψ must be constant equal to i_0.

In the first case, we have $g(\alpha_i)=g(\varepsilon_0-\varepsilon_i)=\varepsilon_{i_0}-\varepsilon_{\psi(i)}$ where $\psi\colon I'\to I\setminus\{i_0\}$ is bijective. We define $\pi\in\operatorname{Sym}(I)$ by $\pi(0)=i_0$ and $\pi|I'=\psi$. Then g satisfies (2) with $\tau=1$. Indeed, this is clear for $0\in\{i,j\}$ while for $0\notin\{i,j\}$ we have $g(\varepsilon_i-\varepsilon_j)=g((\varepsilon_0-\varepsilon_j)-(\varepsilon_0-\varepsilon_i))=(\varepsilon_0-\varepsilon_{\psi(j)})-(\varepsilon_0-\varepsilon_{\psi(i)})=\varepsilon_{\pi(i)}-\varepsilon_{\pi(j)}$. Taking into account 9.1.3, the transformation $f=(\pi,\mathrm{Id})\in\mathrm{O}(\mathit{\Gamma})$ is an obvious extension of g. In the second case, we have $g(\alpha_i)=\varepsilon_{\varphi(i)}-\varepsilon_{i_0}$, and replacing g by $-g$ reduces this case to the first one.

To prove uniqueness of the extension, suppose that $f=(\pi,\sigma)\in\operatorname{Sym}(I)\times\{\pm\mathrm{Id}\}$ acts like the identity on X_0. Since X is spanned by ε_0 and $\mathit{\Gamma}_0$, it suffices to show that $f(\varepsilon_0)=\varepsilon_0$. We have $(\varepsilon_0|\alpha_i)=1$ and hence also

$$(f(\varepsilon_0)|f(\alpha_i))=(\sigma\varepsilon_{\pi(0)}|\varepsilon_0-\varepsilon_i)=1, \tag{6}$$

for all $i\in I'$. Assume $\pi(0)\neq 0$. Choosing for i an element different from 0 and $\pi(0)$, (6) leads to the contradiction $0=1$. Thus $\pi(0)=0$, and then $\sigma=1$, again by (6).

For (3), let $g\in\mathrm{O}(\mathit{\Gamma}_0)$ and let $f\in\operatorname{Stab}_{\mathrm{O}(\mathit{\Gamma})}(\mathit{\Gamma}_0)$ be its unique extension to X. Since $\dot X$ has codimension one in X and the fixed point sets of f and g satisfy $\dot X^g=\dot X\cap X^f$, we deduce from the exact sequence

$$0\longrightarrow \dot X/\dot X^g\longrightarrow X/X^f\longrightarrow X/(\dot X+X^f)\longrightarrow 0$$

that $\operatorname{codim}\dot X^g\leqslant\operatorname{codim}X^f\leqslant 1+\operatorname{codim}\dot X^g$. Hence $g\in\mathrm{GL}(\dot X,\mathbf{c})$ if and only $f\in\mathrm{GL}(X,\mathbf{c})$, and thus (3) follows from (1) and 9.1.4.

9.4. *Characters of* $\mathrm{O}_{\mathrm{fin}}(\mathit{\Gamma})$. By 9.3(a), we have $\mathrm{O}_{\mathrm{fin}}(\mathit{\Gamma})=\mathrm{O}(\mathit{\Gamma},\aleph_0)=\mathfrak{S}\ltimes\mathbf{2}^{(I)}$ where $\mathfrak{S}$ is the group of all finitary permutations of I and $\mathbf{2}^{(I)}$ the group of finitary sign changes as defined in 9.1. The sign $\operatorname{sgn}(\pi)\in\{\pm1\}$ of a finitary permutation π is a well-defined character on $\mathfrak{S}$, with kernel the *alternating group* $\mathfrak{A}=\mathfrak{A}_I$ of I. Since per: $\mathrm{O}_{\mathrm{fin}}(\mathit{\Gamma})\to\mathfrak{S}$ is a homomorphism, we thus have a character $\xi\colon\mathrm{O}_{\mathrm{fin}}(\mathit{\Gamma})\to\{\pm1\}$, given by $\xi(\pi\sigma_F)=\operatorname{sgn}(\pi)$.

There is a second character η on $\mathrm{O}_{\mathrm{fin}}(\Gamma)$ defined by $\eta(\pi\sigma_F) = (-1)^{|F|}$. Indeed, if also $\varrho\sigma_E \in \mathrm{O}_{\mathrm{fin}}(\Gamma)$, then we have $(\varrho\sigma_E)(\pi\sigma_F) = (\varrho\pi)\cdot\sigma_{\pi^{-1}(E)\cdot F}$, and one easily checks that $|\pi^{-1}(E)\cdot F| \equiv |\pi^{-1}(E)| + |F| \equiv |E| + |F| \bmod 2$, for finite subsets E and F of I.

Finally, every $f \in \mathrm{GL}_{\mathrm{fin}}(X) = \mathrm{GL}(X, \aleph_0)$ has a well-defined determinant $\det(f) = \det(\bar f)$, where $\bar f$ is the linear transformation induced by f on the finite-dimensional vector space X/X^f. Using the fact that $\det(f) = \det(\tilde f)$ where $\tilde f$ is the map induced by f on X/Y for any subspace $Y \subset X^f$ of finite codimension, it is easy to see that det is multiplicative on $\mathrm{GL}_{\mathrm{fin}}(X)$. For an element $f = \pi\sigma_F \in \mathrm{O}_{\mathrm{fin}}(\Gamma)$, one checks without difficulty that the determinant is related to ξ and η by

$$\det(f) = \mathrm{sgn}(\pi)\cdot(-1)^{|F|} = \xi(f)\cdot\eta(f).$$

(This could also be used to prove the existence of η). The kernels of these three characters are then normal subgroups of index 2 which we denote by

$$\mathrm{O}^{\mathrm{ev}}_{\mathrm{fin}}(\Gamma) := \mathrm{Ker}(\xi) = \mathfrak{A} \ltimes \mathbf{2}^{(I)}, \tag{1}$$
$$\mathrm{O}^{+}_{\mathrm{fin}}(\Gamma) := \mathrm{Ker}(\eta) = \mathfrak{S} \ltimes \mathbf{2}^{(I)}_{+}, \tag{2}$$
$$\mathrm{SO}_{\mathrm{fin}}(\Gamma) := \mathrm{Ker}(\det) = \big(\mathfrak{A} \ltimes \mathbf{2}^{(I)}_{+}\big) \cup \big((\mathfrak{S}\setminus\mathfrak{A}) \ltimes (\mathbf{2}^{(I)} \setminus \mathbf{2}^{(I)}_{+})\big). \tag{3}$$

Here $\mathbf{2}^{(I)}_{+} := \mathbf{2}^{(I)} \cap \mathrm{O}^{+}_{\mathrm{fin}}(\Gamma)$ denotes the subgroup of $\mathbf{2}^{(I)}$ consisting of all σ_E with $E \subset I$ finite and even. We finally note that $(\xi,\eta)\colon \mathrm{O}_{\mathrm{fin}}(\Gamma) \to \{\pm1\}\times\{\pm1\} \cong (\mathbb{Z}/2\mathbb{Z})^2$ is a surjective homomorphism with kernel $K := \mathfrak{A} \ltimes \mathbf{2}^{(I)}_{+} = \mathrm{SO}^{+}_{\mathrm{fin}}(\Gamma)$.

9.5. Theorem. *Let I be an infinite set of cardinality $\mathbf{d}$ and let R be one of the root systems listed in* 8.1*. We use the notations of* 5.4, 9.1 *and* 9.4*, and let $\mathbf{c}$ denote an infinite cardinal with $\mathbf{c} \leqslant \mathbf{d}^+$, the cardinal successor of $\mathbf{d}$.*

(a) *The automorphism groups $\mathrm{Aut}(R,\mathbf{c})$, the Weyl groups $W(R,\mathbf{c})$ and the outer automorphism groups $\mathrm{Out}(R,\mathbf{c})$ of R are as follows:*

$$\mathrm{Aut}(R,\mathbf{c}) = \left\{\begin{array}{ll} \mathrm{O}(\Gamma_0,\mathbf{c}) & \textit{if } R = \dot{\mathrm{A}}_I \\ \mathrm{O}(\Gamma,\mathbf{c}) & \textit{otherwise} \end{array}\right\}, \tag{1}$$

$$W(R,\mathbf{c}) = \left\{\begin{array}{ll} \mathrm{Sym}(I,\mathbf{c}) & \textit{if } R = \dot{\mathrm{A}}_I \\ \mathrm{O}^{+}_{\mathrm{fin}}(\Gamma) & \textit{if } R = \mathrm{D}_I \textit{ and } \mathbf{c} = \aleph_0 \\ \mathrm{O}(\Gamma,\mathbf{c}) & \textit{otherwise} \end{array}\right\}, \tag{2}$$

$$\mathrm{Out}(R,\mathbf{c}) = \left\{\begin{array}{ll} \mathbb{Z}/2\mathbb{Z} & \textit{if } R = \dot{\mathrm{A}}_I \textit{ and } \mathbf{c} = \mathbf{d}^+ \\ \mathbb{Z}/2\mathbb{Z} & \textit{if } R = \mathrm{D}_I \textit{ and } \mathbf{c} = \aleph_0 \\ \{1\} & \textit{otherwise} \end{array}\right\}. \tag{3}$$

(b) *Every element of the big Weyl group $\overline{W}(R)$ is the product of at most four generalized reflections if $R = \mathrm{D}_I$, and of at most two generalized reflections in the other cases; in particular, $\overline{W}(R) = W(R,\mathbf{c})$ for every infinite cardinal $\mathbf{c} > \dim X$.*

Theorem 9.5 shows that the interesting cardinalities are $\mathbf{c} = \aleph_0$ and $\mathbf{c} = \mathbf{d}^+$. We therefore list the results for these cases in the following table. The notations are as in 9.1 and 9.4.

R	$W(R)$	$\mathrm{Aut}_{\mathrm{fin}}(R)$	$\mathrm{Out}_{\mathrm{fin}}(R)$	$\overline{W}(R)$	$\mathrm{Aut}(R)$	$\mathrm{Out}(R)$
$\dot{\mathrm{A}}_I$	$\mathfrak{S}_I$	$\mathfrak{S}_I$	$\{1\}$	$\mathrm{Sym}(I)$	$\mathrm{Sym}(I) \times \{\pm\mathrm{Id}\}$	$\mathbb{Z}/2\mathbb{Z}$
$\mathrm{B}_I, \mathrm{C}_I$ BC_I	$\mathrm{O}_{\mathrm{fin}}(\Gamma)$	$\mathrm{O}_{\mathrm{fin}}(\Gamma)$	$\{1\}$	$\mathrm{O}(\Gamma)$	$\mathrm{O}(\Gamma)$	$\{1\}$
D_I	$\mathrm{O}^+_{\mathrm{fin}}(\Gamma)$	$\mathrm{O}_{\mathrm{fin}}(\Gamma)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathrm{O}(\Gamma)$	$\mathrm{O}(\Gamma)$	$\{1\}$

Taking into account the well known structure of the Weyl group and automorphism group of the finite classical root systems ([**12**, Planches]), we obtain the following summary, where now I may be finite.

R	$W(R)$	$\overline{W}(R)$	$\mathrm{Aut}(R)$
$\dot{\mathrm{A}}_I,\ \lvert I\rvert \geqslant 2$	$\mathfrak{S}_I$	$\mathrm{Sym}(I)$	$\mathrm{Sym}(I) \times \{\pm\mathrm{Id}\}$ for $\lvert I\rvert \geqslant 3$ $\mathrm{Sym}(I)$ for $\lvert I\rvert = 2$
$\mathrm{B}_I, \mathrm{C}_I$ $\mathrm{BC}_I,\ \lvert I\rvert \geqslant 2$	$\mathfrak{S}_I \ltimes \mathbf{2}^{(I)}$ $= \mathrm{O}_{\mathrm{fin}}(\Gamma)$	$\mathrm{Sym}(I) \ltimes \mathbf{2}^I$ $= \mathrm{O}(\Gamma)$	$\mathrm{Sym}(I) \ltimes \mathbf{2}^I = \mathrm{O}(I) = \overline{W}(R)$
$\mathrm{D}_I,\ \lvert I\rvert \geqslant 5$	$\mathfrak{S}_I \ltimes \mathbf{2}^{(I)}_+$ $= \mathrm{O}^+_{\mathrm{fin}}(\Gamma)$	$\mathrm{Sym}(I) \ltimes \mathbf{2}^I$ (for I infinite)	$\mathrm{Sym}(I) \ltimes \mathbf{2}^I = \mathrm{O}(\Gamma)$

It is remarkable that, with the exception of $\overline{W}(\mathrm{D}_I)$ for an infinite I, the structure of the groups $W(\mathrm{T}_I)$, $\overline{W}(\mathrm{T}_I)$ and $\mathrm{Aut}(\mathrm{T}_I)$ does not depend on the cardinality of I.

Proof. By 4.7, an element $f \in \mathrm{Aut}(R)$ is an orthogonal transformation, and by 7.3 it leaves the weight groups, in particular, the group $\mathcal{P}_{\mathrm{fin}}(R)$ of finite weights, invariant. By 8.7, $\mathcal{P}_{\mathrm{fin}}(\dot{\mathrm{A}}_I) = \Gamma_0$ while $\mathcal{P}_{\mathrm{fin}}(R) = \Gamma$ in the other cases. Thus $\mathrm{Aut}(\dot{\mathrm{A}}_I, \mathbf{c}) \subset \mathrm{O}(\Gamma_0, \mathbf{c})$ and $\mathrm{Aut}(R, \mathbf{c}) \subset \mathrm{O}(\Gamma, \mathbf{c})$ in the other cases. The reverse inclusions follow easily from Prop. 9.3. This proves (1).

Next, we consider the finitary Weyl groups. Simple computations show that the reflections in the roots of R are given as follows:

$$s_{\varepsilon_i - \varepsilon_j}(\varepsilon_k) = \left\{ \begin{array}{ll} \varepsilon_j & \text{for } k = i \\ \varepsilon_i & \text{for } k = j \\ \varepsilon_k & \text{otherwise} \end{array} \right\}, \tag{4}$$

$$s_{\varepsilon_i + \varepsilon_j}(\varepsilon_k) = \left\{ \begin{array}{ll} -\varepsilon_j & \text{for } k = i \\ -\varepsilon_i & \text{for } k = j \\ \varepsilon_k & \text{otherwise} \end{array} \right\}, \tag{5}$$

$$s_{\varepsilon_i}(\varepsilon_k) = s_{2\varepsilon_i}(\varepsilon_k) = \left\{ \begin{array}{ll} -\varepsilon_k & \text{for } k = i \\ \varepsilon_k & \text{for } k \neq i \end{array} \right\}. \tag{6}$$

Since $\mathfrak{S} = \mathrm{Sym}(I, \aleph_0)$ is generated by the transpositions and $\mathbf{2}^{(I)}$ by the single sign changes, these formulas together with Prop. 9.3 and 9.4.2 show that the finitary Weyl groups $W(R) = W(R, \aleph_0)$ are given by (2).

We now consider the root system $\dot{\mathrm{A}}_I$ and claim that $\overline{W}(\dot{\mathrm{A}}_I) \subset \mathrm{Sym}(I)$, identified with a subgroup of $\mathrm{Aut}(\dot{\mathrm{A}}_I) = \mathrm{O}(\Gamma_0)$ via 9.3.3. Indeed, suppose to the contrary that there exists a net $(w_\lambda)_{\lambda\in\Lambda}$ in $W(\dot{\mathrm{A}}_I) \cong \mathfrak{S}$ which converges to $-w = (\pi, -\mathrm{Id})$ where w is induced from a permutation π, and pick three different elements $0, 1, 2 \in I$. Then there exist $\lambda_j \in \Lambda$ $(j = 1, 2)$ such that $w_\lambda(\varepsilon_0 - \varepsilon_j) = -\varepsilon_{\pi(0)} + \varepsilon_{\pi(j)}$, for all $\lambda \succcurlyeq \lambda_j$. On the other hand, w_λ is induced from a permutation π_λ, so that $w_\lambda(\varepsilon_0 - \varepsilon_j) = \varepsilon_{\pi_\lambda(0)} - \varepsilon_{\pi_\lambda(j)}$. Hence $\pi(j) = \pi_\lambda(0)$ for $\lambda \succcurlyeq \lambda_j$, $j = 1, 2$. Since Λ is directed, there exists $\lambda_3 \succcurlyeq \lambda_1, \lambda_2$, so we obtain $\pi_{\lambda_3}(0) = \pi(1) = \pi(2)$, contradicting the fact that π is a permutation.

We show next

$$W(\dot{\mathrm{A}}_I, \mathbf{c}) = \mathrm{Sym}(I, \mathbf{c}). \tag{7}$$

From 9.3.3 we have $W(\dot{\mathrm{A}}_I, \mathbf{c}) \subset \mathrm{Aut}(\dot{\mathrm{A}}_I, \mathbf{c}) \cap \overline{W}(\dot{\mathrm{A}}_I) \subset \mathrm{O}(\Gamma_0, \mathbf{c}) \cap \mathrm{Sym}(I) = \mathrm{Sym}(I, \mathbf{c})$. For the proof of the other inclusion we use the fact that every permutation $\pi \in \mathrm{Sym}(I, \mathbf{c})$ is a product $\pi_1\pi_2$ where $\pi_j \in \mathrm{Sym}(I, \mathbf{c})$ satisfy $\pi_j^2 = \mathrm{Id}$ ([**24**, Lemma 8.1A]). Since π_j contains only 1-cycles and 2-cycles, we can divide the support of π_j into two disjoint subsets K_j and L_j each of which meets every 2-cycle in exactly one point, and then $\pi_j \colon K_j \to L_j$ is bijective. Then $\Omega_j = \{\varepsilon_k - \varepsilon_{\pi_j(k)} : k \in K_j\}$ is an orthogonal system in $\dot{\mathrm{A}}_I$, and it follows easily from 5.3.1 and (4) that $\pi_j = s_{\Omega_j} \in W(\dot{\mathrm{A}}_I, \mathbf{c})$, whence also $\pi \in W(\dot{\mathrm{A}}_I, \mathbf{c})$. This completes the proof of (7). Since $\overline{W}(\dot{\mathrm{A}}_I) \subset \mathrm{Sym}(I) = W(\dot{\mathrm{A}}_I, \mathbf{d}^+) \subset \overline{W}(\dot{\mathrm{A}}_I)$, we conclude $\overline{W}(\dot{\mathrm{A}}_I) = \mathrm{Sym}(I)$. In particular, by what we have shown above, every element of $\overline{W}(\dot{\mathrm{A}}_I)$ is the product of two generalized reflections.

We next consider the root system D_I and claim that

$$W(D_I, \mathbf{c}) = \mathrm{O}(\Gamma, \mathbf{c}) \quad \text{for} \quad \mathbf{c} > \aleph_0. \tag{8}$$

The inclusion from left to right is clear from (1). For the converse we use 9.3.1. As $\dot{\mathrm{A}}_I \subset \mathrm{D}_I$, we have $W(\dot{\mathrm{A}}_I, \mathbf{c}) = \mathrm{Sym}(I, \mathbf{c}) \subset W(\mathrm{D}_I, \mathbf{c})$, so it remains to show that $\mathbf{2}^{(I,\mathbf{c})} \subset W(\mathrm{D}_I, \mathbf{c})$. Let $\sigma = \sigma_J \in \mathbf{2}^{(I,\mathbf{c})}$ so that $J = \mathrm{supp}(\sigma)$, and suppose first that either J is infinite, or finite with an even number of elements. Then we can divide J into two disjoint equipotent subsets K and L, and choose a bijection $\varphi \colon K \to L$. Consider the orthogonal system $\tilde{\Omega} = \{\varepsilon_k \pm \varepsilon_{\varphi(k)} : k \in K\}$ of D_I. From 5.3.1, (4) and (5) one deduces easily that σ is the generalized reflection defined by the orthogonal system $\tilde{\Omega}$, and since $|\Omega| = |K| \leqslant |J| < \mathbf{c}$, we have $\sigma \in W(\mathrm{D}_I, \mathbf{c})$. Next, let J be finite with an odd number of elements. Since I is infinite, there exist countable subsets $M_1 \subset M_2$ of I with $M_2 \setminus M_1 = J$. Then σ_{M_j} has countable support M_j so $\sigma_{M_j} \in W(\mathrm{D}_I, \mathbf{c})$ by what we proved before, and therefore also $\sigma_{M_1}\sigma_{M_2} = \sigma_{M_1 \cdot M_2} = \sigma_{M_2 \setminus M_1} = \sigma_J \in W(\mathrm{D}_I, \mathbf{c})$. This completes the proof of (8). Since $\overline{W}(\mathrm{D}_I) \subset \mathrm{Aut}(\mathrm{D}_I) = W(\mathrm{D}_I, \mathbf{d}^+) \subset \overline{W}(\mathrm{D}_I)$, we conclude again $\overline{W}(\mathrm{D}_I) = W(\mathrm{D}_I, \mathbf{d}^+)$. Also, the proof above combined with 9.3.1 and the fact that every element of $\overline{W}(\dot{\mathrm{A}}_I)$ is a product of at most two generalized reflections, shows that every element of $\overline{W}(\mathrm{D}_I)$ is a product of at most four generalized reflections.

Finally, let R be one of the root systems B_I, C_I and BC_I, and let $\mathbf{c} > \aleph_0$. Since $\mathrm{D}_I \subset R$ and thus $W(\mathrm{D}_I, \mathbf{c}) \subset W(R, \mathbf{c})$, (8) together with the fact that $W(R, \mathbf{c}) \subset \mathrm{Aut}(R, \mathbf{c}) = \mathrm{O}(\Gamma, \mathbf{c})$ (by (1)) show that $W(R, \mathbf{c}) = \mathrm{O}(\Gamma, \mathbf{c})$, establishing (2). From this and (1) it follows easily that the outer automorphism groups are given by (3). It remains to show that every element $f = \pi\sigma \in \mathrm{Aut}(R) = \mathrm{O}(\Gamma)$ is the product of at most two generalized reflections with respect to orthogonal systems contained in R. Decompose $X = \bigoplus_{K \in \mathfrak{Z}} X_K$ as in 9.2 and let f_K be the restriction of f to X_K. Then f_K is an automorphism of the full subsystem $R_K = R \cap X_K$, and these subsystems are orthogonal since this holds for the X_K. By construction, $f \in \prod_{K \in \mathfrak{Z}} \mathrm{Aut}(R_K)$. Hence it suffices to show that each f_K is the product of at most two generalized reflections in R_K. Note also that R_K is of the same type B_K, C_K or BC_K as R, but on the index set K. We now discuss the three possibilities for K as in 9.2, and use the notation established there.

Case 1: K is infinite. Then f_K is the shift $e_i \mapsto e_{i+1}$, $i \in \mathbb{Z}$. Let

$$\mathit{\Omega}_j = \{e_i - e_{j-i} : i \geqslant j\}, \quad j = 1, 2.$$

Since the e_i are up to sign basis vectors ε_l, it is clear that $\mathit{\Omega}_j \subset \mathrm{D}_K \subset R_K$. A straightforward verification shows that $\mathit{\Omega}_j$ is an orthogonal system and that $s_{\mathit{\Omega}_j}(e_i) = e_{j-i}$. Hence $s_{\mathit{\Omega}_2}(s_{\mathit{\Omega}_1}(e_i)) = s_{\mathit{\Omega}_2}(e_{1-i}) = e_{2-(1-i)} = e_{i+1} = f_K(e_i)$, as desired.

Case 2: K is a finite positive cycle of length n. Then f_K is the cyclic shift $e_1 \mapsto e_2 \mapsto \cdots \mapsto e_n \mapsto e_1$, which may be realized as $s_{\mathit{\Omega}_2} s_{\mathit{\Omega}_1}$ for

$$\mathit{\Omega}_j = \{e_i - e_{j-i} : j \leqslant i \leqslant \Big[\frac{n+j-1}{2}\Big]\}, \quad j = 1, 2. \tag{9}$$

In this definition, indices outside $\{1, \ldots, n\}$ are to be taken $\bmod n$.

Case 3: K is a finite negative cycle of length n. Then f_K acts via $e_1 \mapsto e_2 \mapsto \cdots e_n \mapsto -e_1$, cf. 9.2.1. With $\mathit{\Omega}_2$ as in (9), let $\mathit{\Omega}_3 = \{\alpha\} \cup \mathit{\Omega}_2$, where $\alpha = e_1$ or $\alpha = 2e_1$ depending on whether $R \supset \mathrm{B}_I$ or $R \supset \mathrm{C}_I$. Then one verifies that $f_K = s_{\mathit{\Omega}_3} s_{\mathit{\Omega}_1}$ where again $\mathit{\Omega}_1$ is as in (9). This completes the proof of the theorem.

9.6. COROLLARY. *Let $R \subset X$ be a locally finite root system. Then every element of the big Weyl group $\overline{W}(R)$ is the product of at most four generalized reflections, and of at most two involutions.*

(As usual in group theory, an involution here means an element of order two.)

Proof. By a theorem of Carter [**16**, Th. C and Lemma 5], every element in the Weyl group of a finite root system is the product of two generalized reflections. Now the corollary follows from 9.5 and 5.2.3, after decomposing R into irreducible components. Concerning the statement that every element of $\overline{W}(R)$ is a product of two involutions, note that this is clear for $R \neq \mathrm{D}_I$, while for $R = \mathrm{D}_I$ we have $\overline{W}(D_I) = \mathrm{O}(\mathit{\Gamma}) = \overline{W}(\mathrm{B}_I)$. Since by 9.5(b), every element of $\mathrm{O}(\mathit{\Gamma})$ is a product of two generalized reflections of $R = \mathrm{B}_I$, it is a fortiori a product of two involutions.

Remark. Corollary 9.6 indicates that, contrary to the case of finite root systems, not every involution in $\overline{W}(R)$ is a generalized reflection. Indeed, let $R = \mathrm{D}_I$ with I infinite. For a fixed element $0 \in I$ the map σ_0, given by $\sigma_0(\varepsilon_0) = -\varepsilon_0$ and $\sigma_0(\varepsilon_i) = \varepsilon_i$ for $i \neq 0$, is an involution in $\mathrm{O}(\mathit{\Gamma}) = \overline{W}(R)$. In fact, since $\mathrm{supp}(\sigma_0) = \{0\}$ we have $\sigma_0 \in W(\mathrm{D}_I, \mathbf{c})$ for every $\mathbf{c} > \aleph_0$. But σ_0 is not a generalized reflection since none of the nonzero roots of D_I lies in the (-1)-eigenspace of σ_0. Indeed, the eigenspace decomposition of $X = \mathrm{span}(\mathrm{D}_I)$ with respect to σ_0 is $X = X_+ \oplus X_-$ where $\sigma_0 = \mathrm{Id}$ on $X_+ = \mathrm{span}(\mathrm{D}_{I \setminus \{0\}})$ and $\sigma_0 = -\mathrm{Id}$ on $X_- = \mathbb{R}\varepsilon_0 = {X_+}^{\perp}$ so that $R \cap X_- = \{0\}$. This also shows that σ_0 is not a product of generalized reflections in orthogonal systems contained in $X_+^{\perp}$, cf. 5.10.

9.7. COROLLARY. *The assignment $R \mapsto \overline{W}(R)$ is a covariant functor from the category* **RSE** *of root systems and embeddings to the category of groups.*

Proof. This follows from 9.6, 5.2.3 and 5.7.

9.8. *Normal subgroups.* We discuss next the normal subgroup structure of the (finitary) Weyl groups of the infinite irreducible root systems and use the notations of 9.4.

$$\begin{tikzcd}
 & W(\mathrm{B}_I) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathrm{O}^{\mathrm{ev}}_{\mathrm{fin}}(\mathit{\Gamma}) \arrow[dr, dash] \arrow[dd, dash] & \mathrm{O}^{+}_{\mathrm{fin}}(\mathit{\Gamma}) \arrow[d, dash] & \mathrm{SO}_{\mathrm{fin}}(\mathit{\Gamma}) \arrow[dl, dash] \\
 & \mathrm{SO}^{+}_{\mathrm{fin}}(\mathit{\Gamma}) \arrow[dd, dash] & \\
\mathbf{2}^{(I)} \arrow[dr, dash] & & \\
 & \mathbf{2}^{(I)}_{+} \arrow[d, dash] & \\
 & \{1\} &
\end{tikzcd} \tag{1}$$

It is well known [**24**, Th. 8.1A] that the alternating group $\mathfrak{A}$ on I is the only proper normal subgroup of the finitary symmetric group $\mathfrak{S} = W(\dot{\mathrm{A}}_I)$. We claim that the lattice of normal subgroups of $W(\mathrm{B}_I) = W(\mathrm{C}_I) = \mathrm{O}_{\mathrm{fin}}(\mathit{\Gamma}) = \mathfrak{S} \ltimes \mathbf{2}^{(I)}$ is given by diagram (1), while the only normal subgroups of $W(\mathrm{D}_I) = \mathrm{O}^{+}_{\mathrm{fin}}(\mathit{\Gamma}) = \mathfrak{S} \ltimes \mathbf{2}^{(I)}_{+}$ are $\{1\}$, $\mathbf{2}^{(I)}_{+}$, $\mathrm{SO}^{+}_{\mathrm{fin}}(\mathit{\Gamma})$, and $W(\mathrm{D}_I)$ itself. As a first step in the proof, we show:

$$\mathbf{2}^{(I)}_{+} \text{ is the only proper } \mathfrak{A}\text{-invariant or } \mathfrak{S}\text{-invariant subgroup of } \mathbf{2}^{(I)}. \tag{2}$$

Indeed, suppose M is a proper $\mathfrak{A}$-invariant subgroup of $\mathbf{2}^{(I)}$ and, say, $\sigma_F \in M$ where $F \subset I$ is finite and non-empty. Without loss of generality we may assume $\mathbb{N} \subset I$ and $F = \{1, \ldots, n\}$. Since $\mathfrak{A}$ is highly transitive (i.e., n-transitive for any n) on I, there exists $\pi \in \mathfrak{A}$ such that $\pi(F) = \{2, \ldots, n+1\}$. Hence M contains the element $(\pi \sigma_F \pi^{-1})\sigma_F = \sigma_{\pi(F)}\sigma_F = \sigma_{\pi(F)\cdot F} = \sigma_{\{1,n+1\}}$. Using again that $\mathfrak{A}$ is highly transitive, it follows easily that M contains all σ_E where E is an even finite subset of I, so $\mathbf{2}^{(I)}_{+} \subset M$. As $\mathbf{2}^{(I)}/\mathbf{2}^{(I)}_{+} \cong \mathbb{Z}/2\mathbb{Z}$, we see that M must be as claimed, and it is clearly also $\mathfrak{S}$-invariant.

Next, let G be one of the groups $W(\mathrm{B}_I)$ or $W(\mathrm{D}_I)$, and put $K := \mathfrak{A} \ltimes \mathbf{2}^{(I)}_{+} = \mathrm{SO}^{+}_{\mathrm{fin}}(\mathit{\Gamma})$. We claim that

$$N \lhd G \;\text{ and }\; N \not\subset \mathbf{2}^{(I)} \quad \Longrightarrow \quad K \subset N. \tag{3}$$

Indeed, the permutation part $\mathrm{per}(N)$ is then a non-trivial normal subgroup of $\mathrm{per}(G) = \mathfrak{S}$ and hence contains $\mathfrak{A}$. Let $\pi \in \mathfrak{A}$ be any 3-cycle. Since $\pi \in \mathrm{per}(N)$, there exists a finite subset $F \subset I$ such that $\pi\sigma_F \in N$, and as $\pi \in G$ we also have $\sigma_F \in G$. From normality of N and $\sigma_F^2 = 1$ we conclude that N contains the element $(\pi\sigma_F) \cdot \sigma_F(\pi\sigma_F)\sigma_F^{-1} = \pi^2$ and therefore also $(\pi^2)^2 = \pi$. Thus N contains all 3-cycles so N contains $\mathfrak{A}$. Let in particular $\pi = (123)$ and $F = \{2,3\} \subset I$. Then F is an even subset so $\sigma_F \in G$. Now $\pi \in N$ and $N \triangleleft G$ imply $(\sigma_F \pi \sigma_F^{-1})\pi^{-1} = \sigma_F \sigma_{\pi(F)} = \sigma_{F\cdot\pi(F)} = \sigma_{\{1,2\}} \in N \cap \mathbf{2}^{(I)}$. Hence $N \cap \mathbf{2}^{(I)}$ is non-trivial and clearly $\mathfrak{A}$-invariant. From (2) we conclude $\mathbf{2}_+^{(I)} \subset N \cap \mathbf{2}^{(I)}$, and therefore $K \subset N$.

As observed in 9.4, the characters ξ and η on $W(\mathrm{B}_I)$ induce an isomorphism $W(\mathrm{B}_I)/K \cong (\mathbb{Z}/2\mathbb{Z})^2$. Therefore, we have a bijection between the (automatically normal) subgroups N of $W(\mathrm{B}_I)$ with $K \subsetneqq N \subsetneqq W(\mathrm{B}_I)$ and the proper subgroups of $(\mathbb{Z}/2\mathbb{Z})^2$. This yields the three normal subgroups listed in (1) – (3) of 9.4 and, together with (2), establishes (1). In case of $W(\mathrm{D}_I)$, we have $W(\mathrm{D}_I)/K \cong \mathbb{Z}/2\mathbb{Z}$ so the only proper normal subgroups are $\mathbf{2}_+^{(I)}$ and K.

Let us finally remark that K is the derived group of both $W(\mathrm{B}_I)$ and $W(\mathrm{D}_I)$, and that the character group of $W(\mathrm{B}_I)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, generated by ξ and η, while that of $W(\mathrm{D}_I)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, generated by ξ. This follows easily from the above discussion. The details are left to the reader.

9.9. Corollary. *The Weyl group of an uncountable irreducible locally finite root system R is not a Coxeter group.*

Proof. Assume that there exists a Coxeter system (W,S) such that $W = W(R)$. Since R is uncountable and the map $\alpha \mapsto s_\alpha$ has finite fibers by 3.4.2 and 4.3(b), W is uncountable. Hence by 5.14, (W,S) is not irreducible. By [**12**, IV, §1.9], S is the disjoint union of pairwise commuting subsets S_i, and W is the restricted direct product of the subgroups W_i generated by the S_i. For reasons of cardinality, there must be infinitely (in fact, uncountably) many W_i. This contradicts the fact that W has only finitely many normal subgroups by 9.8.

§10. Parabolic subsets and positive systems for symmetric sets in vector spaces

10.1. *Notations.* In this section, we prove a number of elementary properties of parabolic subsets and positive systems in root systems. It turns out that these properties hold in the broader framework of symmetric sets in real vector spaces. Accordingly, in this section we will mainly work in the full subcategory **SSV** of $\mathbf{SV}_{\mathbb{R}}$ whose objects (R, X) are *symmetric* in the sense that $R = -R$. More generally, a subset $S \subset R$ is called symmetric if $S = -S$. Properties specific to parabolic subsets of root systems will be developed in §11, and the classification of parabolic subsets of infinite irreducible root systems will be carried out in §13.

Recall that $\mathbb{N} = \mathbb{Z}_+$ denotes the non-negative integers and $\mathbb{N}_+ = \mathbb{Z}_{++}$ the positive integers, respectively. For $(R, X) \in \mathbf{SSV}$ and a subset A of R we define $\mathbb{N}_+[A]$ as the set of all finite non-empty sums of elements of A, i.e.,

$$\mathbb{N}_+[A] = \bigcup_{n=1}^{\infty} \underbrace{(A + \cdots + A)}_{n}.$$

Thus we always have $A \subset \mathbb{N}_+[A]$ and $A = \emptyset$ if and only if $\mathbb{N}_+[A] = \emptyset$.

For a submonoid M of $(\mathbb{R}, +)$ containing 1 (and also 0 because a monoid by definition has a neutral element), we use the notation $M[A]$ for the additive submonoid of X generated by the set $M \cdot A$, i.e., the set of all (possibly empty) sums of elements of $M \cdot A$. Thus we always have $\{0\} \cup A \subset M[A]$. The cases $M = \mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}_+$ will be important later. We note

$$\mathbb{N}[A] = \{0\} \cup \mathbb{N}_+[A], \quad \mathbb{Z}[A] = \mathbb{N}[A \cup (-A)], \tag{1}$$

$$S \text{ symmetric and nonempty} \quad \Longrightarrow \quad \mathbb{Z}[S] = \mathbb{N}_+[S], \tag{2}$$

since then $0 = \alpha + (-\alpha) \in \mathbb{N}_+[S]$.

10.2. *Additively closed subsets and the partial sum property.* Let $(R, X) \in \mathbf{SSV}$ and let $A \subset B \subset R$. We say A is *additively closed in* B if for any finite non-empty family $(\alpha_i)_{i \in I}$ of elements of A with $\beta := \sum_{i \in I} \alpha_i \in B$ we have $\beta \in A$, in other words:

$$A \subset B \subset R \quad \text{is additively closed in } B \quad \Longleftrightarrow \quad A = B \cap \mathbb{N}_+[A]. \tag{1}$$

In case $B = R$, we will usually simply speak of an additively closed subset, or even just of a closed subset. We note that

$$A \subset R \quad \text{closed} \quad \Longrightarrow \quad (-A) \cap \mathbb{N}_+[A] = (-A) \cap A \tag{2}$$

which follows immediately from (1). We also note that for a subset A of $R^\times$,

$$A \text{ is closed in } R^\times \quad \Longleftrightarrow \quad A \cup \{0\} \text{ is closed in } R. \tag{3}$$

This follows easily from the definitions. On the other hand, a subset A of $R^\times$ which is closed in $R^\times$ is not necessarily closed in R; for example, the set $A = \{\alpha, -\alpha\}$ provided $\alpha \in R^\times$ and $n\alpha \notin R$ for all $n \in \mathbb{Z}$, $n \neq \pm 1, 0$. See Lemma 10.10(a) for a characterization of those subsets of $R^\times$ which are closed in R.

Obviously R is closed in R, and the intersection of closed subsets is closed. Hence, for any subset A of R there exists a smallest closed subset A^c containing A, namely the intersection of all closed subsets containing A, which is easily seen to be

$$A^c = R \cap \mathbb{N}_+[A], \tag{4}$$

called the *additive closure* of A. Also, if $(R, X) = \coprod (R_i, X_i)$ is a direct sum in **SSV** then $A \subset R$ is closed if and only if all $A \cap R_i$ are closed in R_i.

We now show that for a root system R, a subset A of $R^\times$ is closed in $R^\times$ in the above sense if and only if it is closed in the usual sense, as defined for example in [**12**, VI, §1.7, Déf. 4] by using sums of two roots.

In somewhat greater generality, let us say that $(R, X) \in$ **SSV** has the *partial sum property* if for all $n \geqslant 1$ and all $\alpha_1, \ldots, \alpha_n \in R$ such that $\beta := \alpha_1 + \cdots + \alpha_n \in R$, there exists a permutation $\pi \in \mathfrak{S}_n$ such that all partial sums $\alpha_{\pi(1)} + \cdots + \alpha_{\pi(i)}$ belong to R, for all $i = 1, \ldots, n$. We note that

$$\textit{root systems have the partial sum property.} \tag{5}$$

This is usually only formulated for positive roots, see e.g. A.14. The proof of (5) is by induction on n, the cases $n = 1, 2$ being obvious. If $\beta = 0$ then $\alpha_1 + \cdots + \alpha_{n-1} = -\alpha_n$, so the assertion holds by induction hypothesis. If $\beta \neq 0$, we have $2 = \langle \beta, \beta^\vee \rangle = \sum_{i=1}^n \langle \alpha_i, \beta^\vee \rangle$, so $\langle \alpha_i, \beta^\vee \rangle > 0$ for some i, and we may assume $i = n$ after renumbering. Then $\alpha_1 + \cdots + \alpha_{n-1} = \beta - \alpha_n \in R$ by A.3(a), and again we are done by induction. — The aforementioned consistency of the definitions of a closed subset is now a consequence of (4) and the following lemma.

10.3. Lemma. *Let $(R, X) \in$ **SSV** have the partial sum property. Then a subset A of R is closed if and only if it is closed with respect to sums of two roots in A, i.e., $(A + A) \cap R \subset A$. Similarly, a subset A of $R^\times$ is closed in $R^\times$ if and only if $(A + A) \cap R^\times \subset A$.*

Proof. We prove the second statement; the proof of the first is similar but simpler. If A is closed in $R^\times$ it is in particular closed with respect to sums of two roots in A. Conversely, suppose $\beta := \alpha_1 + \cdots + \alpha_n \in R^\times$ where $\alpha_i \in A$ and $n \geqslant 3$. By the partial sum property, we may renumber the α_i in such a way that $\gamma := \alpha_1 + \cdots + \alpha_{n-1}$ belongs to R. If $\gamma = 0$ then $\beta = \alpha_n \in A$. Otherwise, $\gamma \in A$ by induction, and hence $\beta = \gamma + \alpha_n \in A$ because A is closed with respect to sums of two roots.

10.4. Lemma. *Let $(R, X) \in$ **SSV**. For a nonempty subset S of R, the following conditions are equivalent:*

(i) *S is closed and symmetric,*
(ii) *$S = R \cap \mathbb{Z}[S]$.*

If R is a root system, then these conditions are also equivalent to

(iii) *S is a closed subsystem.*

Proof. The equivalence of (i) and (ii) follows from 10.1.2 and 10.2.1. Now let R be a root system. Since subsystems are symmetric, we have (iii) $\Longrightarrow$ (i). Next, suppose (ii). Then for $\alpha, \beta \in S^\times = S \cap R^\times$ we have $s_\alpha(\beta) \in R \cap \mathbb{Z}[S] = R \cap \mathbb{N}_+[S] = S$, by 10.1.2 and 10.2.1.

10.5. Definition. Let $(R, X) \in \mathbf{SSV}$ and let $A \subset R$ be an additively closed subset. We call A

(i) *positive* if $A \cap (-A) \subset \{0\}$,
(ii) *parabolic* if $A \cup (-A) = R$,
(iii) a *positive system of* R if A is both positive and parabolic; i.e., if $A \cup (-A) = R$ and $A \cap (-A) = \{0\}$,
(iv) *unipotent* if $R \setminus A$ is parabolic.

We note here that it would not make sense to define these concepts for sets in vector spaces over fields of characteristic $p > 0$. Indeed, a subset $A \subset R$ would be closed and positive in the sense above if and only if $A \subset \{0\}$, because $\alpha \in A$ implies $-\alpha = (p-1)\alpha \in R \cap \mathbb{N}_+[A] = A$ and thus $-\alpha \in A \cap (-A) = \{0\}$. Similarly, the only parabolic subset of R would be R itself.

A concept of a positive set of roots in the setting of Kac-Moody algebras appears in Tits [**73**, 3.2] where it is called a nilpotent set of roots. For finite root systems, Tits' definition is equivalent to the one given here, as one easily sees from [**12**, VI, §1.7, Prop. 22] and [**12**, VI, §1.6, Cor. 3 of Prop. 17]. Closed subsets of finite root systems whose complement is again closed ("invertible" subsets) are classified in [**26**]. They include the parabolic subsets. The notion of parabolic subsets and positive systems is standard for finite root systems, see e.g. [**12**, VI, §1.7 Déf. 4]. Positive systems in affine root systems were described by Jakobsen-Kac [**34**] and by Futorny [**28**]. The concept of a positive system is defined differently in Neeb [**50**, I.1]. Lemma 10.10(b) below shows that our definition coincides with Neeb's.

We will see in 10.8(c) that positive systems always exist. If (R, X) is a root system admitting a root basis B then the set $R \cap \mathbb{N}[B]$ of all roots which are linear combinations of B with non-negative coefficients is a positive system. In finite root systems this establishes a bijection between positive systems and root bases [**12**, VI, §1.7, Cor. 1 of Prop. 20]. This is no longer true for arbitrary root systems. Indeed, an uncountable irreducible root system does not have a root basis (see 6.9), and even when R does admit root bases, there may well be positive systems not determined by a root basis, see 14.15.

We note that the intersection of a positive (parabolic) subset A with any symmetric subset S of R is a positive (parabolic) subset of S, and the same is true for a positive system. Also, if $(R, X) = \coprod (R_i, X_i)$ is a direct sum in $\mathbf{SSV}$ then $A \subset R$ is positive (parabolic) if and only if all $A \cap R_i$ are positive (parabolic) in R_i.

A necessary condition for a subset U of R to be unipotent is of course that U be positive and $0 \notin U$, but this is not sufficient, as the example $\{\varepsilon_2 \pm \varepsilon_1\}$ in the root system B_2 shows.

10.6. *The symmetric and unipotent part of a parabolic subset.* Let $(R, X) \in \mathbf{SSV}$. For a parabolic subset P of R, we introduce the *symmetric* and *unipotent* part of P, respectively, by

$$P_s := P \cap (-P) \quad \text{and} \quad P_u := P \setminus P_s.$$

Then P_s is clearly symmetric and additively closed as the intersection of two additively closed sets, and we have the disjoint decomposition

$$R = P_u \mathbin{\dot\cup} P_s \mathbin{\dot\cup} (-P_u). \tag{1}$$

Also, P_u is additively closed (and hence positive), more precisely,

$$\alpha = \sum_{i=1}^{n} \alpha_i \in R \text{ where } \alpha_i \in P \text{ and } \alpha_1 \in P_u \quad \Longrightarrow \quad \alpha \in P_u. \tag{2}$$

Indeed, $\alpha \in P$ since P is closed. Suppose $\alpha \in P_s$. Then $-\alpha \in P$ and $-\alpha_1 = -\alpha + \sum_{i=2}^{n} \alpha_i \in R \cap \mathbb{N}_+[P] = P$, so $\alpha_1 \in P_u \cap (-P) = \emptyset$, contradiction. Finally, P_u is indeed unipotent since $R \setminus P_u = P_s \mathbin{\dot\cup} (-P_u) = -P$ is parabolic.

10.7. *The preorder induced by a subset of R.* It will be useful later to describe some of the concepts introduced above in terms of preorders on R. Therefore, we first recall the relevant terminology.

Given a set A with a relation $\preccurlyeq$, we will also write $\alpha \succcurlyeq \beta$ for $\beta \preccurlyeq \alpha$, and $\alpha \prec \beta$ for $\alpha \preccurlyeq \beta$ but $\alpha \neq \beta$. The relation $\preccurlyeq$ is called a *preorder* if it is transitive and reflexive. A preorder $\preccurlyeq$ is a *partial order* if it is symmetric, i.e., $\alpha \preccurlyeq \beta$ and $\beta \preccurlyeq \alpha$ implies $\alpha = \beta$. A partial order is a *total order* if for every $\alpha, \beta \in A$ we have $\alpha \preccurlyeq \beta$ or $\beta \preccurlyeq \alpha$. A subset A of a vector space X will be called *pointed* if $0 \in A$.

Let now $(R, X) \in \mathbf{SSV}$. Any $A \subset R$ induces a preorder $\preccurlyeq_A$ on X by

$$x \preccurlyeq_A y \quad \Longleftrightarrow \quad y - x \in \mathbb{N}[A] \tag{1}$$

which satisfies $0 \preccurlyeq_A A$ and makes $(X, +)$ a preordered abelian group in the sense that

$$x \preccurlyeq_A y \quad \Longrightarrow \quad x + z \preccurlyeq_A y + z \quad \text{for all} \quad z \in X. \tag{2}$$

Since A and $A \cup \{0\}$ determine the same preorder, we will consider $\preccurlyeq_A$ for pointed subsets only. Then we have $\{\alpha \in R : 0 \preccurlyeq_A \alpha\} = R \cap \mathbb{N}_+[A]$, whence the additive closure A^c of a pointed A is

$$A^c = \{\alpha \in R : 0 \preccurlyeq_A \alpha\}; \tag{3}$$

in particular, a closed and pointed A can be recovered from $\preccurlyeq_A$.

Conversely, to every preorder $\preccurlyeq$ on X satisfying (2) we can associate the pointed set $A = \{\alpha \in R : 0 \preccurlyeq \alpha\}$. Its associated preorder is weaker than the given $\preccurlyeq$: For $x, y \in X$ we have

$$x \preccurlyeq_A y \quad \Longrightarrow \quad x \preccurlyeq y. \tag{4}$$

Indeed, we can write y in the form $y = x + \alpha_1 + \cdots + \alpha_n$ with $\alpha_i \in R$ satisfying $\alpha_i \succcurlyeq 0$. Applying (2) n times we obtain $x \preccurlyeq x + \alpha_1 \preccurlyeq x + \alpha_1 + \alpha_2 \preccurlyeq \cdots \preccurlyeq y$. Moreover, A is closed since $R \cap \mathbb{N}[A] = \{\alpha \in R : 0 \preccurlyeq_A \alpha\} \subset A$ by (4).

If P is parabolic the corresponding preorder $\preccurlyeq_P$ satisfies $0 \preccurlyeq_P \alpha$ or $0 \preccurlyeq_P -\alpha$ for any $\alpha \in R$. Conversely, for any preorder $\preccurlyeq$ on X with this property the positive elements form a parabolic subset. The preorder $\preccurlyeq_A$ is not necessarily compatible with the vector space structure of X. We next consider preorders which do have this property.

Recall [**11**, II, §2.5] that a preorder $\geqslant$ on X is said to be *compatible with the vector space structure of* X if

(i) $x \geqslant y$ implies $x+z \geqslant y+z$ for every $z \in X$, and
(ii) $x \geqslant 0$ implies $sx \geqslant 0$ for every $s \in \mathbb{R}_+$.

Note that compatible partial orders always exist, even total orders; for example, the lexicographic order with respect to any vector space basis of X.

These concepts are intimately related with convex cones. We will say that a convex cone $C \subset X$ (with vertex 0) is *proper* if C does not contain an entire line, equivalently, $C \cap (-C) \subset \{0\}$. Given a compatible preorder $\leqslant$ on X, the subset $X_+ = \{x \in X : x \geqslant 0\}$ is a pointed convex cone. Conversely, any pointed convex cone $C \subset X$ induces a compatible preorder $\leqslant$ on X by $x \geqslant y \iff x - y \in C$, and this is a partial order if and only if C is proper.

Using the concepts above, a typical way of constructing parabolic subsets is as follows.

10.8. Lemma. *Let* $(R, X) \in \mathbf{SSV}$.

(a) *Consider a linear map* $f\colon X \to Y$ *where* $(Y, \geqslant)$ *is a partially ordered real vector space, and assume that* $f(R) \subset Y_+ \cup (-Y_+)$ *(which is always the case if* $\geqslant$ *is a total order). Then* $x \leqslant y :\iff f(x) \leqslant f(y)$ *defines a compatible preorder on* X *whose associated cone is* $f^{-1}(Y_+)$*, and*

$$P = R_+(f) := \{\alpha \in R : f(\alpha) \geqslant 0\} = R \cap f^{-1}(Y_+) \tag{1}$$

is a parabolic subset, with symmetric and unipotent part given, respectively, by

$$P_s = R_0(f) := R \cap \operatorname{Ker}(f), \quad P_u = R_{++}(f) := \{\alpha \in R : f(\alpha) > 0\}. \tag{2}$$

In particular, the symmetric part of a parabolic subset of this type is a full subset (but in general $\operatorname{Ker}(f)$ *is not a tight subspace, i.e., it is not spanned by* P_s*).*

(b) *Conversely, every full* $S \subset R$ *is the symmetric part of a parabolic subset.*

(c) (R, X) *contains positive systems.*

Proof. (a) That P is parabolic is a special case of the construction in 10.7. The remaining statements are straightforward.

(b) Consider the quotient $Y = X/\operatorname{span}(S)$, let $\geqslant$ be any total order on Y compatible with the vector space structure, and let $f = \mathrm{can}\colon X \to Y$. Then $P = R_+(f)$ is parabolic by (a), and $\alpha \in P_s$ if and only if $\alpha \in R \cap \operatorname{Ker}(\mathrm{can}) = S$, by fullness of S.

(c) It suffices to apply (b) to the special case $S = \{0\}$. (This result will also follow from Prop. 10.13.)

10.9. *Parabolic subsets of scalar type.* We say a parabolic subset P of some $(R, X) \in \mathbf{SSV}$ is *scalar* or *of scalar type* if $P = R_+(f)$ for some $f \in X^*$. In a finite root system every parabolic subset is of scalar type, as Lemma 11.1 shows. For infinite root systems this is no longer true. Indeed, for R an irreducible root system and $f\colon X \to \mathbb{R}$ a linear form, $\operatorname{rank}(f) := \operatorname{codim}\operatorname{span}(R_0(f)) \leqslant \operatorname{Card}(\mathbb{R})$, by 8.11. On the other hand, $P_s = \{0\}$ for a positive system P, so $\operatorname{codim}\operatorname{span}(P_s) = \dim X$ can be arbitrarily large. However, by 15.6.2, every parabolic subset is an intersection of parabolic subsets of scalar type. Also, we will see below in 10.17 that all parabolic subsets are of type $R_+(f)$ where f takes values in a suitable partially ordered vector space, provided (R, X) satisfies a rationality condition introduced in 10.15, and we will in 13.7 characterize scalar parabolic subsets of root systems. — The following observations will be useful.

If $\varphi \in \mathrm{Aut}(R)$ then P is of scalar type if and only if $\varphi(P)$ is so. Indeed, for any $f \in X^*$ we have

$$\varphi(R_+(f)) = R_+(f \circ \varphi^{-1}). \tag{1}$$

Suppose $(R, X) = \coprod (R_i, X_i) = (\bigcup R_i, \bigoplus X_i)$ is a direct sum. As remarked in 10.5, we then have $P = \bigcup P_i$ where the $P_i = P \cap R_i$ are parabolic in R_i. Moreover,

$$P \text{ is of scalar type} \quad \Longleftrightarrow \quad \text{every } P_i \text{ is of scalar type.} \tag{2}$$

Indeed, if $P = R_+(f)$ then $P_i = (R_i)_+(f|X_i)$. Conversely, if $P_i = (R_i)_+(f_i)$ for $f_i \in X_i^*$, then $P = R_+(f)$ where $f = \prod f_i \in X^* = \prod X_i^*$.

10.10. Lemma. *Let $(R, X) \in$ **SSV**.*

(a) *For an arbitrary subset A of R, the following conditions are equivalent:*

(i) *A is positive,*

(ii) *$A^\times := A \setminus \{0\}$ is closed in R,*

(iii) *A is closed and $\alpha_1 + \cdots + \alpha_n \neq 0$ whenever $\alpha_1, \ldots, \alpha_n \in A^\times$,*

(iv) *A is closed and $\mathbb{N}_+[A] \cap \mathbb{N}_+[-A] \subset \{0\}$;*

in particular, A is positive if and only if $A \cup \{0\}$ is positive, and the positive subsets of R contained in $R^\times$ are precisely the subsets of $R^\times$ which are closed in R.

If these conditions hold for a subset A of R then the preorder $\preccurlyeq_A$ of 10.7 is a partial order.

(b) *Let P be a subset of R with $P \cup (-P) = R$. Then the following conditions are equivalent:*

(i) *P is a positive system,*

(ii) *$\mathbb{N}_+[P] \cap \mathbb{N}_+[-P] = \{0\}$,*

(iii) *$\mathbb{N}_+[P] \cap (-P) = \{0\}$.*

The equivalence of (ii) and (iii) in part (b) is also proven in [**50**, I.2] for root systems with a different proof.

Proof. (a) (i) $\Longrightarrow$ (ii): Let $\alpha_1, \ldots, \alpha_n \in A^\times$ and $\beta := \alpha_1 + \cdots + \alpha_n \in R$. Then $\beta \in A$ because A is positive and thus in particular closed, so we only must show that $\beta = 0$ is impossible. Assuming $\beta = 0$ we have $n \geqslant 2$ and $0 \neq \alpha_2 + \cdots + \alpha_n = -\alpha_1 \in (-A) \cap \mathbb{N}_+[A] = (-A) \cap A$ (by 10.2.2) $\subset \{0\}$ (because A is positive), contradiction.

(ii) $\Longrightarrow$ (iii): Clearly $A^\times \cup \{0\}$ is closed along with $A^\times$ which implies that A is closed. Assuming $\alpha_1 + \cdots + \alpha_n = 0$ for $\alpha_i \in A^\times$, we conclude $0 \in A^\times$ because $0 \in R$ and $A^\times$ is closed in R, contradiction.

The remaining implications are obvious, and the statement about $\succcurlyeq_A$ follows from (iv).

(b) (i) $\Longrightarrow$ (ii) is clear from (a), since a positive system is positive, and (ii) $\Longrightarrow$ (iii) follows from $0 \in P \subset \mathbb{N}_+[P]$. If (iii) holds, then clearly $P \cap (-P) = \{0\}$ and P is closed since $P^c = R \cap \mathbb{N}_+[P]$ (by 10.2.4) $= P \cap \mathbb{N}_+[P]$ (by $R = P \cup (-P)$ and (iii)) $= P$. Hence P is a positive system.

10.11. PROPOSITION. *Let* $(R, X) \in \mathbf{SSV}$ *have the partial sum property, and let* $P \subsetneqq R$ *be a parabolic subset. Also, let* $\preccurlyeq$ *be the partial order on* P_u *induced by* $\preccurlyeq_{P_u}$ *as in* 10.7.1*, and let* $\alpha \in P_u$*. Then the following conditions are equivalent:*

(i) α *is minimal (resp. maximal) with respect to* $\preccurlyeq$*,*
(ii) α *is not the sum (resp. difference) of two elements of* P_u*,*
(iii) $\alpha - \beta$ *(resp.* $\alpha + \beta$*) is not in* P_u*, for all* $\beta \in P_u$*.*

We denote the subsets of minimal and maximal elements of P_u, respectively, by $P_{\min}$ and $P_{\max}$. Note that either or both of these sets may be empty.

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are clear. For (iii) $\Longrightarrow$ (i) in case $\alpha - \beta \notin P_u$ for all $\beta \in P_u$, assume $\beta \prec \alpha$ for some $\beta \in P_u$. Then $\alpha - \beta = \alpha_1 + \cdots + \alpha_n$ where $\alpha_i \in P_u$, so putting $\alpha_{n+1} := \beta$, we have $\alpha = \alpha_1 + \cdots + \alpha_{n+1}$. By the partial sum property (10.2), there exists a permutation $\pi \in \mathfrak{S}_{n+1}$ such that, in particular, $\gamma := \alpha_{\pi(1)} + \cdots + \alpha_{\pi(n)} \in R$ and hence, by 10.6.2, $\gamma \in P_u$. Thus $\alpha = \gamma + \alpha_{\pi(n+1)}$ is the sum of two elements of P_u, contradiction.

To prove (iii) $\Longrightarrow$ (i) in case $\alpha + \beta \notin P_u$ for all $\beta \in P_u$, assume $\alpha \prec \beta \in P_u$. Then $\beta - \alpha = \alpha_1 + \cdots + \alpha_n$ where $\alpha_i \in P_u$. Putting $\alpha_{n+1} := \alpha$, we have $\beta = \alpha_1 + \cdots + \alpha_{n+1}$, and the partial sum property yields a permutation $\pi \in \mathfrak{S}_{n+1}$ such that all $\alpha_{\pi(1)} + \cdots + \alpha_{\pi(i)}$ are in R and hence in P_u, by 10.6.2. Let $n+1 = \pi(j)$. If $j = 1$ then $\alpha_{\pi(1)} + \alpha_{\pi(2)} = \alpha + \alpha_{\pi(2)} \in P_u$, contradiction. If $j > 1$, then $\gamma := \alpha_{\pi(1)} + \cdots + \alpha_{\pi(j-1)} \in P_u$, and $\gamma + \alpha_{\pi(j)} = \gamma + \alpha \in P_u$, contradiction.

10.12. COROLLARY. *With the assumptions and notations of Prop.* 10.11*, we have* $P_u = P_{\min} \iff P_u = P_{\max}$.

Proof. This follows from the equivalences

$$\begin{aligned} P_u \setminus P_{\min} \neq \emptyset \quad &\iff \quad \text{there exists } \alpha, \beta, \gamma \in P_u \text{ such that } \alpha = \beta + \gamma \\ &\iff \quad P_u \setminus P_{\max} \neq \emptyset. \end{aligned}$$

Indeed, by 10.11, we have $\alpha \in P_u \setminus P_{\min}$ in the first equivalence, while $\gamma \in P_u \setminus P_{\max}$ in the second.

10.13. PROPOSITION. *Let* $(R, X) \in \mathbf{SSV}$*.*

(a) *Any positive subset is contained in a positive system.*

(b) *A subset* P *of* R *is a positive system if and only if* P *is a maximal positive subset (with respect to inclusion).*

The equivalence (b) is also proven in [**50**, I.9] using Lie algebra techniques.

Proof. (a) We fix a positive subset A_0 of R and consider the non-empty set $\mathfrak{A} = \{A : A_0 \subset A \text{ and } A \text{ is positive}\}$. This is an inductively ordered set with respect to inclusion. Hence, by Zorn's Lemma, $\mathfrak{A}$ contains a maximal element P. Note that $0 \in P$ since $A \cup \{0\}$ is again positive, for any positive set A. To show that P is a positive system it remains to verify $P \cup (-P) = R$.

Assume to the contrary that there exists $\alpha \in R \setminus \big(P \cup (-P)\big)$, so in particular $\alpha \neq 0$. By maximality of P, the two closed subsets $P_+ = (P \cup \{\alpha\})^c$ and $P_- = (P \cup \{-\alpha\})^c$ are then not positive, whence there exist $0 \neq \beta_\varepsilon \in P_\varepsilon \cap (-P_\varepsilon)$ for $\varepsilon = +$ and $\varepsilon = -$. The description of the closure in 10.2.4 implies that $\pm\beta_\varepsilon$ are finite sums of roots in $\{\varepsilon\alpha\} \cup P$. Thus we can write

$$\beta_+ = p\alpha + \sum_{i=1}^{n} \alpha_i, \qquad -\beta_+ = q\alpha + \sum_{i=1}^{n'} \alpha'_i, \tag{1}$$

$$\beta_- = -r\alpha + \sum_{j=1}^{m} \beta_j, \qquad -\beta_- = -s\alpha + \sum_{j=1}^{m'} \beta'_j, \tag{2}$$

where the $\alpha_i, \alpha'_i, \beta_j, \beta'_j$ are in $P^\times$ and $m, n, m', n', p, q, r, s \in \mathbb{N}$. From (1) we obtain

$$0 = (p+q)\alpha + \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n'} \alpha'_i. \tag{3}$$

If $p+q=0$ then $p=q=0$, so (3) and Lemma 10.10(a) show that $n=n'=0$ and hence $\beta_+=0$, contradiction. Thus we have $k := p+q \in \mathbb{N}_+$. Similarly, (2) implies

$$0 = -(r+s)\alpha + \sum_{j=1}^{m} \beta_j + \sum_{j=1}^{m'} \beta'_j, \tag{4}$$

where $l := r+s \in \mathbb{N}_+$. But then (3) and (4) show that $kl\alpha \in \mathbb{N}_+[P] \cap \mathbb{N}_+[-P] = \{0\}$ (by Lemma 10.10(a)) and hence $\alpha = 0$, contradiction.

(b) It is clear that a positive system is a maximal positive set, while the other direction follows from (a).

10.14. Proposition. *Let $(R, X) \in$ **SSV** and let $P \subset R$ be a parabolic subset, decomposed into symmetric part P_s and unipotent part P_u as in* 10.6.1.

(a) *The positive systems of R contained in P are precisely the sets $P_s^+ \cup P_u$ where P_s^+ is a positive system of P_s. In particular, P does contain positive systems, and the positive systems of R are precisely the minimal parabolic subsets of R.*

(b) *Let $A \subset P$ be a positive subset of R. Then there exists a positive system R^+ of R with $A \subset R^+ \subset P$.*

That every parabolic subset of a root system contains a positive system is also shown in [**50**, I.9], using Lie algebra techniques.

Proof. (a) Let P_s^+ be a positive system of P_s and define $R^+ = P_s^+ \cup P_u$. It is then easily seen that $R^+ \cap (-R^+) = \{0\}$ and $R^+ \cup (-R^+) = R$. Hence R^+ is a positive system as soon as we know that it is closed. But this follows from 10.6.2 and closedness of P_s^+ in R, which in turn is a consequence of closedness of P_s^+ in P_s and the fact that $P_s = P \cap (-P)$ is closed in R by 10.6. Conversely, let R^+ be a positive system of R contained in P. Then $P_u \subset R^+$, otherwise there would exist $\alpha \in P_u \cap (-R^+) \subset P_u \cap (-P) = \emptyset$, by 10.6.1. Hence $R^+ = P_s^+ \cup P_u$ where $P_s^+ = R^+ \cap P_s$ is a positive system in P_s.

(b) As noted in 10.5, $A \cap P_s$ is a positive subset of P_s. By 10.13(a), there exists a positive system P_s^+ of P_s containing $A \cap P_s$. Then $A = (A \cap P_s) \cup (A \cap P_u) \subset P_s^+ \cup P_u = R^+ \subset P$, as desired.

10.15. *Rationality.* Let $(R, X) \in \mathbf{SSV}$, and let $X_{\mathbb{Q}} = \operatorname{span}_{\mathbb{Q}}(R)$ be the $\mathbb{Q}$-subspace of X spanned by R. We say (R, X) is *rational* if $X_{\mathbb{Q}}$ is a $\mathbb{Q}$-structure on X, i.e., if the canonical map $X_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \to X$ is an isomorphism [**13**, II, §8.1]. Thus, $(R, X) \in \mathbf{SSV}$ is rational if and only if $X_{\mathbb{Q}}$ has a basis that is $\mathbb{R}$-free, i.e., $X_{\mathbb{Q}}$ admits a $\mathbb{Q}$-basis in the sense of 2.7.

Examples. (1) If $(R, X) \in \mathbf{SSV}$ admits an A-basis, where A is a subring of $\mathbb{Q}$, then (R, X) is rational. Examples of (R, X) containing integral bases ($A = \mathbb{Z}$) are quotients of root systems by full subsystems, and hence a fortiori root systems (Th. 6.4), extended affine root systems or the root system of a Kac-Moody Lie algebra (6.1).

(2) If X is finite-dimensional and $\mathbb{Z}[R]$ is a lattice in X, then (R, X) is rational (even if R may not contain an integral basis). For example, the (real) roots $R = \Sigma \cup \{0\}$ of a set of root data over $\mathbb{R}$ in the sense of [**47**, 5.1] satisfy this criterion with $X = \operatorname{span}_{\mathbb{R}}(R)$. Indeed, that R is symmetric follows from [**47**, 5.1, Prop. 4], while the lattice property is part of axiom (RD4).

Recall that a real subspace $V \subset X$ is rational (or defined over $\mathbb{Q}$) if and only if $V = \operatorname{span}(V \cap X_{\mathbb{Q}})$, in which case $V_{\mathbb{Q}} := V \cap X_{\mathbb{Q}}$ is a $\mathbb{Q}$-structure on V [**13**, II, §8.2, Prop. 2]. If $R' = -R'$ is a symmetric subset of a rational (R, X) then $(R', \operatorname{span}(R'))$ is rational in $\mathbf{SSV}$ and $\operatorname{span}(R')$ is a rational subspace. We also remark that the quotient of a rational $(R, X) \in \mathbf{SSV}$ by a full subsystem (R', X') is again rational, which follows from $(X_{\mathbb{Q}}/X'_{\mathbb{Q}}) \otimes \mathbb{R} \cong X/X'$.

10.16. Lemma. *Let $(R, X) \in \mathbf{SSV}$ be rational, and suppose $v \in X_{\mathbb{Q}}$ and $\alpha_1, \ldots, \alpha_n \in R$ satisfy a relation $v = \sum_{i=1}^n c_i \alpha_i$ where $c_i \in \mathbb{R}$. Then there exist $r_i \in \mathbb{Q}$ such that*

$$v = \sum_{i=1}^{n} r_i \alpha_i. \tag{1}$$

If all c_i are positive then the r_i may be chosen positive as well.

Proof. Consider the linear map $f \colon \mathbb{R}^n \to X$ sending the standard basis e_i to α_i. Then f is defined over $\mathbb{Q}$, so its kernel $Z = \operatorname{Ker}(f)$ and image $V = \operatorname{span}\{\alpha_1, \ldots, \alpha_n\}$ are defined over $\mathbb{Q}$ as well, the $\mathbb{Q}$-structure of V being $V_{\mathbb{Q}} = f(\mathbb{Q}^n) = \operatorname{span}_{\mathbb{Q}}\{\alpha_1, \ldots, \alpha_n\}$ [**13**, II, §8.3]. Hence $v \in V \cap X_{\mathbb{Q}} = V_{\mathbb{Q}}$ is a rational linear combination of the α_i.

By choosing a $\mathbb{Q}$-basis of Z and extending it to a $\mathbb{Q}$-basis of $\mathbb{Q}^n$, one sees that $Z_{\mathbb{Q}} = Z \cap \mathbb{Q}^n$ is dense in Z (in the topology induced from $\mathbb{R}^n$). Since $v \in f(\mathbb{Q}^n)$, the affine subspace $L = f^{-1}(v) \subset \mathbb{R}^n$ of real solutions of (1) is defined over $\mathbb{Q}$, namely $L = r + Z$ where $r = (r_1, \ldots, r_n) \in \mathbb{Q}^n$ is a rational solution of (1). It follows that the space $L_{\mathbb{Q}} = r + Z_{\mathbb{Q}}$ of rational solutions of (1) is dense in the real affine space $L = r + Z$.

Now suppose the c_i are positive, and let $C = \mathbb{R}_{++}[e_1, \ldots, e_n]$ be the open convex cone spanned by the standard basis of $\mathbb{R}^n$. Then $C \cap L$ is an open and non-empty subset of L, because it contains $(c_1, \ldots, c_n)$. Hence $C \cap L_{\mathbb{Q}} \neq \emptyset$, showing that (1) has positive rational solutions.

10.17. Proposition. *Let $(R, X) \in \mathbf{SSV}$ be rational and let $P \subset R$ be parabolic, decomposed into symmetric and unipotent part as in* 10.6.1. *We put $Y :=$*

$X/\operatorname{span}(P_s)$ and denote by can: $X \to Y$ *the canonical map. With P we associate the pointed convex cones*

$$K_u := \mathbb{R}_+[P_u] \subset K := \mathbb{R}_+[P] \subset X, \tag{1}$$
$$C := \operatorname{can}(K) = \mathbb{R}_+[\operatorname{can}(P)] \subset Y. \tag{2}$$

(a) *P and P_u can be reconstructed from K and K_u, respectively, by*

$$K \cap R = P, \tag{3}$$
$$K_u \cap R^\times = P_u. \tag{4}$$

(b) *P_s and K are related by*

$$K \cap (-K) \cap R = K \cap (-P) = P_s, \tag{5}$$
$$K \cap (-K) = \operatorname{span}(P_s). \tag{6}$$

In particular, P_s is a full subset of R.

(c) *C is a proper convex cone and $K = \operatorname{can}^{-1}(C)$ whence $P = R \cap K = R \cap \operatorname{can}^{-1}(C) = R_+(\operatorname{can})$ is obtained by the construction given in* 10.8.1. *Likewise, K_u is proper, and we have*

$$\operatorname{can}(K_u \setminus \{0\}) = C \setminus \{0\}. \tag{7}$$

(d) *P is a positive system if and only if K is proper.*

Proof. (a) The inclusion from right to left in (3) and (4) is obvious. Conversely, let $\alpha \in K \cap R$ (resp. $\alpha \in K_u \cap R^\times$), so $\alpha = \sum_{i=1}^n c_i\alpha_i$ where $0 < c_i \in \mathbb{R}$ and $\alpha_i \in P$ (resp. $\alpha_i \in P_u$). By Lemma 10.16, there exist positive rational numbers $r_i = p_i/q_i$ (where $p_i, q_i \in \mathbb{N}_+$) such that $\alpha = \sum_{i=1}^n r_i\alpha_i$. Let m be the product of the denominators q_i and put $m_i = mp_i$. Then $m, m_i \in \mathbb{N}_+$ and we have $m\alpha = \sum_{i=1}^n m_i\alpha_i$. Assume $\alpha \notin P$ (resp. $\alpha \notin P_u$). Then by 10.6.1, $-\alpha \in P$, and it follows that $\alpha = (m-1)(-\alpha) + \sum_{i=1}^n m_i\alpha_i \in R \cap \mathbb{N}_+[P] = P$, since P is closed (resp. $\alpha \in P_u$, by 10.6.2), contradiction.

(b) By (3) and $R = -R$ we have $-P = (-K) \cap R$ and hence formula (5). For (6), observe first that $K \cap (-K)$ is a vector subspace of X (in fact, the largest vector subspace contained in K [**11**, II, §2.4, Cor. 2 of Prop. 10]) and it contains P_s by (5). Hence it contains $\operatorname{span}(P_s)$. Conversely, let $0 \neq x \in K \cap (-K)$. Then there exist $c_i > 0$ and $\alpha_i \in P$ such that $-x = \sum_{i=1}^n c_i\alpha_i$. As K is a convex cone, this implies $-\alpha_i = c_i^{-1}(x + \sum_{j\neq i} c_j\alpha_j) \in K$, and therefore $-\alpha_i \in K \cap R = P$, by (3). Thus $\alpha_i \in P \cap (-P) = P_s$, showing $x \in \operatorname{span}(P_s)$.

(c) We have $Y = X/(K \cap (-K))$ by (6). Hence [**11**, II, §2.5] shows that C is a proper convex cone in Y satisfying $K = f^{-1}(C)$ for $f = \operatorname{can}$. Now $P = R_+(f)$ follows from (3) and the definition of $R_+(f)$ in 10.8.1.

From $P = P_s \mathbin{\dot\cup} P_u$ and $f(P_s) = \{0\}$ it is clear that $C = f(K_u)$. Let $\gamma \in P_u$. Then $\gamma \notin P_s$ whence $f(\gamma) \neq 0$ by (5). Hence $f(\gamma) > 0$ with respect to the partial order $\geqslant$ on Y determined by C. It follows that any positive linear combination $x = \sum c_j\gamma_j$ of elements of P_u also satisfies $f(x) = \sum c_j f(\gamma_j) > 0$, so we have (7). Now $K_u \cap (-K_u) = \{0\}$ follows from $C \cap (-C) = \{0\}$.

(d) Since P is a positive system if and only if $P_s = \{0\}$ this is immediate from (6).

10.18. COROLLARY. *Let (R, X) be a root system. Then the map $P \mapsto P^\vee := \{\alpha^\vee : \alpha \in P\}$ is a bijection between the set of parabolic subsets of R and those of $R^\vee$, which satisfies $(P_s)^\vee = (P^\vee)_s$ and $(P_u)^\vee = (P^\vee)_u$ and under which positive systems correspond to positive systems.*

Proof. We clearly have $P^\vee \cup (-P^\vee) = R^\vee$ (because of $(-\alpha)^\vee = -\alpha^\vee$) and $(P \cap (-P))^\vee = P^\vee \cap (-P^\vee)$, so it remains to show that $P^\vee$ is additively closed. Let $(\ |\ )$ be an invariant inner product on X, let $\flat\colon X \to X^\vee$ be the vector space isomorphism induced by $(\ |\ )$ (cf. Lemma 4.8), and let K be the convex cone spanned by P. Then $\flat(K)$ is a convex cone in $X^\vee$, and we have $P^\vee = R^\vee \cap \flat(K)$. Indeed, the inclusion from left to right is clear from the formula 4.8.2. Conversely, if $\alpha^\vee = 2\alpha^\flat/(\alpha|\alpha) \in R^\vee \cap \flat(K)$ then $2\alpha/(\alpha|\alpha) \in K$ whence $\alpha \in R \cap K = P$ by 10.17.3, so $\alpha^\vee \in P^\vee$. Since $R^\vee \cap \flat(K)$ is obviously additively closed, the assertion follows.

Remark. For a closed but not parabolic subset A of R, it is in general not true that $A^\vee$ is again a closed subset of $R^\vee$.

10.19. PROPOSITION. *Let $(R, X) \in$ **SSV** be rational, and let $R' \subset R$ be a full subsystem, with linear span $X' = \operatorname{span}(R')$, and quotient $(\bar{R}, \bar{X}) = (R/R', X/X')$. We denote by $g\colon X \to \bar{X}$ the canonical map, and put $g^*(T) := R \cap g^{-1}(T)$, for a subset T of $\bar{R}$.*

(a) *Let $P \subset R$ be parabolic with $R' \subset P_s$. Then $g(P) \subset \bar{R}$ is parabolic and*

$$g^*(g(P)) = P, \tag{1}$$

$$g^*(g(P_s)) = P_s, \qquad g(P)_s = g(P_s), \tag{2}$$

$$g^*(g(P_u)) = P_u, \qquad g(P)_u = g(P_u). \tag{3}$$

(b) *Conversely, if $Q \subset \bar{R}$ is parabolic then $g^*(Q) \subset R$ is parabolic and satisfies $R' \subset g^*(Q)$.*

(c) *The maps $P \mapsto g(P)$ and $Q \mapsto g^*(Q)$ are inverse bijections between the set of all parabolic subsets P of R satisfying $R' \subset P_s$, and the set of all parabolic subsets of $\bar{R}$. Moreover, $R' = P_s$ if and only if $g(P)$ is a positive system in $\bar{R}$, and P is of scalar type if and only if $g(P)$ is of scalar type.*

Proof. (a) Clearly $g(P) \cup -(g(P)) = \bar{R}$. To show that $g(P)$ is additively closed, let $(\alpha_i)_{i\in I} \subset P$ be a finite family such that $\sum_{i\in I} g(\alpha_i) = g(\beta) \in \bar{R}$. Let $Y = X/\operatorname{span}(P_s)$ and let $f\colon X \to Y$ be the canonical map. Since $R' \subset P_s$, we have $X' \subset \operatorname{span}(P_s)$ and hence a linear map $h\colon \bar{X} \to Y$ satisfying $f = h \circ g$. Let $C = \mathbb{R}_+[f(P)]$ as in Prop. 10.17. Then $f(\beta) = h(g(\beta)) = \sum_i h(g(\alpha_i)) = \sum_i f(\alpha_i) \in C$ because C is additively closed. Thus $\beta \in R \cap f^{-1}(C) = R \cap K = P$ by Prop. 10.17, and hence $g(\beta) \in g(P)$. This shows $g(P)$ is parabolic.

The inclusion from right to left in (1) is obvious. Conversely, let $\alpha \in g^*(g(P))$, so $\alpha \in R$ and $g(\alpha) \in g(P)$. Then $f(\alpha) = h(g(\alpha)) \in h(g(P)) = f(P) \subset C$, whence $\alpha \in R \cap f^{-1}(C) = R \cap K = P$.

In (2), $P_s \subset g^*(g(P))$ is clear. Conversely, let $\alpha \in g^*(g(P_s))$. Then $g(\alpha) \in g(P_s)$ so $g(\alpha) = g(\beta)$ for some $\beta \in P_s$. Hence $\alpha - \beta \in X' \subset \operatorname{span}(P_s)$ and therefore $\alpha \in R \cap \operatorname{span}(P_s) = P_s$ since P_s is full by Prop. 10.17(b). For the second formula, let

$\alpha \in P_s$. Then $\pm\alpha \in P$, so $\pm g(\alpha) \in g(P)$ or $g(\alpha) \in g(P)_s$, proving the inclusion from right to left. Conversely, let $\alpha \in R$ and $g(\alpha) \in g(P)_s$. Then $\pm g(\alpha) = g(\pm\alpha) \in g(P)$, so $\pm\alpha \in g^*(g(P)) = P$ (by (1)) or $\alpha \in P_s$, which proves the inclusion from left to right. Now (3) follows from (2) and the fact that a parabolic subset is the disjoint union of its symmetric and unipotent part.

(b) This is immediately verified from the definitions.

(c) Since $g\colon R \to \bar{R}$ is surjective, we have $g(g^*(Q)) = Q$, so the first assertion follows from (1). The second statement follows from (2) and the fact that a parabolic subset is a positive system if and only if its symmetric part is $\{0\}$.

For the last statement, let $P = R_+(l)$ be of scalar type (for some linear form l on X) with $R' \subset P_s$. Since $P_s = R_0(l)$ by 10.8.2, we have $X' \subset \mathrm{Ker}(l)$, so l induces a linear form $\bar{l}$ on $\bar{X}$, and then it is easy to see that $g(P) = \bar{R}_+(\bar{l})$. Conversely, if $Q = \bar{R}_+(m)$ for some linear form m on $\bar{X}$, one checks that $g^*(Q) = R_+(m \circ g)$.

§11. Parabolic subsets of root systems and presentations of the root lattice and the Weyl group

In this section, (R, X) is a root system unless specified otherwise.

11.1. Lemma. *Let (R, X) be a finite root system. For a subset P of R the following conditions are equivalent:*

(i) *P is parabolic,*

(ii) *there exists a root basis B of R and a partition $B = B_u \dot\cup B_s$ of B such that, denoting by q_β ($\beta \in B$) the dual coweights determined by B as in* 7.10.3*,*

$$P = \bigcap_{\beta \in B_u} R_+(q_\beta), \tag{1}$$

(iii) *there exists a coweight q of R such that $P = R_+(q)$,*

(iv) *there exists $f \in X^*$ such that $P = R_+(f)$.*

In this case, B_s is a root basis of P_s, the symmetric part of P, and we have $B_s = B \cap P_s$ and $B_u = B \cap P_u$.

Proof. (i) $\Longrightarrow$ (ii): This follows from A.16.

(ii) $\Longrightarrow$ (iii): Let $q = \sum_{\beta \in B_u} q_\beta$. Then q is a coweight of R, and clearly $\alpha \in P$ implies $\langle \alpha, q \rangle = \sum_{\beta \in B_u} \langle \alpha, q_\beta \rangle \geqslant 0$. Conversely, if $\langle \alpha, q \rangle \geqslant 0$ but $\langle \alpha, q_\gamma \rangle < 0$ for some $\gamma \in B_u$ then, since all $\langle \alpha, q_\beta \rangle$ ($\beta \in B$) are of the same sign, it would follow that all $\langle \alpha, q_\beta \rangle \leqslant 0$, and hence $\langle \alpha, q \rangle < 0$, contradiction. Therefore, $\langle \alpha, q \rangle \geqslant 0$ implies $\alpha \in P$ by (1), and (iii) follows. The remaining implications (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i) and the statements concerning B_s and B_u are obvious.

11.2. Proposition. *Let P be a parabolic subset of a root system (R, X), let $\mathcal{Q}(R) = \mathbb{Z}[R]$ be the group of radicial weights of R, and $K = \mathbb{R}_+[P]$ the convex cone spanned by P. Then*

$$K \cap \mathbb{Z}[R] = \mathbb{N}[P]. \tag{1}$$

Hence, the restriction of the preorder $\preccurlyeq_P$ of 10.7 *to $\mathbb{Z}[R]$ coincides with the restriction of the preorder $\preccurlyeq_K$ determined by the cone K.*

Proof. The inclusion from right to left is obvious. Conversely, let $x = \sum_{i=1}^m n_i \alpha_i = \sum_{j=1}^k c_j \gamma_j$, where $n_i \in \mathbb{Z}$, $\alpha_i \in R$, $c_j > 0$ and $\gamma_j \in P$. Let X' be the span of the α_i and γ_j, and put $R' = R \cap X'$ and $P' = P \cap R'$. Then R' is a finite root system in X', and P' is a parabolic subset of R'. Choose a basis B' of R' and a partition $B' = B'_u \dot\cup B'_s$ of B' describing P' as in 11.1.1, and let $n_\beta = \langle x, q_\beta \rangle$. Then $x = \sum_{\beta \in B'} n_\beta \beta$, and since $x \in \mathbb{Z}[R']$, the n_β are integers. If $\beta \in B'_u$, we have $\langle \gamma_j, q_\beta \rangle \geqslant 0$ because $\gamma_j \in P'$, and therefore $n_\beta = \sum_j c_j \langle \gamma_j, q_\beta \rangle \geqslant 0$. Also, 11.1.1 shows $-B'_s \subset P'$. Hence, for a suitable choice of signs at the $\beta \in B'_s$,

$$x = \sum_{\beta \in B'} n_\beta \beta = \sum_{\beta \in B'_s} |n_\beta| (\pm\beta) + \sum_{\beta \in B'_u} n_\beta \beta$$

shows that $x \in \mathbb{N}[P]$.

11.3. LEMMA. *Let R_0 be a closed subsystem of a root system (R, X). Then the full subsystem*

$$S := R \cap \operatorname{span}(R \setminus R_0)$$

is a direct summand of R:

$$R = S \oplus (R_0 \cap (R \setminus R_0)^\perp). \tag{1}$$

Every element of $S \cap R_0^\times$ is the difference of two roots in $R_{\text{ind}} \setminus R_0$.

Proof. We prove the first statement by applying Lemma 3.11. Since S is full it remains to show that any $\gamma \in R^\times$ which is not perpendicular to S is already in S. Thus let $\gamma \not\perp S$. As $R \setminus R_0 \subset S$, we may assume $\gamma \in R_0$. Since every element of S is a linear combination of $R \setminus R_0$, there exists $\beta \in R \setminus R_0$ such that $\gamma \not\perp \beta$, and since also $-\beta \in R \setminus R_0$, it is no restriction to assume $\langle \gamma, \beta^\vee \rangle < 0$. Moreover, if β is divisible then $\beta' = \beta/2 \notin R_0$, so we may assume that $\beta \in R_{\text{ind}}$. If $\gamma + \beta = 0$ then $\gamma = -\beta \in R_0 \cap (R \setminus R_0)$ which is impossible. Hence $\alpha := \gamma + \beta \in R_{\text{ind}}^\times$ by A.3. Here $\alpha \in R \setminus R_0$ else $\beta = \alpha - \gamma$ would be in R_0 because R_0 is an additively closed subsystem. It follows that $\gamma = \alpha - \beta \in R \cap \operatorname{span}(R \setminus R_0) = S$. The decomposition (1) is now immediate from 3.11. The last statement was obtained in the proof just given.

11.4. PROPOSITION. *Let (R, X) be an irreducible root system, and let $f \colon X \to Y$ be a surjective linear map onto some real vector space Y. Then $f(R)$ is irreducible in the following sense: If $Y = Y_1 \oplus Y_2$ is the direct sum of two subspaces and $f(R) \subset Y_1 \cup Y_2$, then Y_1 or Y_2 is trivial. In particular,*

(i) *if $(f(R), Y)$ is a root system then $f(R)$ is irreducible in the usual sense,*
(ii) *if Y is a Euclidean space, $f(R)^\times$ cannot be written as the union of two nonempty orthogonal subsets.*

This is a straightforward generalization of a result by Allison, Berman and Pianzola [**2**, Lemma 3.34] who considered a finite root system R and a Euclidean space Y. The special case (i) for a finite R had been proven before by Doković and Thăńg [**25**, Prop. 1, Prop. 2].

Proof. Suppose $Y = Y_1 \oplus Y_2$ and $f(R) \subset Y_1 \cup Y_2$. We may assume that $Y \neq 0$. Then $f(R)^\times \neq \emptyset$ since $f(R)$ spans Y. For $i = 1, 2$ we put $R_i = \{\alpha \in R : 0 \neq f(\alpha) \in Y_i\} \subset R^\times$, and claim

$$R_1 \perp R_2. \tag{1}$$

Indeed, suppose to the contrary that there exist roots $\alpha_i \in R_i$ such that $\alpha_1 \not\perp \alpha_2$. Since $R_i = -R_i$ we may assume $\langle \alpha_1, \alpha_2^\vee \rangle < 0$ whence $\alpha_1 + \alpha_2 \in R$. Because $0 \neq f(\alpha_i) \in Y_i$ this contradicts $f(\alpha_1) + f(\alpha_2) = f(\alpha_1 + \alpha_2) \in Y_1 \cup Y_2$. Thus $R_1 \perp R_2$.

Since $Y \neq 0$ the closed subsystem $R_0 = R \cap \operatorname{Ker} f$ is proper, hence $R = R \cap \operatorname{span}(R \setminus R_0)$ by 11.3 and irreducibility of R. Moreover, again by 11.3, every $\alpha \in R_0^\times$ has the form $\alpha = \beta - \gamma$ for suitable roots $\beta, \gamma \in R \setminus R_0 = R_1 \cup R_2$. Since $0 = f(\alpha) = f(\beta) - f(\gamma)$ we in fact have $\beta, \gamma \in R_1$ or $\beta, \gamma \in R_2$. Thus $R = S_1 \cup S_2$ for $S_i = R_i \cup (R_0 \cap (R_i - R_i))$, where $S_1^\times \perp S_2^\times$ by (1). Irreducibility of R implies $S_i = 0$ for a suitable i, hence also $Y_i = \operatorname{span} f(R_i) = 0$.

The special case (i) is obvious. For (ii) it suffices to note that any decomposition $f(R)^\times = T_1 \cup T_2$, where $T_1 \perp T_2$ and $T_i \neq \emptyset$, gives rise to a decomposition $Y = Y_1 \oplus Y_2$ with nontrivial orthogonal subspaces $Y_i = \operatorname{span}(T_i)$ and $f(R) \subset Y_1 \cup Y_2$.

11.5. Definition. From Lemma 11.3, it is clear that for an additively closed subsystem R_0 of a root system (R, X), the following conditions are equivalent:

(i) $R \setminus R_0$ spans X,
(ii) R_0 contains no connected component of R,
(iii) every $\gamma \in R_0^\times$ is of the form $\gamma = \alpha - \beta$ for suitable $\alpha, \beta \in R \setminus R_0$.

A subsystem satisfying these conditions is said to be *effective*. If R is irreducible, it is clear that any proper closed subsystem is effective. In general, Lemma 11.3 shows that R splits into an effective part S and a ("totally ineffective") direct summand $T \subset R_0$.

By abuse of language, we will call a parabolic subset P of R effective if its symmetric part $P_s = P \cap (-P)$ (which is additively closed, being the intersection of two such subsets) is effective. Note that by Proposition 10.17(b), P_s is even a full subsystem of R.

11.6. Lemma. *For a parabolic subset P of a root system (R, X), the following conditions are equivalent:*

(i) *P is effective,*
(ii) *P_u spans X,*
(iii) *every $\mu \in P_s^\times$ is of the form $\mu = \alpha - \beta$ where $\alpha, \beta \in P_u$ and $c_{\alpha\beta} \leqslant 1$ (cf. 4.4),*
(iv) *every $\mu \in P_s^\times$ has the form $\mu = s_\beta(\gamma)$ for suitable $\beta, \gamma \in P_u$.*

Proof. (i) $\iff$ (ii): This follows immediately from $R \setminus P_s = P_u \cup (-P_u)$ and 11.5.

(ii) $\Longrightarrow$ (iii): Since P_s is effective in the sense of 11.5, it is clear that every $\mu \in P_s^\times$ is the difference of two elements in $R \setminus P_s = P_u \cup (-P_u)$. But $(P_u + P_u) \cap R \subset P_u$ by 10.6.2, so μ is the difference of two elements in P_u, say, $\mu = \alpha - \beta$. In particular, then, α, β and μ all belong to the same connected component of R, so $c_{\alpha\beta}$ is well defined. We also have $\mu = -s_\mu(\mu) = s_\mu(\beta) - s_\mu(\alpha)$ where $s_\mu(\alpha) = \alpha - \langle \alpha, \mu^\vee \rangle \mu \in P_u$ by 10.6.2, and similarly $s_\mu(\beta) \in P_u$. Since $c_{s_\mu(\beta)s_\mu(\alpha)} = c_{\beta\alpha}$ (by 4.4.3) $= c_{\alpha\beta}^{-1}$, it follows that every $\mu \in P_s^\times$ has a representation $\mu = \alpha - \beta$ where $c_{\alpha\beta} \leqslant 1$.

(iii) $\Longrightarrow$ (iv): Let $\mu = \alpha - \beta$ where $\alpha, \beta \in P_u$. Note that β and α must be linearly independent, for otherwise $\alpha = 2\beta$ or $\beta = 2\alpha$, which would result in $\mu = \beta$ or $\mu = -\alpha$. Hence by A.2, we have $n := \langle \alpha, \beta^\vee \rangle \in \{0, 1, -1\}$. It follows that $\gamma := s_\beta(\mu) = \alpha - n\beta + \beta = \alpha + (1-n)\beta \in P_u$ by 10.6.2, and then $\mu = s_\beta(\gamma)$.

(iv) $\Longrightarrow$ (ii): From $\mu = \gamma - \langle \gamma, \beta^\vee \rangle \beta$ it follows that $P_s \subset \operatorname{span}(P_u)$ and hence $R \subset \operatorname{span}(P_u)$, showing P_u spans X.

Remark. A sharper result concerning the representation (iii) of a root $\mu \in P_s$ will be given below in Prop. 11.14.

11.7. Proposition. *Let (R, X) and (S, Y) be root systems and let P be an effective parabolic subset of R. Assume that $f\colon P_u \to S$ is a map satisfying*

$$\langle f(\alpha), f(\beta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle \tag{1}$$

for all $\alpha, \beta \in P_u$. Then f extends uniquely to an embedding $f\colon (R, X) \to (S, Y)$ of root systems in the sense of 3.6.

Proof. Uniqueness of f is clear from the fact that P_u spans X by 11.6(ii). To prove existence, we will use Cor. 7.7 and therefore must extend f to a map $f\colon R \to S$ which satisfies (1) for all $\alpha, \beta \in R$. This is done as follows. Put $f(0) := 0$ and $f(\alpha) := -f(-\alpha)$ for $\alpha \in -P_u$. For every $\mu \in P_s^\times$ choose a representation $\mu = s_\gamma(\delta)$ as in Lemma 11.6(iv), and define $f(\mu) := s_{f(\gamma)}(f(\delta))$.

Relation (1) is obvious if α or β is zero, and follows for $\alpha, \beta \in P_u \,\dot\cup\, (-P_u)$ from our assumption on f and the fact that $(-\beta)^\vee = -\beta^\vee$. Thus it remains to deal with the following cases.

Case 1: $\alpha \in \pm P_u$ and $\beta \in P_s^\times$. By definition of f on $-P_u$ it is no restriction to assume $\alpha \in P_u$. Let $\beta = s_\gamma(\delta) \in P_s^\times$ as in 11.6(iv). Then

$$\begin{aligned}\langle \alpha, \beta^\vee \rangle &= \langle \alpha, s_\gamma(\delta)^\vee \rangle = \langle s_\gamma(\alpha), \delta^\vee \rangle = \langle \alpha - \langle \alpha, \gamma^\vee \rangle \gamma, \delta^\vee \rangle \\ &= \langle f(\alpha) - \langle f(\alpha), f(\gamma)^\vee \rangle f(\gamma), f(\delta)^\vee \rangle = \langle s_{f(\gamma)}(f(\alpha)), f(\delta)^\vee \rangle \\ &= \langle f(\alpha), s_{f(\gamma)}(f(\delta))^\vee \rangle = \langle f(\alpha), f(\beta)^\vee \rangle.\end{aligned}$$

Case 2: $\alpha \in P_s^\times$ and $\beta \in \pm P_u$. This is handled by a similar but simpler computation as Case 1.

Case 3: Both α and β are in $P_s^\times$. Let $\alpha = s_\gamma(\delta)$ for $\gamma, \delta \in P_u$. Then we have, using what we proved in Case 1 above,

$$\begin{aligned}\langle \alpha, \beta^\vee \rangle &= \langle \delta - \langle \delta, \gamma^\vee \rangle \gamma, \beta^\vee \rangle = \langle f(\delta) - \langle f(\delta), f(\gamma)^\vee \rangle f(\gamma), f(\beta)^\vee \rangle \\ &= \langle s_{f(\gamma)}(f(\delta)), f(\beta)^\vee \rangle = \langle f(\alpha), f(\beta)^\vee \rangle,\end{aligned}$$

as claimed.

11.8. Lemma. *Let P be a parabolic subset of a root system R, and let $\preccurlyeq$ be the preorder induced by P on R as in 10.7. For $\alpha \in R^\times$ let $\mathrm{C}(\alpha)$ be the connected component of R containing α. Then $\beta \in P_u$, $\gamma \in R$ and $\beta \preccurlyeq \gamma$ imply $\gamma \in P_u$ and* $\mathrm{C}(\beta) = \mathrm{C}(\gamma)$.

Proof. We have

$$\gamma = \beta + \alpha_1 + \cdots + \alpha_n \tag{1}$$

where $\alpha_i \in P$, hence $\gamma \in P_u$ by 10.6.2. If $\beta \not\perp \gamma$ the assertion is clear, so we may assume $\beta \perp \gamma$, in particular, $\gamma \neq \beta$. Then (1) implies

$$0 = \langle \gamma, \beta^\vee \rangle = \langle \beta, \beta^\vee \rangle + \sum_{i=1}^{n} \langle \alpha_i, \beta^\vee \rangle = 2 + \sum_{i=1}^{n} \langle \alpha_i, \beta^\vee \rangle. \tag{2}$$

We prove $\mathrm{C}(\gamma) = C(\beta)$ by induction on n. For $n = 1$, (2) shows $\langle \alpha_1, \beta^\vee \rangle < 0$. Hence $\alpha_1 \in \mathrm{C}(\beta)$, and since connected components are additively closed, it follows that also $\gamma = \beta + \alpha_1 \in \mathrm{C}(\beta)$. As $\gamma \in P_u$ is not zero, we see that $\mathrm{C}(\gamma) = \mathrm{C}(\beta)$.

In general, (2) implies $\langle \alpha_i, \beta^\vee \rangle < 0$ for some i, say, $\langle \alpha_1, \beta^\vee \rangle < 0$. By A.3, $\beta' := \beta + \alpha_1 \in R$, and even $\beta' \in P_u$ by 10.6.2. Now $\gamma = \beta' + \sum_{i=2}^n \alpha_i \succcurlyeq \beta'$, thus $\mathrm{C}(\gamma) = \mathrm{C}(\beta')$ (by induction) $= \mathrm{C}(\beta)$ (by the case $n = 1$).

11.9. PROPOSITION. *Let $P \subset R$ be parabolic with symmetric part P_s and unipotent part P_u. Then the following conditions are equivalent:*

(i) *R is irreducible,*

(ii) *P is connected (in the sense of* 3.12*).*

If P is effective (in particular, if P is a positive system), then these conditions are also equivalent to:

(iii) *P_u is connected,*

(iv) *R is directed with respect to the relation $\preccurlyeq$ induced by P as in* 10.7*.*

In these conditions hold, then in both sets, P and P_u, two elements can always be connected by a chain of length at most 2.

Proof. (i) $\Longleftrightarrow$ (ii): Suppose that R is irreducible, and let $\alpha, \beta \in P$ be orthogonal. By 3.13 and 3.12 there exists a connecting chain α, γ, β in R of length 2. Since $\alpha, -\gamma, \beta$ is also a connecting chain and since γ or $-\gamma$ lies in P, one of the two chains lies in P, proving that P is connected. Conversely, if P is connected, it is contained in an irreducible component C of R, but then also $R = P \cup (-P) \subset C$, showing $R = C$ is irreducible.

Now assume P effective.

(ii) $\Longrightarrow$ (iii): Let $\alpha, \beta \in P_u$. If $\alpha \not\perp \beta$ we are done so assume $\alpha \perp \beta$. Then we can choose a connecting chain $\alpha \not\perp \gamma \not\perp \beta$ in P where we may assume $\gamma \notin P_u$ whence $\gamma \in P_s$. Since P_s is symmetric we may replace γ by $-\gamma$ if necessary and thus assume $\langle \gamma, \alpha^\vee \rangle < 0$. Then $\delta = \alpha + \gamma \in R$, and so $\delta \in P_u$ by 10.6.2. Note that $\langle \delta, \beta^\vee \rangle = \langle \alpha + \gamma, \beta^\vee \rangle = \langle \gamma, \beta^\vee \rangle \neq 0$. Therefore $\alpha \not\perp \delta \not\perp \beta$ is a connecting chain in P_u unless $\alpha \perp \delta$. But in this case, $0 = \langle \alpha + \gamma, \alpha^\vee \rangle$ so $\langle \gamma, \alpha^\vee \rangle = -2$, and hence $\varepsilon = s_\alpha(\gamma) = \gamma + 2\alpha = \delta + \alpha \in P_u$ since P_u is additively closed, so $\alpha \not\perp \varepsilon \not\perp \beta$ is a connecting chain in P_u.

(iii) $\Longrightarrow$ (i): If P_u is connected, then it is contained in a connected component, say C, of R, and since P_u spans X by 11.6, it follows that $C = R$ is connected.

(i) $\Longrightarrow$ (iv): Let R be irreducible and let $\alpha, \beta \in R$. By 3.15(b), there exists a finite full irreducible subsystem F of R containing α and β. Then $P \cap F$ is a parabolic subset in F and hence can be described by a root basis B of F and a partition $B = B_u \dot\cup B_s$ of B as in 11.1. By [**12**, VI, §1.8, Prop. 25] F has a maximal (highest) root $\tilde\alpha$ with the property that $\tilde\alpha - \gamma \in \mathbb{N}[B]$ for all $\gamma \in F$. As $B \subset P$, it follows that $\tilde\alpha \succcurlyeq \alpha$ and $\tilde\alpha \succcurlyeq \beta$.

(iv) $\Longrightarrow$ (i): Let R be directed, and let $\alpha, \beta \in P_u$. Then there exists $\gamma \in R$ with $\gamma \succcurlyeq \alpha$ and $\gamma \succcurlyeq \beta$. By Lemma 11.8, we therefore have $\mathrm{C}(\alpha) = \mathrm{C}(\gamma) = \mathrm{C}(\beta)$. Hence P_u is contained in a connected component, say C, of R, and since P_u spans X by 11.6, it follows that $C = R$ is connected.

Remark. Condition (iv) is in a sense the infinite analogue to the existence of a highest root in finite irreducible root systems. Indeed, if R is finite we recover the existence of the highest root by applying the proposition to the case where P is a positive system. Then R is a finite partially ordered directed set and therefore has a maximum.

11.10. LEMMA. *Let $\alpha_1, \alpha_2, \alpha_3 \in R$ with $\sum_{i=1}^3 \alpha_i \in R$ and $\alpha_i + \alpha_j \neq 0$ for $i \neq j$. Then at least two of the three partial sums $\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3$, $\alpha_1 + \alpha_3$ belong to $R^\times$.*

Proof. Let $\beta = \sum_{i=1}^3 \alpha_i \in R$. If $\beta = 0$ then all three partial sums belong to R, so we may assume $\beta \neq 0$. Then $2 = \langle \beta, \beta^\vee \rangle = \sum_{i=1}^3 \langle \alpha_i, \beta^\vee \rangle$ shows that, say, $\langle \alpha_1, \beta^\vee \rangle > 0$ and hence $\beta - \alpha_1 = \alpha_2 + \alpha_3 \in R$, by A.3. Now let $(\mid)$ be an invariant inner product. Then $0 < \|\alpha_2 + \alpha_3\|^2 = (\beta - \alpha_1 | \alpha_2 + \alpha_3) = (\beta|\alpha_2) + (\beta|\alpha_3) - (\alpha_1|\alpha_2) - (\alpha_1|\alpha_3)$, so one of these four terms must be positive. If $(\beta|\alpha_2) > 0$ or $-(\alpha_1|\alpha_3) > 0$ then $\beta - \alpha_2 = \alpha_1 + \alpha_3 \in R$ by A.3, and similarly $(\beta|\alpha_3) > 0$ or $-(\alpha_1|\alpha_2) > 0$ implies $\beta - \alpha_3 = \alpha_1 + \alpha_2 \in R$.

11.11. Lemma. *Let P be a parabolic subset of a root system R, and let $\kappa, \lambda \in P_s^\times$ such that also $\mu := \kappa + \lambda \in P_s^\times$. If $\kappa = \alpha - \beta$ and $\lambda = \gamma - \delta$ for $\alpha, \beta, \gamma, \delta \in P_u$, then either $\alpha - \beta + \gamma$ or $\delta - \alpha + \beta$ is in P_u.*

Proof. No two of the three roots $(\alpha_1, \alpha_2, \alpha_3) := (\kappa, \gamma, -\delta)$ have sum zero. Indeed, $\alpha_1 + \alpha_2 = 0$ would imply $\gamma = -\kappa \in P_u \cap P_s = \emptyset$. Similarly, $\alpha_1 + \alpha_3 = 0$ is impossible, and $\alpha_2 + \alpha_3 = 0$ would imply $\kappa = \mu$ or $\lambda = 0$. Also, $\kappa + \gamma - \delta = \mu \in R$, so by Lemma 11.10, $\kappa + \gamma = \alpha - \beta + \gamma$ or $\kappa - \delta = \alpha - \beta - \delta$ is in $R^\times$, and then even in P_u, by 10.6.2.

11.12. Proposition. *Let P be an effective parabolic subset of a root system R. Then the group $\mathcal{Q}(R)$ of radicial weights is isomorphic to the abelian group presented by generators x_α ($\alpha \in P_u$) and relations*

$$x_\alpha + x_\beta = x_{\alpha+\beta} \quad \text{whenever } \alpha, \beta \text{ and } \alpha + \beta \in P_u, \tag{1}$$
$$x_\alpha - x_\beta = x_\gamma - x_\delta \quad \text{for all } \alpha, \beta, \gamma, \delta \in P_u \text{ with } \alpha - \beta = \gamma - \delta \in P_s^\times. \tag{2}$$

Proof. Let G be the group defined in the statement above. Since the relations (1) and (2) hold in $\mathcal{Q}(R)$ and since P_u generates $\mathcal{Q}(R)$ by 11.6, we have a well-defined epimorphism $G \to \mathcal{Q}(R)$ mapping $x_\alpha \mapsto \alpha$. To construct a map in the opposite direction, we first define x_ξ for all $\xi \in R$ by

$$x_0 = 0, \tag{3}$$
$$x_\alpha = -x_{-\alpha} \qquad \text{for } \alpha \in (-P_u), \tag{4}$$
$$x_\mu = x_\alpha - x_\beta \qquad \text{for } \mu = \alpha - \beta \in P_s^\times \text{ and } \alpha, \beta \in P_u. \tag{5}$$

Note that x_μ for $\mu \in P_s^\times$ is well-defined by Lemma 11.6(iii) and (2). Also, from (3) – (5) it is clear that

$$x_{-\xi} = -x_\xi \quad \text{for all } \xi \in R. \tag{6}$$

In order to show that $\alpha \mapsto x_\alpha$ extends to a homomorphism $\mathcal{Q}(R) \to G$, we use the presentation of $\mathcal{Q}(R)$ given in 7.6 and thus have to show that

$$x_{\xi+\eta} = x_\xi + x_\eta \tag{7}$$

whenever ξ, η and $\xi + \eta$ belong to R. Obviously (7) holds for $\xi = 0$ or $\eta = 0$. By symmetry of (7) in ξ and η and because of (6), it suffices to consider the two cases $\xi \in P_u$, $\eta \in R^\times$ arbitrary, and $\xi, \eta \in P_s^\times$.

Case $\beta = \xi \in P_u$: If also $\eta \in P_u$ we are done by (1). Next, let $\eta \in P_s^\times$, written as $\eta = \gamma - \delta$ for $\gamma, \delta \in P_u$. By assumption $\xi + \eta = \beta + \gamma - \delta =: \alpha \in R$, and then $\alpha \in (P_u + P_s) \cap R \subset P_u$. Since $\alpha - \beta = \gamma - \delta \in P_s^\times$, relation (2) and the definition (5) imply $x_{\xi+\eta} - x_\xi = x_\alpha - x_\beta = x_\gamma - x_\delta = x_\eta$, as desired.

Now let $\eta \in (-P_u)$, so $-\eta \in P_u$. If $\xi + \eta = 0$ we are done by (3) and (6), so we may assume $\xi + \eta \in R^\times = P_u \dot\cup P_s^\times \dot\cup (-P_u)$ and accordingly have to consider the following three subcases:

Subcase $\xi + \eta \in P_u$*:* Then $\xi = (\xi + \eta) + (-\eta)$ where all three terms are in P_u, so $x_\xi = x_{\xi+\eta} + x_{-\eta}$ (by (1)) $= x_{\xi+\eta} - x_\eta$ (by (6)).

Subcase $\xi+\eta \in P_s^\times$*:* Then $\xi+\eta = \xi-(-\eta)$, so by (5) and (6), $x_{\xi+\eta} = x_\xi - x_{-\eta} = x_\xi + x_\eta$.

Subcase $\xi + \eta \in (-P_u)$*:* Then $-\eta = \xi + (-(\xi + \eta))$ where all three terms are in P_u, so the required relation follows again from (1) and (6).

Case $\xi, \eta \in P_s^\times$*:* Here we write $\xi = \alpha - \beta$ and $\eta = \gamma - \delta$ and use 11.11 and the cases already established. The straightforward verification is left to the reader.

11.13. Theorem. *Let (R, X) be a root system, and let $P \subset R$ be an effective parabolic subset with unipotent part P_u and symmetric part P_s. Then the Weyl group $W(R)$ is presented by generators $\{h_\alpha : \alpha \in P_u\}$ and the following relations:*

(R1) $\quad h_\alpha = h_{2\alpha}$ *if α and 2α are in P_u,*

(R2) $\quad h_\alpha h_\beta h_\alpha = h_{\pm s_\alpha(\beta)}$ *if α, β, and $\pm s_\alpha(\beta) \in P_u$,*

(R3) $\quad h_\alpha h_\beta h_\alpha = h_\gamma h_\delta h_\gamma =: h_\mu$ *if $\alpha, \beta, \gamma, \delta \in P_u$ and $\mu := s_\alpha(\beta) = \pm s_\gamma(\delta) \in P_s$,*

(R4) $\quad h_\beta \cdot h_\alpha h_\gamma h_\alpha = h_\alpha h_\gamma h_\alpha \cdot h_\beta$ *if $\alpha, \beta, \gamma \in P_u$ satisfy $\beta \perp s_\alpha(\gamma) = -s_\gamma(\alpha) \in P_s$, $s_\alpha(\beta) \in P_s$, and $s_\gamma(\beta) \in P_s$.*

We will later in 11.17 evaluate these relations more precisely, using the standard representation derived in 11.14, and also (in 18.12) for the special case of a 3-graded root system (R, R_1) where $P = R_0 \dot\cup R_1$. Note also that by Lemma 11.6(iv), every $\mu \in P_s^\times$ has the form $\mu = s_\alpha(\beta)$ for suitable $\alpha, \beta \in P_u$.

Proof. (a) Let H be the group with the presentation above. We first show that there is a unique homomorphism $H \to W(R)$ mapping h_α to s_α, for all $\alpha \in P_u \cup (-P_u)$. Indeed, this amounts to showing that the relations (R1) – (R4) hold in $W(R)$, when we replace the h_α by s_α. For (R1) this is clear from 4.3(b), while the remaining relations follow from 3.9.2 and 3.9.4.

(b) To construct a homomorphism $W(R) \to H$ in the opposite direction, we use the presentation of $W(R)$ given in Theorem 5.12. Thus we define h_μ for $\mu \in P_s^\times$ as in (R3) and put

$$h_{-\alpha} := h_\alpha \quad \text{for } \alpha \in P_u. \tag{1}$$

Note that the relations (R2) and (R3) then hold for all α, β, γ, δ in $P_u \cup (-P_u)$. Now we must show that the h_ξ ($\xi \in R^\times$) satisfy the relations of 5.12, i.e.,

$$h_\xi = h_\eta \quad \text{if } \xi \text{ and } \eta \text{ are linearly dependent,} \tag{2}$$

$$h_{s_\xi(\eta)} = h_\xi h_\eta h_\xi \quad \text{for all } \xi, \eta \in R^\times. \tag{3}$$

We first establish

$$h_\xi^2 = 1 \quad \text{for all } \xi \in R^\times. \tag{4}$$

Indeed, putting $\alpha = \beta$ in (R2) and observing (1) yields $h_\alpha^3 = h_{-\alpha} = h_\alpha$ (by (R1)) for all $\alpha \in P_u \cup (-P_u)$, and then $h_\mu^2 = 1$ follows immediately from the definition of h_μ.

Proof of (2)*:* If ξ and η are in $P_u \cup (-P_u)$ then (2) holds by (R1) and (1). If $\mu \in P_s^\times$ then $h_\mu = h_{-\mu}$ holds by (R3), so it remains to show $h_\mu = h_{2\mu}$ for $\mu, 2\mu \in P_s^\times$. Write $\mu = s_\alpha(\beta)$ for $\alpha, \beta \in P_u$. Then $2\beta = s_\alpha(2\mu) \in P_u$ and $2\mu = s_\alpha(2\beta)$, so $h_{2\mu} = h_\alpha h_{2\beta} h_\alpha = h_\alpha h_\beta h_\alpha = h_\mu$ by (R1) and (R3).

Proof of (3)*:* We distinguish the following cases:

(i) $\xi = \alpha$ and $\eta = \beta$ are in $\pm P_u$: Then (3) holds by (R2) and (R3).

(ii) $\xi \in P_u$, $\eta \in P_s$: This case will be proved below.

(iii) $\xi \in P_s^\times$, $\eta \in R^\times$: Assuming that (ii) has been established, we show (iii). Indeed, let $\xi = s_\alpha(\beta) \in P_s^\times$ where $\alpha, \beta \in P_u$ as in Lemma 11.6(iv). Then $s_\xi = s_\alpha s_\beta s_\alpha$ by 3.9.2. For easier notation, put $a = h_\alpha$, $b = h_\beta$. Then $h_\xi = aba$ by (R3), and using (i) and (ii) repeatedly, we have

$$h_{s_\xi(\eta)} = h_{s_\alpha s_\beta s_\alpha(\eta)} = a h_{s_\beta s_\alpha(\eta)} a = ab h_{s_\alpha(\eta)} ba = aba \cdot h_\eta \cdot aba = h_\xi h_\eta h_\xi.$$

We now come to the proof of (3) in case (ii), and henceforth assume $\xi = \gamma \in P_u$ and $\eta = \mu \in P_s^\times$. Then we must show $h_{s_\gamma(\mu)} = h_\gamma h_\mu h_\gamma$. If $\langle \mu, \gamma^\vee \rangle \neq 0$ then $\delta := s_\gamma(\mu) = \mu - \langle \mu, \gamma^\vee \rangle \gamma \in \pm P_u$ by 10.6.2, and hence $\mu = s_\gamma(\delta)$ implies $h_\mu = h_\gamma h_\delta h_\gamma$ by (R3), and therefore $h_\gamma h_\mu h_\gamma = h_\gamma^2 h_\delta h_\gamma^2 = h_\delta$ (by (4)) $= h_{s_\gamma(\mu)}$, as claimed.

We therefore assume $\gamma \perp \mu$ from now on. Then $s_\gamma(\mu) = \mu$, and thus we must show $[h_\gamma, h_\mu] = 1$ where the brackets denote the group commutator.

Write $\mu = s_\alpha(\beta)$ (where $\alpha, \beta \in P_u$) as in Lemma 11.6(iv), put

$$\tilde\alpha := s_\gamma(\alpha), \quad \tilde\beta := s_\gamma(\beta),$$

and note that, by 3.9.2,

$$\mu = s_\gamma(\mu) = s_\gamma(s_\alpha(\beta)) = s_\gamma s_\alpha s_\gamma(\tilde\beta) = s_{\tilde\alpha}(\tilde\beta). \tag{5}$$

Furthermore, we have the following alternative:

$$\textit{Either } \tilde\alpha \textit{ and } \tilde\beta \textit{ are both in } P_u \cup (-P_u) \textit{ or they are both in } P_s. \tag{6}$$

Indeed, let $n = \langle \beta, \alpha^\vee \rangle = \langle \tilde\beta, \tilde\alpha^\vee \rangle$. Then $n \neq 0$, because otherwise $\mu = s_\alpha(\beta) = \beta - n\alpha = \beta \in P_u$. Now $\mu = \tilde\beta - n\tilde\alpha \in P_s$ and the fact that P_s is a full subsystem by 10.17(b) show that $\tilde\alpha \in P_s$ if and only if $\tilde\beta \in P_s$, proving (6).

We put $a = h_\alpha$, $b = h_\beta$, as well as $m = h_\mu$ and $c = h_\gamma$, for easier notation. By (R3) we then have

$$m = aba. \tag{7}$$

Case 1: Both $\tilde\alpha$ and $\tilde\beta$ are in $P_u \cup (-P_u)$. Then, putting $\tilde a = h_{\tilde\alpha}$ and $\tilde b = h_{\tilde\beta}$, we have by (5), (R3) and (7) that $m = aba = \tilde a \tilde b \tilde a$, $\tilde a = cac$ and $\tilde b = cbc$. Since $c^2 = m^2 = 1$ by (4), it follows that $[c, m] = cmc \cdot m = c \cdot aba \cdot c \cdot m = cac \cdot cbc \cdot cac \cdot m = \tilde a \tilde b \tilde a \cdot m = m^2 = 1$.

Case 2: Both $\tilde\alpha$ and $\tilde\beta$ are in P_s. Then α and γ must be linearly independent, otherwise a multiple of α would be in P_s, and therefore so would be α, since P_s is full. We let $\delta := s_\alpha(\gamma)$ and note first that $\mu \perp \gamma$ implies $\beta = s_\alpha(\mu)$ is orthogonal to $s_\alpha(\gamma) = \delta$. Now we distinguish the following two subcases:

Subcase 2.1: $\delta \in \pm P_u$. Putting $d := h_\delta$, and using (i) and $s_\delta(\beta) = \beta$, which follows from $\delta \perp \beta$, we have $dbd = b$ or $[d, b] = 1$. Also, $d = aca$ by (R2). This implies $[c, m] = [c, aba] = [aca, b]$ (by (4)) $= [d, b] = 1$.

Subcase 2.2: $\delta \in P_s$. Let $p = \langle \alpha, \gamma^\vee \rangle$ and $q = \langle \gamma, \alpha^\vee \rangle$. Since both $\tilde{\alpha} = \alpha - p\gamma$ and $\delta = \gamma - q\alpha$ are in P_s, 10.6.2 implies that $p \geqslant 1$ and $q \geqslant 1$. Furthermore, $-(\tilde{\alpha} + \delta) = (q-1)\alpha + (p-1)\gamma \in K_u \cap \operatorname{span}(P_s) = \{0\}$ by 10.17.7. From linear independence of α and γ we conclude that $p = q = 1$ and that $\delta = -\tilde{\alpha}$. Hence we are in the situation of relation (R4), so $[aca, b] = 1$, and therefore $[c, m] = [c, aba] = [aca, b] = 1$. This completes the proof.

11.14. Proposition. *Let P be an effective parabolic subset of a root system R.*

(a) *Every root $\mu \in P_s^\times$ can be represented in the form $\mu = \alpha - \beta$ where $\alpha, \beta \in P_u$ satisfy one of the following conditions:*

Type I_n*:* $-\langle \mu, \beta^\vee \rangle = \langle \alpha, \beta^\vee \rangle = 1$ *and hence* $\mu = s_\beta(\alpha)$, $\langle \beta, \alpha^\vee \rangle = n \in \{1, 2, 3\}$,

Type II*:* $-\langle \mu, \beta^\vee \rangle = 2$, *$\alpha$ and β are* weakly orthogonal *in the sense that $\alpha \perp \beta$ but $\alpha \pm \beta \in R^\times$.*

For such a representation, the coroot of μ is given by

$$\mu^\vee = \left\{ \begin{array}{ll} \alpha^\vee - n\beta^\vee & \textit{in type } \mathrm{I}_n \\ \frac{1}{2}(\alpha^\vee - \beta^\vee) & \textit{in type } \mathrm{II} \end{array} \right\} \tag{1}$$

and we have

$$s_\beta(\mu) = \left\{ \begin{array}{ll} \alpha & \textit{in type } \mathrm{I} \\ \alpha + \beta & \textit{in type } \mathrm{II} \end{array} \right\} \in P_u. \tag{2}$$

(b) *The following conditions are equivalent for a root $\mu \in P_s^\times$:*

(i) *μ has a type* I *representation,*

(ii) *μ can be written in the form $\mu = \gamma - \delta$ where $\gamma, \delta \in P_u$ are not orthogonal,*

(iii) *$\langle \mu, \beta^\vee \rangle = -1$ for some $\beta \in P_u$.*

(c) *Let μ and 2μ be in $P_s^\times$. Then μ admits only a type* I *representation, while 2μ admits only a type* II *representation. If $\mu = \alpha - \beta$ is a type* I *representation, then $2\alpha - \beta = -s_\alpha(\beta) \in P_u$ and $2\mu = (2\alpha - \beta) - \beta$ is a type* II *representation.*

Representations of type I or II are called *standard representations.* They are by no means unique. Not only may it happen that μ has several representations of type I or of type II, it may even happen that μ has representations of both types. Example: In $R = \mathrm{B}_3$ consider the coweight q defined by $q(\varepsilon_1) = 2 = q(\varepsilon_2)$ and $q(\varepsilon_3) = 1$, and let $P = R_+(q)$ be the corresponding parabolic subset as in 10.8.1. Then $\mu = \varepsilon_1 - \varepsilon_2 = (\varepsilon_1 - \varepsilon_3) - (\varepsilon_2 - \varepsilon_3)$ is represented in both ways. Nevertheless, standard representations will be an important tool in the following.

Proof. (a) Write $\mu = \alpha - \beta$ as in Lemma 11.6(iii) and let $(\ |\ )$ be an invariant inner product. Then $c_{\alpha\beta} \leqslant 1$ means $\|\alpha\| \leqslant \|\beta\|$, so by Lemma A.4 there are the following of possibilities:

Case	$(\|\alpha\|^2 : \|\mu\|^2 : \|\beta\|^2)$	$-\langle \mu, \beta^\vee \rangle$	$\langle \alpha, \beta^\vee \rangle$	$\langle \beta, \alpha^\vee \rangle$	$\mu^\vee$
I_1	$(1:1:1)$	1	1	1	$\alpha^\vee - \beta^\vee$
I_2	$(1:1:2)$	1	1	2	$\alpha^\vee - 2\beta^\vee$
I_3	$(1:1:3)$	1	1	3	$\alpha^\vee - 3\beta^\vee$
II	$(1:2:1)$	2	0	0	$(\alpha^\vee - \beta^\vee)/2$
III	$(1:3:1)$	3	-1	-1	$(\alpha^\vee - \beta^\vee)/3$

Clearly, the cases $\mathrm{I}_1 - \mathrm{I}_3$ are type I representations, and case II is a type II representation. In case III, we have $\langle \mu, \beta^\vee \rangle = -3$ and $\langle \beta, \mu^\vee \rangle = -1$, by A.2. Hence $\beta' := s_\beta(\mu) = \mu + 3\beta \in R$ and even $\beta' \in P_u$, by 10.6.2. Also, $\langle \beta', \mu^\vee \rangle = \langle \mu + 3\beta, \mu^\vee \rangle = 2 - 3 = -1$, so $\alpha' := s_\mu(\beta') = \beta' + \mu \in R$ and again $\alpha' \in P_u$. It follows that $\mu = \alpha' - \beta'$ where now $\|\mu\| = \|\beta'\| = \|\alpha'\|$, and therefore the representation $\mu = \alpha' - \beta'$ falls under case I_1 and is of type I. Now (a) is evident from the table.

For (b), the implications (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii) are clear. For (ii) $\Longrightarrow$ (i), let $\mu = \gamma - \delta$ and $\gamma \not\perp \delta$. If $\|\gamma\| \leqslant \|\delta\|$ then the assertion follows from the proof of (a). If $\|\gamma\| > \|\delta\|$, write $\mu = -s_\mu(\mu) = s_\mu(\delta) - s_\mu(\gamma)$. Then still $s_\mu(\delta) \not\perp s_\mu(\gamma)$, and now $\|s_\mu(\delta)\| < \|s_\mu(\gamma)\|$, so we are back to the case already dealt with.

If (iii) holds then $\alpha := s_\beta(\mu) = \mu + \beta \in R \cap (P_s + P_u) \subset P_u$ and $\mu = \alpha - \beta$, as well as $s_\beta(\alpha) = s_\beta^2(\mu) = \mu$ and $\langle \alpha, \beta^\vee \rangle = \langle s_\beta(\mu), \beta^\vee \rangle = -\langle \mu, \beta^\vee \rangle = 1$, so $\mu = \alpha - \beta$ is a type I representation.

(c) We have $\langle 2\mu, \beta^\vee \rangle \in 2\mathbb{Z}$, so 2μ cannot have a type I representation. Assume that $\mu = \alpha - \beta$ is a type II representation. Then $\langle \mu, \beta^\vee \rangle = -2$ whence $\langle 2\mu, \beta^\vee \rangle = -4$, which implies $-2\mu = \beta \in P_s^\times \cap P_u$, contradiction. Hence μ and 2μ admit only representations of type I and II, respectively. Let $\mu = \alpha - \beta$ be a type I representation. Then $\langle 2\mu, \beta^\vee \rangle = -2$. Since $2\mu \neq -\beta$, it follows that $\langle \beta, (2\mu)^\vee \rangle = -1 = (1/2)\langle \beta, \mu^\vee \rangle$, and therefore $-2 = \langle \beta, \mu^\vee \rangle = \langle \beta, (s_\beta(\alpha))^\vee \rangle = \langle s_\beta(\beta), \alpha^\vee \rangle = -\langle \beta, \alpha^\vee \rangle$. Then $-s_\alpha(\beta) = 2\alpha - \beta = \mu + \alpha \in P_u$, and $2\mu = (2\alpha - \beta) - \beta$ is a type II representation of 2μ.

11.15. Corollary. *Let P be a proper parabolic subset of an irreducible root system R, and let $(\ |\ )$ be an invariant inner product. Then*

$$\{(\mu|\mu) : \mu \in P_s^\times\} \subset \{(\alpha|\alpha) : \alpha \in P_u\} = \{(\alpha|\alpha) : \alpha \in R^\times\}. \tag{1}$$

In particular, all roots in P_u have the same length if and only if R is simply laced.

Proof. Let $\mu = \alpha - \beta$ be a standard representation of a root in $P_s^\times$. If $\langle \alpha, \beta^\vee \rangle = 1$ then the roots $\mu = s_\beta(\alpha)$ and $\alpha \in P_u$ have the same length. If $\alpha \perp \beta$ this is so for μ and $\alpha + \beta \in P_u$. This proves the inclusion in (1), and then the equality is immediate from the decomposition $R = P_s \,\dot\cup\, P_u \,\dot\cup\, (-P_u)$.

11.16. *Elementary relations.* Let (R, X) be a root system. It will be useful to introduce special names and symbols for some of the possible relations between two roots, besides orthogonality (3.5.4). For $\alpha, \beta \in R^\times$ we define

$$\begin{array}{llll}
\alpha \top \beta & (\alpha \textit{ collinear} \text{ to } \beta) & \iff & \langle \alpha, \beta^\vee \rangle = 1 = \langle \beta, \alpha^\vee \rangle \\
& & \iff & s_\alpha(\beta) = -s_\beta(\alpha) \\
& & \iff & \angle(\alpha, \beta) = \dfrac{\pi}{3}, \\
\alpha \vdash \beta & (\alpha \textit{ governs } \beta) & \iff & \langle \alpha, \beta^\vee \rangle = 1 \text{ and } \langle \beta, \alpha^\vee \rangle = 2 \\
& & \iff & \angle(\alpha, \beta) = \dfrac{\pi}{4} \text{ and } \|\alpha\| < \|\beta\|.
\end{array}$$

Here angles and lengths are understood with respect to some invariant inner product. The symbol $\beta \dashv \alpha$ is equivalent to $\alpha \vdash \beta$. We will refer to the relations $\perp$, $\top$ and $\vdash$ as the *elementary relations*.

We will also use these symbols in sequence. For example, if α and β are weakly orthogonal roots we have $\alpha \vdash (\alpha + \beta) \dashv \beta \perp \alpha$.

11.17. Theorem. *Let P be an effective parabolic subset of a root system (R, X), with unipotent part P_u and symmetric part P_s. Then the Weyl group $W(R)$ is presented by generators $\{t_\alpha : \alpha \in P_u\}$, and the following relations, where always $\alpha, \beta, \gamma, \delta \in P_u$:*

(S1) $\quad t_\alpha = t_{2\alpha}$ *if α and $2\alpha \in P_u$,*

(S2) $\quad t_\alpha t_\beta t_\alpha = t_{\pm s_\alpha(\beta)}$ *if $\pm s_\alpha(\beta) \in P_u$,*

(S3) $\quad t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$ *if $\alpha \top \beta$ and $\alpha - \beta \in P_s$,*

(S4) $\quad t_\alpha t_{\alpha+\beta} t_\alpha = t_\beta t_{\alpha+\beta} t_\beta$ *if α and β are weakly orthogonal and $\alpha - \beta \in P_s$,*

(S5) $\quad t_\beta t_{s_\beta(\mu)} t_\beta = t_\delta t_{s_\delta(\mu)} t_\delta =: t_\mu$ *if $\mu = \alpha - \beta = \gamma - \delta \in P_s$ are two standard representations,*

(S6) $\quad t_\beta \cdot t_\alpha t_\gamma t_\alpha = t_\alpha t_\gamma t_\alpha \cdot t_\beta$ *if $\alpha \top \gamma$, $\alpha - \gamma \in P_s$, $s_\gamma(\beta) \in P_s$ and one of the following holds: $\alpha \top \beta \top \gamma$, or $\alpha \vdash \beta \dashv \gamma$, or $\alpha \dashv \beta \vdash \gamma$.*

Concerning (S5), we recall from 11.14.2 that

$$s_\beta(\mu) = \left\{ \begin{array}{ll} \alpha & \text{in type I} \\ \alpha + \beta & \text{in type II} \end{array} \right\} \in P_u$$

for all standard representations $\mu = \alpha - \beta$, so $t_{s_\beta(\mu)}$ makes sense.

Proof. Let T be the group with the presentation above. Mapping the generators t_α onto the generators h_α of the presentation 11.13 of $W(R)$ induces an epimorphism $T \to W(R)$, since the relations (S1) – (S6) hold in $W(R)$. Indeed, (S1) and (R1) are clearly equivalent, while (S2) – (S6) are special cases of the relation 11.13.3.

It remains to show that the map $h_\alpha \mapsto t_\alpha$ (for $\alpha \in P_u$) extends to a homomorphism $W(R) \to T$. For this, it suffices to verify that the relations (R1) – (R4) of 11.13 are satisfied by the t_α. It is straightforward to see that (S1) and (S2) imply (R1) and (R2), respectively. Also note that (S2) implies, by setting $\alpha = \beta$, the relation

$$t_\alpha^2 = 1 \quad \text{for all } \alpha \in P_u. \tag{1}$$

We define t_μ for $\mu \in P_s^\times$ by (S5). The relation (R3) then becomes a combination of two relations, namely

$$t_\alpha t_\beta t_\alpha = t_\mu \quad \text{for } \alpha, \beta \in P_u \text{ and } s_\alpha\beta = \mu \in P_s^\times, \tag{2}$$

and the relation

$$t_\mu = t_{-\mu} \quad \text{for } \mu \in P_s^\times. \tag{3}$$

We first prove (2) and thus assume $\mu = s_\alpha\beta \in P_s$. Then by 10.6.2 we necessarily have $\langle\beta, \alpha^\vee\rangle \in \{1, 2, 3\}$. If $\langle\beta, \alpha^\vee\rangle = 1$ then $s_\alpha\beta = \beta - \alpha$ is a standard representation of type I and hence (2) holds by definition of t_μ. In case $\langle\beta, \alpha^\vee\rangle = 2$ we have $s_\alpha(\beta) = \beta - 2\alpha \in P_s$ whence $\beta - \alpha = (\beta - 2\alpha) + \alpha \in P_u$ and $s_\alpha\beta = (\beta - \alpha) - \alpha$ is a standard representation of type II. Therefore, by definition, $t_\mu = t_\alpha t_\beta t_\alpha$. Finally, if $\langle\beta, \alpha^\vee\rangle = 3$ we have $s_\alpha(\beta) = \beta - 3\alpha \in P_s$. Since then $2\beta - 3\alpha = \beta + (\beta - 3\alpha) \in P_u$ (because of $\langle\beta - 3\alpha, \beta^\vee\rangle = 2 - 3\cdot 1 = -1$ and A.3) it follows that $\beta - 3\alpha = (2\beta - 3\alpha) - \beta$ is a standard representation of type I, whence (2) becomes

$$t_\beta\, t_{2\beta-3\alpha}\, t_\beta = t_\alpha\, t_\beta\, t_\alpha. \tag{4}$$

We have $-s_\beta(\alpha) = \beta - \alpha = (\beta - 3\alpha) + 2\alpha \in P_u$ by 10.6.2, whence $t_\beta t_\alpha t_\beta = t_{\beta-\alpha}$ by (S2). By (1), this is equivalent to $t_\beta\, t_{\beta-\alpha}\, t_\beta = t_\alpha$. Moreover, $-s_{\beta-\alpha}(\beta) = -s_{s_\beta(\alpha)}(\beta) = -s_\beta s_\alpha s_\beta(\beta) = s_\beta(\beta - 3\alpha) = -\beta + 3(\beta - \alpha) = 2\beta - 3\alpha \in P_u$ which, again by (S2), gives $t_{\beta-\alpha}\, t_\beta\, t_{\beta-\alpha} = t_{2\beta-3\alpha}$. Now (4) follows from

$$t_\beta t_{2\beta-3\alpha} t_\beta = t_\beta t_{\beta-\alpha} t_\beta t_{\beta-\alpha} t_\beta = t_\alpha t_{\beta-\alpha} t_\beta = t_\alpha t_\beta t_\alpha.$$

We now verify (3) and write $\mu = \alpha - \beta$ in standard representation. If the type is II then $-\mu = \beta - \alpha$ is again a standard representation of type II, and we have $s_\beta(\mu) = \alpha + \beta$ and $s_\alpha(-\mu) = -(\alpha + \beta)$ by 11.14.2. Hence, by (S4), $h_{-\mu} = h_\alpha h_{-(\alpha+\beta)} h_\alpha = h_\beta h_{\alpha+\beta} h_\beta = h_\mu$, as desired.

If the type is I there are three subcases I_n where $n = \langle\beta, \alpha^\vee\rangle \in \{1, 2, 3\}$. Before dealing with them in turn, note that $\langle\mu, \beta^\vee\rangle = -1$ and $\langle\beta, \mu^\vee\rangle = -n$ by 11.14.1. Hence, putting

$$\beta' := s_\mu(\beta) = \beta + n\mu \quad \text{and} \quad \alpha' := s_\mu(\alpha) = s_\mu(\mu + \beta) = \beta + (n-1)\mu,$$

we have $\alpha', \beta' \in P_u$ by 10.6.2, and $-\mu = s_\mu(\mu) = \alpha' - \beta'$ is a standard representation of type I_n of $-\mu$. Explicitly,

$$(\alpha', \beta') = \left\{ \begin{array}{ll} (\beta, \alpha) & \text{if } n = 1 \\ (\alpha, 2\alpha - \beta) & \text{if } n = 2 \\ (2\alpha - \beta, 3\alpha - 2\beta) & \text{if } n = 3 \end{array} \right\}.$$

For easier notation, let $a = h_\alpha$, $b = h_\beta$, $a' = h_{\alpha'}$ and $b' = h_{\beta'}$. Then we must show

$$bab = b'a'b'. \tag{5}$$

Subcase $n = 1$: Then $\alpha' = \beta$ and $\beta' = \alpha$, so (5) becomes $bab = aba$ which is just (S3).

Subcase $n = 2$: Then $\beta' = 2\alpha - \beta = -s_\alpha(\beta) \in P_u$, and $\beta \perp \beta'$. Now (S2) and (1) imply $b' = aba$ and $bb' = b'b$. Also, $\alpha' = \alpha$ so $a' = a$. Hence, using again (1),

$$b'a'b' = aba \cdot a \cdot b' = a \cdot bb' = ab'b = a \cdot aba \cdot b = bab.$$

Subcase $n = 3$: Let $\gamma := -s_\alpha(\beta) = 3\alpha - \beta = \mu + 2\alpha \in P_u$, and put $c := h_\gamma = aba$ (by (S2)). Then $\langle \alpha, \beta^\vee \rangle = 1$ implies $s_\beta(\alpha) = \alpha - \beta$ and

$$-s_\gamma(\alpha) = -s_{s_\alpha(\beta)}(\alpha) = s_\alpha(s_\beta(\alpha)) = s_\alpha(\alpha - \beta) = -\alpha + (3\alpha - \beta) = \alpha'.$$

Hence by (S2), $a' = cac$. We also have $s_{\beta'}(\beta) = 3\alpha - \beta = \gamma$ by an easy computation, and hence, by (S2), $c = b'bb'$, as well as $b'a = ab'$, because of (S2) and $\langle \beta', \alpha^\vee \rangle = \langle 3\alpha - 2\beta, \alpha^\vee \rangle = 3 \cdot 2 - 2 \cdot 3 = 0$. Now we compute, using again (1):

$$b'a'b' = b' \cdot cac \cdot b' = b' \cdot b'bb' \cdot a \cdot b'bb' \cdot b' = bb'ab'b = ba \cdot b'b' \cdot b = bab.$$

Finally, in the situation of (R4) we have $\alpha \top \gamma$ from $s_\alpha(\gamma) = -s_\gamma(\alpha)$ and 11.16. Moreover, $\beta \perp s_\alpha(\gamma) = \gamma - \alpha$ is equivalent to $\langle \alpha, \beta^\vee \rangle = \langle \gamma, \beta^\vee \rangle$ and $\langle \beta, \alpha^\vee \rangle = \langle \beta, \gamma^\vee \rangle$. Since $s_\gamma(\beta) \in P_s$, these Cartan integers must be positive. We exclude the cases $\langle \alpha, \beta^\vee \rangle = 3$ and $\langle \beta, \alpha^\vee \rangle = 3$ as follows. Let $(\alpha, \beta, \gamma) = (\alpha_1, \alpha_2, \alpha_3)$ and consider the determinant of the Cartan matrix:

$$\det\big(\langle \alpha_i, \alpha_j^\vee \rangle\big) = \det \begin{pmatrix} 2 & \langle \alpha, \beta^\vee \rangle & 1 \\ \langle \beta, \alpha^\vee \rangle & 2 & \langle \beta, \alpha^\vee \rangle \\ 1 & \langle \alpha, \beta^\vee \rangle & 2 \end{pmatrix} = 2\big(3 - \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle\big).$$

Since $\langle \alpha, \beta^\vee \rangle = 3$ implies $\langle \beta, \alpha^\vee \rangle = 1$ and vice versa by A.2, we see that if either $\langle \alpha, \beta^\vee \rangle = 3$ or $\langle \beta, \alpha^\vee \rangle = 3$, the α_i must be linearly dependent and hence $\beta = x\alpha + y\gamma$ must be a linear combination of α and γ. But then $s_\alpha(\beta) = -x\alpha + ys_\alpha(\gamma)$ and $s_\gamma(\beta) = xs_\gamma(\alpha) - y\gamma$. Since $s_\alpha(\beta)$, $s_\gamma(\beta)$ and $s_\alpha(\gamma) = -s_\gamma(\alpha)$ all belong to the full subsystem P_s and x and y do not both vanish, it follows that either α or γ is in P_s, contradiction. Therefore, $(\langle \alpha, \beta^\vee \rangle, \langle \beta, \alpha^\vee \rangle) = (\langle \gamma, \beta^\vee \rangle, \langle \beta, \gamma^\vee \rangle) = (1, 1)$, $(1, 2)$ or $(2, 1)$ which yields the three cases in (S6).

§12. Closed and full subsystems of finite and infinite classical root systems

12.1. *Notations and conventions.* In this section, we study the closed subsystems of the irreducible infinite root systems classified in 8.4, with special emphasis on the full subsystems for which we describe their orbit spaces under the big Weyl group and their quotients.

In the finite case, the description of the full subsystems is well known in terms of subsets of the Dynkin diagram, see [**12**, VI, §1.7, Prop. 24]: Every full subsystem S of a finite R is of the form $R \cap \operatorname{span}(\Sigma)$ where Σ is a subset of some root basis B of R. The conjugacy classes of full subsystems (also called "subsystems of parabolic type") under the Weyl group and the automorphism group were determined by Bala-Carter [**3**]. An efficient method to determine the W-action on subsets of B was described by Richardson [**64**], see for example the exposition in [**36**, chap. 28]. The maximal closed subsystems of a finite irreducible root system are described by a theorem of Borel-Siebenthal [**7**], see [**12**, VI, §4, Exerc. 4] or [**36**, chap. 12].

Although our main interest is in the infinite case, it turns out that our methods work equally well for the finite classical root systems, which are therefore included in our setting.

Let thus I be an arbitrary set, $X = \bigoplus_{i\in I} \mathbb{R}\varepsilon_i$ the free vector space on I, and $t\colon X \to \mathbb{R}$ the trace form given by $t(\varepsilon_i) = 1$ for all $i \in I$, with kernel $\dot{X} := \operatorname{Ker}(t)$, cf. 8.1. For a subset J of I, we let

$$X_J := \bigoplus_{j\in J} \mathbb{R}\varepsilon_j \quad \text{and} \quad \dot{X}_J := \dot{X} \cap X_J.$$

Throughout this section, R will denote one of the root systems $R = \mathrm{T}_I$ introduced in 8.1, where $\mathrm{T} \in \mathfrak{T} = \{\dot{\mathrm{A}}, \mathrm{B}, \mathrm{C}, \mathrm{BC}, \mathrm{D}\}$. Then $\dot{\mathrm{A}}_I$ is a root system in $\dot{X}$, and T_I is a root system in X for $\mathrm{T} \neq \dot{\mathrm{A}}$, with the exception of the case $|I| = 1$ and $\mathrm{T} = \mathrm{D}$, where $\mathrm{D}_1 = \{0\}$ does not span X. We emphasize that for the realizations in 8.1 we have the inclusions of subsystems

$$\dot{\mathrm{A}}_I \subset \mathrm{D}_I \subset \mathrm{B}_I \subset \mathrm{BC}_I \quad \text{and} \quad \dot{\mathrm{A}}_I \subset \mathrm{D}_I \subset \mathrm{C}_I \subset \mathrm{BC}_I\,.$$

For a subset $J \subset I$ we let $\mathrm{T}_J = \mathrm{T}_I \cap X_J$ as in 8.9, and note that T_J is a full subsystem of T_I. We recall that for small I it may happen that (the isomorphism class of) the root system T_I does not determine the type T and the cardinality of the set I, see the list of exceptional isomorphisms in 8.2.1.

The reader should be aware that several of our constructions, for example in Lemma 12.3, depend not only on the isomorphism class of $R = \mathrm{T}_I$, but on the concrete realization of R in the form T_I, i.e., on the *pair* (T, I); in particular, on the vector space basis $(\varepsilon_i)_{i\in I}$ of X which we consider as fixed in the following. This will usually (but not always) be indicated by the notation (T, I) instead of T_I.

If $\sim$ is an equivalence relation on a set I we denote by $I/{\sim}$ the set of equivalence classes of $\sim$. Recall that, by definition, equivalence classes are non-empty. Also, we denote by M/G the set of orbits of a group G acting on a set M (on the left or on the right).

12.2. *Outline.* With any subsystem S of a root system $R = \mathrm{T}_I$ we associate combinatorial invariants consisting of a subset $I_0(S)$ of I and two equivalence relations $\sim_S$ and $\approx_S$ on I (Lemma 12.3). For closed subsystems these invariants, together with $S \cap X_{I_0(S)}$, determine S completely (Prop. 12.5). The condition $\sim_S = \approx_S$ singles out the subclass of so-called *pure* closed subsystems described in Cor. 12.6. The full subsystems are those closed subsystems for which $S \cap X_{I_0(S)} = T_{I_0(S)}$ (Prop. 12.11).

Let $G = \mathrm{Sym}(I) \ltimes N \subset \mathrm{Aut}(R)$ where N is a group of sign changes, defined in 12.7.1. The set $\mathfrak{C}_0$ of pure closed subsystems is a fundamental domain for the action of N on the set $\mathfrak{C}$ of all closed subsystems of R, and the analogous statement holds for the set $\mathfrak{F}_0$ of all pure full subsystems in the set $\mathfrak{F}$ of all full subsystems of R (Prop. 12.10). As an application of these results we classify the maximal closed subsystems in 12.13.

The invariants $I_0(S)$ and $\sim_S$ describing an $S \in \mathfrak{F}_0$ satisfy certain restrictions. Taking them as definitions, we obtain a set $\mathbb{F}_0$ of combinatorial data. The map $S \mapsto (I_0(S), \sim_S)$ is a $\mathrm{Sym}(I)$-equivariant bijection $\mathfrak{F}_0 \cong \mathbb{F}_0$, and combining this with the bijection $\mathfrak{F}/N \cong \mathfrak{F}_0$, we obtain the description $\mathfrak{F}/G \cong \mathbb{F}_0/\mathrm{Sym}(I)$ of the orbit space of $\mathfrak{F}$ under G (Th. 12.17).

From the explicit description of full subsystems in terms of their invariants, it is then easy to determine the quotients R/S (Prop. 12.19). As an application, we show that the necessary condition of 8.11 for a full subsystem S to be of scalar type is also sufficient (Cor. 12.20).

The following lemma introduces the combinatorial data which will form the basis of our description of closed and full subsystems.

12.3. LEMMA. *With the notations of* 12.1*, let* $\mathrm{T} \in \mathfrak{T} = \{\dot{\mathrm{A}}, \mathrm{B}, \mathrm{C}, \mathrm{BC}, \mathrm{D}\}$*, let* $R = \mathrm{T}_I$*, and let* $S \subset R$ *be a not necessarily closed subsystem. Define relations* $\sim_S$ *and* $\approx_S$ *on* I *by*

$$i \sim_S j \quad :\Longleftrightarrow \quad \varepsilon_i - \varepsilon_j \in S, \tag{1}$$
$$i \approx_S j \quad :\Longleftrightarrow \quad \varepsilon_i - \varepsilon_j \in S \text{ or } \varepsilon_i + \varepsilon_j \in S, \tag{2}$$

as well as the subset

$$I_0(S) := \{j \in I : \varepsilon_j \in \mathrm{span}(S)\}. \tag{3}$$

Then:

(a) $\sim_S$ *is an equivalence relation on* I*, and*

$$S \cap \dot{\mathrm{A}}_I = \bigcup_{J \in I/\sim_S} \dot{\mathrm{A}}_J \,. \tag{4}$$

(b) *If* $\mathrm{T} = \dot{\mathrm{A}}$ *then*

$$S = \bigcup_{J \in I/\sim_S} \dot{\mathrm{A}}_J \,. \tag{5}$$

Conversely, for every equivalence relation $\sim$ *on* I*, the right hand side of* (5) *(with* $\sim_S$ *replaced by* $\sim$*) defines a subsystem of* $\dot{\mathrm{A}}_I$*. Every subsystem of* $\dot{\mathrm{A}}_I$ *is full and hence in particular closed.*

(c) $\approx_S$ *is an equivalence relation on I which induces the decomposition*

$$S = \bigcup_{J \in I/\approx_S} S \cap X_J\,. \tag{6}$$

(d) *Every equivalence class J of $\approx_S$ is either also an equivalence class of $\sim_S$ or a union of two equivalence classes of $\sim_S$. In the second case let, say, $J = J_1 \,\dot\cup\, J_2$ for J_1, J_2 equivalence classes of $\sim_S$. Then we have $\mathrm{T} \neq \dot{\mathrm{A}}$, hence $\mathrm{D}_I \subset R$, and*

$$S \cap \mathrm{D}_J = \dot{\mathrm{A}}_{J_1} \cup \dot{\mathrm{A}}_{J_2} \cup \{\pm(\varepsilon_i + \varepsilon_j) : i \in J_1,\ j \in J_2\}. \tag{7}$$

(e) *$I_0(S)$ is a (possibly empty) union of equivalence classes of $\approx_S$. Moreover, $I_0(S) = \emptyset$ if $\mathrm{T} = \dot{\mathrm{A}}$, while $\operatorname{Card} I_0(S) \neq 1$ if $\mathrm{T} = \mathrm{D}$.*

Remarks. (i) The notations $\sim_S$, $\approx_S$ and $I_0(S)$ are incomplete insofar as these invariants do not only depend on S and R but also on the realization of R as T_I, i.e., they really depend on the triple (S, T, I). For example, $I_0(S) = \emptyset$ for every $S \subset \dot{\mathrm{A}}_4$, while this need not be the case for $S \subset \mathrm{D}_3$, even though $\dot{\mathrm{A}}_4 \cong \mathrm{D}_3$ by 8.2.1.

(ii) Although (b) completely describes the subsystems of $\dot{\mathrm{A}}_I$, we will in the following not exclude this case since this would not lead to a simplification. The decomposition (5) is the decomposition of S into irreducible components. For an arbitrary R, the decomposition (6) is an orthogonal decomposition of S but the intersections $S \cap X_J$ are in general not connected.

Proof. (a) Reflexivity and symmetry of $\sim_S$ follow from $0 \in S = -S$. To check transitivity, let $i \sim_S j$ and $j \sim_S k$. We can assume that i, j, k are pairwise distinct. Then $\varepsilon_i - \varepsilon_j$ and $\varepsilon_j - \varepsilon_k$ are two roots in S with $(\varepsilon_i - \varepsilon_j | \varepsilon_j - \varepsilon_k) = -1$ where $(\ |\ )$ is the canonical invariant inner product of 8.1. By A.3 applied to the root system S, we then have $\varepsilon_i - \varepsilon_k = (\varepsilon_i - \varepsilon_j) + (\varepsilon_j - \varepsilon_k) \in S$. For the proof of (4) we note that, by definition of $\sim_S$, we have $\dot{\mathrm{A}}_J \subset S$ for any $J \in I/\sim_S$. Conversely, if $\alpha \in S \cap \dot{\mathrm{A}}_I$ then $\alpha = \varepsilon_i - \varepsilon_j$ for some $i, j \in I$, hence $i \sim_S j$ and $\alpha \in \dot{\mathrm{A}}_J$ for some $J \in I/\sim_S$.

(b) Formula (5) is a special case of (4). That, conversely, any subset of the form (5) is a subsystem, is immediate. Any subsystem S of $\dot{\mathrm{A}}_I = R$ is a full subsystem since $S = R \cap Y$ for $Y = \bigoplus_{J \in I/\sim_S} \dot{X}_J$ where $\dot{X}_J = X_J \cap \dot{X}$. Indeed, by (5), we have $S \subset R \cap Y$. Conversely, any $\alpha \in R \cap Y$ has the form $\alpha = \varepsilon_j - \varepsilon_k = \sum_J x_J$ for $x_J \in \dot{X}_J$. Let J and K be the $\sim_S$-equivalence classes of j and k, respectively. If $J \neq K$ we obtain $\varepsilon_j = x_J$ and $-\varepsilon_k = x_K$ by comparing components in the direct sum decomposition of Y, leading to the contradiction $1 = t(\varepsilon_j) = t(x_J) = 0$. Thus $J = K$ and $\alpha \in \dot{\mathrm{A}}_J \subset S$.

(c) The proof that also $\approx_S$ is an equivalence relation is similar to the one given in (a). In (6) the inclusion from right to left is obvious. For the other inclusion we consider the following two cases: If $\alpha \in S \cap \mathbb{Z}\varepsilon_i$ for some $i \in I$ then $\alpha \in S \cap X_J$ where J is the equivalence class of i with respect to $\approx_S$. If $\alpha \in S$ has the form $\pm\varepsilon_i \pm \varepsilon_j$ for $i \neq j$ then $i \approx_S j$, so i, j belong to the same class $J \in I/\approx_S$, and $\alpha \in S \cap X_J$.

(d) Since by definition $i \sim_S j$ implies $i \approx_S j$, it is clear that every equivalence class J of $\approx_S$ is a union of equivalence classes of $\sim_S$. Suppose J is not a full equivalence class of $\sim_S$. Then there exist two elements in J, say 1 and 2, which are inequivalent modulo $\sim_S$, i.e., $\varepsilon_1 + \varepsilon_2 \in S$ but $\varepsilon_1 - \varepsilon_2 \notin S$. We claim that then $J = J_1 \dot\cup J_2$ where J_1 and J_2 are the equivalence classes of 1 and 2 with respect to $\sim_S$. Indeed, assume $i \in J \setminus (J_1 \cup J_2)$. Then $i \approx_S 1$ and $i \approx_S 2$ imply $\varepsilon_i + \varepsilon_1 \in S$ and $\varepsilon_i + \varepsilon_2 \in S$, and therefore $\varepsilon_1 - \varepsilon_2 = (\varepsilon_i + \varepsilon_1) - (\varepsilon_i + \varepsilon_2) \in S$ by A.3(a). This yields the contradiction $1 \sim_S 2$. Thus $J = J_1 \dot\cup J_2$. Observe that $\varepsilon_1 + \varepsilon_2 \in R$ implies $R \neq \dot{\mathrm{A}}_I$ and therefore $\mathrm{D}_I \subset R$. In (7), the inclusion from right to left holds by definition of $\approx_S$ and $J_1 \not\approx_S J_2$. To show the other inclusion, let $m, n \in J$ and $\varepsilon_m + \varepsilon_n \in S$. It remains to prove that the assumption $m, n \in J_1$ or $m, n \in J_2$ leads to a contradiction. By symmetry, we may assume $m, n \in J_1$. Then $\varepsilon_n + \varepsilon_2 \in S$. By A.3, we have $(\varepsilon_m + \varepsilon_n) - (\varepsilon_n + \varepsilon_2) = \varepsilon_m - \varepsilon_2 \in S$ leading to the contradiction $m \sim_S 2$.

(e) Let $i \in I_0(S)$ and suppose $i \approx_S j$, i.e., $\alpha = \varepsilon_i \pm \varepsilon_j \in S$ for some $j \in I$ and a suitable sign. Then $\pm\varepsilon_j = \alpha - \varepsilon_i \in \mathrm{span}(S)$, proving $j \in I_0(S)$. Thus $I_0(S)$ is a union of equivalence classes of $\approx_S$. Next, let $R = \dot{\mathrm{A}}_I$. Then $S \subset \dot{\mathrm{A}}_I \subset \mathrm{Ker}(t)$ hence also $\mathrm{span}(S) \subset \mathrm{Ker}(t)$. Since $t(\varepsilon_j) = 1$ this proves $I_0(S) = \emptyset$. Finally, let $R = \mathrm{D}_I$ and suppose $I_0(S) \neq \emptyset$, say, $j \in I_0(S)$. Then $\varepsilon_j = \sum_{\nu=1}^n c_\nu \alpha_\nu$ where $0 \neq c_\nu \in \mathbb{R}$ and $\alpha_\nu \in S$. Since $S = -S$ we may assume $c_\nu > 0$ for all ν. Let f be the linear form defined by $f(\varepsilon_i) = \delta_{ij}$. Then $1 = f(\varepsilon_j) = \sum_\nu c_\nu f(\alpha_\nu)$ implies that $f(\alpha_\mu) > 0$ for some μ, and hence, because D_I does not contain any roots of the form $c\varepsilon_i$, that $\alpha_\mu = \varepsilon_j \pm \varepsilon_i$ for some $i \neq j$. It follows that $\pm\varepsilon_i = \alpha_\mu - \varepsilon_j \in \mathrm{span}(S)$ and therefore $i \in I_0(S)$, so $\mathrm{Card}\, I_0(S) \geqslant 2$.

12.4. Definition. We keep the notations of Lemma 12.3 and introduce the following terminology. An equivalence class J of $\approx_S$ will be called *mixed* if it decomposes into two equivalence classes of $\sim_S$, and *pure* otherwise. By Lemma 12.3(d),

$$J \in I/{\approx_S} \text{ is pure} \quad \iff \quad \dot{\mathrm{A}}_J \subset S. \tag{1}$$

A subsystem S of $R = \mathrm{T}_I$ will be called *pure* if it satisfies the following equivalent conditions:

(i) $\varepsilon_i + \varepsilon_j \in S$ for $i \neq j$ implies $\varepsilon_i - \varepsilon_j \in S$,
(ii) the equivalence relations $\sim_S$ and $\approx_S$ agree,
(iii) every equivalence class of $\approx_S$ is pure.

Clearly, every subsystem of $\dot{\mathrm{A}}_I$ is pure, and every one-element class of $\approx_S$ is automatically pure. The smallest example of a mixed full subsystem is $S = \{0, \pm(\varepsilon_1 + \varepsilon_2)\} \subset \mathrm{D}_2$.

The reader should realize that, just as in Remark (i) of 12.3, the property of a subsystem S being pure depends not only on S and R but on the triple (S, T, I). For example, every subsystem of A_3 is pure, but the isomorphic root system D_3 has subsystems which are not pure.

We now turn to closed subsystems $S \subset R$ and describe the subsystems $S \cap X_J$ in the decomposition 12.3.6.

12.5. PROPOSITION. *Let* $\mathrm{T} \in \mathfrak{T}$, *let* S *be a closed subsystem of one of the root systems* $R = \mathrm{T}_I$, *and let*

$$S = \bigcup_{J \in I/\approx_S} S \cap X_J \tag{1}$$

be the decomposition established in 12.3.6. *Then all classes* $J \in I_0(S)/\approx_S$ *are pure. For every* $J \in I/\approx_S$ *the intersections* $S \cap X_J$ *are closed subsystems of* T_J *which have the following descriptions:*

(a) *If* $J \subset I_0(S)$ *and* $\mathbb{Z}\varepsilon_j \cap S \neq \{0\}$ *for some* $j \in J$ *then*

$$S \cap X_J = \left\{ \begin{array}{ll} \mathrm{B}_J & \textit{if } \mathrm{T} = \mathrm{B} \\ \mathrm{C}_J & \textit{if } \mathrm{T} = \mathrm{C} \\ \mathrm{C}_J \textit{ or } \mathrm{BC}_J & \textit{if } \mathrm{T} = \mathrm{BC} \end{array} \right\}. \tag{2}$$

If $\mathrm{T} = \mathrm{B}$ *or* BC *there exists at most one* $J \in I_0(S)/\approx_S$ *with* $S \cap X_J = \mathrm{T}_J$.

(b) *If* $J \subset I_0(S)$ *and* $\mathbb{Z}\varepsilon_j \cap S = \{0\}$ *for all* $j \in J$ *then*

$$S \cap X_J = \mathrm{D}_J. \tag{3}$$

In this case, $|J| \geqslant 2$ *and* $\mathrm{T} = \mathrm{B}$ *or* D.

(c) *If* J *is a pure equivalence class with* $J \cap I_0(S) = \emptyset$ *then*

$$S \cap X_J = \dot{\mathrm{A}}_J. \tag{4}$$

(d) *If* J *is a mixed equivalence class, say,* $J = J_1 \mathbin{\dot\cup} J_2$ *as in* 12.3(d), *with* $J \cap I_0(S) = \emptyset$, *then*

$$S \cap X_J = \dot{\mathrm{A}}_{J_1} \cup \dot{\mathrm{A}}_{J_2} \cup \{\pm(\varepsilon_i + \varepsilon_j) : \ i \in J_1, \ j \in J_2\}. \tag{5}$$

Proof. Purity of all equivalence classes $J \subset I_0(S)$ will follow from the descriptions of $S \cap X_J$ in (a) and (b). Also, an intersection $S \cap Y$ of the closed subsystem $S \subset R$ with a subspace Y of X is again closed in $R \cap Y$, so all $S \cap X_J$ are closed in T_J.

(a) We have $\mathbb{Z}\varepsilon_j \cap S = Z\varepsilon_j$ where $Z = \{0, \pm 1\}$ or $\{0, \pm 2\}$ or $\{0, \pm 1, \pm 2\}$. Clearly (2) holds in case $|J| = 1$, so suppose there exists $i \in J$, $i \neq j$. Then $\alpha = \varepsilon_i \pm \varepsilon_j \in S$ for a suitable sign. Since $s_\alpha(\varepsilon_j) = \mp\varepsilon_i$, see 9.5.4 and 9.5.5, we have $Z\varepsilon_i = s_\alpha(Z\varepsilon_j) \subset S$ which implies $\mathbb{Z}\varepsilon_i \cap S = Z\varepsilon_i$ for all $i \in J$. From $s_{\varepsilon_j}(\varepsilon_i \pm \varepsilon_j) = \varepsilon_i \mp \varepsilon_j$ it follows that $D_J \subset S \cap X_J$, and therefore

$$S \cap X_J = \left\{ \begin{array}{ll} \mathrm{B}_J & \text{if } Z = \{0, \pm 1\} \\ \mathrm{C}_J & \text{if } Z = \{0, \pm 2\} \\ \mathrm{BC}_J & \text{if } Z = \{0, \pm 1, \pm 2\} \end{array} \right\}.$$

In particular, by 12.4.1, J is pure. Clearly, if the type is B or C then only the first or second possibility occurs. For $\mathrm{T} = \mathrm{BC}$ the assumption $\varepsilon_j \in S$ implies $2\varepsilon_j \in S$ by closedness of S, and therefore $Z = \{0, \pm 2\}$ or $Z = \{0, \pm 1, \pm 2\}$ in case $\mathrm{T} = \mathrm{BC}$. Finally, suppose $\mathrm{T} \in \{\mathrm{B}, \mathrm{BC}\}$ and let $J_1, J_2 \in I_0/\approx_S$ with $S \cap X_{J_i} = \mathrm{T}_{J_i}$ for $i = 1, 2$. Then there exists $j_i \in J_i$ such that $\varepsilon_{j_i} \in S$ and therefore also $\varepsilon_{j_1} + \varepsilon_{j_2} \in \mathrm{T}_I \cap (S + S) \subset S$, whence $J_1 = J_2$.

(b) Observe that (1) implies $\operatorname{span}(S) = \bigoplus_{J\in I/\approx_S} \operatorname{span}(S \cap X_J)$. Since $X_J \subset \operatorname{span}(S)$ for any $J \subset I_0(S)$ we therefore have

$$\operatorname{span}(S \cap X_J) = X_J. \tag{6}$$

By assumption $S\cap X_J \subset \mathrm{D}_J$, so $S\cap X_J = S\cap \mathrm{D}_J$. Suppose that $J = J_1 \dot\cup J_2 \subset I_0(S)$ decomposes into two equivalence classes with respect to $\sim_S$. Then 12.3.7 implies

$$S \cap X_J = S \cap \mathrm{D}_J = \dot{\mathrm{A}}_{J_1} \cup \dot{\mathrm{A}}_{J_2} \cup \{\pm(\varepsilon_i + \varepsilon_j) : i \in J_1,\ j \in J_2\}.$$

The linear form f on X_J defined by $f(\varepsilon_i) = 1$ for $i \in J_1$ and $f(\varepsilon_i) = -1$ for $i \in J_2$ vanishes on $S \cap X_J$, so $\operatorname{span}(S \cap X_J)$ has codimension $\geqslant 1$ in X_J, contradicting (6). Therefore J is pure, and then $\dot{\mathrm{A}}_J \subset S \cap X_J$ by 12.4.1. Because of (6) this must be a proper inclusion, so there exists a root $\varepsilon_i + \varepsilon_j \in S \cap X_J$. Since $s_\alpha(\varepsilon_i + \varepsilon_j) \in S$ for any $\alpha \in \dot{\mathrm{A}}_J$ it follows from 9.5.4 that $\varepsilon_m + \varepsilon_n \in S$ for all $m, n \in J$, $m \neq n$. Therefore $S \cap X_J = \mathrm{D}_J$. Because $\mathrm{D}_1 = \{0\}$, (6) implies $|J| \geqslant 2$. Also, since for two distinct elements $i, j \in J$ we have $2\varepsilon_i = (\varepsilon_i + \varepsilon_j) + (\varepsilon_i - \varepsilon_j) \in S + S$ and since $S \cap X_J$ is closed, it follows that $\mathrm{T} \neq \mathrm{C}, \mathrm{BC}$.

We will prove (c) and (d) simultaneously. We cannot have $c\varepsilon_j \in S \cap X_J$ for some non-zero $c \in \mathbb{Z}$ because then $j \in I_0(S)$. Thus $S \cap X_J \subset \mathrm{D}_J$, which implies $S \cap X_J = S \cap \mathrm{D}_J$. Hence (d) follows from (d) of Lemma 12.3. In case (c) we may assume that $|J| \geqslant 2$, say $i, j \in J$, $i \neq j$. Suppose that also $\varepsilon_i + \varepsilon_j \in S$. Since $i \sim_S j$ implies $\varepsilon_i - \varepsilon_j \in S$ it follows that $2\varepsilon_i = (\varepsilon_i + \varepsilon_j) + (\varepsilon_i - \varepsilon_j) \in \operatorname{span}(S)$, so $i \in I_0(S)$ which is excluded by assumption. Therefore $S \cap X_J \subset \dot{\mathrm{A}}_J$ and then (c) follows from (a) of Lemma 12.3.

12.6. Corollary. *Let S be a closed subsystem of $R = \mathrm{T}_I$. We introduce the notations*

$$\bar I(S) = \{J \in I/\approx_S : J \cap I_0(S) = \emptyset\}, \quad \bar I_2(S) = \{J \in \bar I(S) : |J| \geqslant 2\}. \tag{1}$$

Then the following conditions are equivalent:

(i) *S is pure,*
(ii) *$S \cap X_J = \dot{\mathrm{A}}_J$ for all $J \in \bar I_2(S)$,*
(iii) *S is given by*

$$S = (S \cap X_{I_0(S)}) \cup \bigcup_{J\in \bar I_2(S)} \dot{\mathrm{A}}_J. \tag{2}$$

Proof. As observed in 12.4, a one-element $J \in \bar I(S)$ is automatically pure. Also, $\dot{\mathrm{A}}_J = \{0\}$ in this case, so the corresponding term in 12.3.6 may be omitted. Now the equivalence of (i) – (iii) is immediate from Prop. 12.5.

12.7. *Notations.* Let $\mathrm{T} \in \mathfrak{T}$ and $R = \mathrm{T}_I$. We denote the set of all closed subsystems of R by $\mathfrak{C} = \mathfrak{C}(R)$ and the set of pure closed subsystems by $\mathfrak{C}_0 = \mathfrak{C}_0(\mathrm{T}, I)$ (recall from 12.4 that the property of being a pure subsystem depends not only on R but on the pair (T, I)). Similarly, the set of all full subsystems of R will be denoted by $\mathfrak{F} = \mathfrak{F}(R)$ and the set of pure full subsystems by $\mathfrak{F}_0 = \mathfrak{F}_0(\mathrm{T}, I)$. Since a full subsystem is in particular closed, we have $\mathfrak{F} \subset \mathfrak{C}$ and $\mathfrak{F}_0 = \mathfrak{F} \cap \mathfrak{C}_0$. Our next aim is to show (Prop. 12.10) that $\mathfrak{C}_0 \subset \mathfrak{C}$ and $\mathfrak{F}_0 \subset \mathfrak{F}$ are fundamental domains for the action of a group N of automorphisms of R which we define next.

Recall from 9.1 the action of the group $\mathrm{Sym}(I) \ltimes \mathbf{2}^I$ on X: A permutation π and a sign change $\sigma = \sigma_L$, corresponding to a subset L of I, act by

$$\pi(\varepsilon_i) = \varepsilon_{\pi(i)}, \qquad \sigma_L(\varepsilon_i) = \begin{Bmatrix} -\varepsilon_i & \text{if } i \in L \\ \varepsilon_i & \text{if } i \notin L \end{Bmatrix}.$$

We put

$$N := N(\mathrm{T}, I) := \begin{Bmatrix} \{\mathrm{Id}\} & \text{if } \mathrm{T} = \dot{\mathrm{A}} \\ \mathbf{2}^I & \text{if } \mathrm{T} \neq \dot{\mathrm{A}} \end{Bmatrix}, \qquad G := G(\mathrm{T}, I) := \mathrm{Sym}(I) \ltimes N. \tag{1}$$

The example $\dot{\mathrm{A}}_4 \cong \mathrm{D}_3$ where $N(\dot{\mathrm{A}}, 4) = \{\mathrm{Id}\}$ but $N(\mathrm{D}, 3) = \mathbf{2}^3 \cong \mathbb{Z}_2^3$, shows that for small I the group N does indeed depend on (T, I) and not only on $R = \mathrm{T}_I$.

In all cases, G acts faithfully on X resp. $\dot{X}$ by automorphisms of R. Also, by Theorem 9.5, G induces the big Weyl group of R except in the finite case for $R = \mathrm{D}_n$ where $W(\mathrm{D}_n)$ has index 2 in G. Indeed, we have in all cases

$$\overline{W}(R) = \mathrm{Sym}(I) \ltimes N_+ \quad \text{where} \quad N_+ = \begin{Bmatrix} \mathbf{2}^n_+ & \text{if } \mathrm{T} = \mathrm{D} \text{ and } |I| = n < \infty \\ N & \text{otherwise} \end{Bmatrix}. \tag{2}$$

Both $\mathfrak{C}$ and $\mathfrak{F}$ are invariant under the action of the full automorphism group $\mathrm{Aut}(R)$, hence in particular under the action of G. From Prop. 12.5 and from 12.8.2, 12.8.3 below it will become clear that $\mathfrak{C}_0$ and hence also $\mathfrak{F}_0$ is stable under $\mathrm{Sym}(I)$. This is not so under sign changes, see 12.10.

12.8. LEMMA. *Let S be a subsystem of $R = \mathrm{T}_I$. Then for a permutation $\pi \in \mathrm{Sym}(I)$ and a sign change $\sigma \in N$, we have*

$$\begin{aligned} I_0(\pi(S)) &= \pi(I_0(S)), & (1)\\ \sim_{\pi(S)} &= (\pi \times \pi)(\sim_S), & (2)\\ \approx_{\pi(S)} &= (\pi \times \pi)(\approx_S), & (3)\\ I_0(\sigma(S)) &= I_0(S), & (4)\\ \approx_{\sigma(S)} &= \approx_S. & (5) \end{aligned}$$

If S is closed then with the notation of 12.6.1*,*

$$\bar{I}(\sigma(S)) = \bar{I}(S), \quad \bar{I}_2(\sigma(S)) = \bar{I}_2(S). \tag{6}$$

(In (2) and (3), $\sim_S$ and $\approx_S$ are of course considered as subsets of $I \times I$.)

Proof. Formula (1) is clear from the definitions and the action of π recalled in 12.7. Then (2) follows from $\pi(i) \sim_{\pi(S)} \pi(j) \iff \varepsilon_{\pi(i)} - \varepsilon_{\pi(j)} = \pi(\varepsilon_i - \varepsilon_j) \in \pi(S)$ $\iff \varepsilon_i - \varepsilon_j \in S \iff i \sim_S j$, and the proof of (3) is analogous. For (4), note that $i \in I_0(\sigma(S)) \iff \varepsilon_i \in \mathrm{span}\,\sigma(S) \iff \sigma(\varepsilon_i) = \pm\varepsilon_i \in \mathrm{span}\, S \iff i \in I_0(S)$. Formula (5) follows similarly. The remaining formulas are immediate from (4) and (5).

12.9. Lemma. *Let S be a closed subsystem of $R = \mathrm{T}_I$ and let $\sigma = \sigma_L \in N$ be a sign change defined by the subset L of I as in* 12.7. *By* 12.8.4 *and* 12.8.5, *the invariants $I_0(S)$ and $\approx_S$ are the same for S and $\sigma(S)$, so we denote them simply by I_0 and $\approx$. Then $\sigma(S) \cap X_J$ for $J \in I/{\approx}$ is described as follows:*

(a) *If $J \subset I_0$ then*

$$\sigma(S) \cap X_J = S \cap X_J. \tag{1}$$

(b) *If $J \in \bar{I}(S)$ is a pure equivalence class then*

$$\sigma(S) \cap X_J = \dot{\mathrm{A}}_{J\cap L} \cup \dot{\mathrm{A}}_{J\setminus L} \cup \{\pm(\varepsilon_i + \varepsilon_j) : i \in J \cap L,\ j \in J \setminus L\}. \tag{2}$$

(c) *If J is a mixed equivalence class and $J = J_1 \dot\cup J_2$ as in* 12.3(d) *then*

$$\sigma(S) \cap X_J = \dot{\mathrm{A}}_{J_1'} \cup \dot{\mathrm{A}}_{J_2'} \cup \{\pm(\varepsilon_i + \varepsilon_j) : i \in J_1',\ j \in J_2'\}, \tag{3}$$

where $J_1' = (J_1 \setminus L) \cup (J_2 \cap L)$ and $J_2' = (J_2 \setminus L) \cup (J_1 \cap L)$.

(d) *For a pure closed subsystem S and $\sigma = \sigma_L \in N$ as above, the following conditions are equivalent (notation as in* 12.6.1*):*

(i) *$\sigma(S) = S$,*
(ii) *$\sigma(S)$ is again pure,*
(iii) *for all $J \in \bar{I}_2(S)$, either $L \cap J = \emptyset$ or $J \subset L$.*

Proof. A sign change σ satisfies $\sigma(X_J) = X_J$ for any subset J of I, so we have $\sigma(S) \cap X_J = \sigma(S \cap X_J)$. Now (a) – (c) follow easily from Prop. 12.5. The details are left to the reader.

It remains to prove (d), where the implication (i) $\Longrightarrow$ (ii) is trivial.

(ii) $\Longrightarrow$ (iii): By Cor. 12.6, $\sigma(S)$ is pure if and only if $\sigma(S) \cap X_J = \dot{\mathrm{A}}_J$ for all $J \in \bar{I}_2(S)$. Now (iii) follows from (2).

(iii) $\Longrightarrow$ (i): The conditions on L imply that $\sigma\big|X_J$ is $\pm\mathrm{Id}$ and therefore $\sigma(S) \cap X_J = S \cap X_J$ for each $J \in \bar{I}_2(S)$. Now it follows from (1) and Cor. 12.6(iii) that $\sigma(S) = S$.

We recall that a fundamental domain for a group G acting on a set M is a subset M_0 of M which intersects each orbit of G in exactly one point; equivalently, M_0 is a set of representatives for M/G, or the map $M_0 \hookrightarrow M \xrightarrow{\text{can}} M/G$ is bijective.

12.10. Proposition. *Let $R = \mathrm{T}_I$. We use the notations introduced in* 12.7.

(a) *$\mathfrak{C}_0$ and $\mathfrak{F}_0$ are fundamental domains for the action of N on $\mathfrak{C}$ and $\mathfrak{F}$, respectively.*

(b) $\mathrm{Sym}(I) \cong G/N$ *acts naturally on $\mathfrak{C}/N$ and $\mathfrak{F}/N$, and the maps*

$$\Phi\colon \mathfrak{C}_0 \hookrightarrow \mathfrak{C} \xrightarrow{\text{can}} \mathfrak{C}/N, \qquad \Phi\big|\mathfrak{F}_0\colon \mathfrak{F}_0 \hookrightarrow \mathfrak{F} \xrightarrow{\text{can}} \mathfrak{F}/N \tag{1}$$

are bijective and $\mathrm{Sym}(I)$*-equivariant.*

Proof. It suffices to prove this for $\mathfrak{C}$; the corresponding statements for $\mathfrak{F}$ then follow from invariance of $\mathfrak{F}$ under N and $\mathfrak{F}_0 = \mathfrak{F} \cap \mathfrak{C}_0$.

(a) By Lemma 12.9(d), an N-orbit intersects $\mathfrak{C}_0$ in at most one point. It remains to show that for every $S \in \mathfrak{C}$ there exists $\sigma \in N$ such that $\sigma(S) \in \mathfrak{C}_0$. Let $\mathfrak{M}$ be the set of all mixed equivalence classes of $\approx_S$, and decompose each $J \in \mathfrak{M}$ in $J = J_1 \dot\cup J_2$ as in Lemma 12.3(d). Put $L = \bigcup_{J\in\mathfrak{M}} J_2$ and $\sigma = \sigma_L$. Then $L \cap J = \emptyset$ for all pure class $J \in \bar{I}(S)$, so $\sigma(S) \cap X_J = S \cap X_J = \dot{\mathrm{A}}_J$ by 12.9.2. For a mixed class $J = J_1 \dot\cup J_2$ we have, with the notation of Lemma 12.9(c), that $J_1' = J$ and $J_2' = \emptyset$ and therefore $\sigma(S) \cap X_J = \dot{\mathrm{A}}_J$ by 12.9.3. Since $\bar{I}_2(S) = \bar{I}_2(\sigma(S))$ by 12.8.6, Cor. 12.6(ii) shows that $\sigma(S)$ is pure.

(b) Bijectivity of Φ is clear by (a). As noted in 12.7, $\mathfrak{C}_0$ is stable under $\mathrm{Sym}(I)$. Since N is a normal subgroup of G, the quotient G/N acts naturally on $\mathfrak{C}/N$, and hence so does $\mathrm{Sym}(I) \cong G/N$. It follows that Φ is $\mathrm{Sym}(I)$-equivariant.

We now turn to full subsystems and first characterize them within $\mathfrak{C}$.

12.11. Proposition. *A closed subsystem S of $R = \mathrm{T}_I$ is full if and only if*

$$S \cap X_{I_0(S)} = \mathrm{T}_{I_0(S)}. \tag{1}$$

In this case, for $i \neq j$ we have

$$\varepsilon_i + \varepsilon_j \text{ and } \varepsilon_i - \varepsilon_j \in S \quad\iff\quad i, j \in I_0(S). \tag{2}$$

Moreover, $I_0(S)$ is either empty or a pure equivalence class of $\approx_S$. A pure full subsystem S is given by

$$S = \mathrm{T}_{I_0(S)} \cup \bigcup_{J\in\bar{I}_2(S)} \dot{\mathrm{A}}_J. \tag{3}$$

Remark. Recall from 10.8(b) that every full subset S of R is the symmetric part of a parabolic subset P of R. In the setting of root reductive direct limit Lie algebras, P gives rise to a parabolic subalgebra whose semisimple part has root system S. In this context, a (less precise) version of (3) for $\mathrm{T} \neq \mathrm{BC}$ was given by Dimitrov-Penkov in [**23**, Prop. 5].

Proof. Suppose S is full, so $S = R \cap \mathrm{span}(S)$. Also $X_{I_0(S)} \subset \mathrm{span}(S)$ holds by definition of $I_0(S)$. Since $R \cap X_J = \mathrm{T}_J$ for any subset J of I, we have $S \cap X_{I_0(S)} = R \cap \mathrm{span}(S) \cap X_{I_0(S)} = R \cap X_{I_0(S)} = T_{I_0(S)}$, i.e., (1).

Conversely, assume that S satisfies (1). By 12.9.1, this condition is invariant under the action of N, and this is also true for the property of being full. By Prop. 12.10(a) we may therefore assume that S is pure. Let

$$Y = X_{I_0(S)} \oplus \bigoplus_{J\in\bar{I}_2(S)} \dot{X}_J \tag{4}$$

where $\dot{X}_J = X_J \cap \dot{X}$ is the subspace of trace zero elements in X_J. We claim that $S = R \cap Y$. By 12.6.2 and (1) we have (3), and therefore $S \subset R \cap Y$. For the reverse inclusion, let $0 \neq \alpha \in R \cap Y$ and let $\alpha = y_0 + \sum_{J\in\bar{I}_2(S)} y_J$ be the decomposition of α with respect to (4). If $\alpha \in \mathbb{Z}\varepsilon_i$ then $t(\alpha) \neq 0$, so $i \in I_0(S)$ and hence $\alpha \in R \cap X_{I_0(S)} = \mathrm{T}_{I_0(S)} \subset S$. It remains to consider $\alpha = \pm\varepsilon_k \pm \varepsilon_m$ for $k \neq m$. If $k, m \in I_0(S)$ then $\alpha \in R \cap X_{I_0(S)} \subset S$ as before. We thus may assume that at least one of k, m does not lie in $I_0(S)$. Let K and M be the $\approx_S$-equivalence classes of k and m, respectively. If $K \neq M$ we have $\pm\varepsilon_k = x_K$ and $\pm\varepsilon_m = x_M$ where at least one of x_K, x_M has trace zero by definition of Y, contradiction. Thus $K = M \in \bar{I}_2(S)$, and $\alpha = x_K$ has trace zero by definition of Y. But then $\alpha = \pm(\varepsilon_k - \varepsilon_m) \in \dot{\mathrm{A}}_K \subset S$ by (3). This proves that $S = R \cap Y$ is full.

Let now S be a full subsystem. For (2), let $\varepsilon_i + \varepsilon_j$ and $\varepsilon_i - \varepsilon_j$ be in S. Then $2\varepsilon_i = (\varepsilon_i + \varepsilon_j) + (\varepsilon_i - \varepsilon_j) \in \operatorname{span}(S)$, and similarly $2\varepsilon_j \in \operatorname{span}(S)$, whence $i, j \in I_0(S)$. Conversely, $i, j \in I_0(S)$ implies first of all $R \neq \dot{\mathrm{A}}_I$ by 12.3(e), so $\varepsilon_i \pm \varepsilon_j \in R$, and hence $\varepsilon_i \pm \varepsilon_j \in R \cap \operatorname{span}(S) = S$. Thus (2) holds, and this immediately implies that $I_0(S)$, if not empty, is a pure equivalence class of $\approx_S$.

We next use the results obtained so far to describe the maximal closed subsystems S of root systems. In the finite case, this is due to Borel-de Siebenthal [**7**, **36**]. Their description uses root bases in an essential way, but is more precise as it allows to determine easily the isomorphism class of S.

12.12. Lemma. *Let S be a maximal proper closed subsystem of a root system (R, X) and assume* $\operatorname{span}(S) = X$. *Then there exists a full subsystem R_0 of R of corank one with $R_0 \subset S$.*

Proof. Decompose $R = \coprod R_\lambda$ into irreducible components, and let $S_\lambda = S \cap R_\lambda$. Then it is clear that there is a unique μ such that $S_\lambda = R_\lambda$ for all $\lambda \neq \mu$, while S_μ is a maximal closed subsystem of R_μ with $\operatorname{span}(S_\mu) = R_\mu$. It is therefore no restriction to assume R irreducible.

If R is finite, the existence of R_0 follows easily from the Borel-de Siebenthal theorem [**36**, Th. 12.1]. We thus assume $R = \mathrm{T}_I$ where I is infinite and $\mathrm{T} \in \mathfrak{T}$. Clearly, S is not full so by 12.3(b), $\mathrm{T} \neq \dot{\mathrm{A}}$. From 12.5 and $\operatorname{span}(S) = X$ it follows that $I = I_0(S)$ because $\operatorname{span}(S \cap X_J)$ has codimension 1 for every class $J \in (I \setminus I_0(S))/{\approx_S}$, by (c) and (d) of 12.5. We write $\approx$ instead of $\approx_S$ for short and distinguish the following cases:

Case 1: $\mathrm{T} = \mathrm{B}$ or $\mathrm{T} = \mathrm{BC}$: By (a) and (b) of 12.5, there exists at most one $J \in I/{\approx}$ with $S \cap X_J = \mathrm{T}_J$, and then $S \cap X_K = \mathrm{T}'_K$ for $K \neq J$ where

$$\mathrm{T}' = \left\{ \begin{matrix} \mathrm{D} & \text{if } \mathrm{T} = \mathrm{B} \\ \mathrm{C} & \text{if } \mathrm{T} = \mathrm{BC} \end{matrix} \right\}.$$

Note that $\mathrm{T}'_K = \{\alpha \in \mathrm{T}_K : q_K(\alpha) \in 2\mathbb{Z}\}$, where q_K is defined as in B.3.1. By 12.5.1, there are the following subcases:

(a) $S = \mathrm{T}_J \oplus \bigoplus_{K \in (I \setminus J)/\approx} \mathrm{T}'_K$. Then $I \setminus J$ must be a single equivalence class, otherwise $S' = \mathrm{T}_J \oplus \mathrm{T}'_{I \setminus J} = \{\alpha \in \mathrm{T}_I : q_{I \setminus J}(\alpha) \in 2\mathbb{Z}\}$ would be a closed subsystem with $S \subsetneqq S' \subsetneqq R$, contradicting maximality of S. Now $S = \mathrm{T}_J \oplus \mathrm{T}'_K$ for $K = I \setminus J$, and we may put $R_0 = \mathrm{T}_J \oplus \dot{\mathrm{A}}_K = R_0(q_K)$.

(b) $S = \bigoplus_{K \in I/\approx} \mathrm{T}'_K$. Then $\mathrm{T}'_I \supset S$ is a closed proper subsystem so $S = \mathrm{T}'_I$ by maximality, and therefore $R_0 = \dot{\mathrm{A}}_I = R_0(q_I)$ meets our requirements.

Case 2: $\mathrm{T} = \mathrm{C}$ or $\mathrm{T} = \mathrm{D}$: If $\mathrm{T} = \mathrm{C}$ we are never in the situation of 12.5(b), while for $\mathrm{T} = \mathrm{D}$ we always are. Hence we have $S = \bigoplus_{J \in I/\approx} \mathrm{T}_J$ by 12.5. From maximality of S it follows easily that $I/{\approx}$ must have 2 elements, so $S = \mathrm{T}_J \oplus \mathrm{T}_K$ for a disjoint nontrivial decomposition $I = J \,\dot\cup\, K$, with $|J|, |K| \geqslant 2$ in case $\mathrm{T} = \mathrm{D}$. Then $R_0 = \mathrm{T}_J \oplus \dot{\mathrm{A}}_K = R_0(q_K)$ has the desired properties.

12.13. THEOREM. *Let (R, X) be a root system. For a basic coweight f of R we denote by $m = m(f)$ the unique positive integer such that $f(R) = [-m, m] \cap \mathbb{Z}$ as in Prop.* 7.12. *Then a subsystem S of R is maximal among the proper closed subsystems of R if and only if*

(i) *either $S = R_0(f) = \{\alpha \in R : f(\alpha) = 0\}$ for a basic coweight f with $m(f) = 1$, i.e., f is minuscule,*
(ii) *or $S = R_{[p]}(f) = \{\alpha \in R : f(\alpha) \in p\mathbb{Z}\}$ for a basic coweight f with $m(f) > 1$, and p a prime number with $p \leqslant m(f)$.*

Subsystems of type (i) *are full while those of type* (ii) *are not.*

Remarks. (a) We have $1 \leqslant m \leqslant 6$ by 7.12, so $p \in \{2, 3, 5\}$.

(b) The cases (i) and (ii) of the theorem correspond to the cases (i) and (ii) of [**36**, Th. 12.1].

(c) The subsystems of case (i) will be classified in 17.8.

Proof. From Prop. 7.16 it follows immediately that subsystems of type (i) and (ii) are maximal proper closed subsystems. Conversely, let S be a maximal proper closed subsystem. We first show that $\mathrm{span}(S)$ has codimension at most 1. Indeed, assume $\mathrm{span}(S)$ has codimension $\geqslant$2, and choose $\alpha \in R \setminus S$. Then $S' = R \cap \mathrm{span}(S \cup \{\alpha\})$ is a proper closed (even full) subsystem strictly bigger than S which is impossible. If $\mathrm{span}(S) = H$ is a hyperplane, then $R \cap H$ is a proper closed subsystem containing S so by maximality, $S = R \cap H$. Let $f \in X^*$ with $\mathrm{Ker}(f) = H$. Then $S = R_0(f)$. Also, f is a linear form of rank 1, so after replacing f by a suitable scalar multiple, we may assume f is a basic coweight (cf. Prop. 7.12). Assume $m(f) > 1$. Then S is not maximal by Prop. 7.16, so we must have $m(f) = 1$ and S is of type (i). If $\mathrm{span}(S) = X$ then by Lemma 12.12, S contains a full subsystem R_0 of R of corank 1, so $R_0 = R_0(f)$ for a basic coweight f of R. Now it follows from Prop. 7.16 that $S = R_{[p]}(f)$ for a prime number $p \leqslant m(f)$ so S is of type (ii).

12.14. DEFINITION. Our next aim is to give a purely combinatorial description of the set $\mathfrak{F}_0 = \mathfrak{F}_0(\mathrm{T}, I)$ of pure full subsystems of $R = \mathrm{T}_I$ and its $\mathrm{Sym}(I)$-action. Prop. 12.11 shows that an $S \in \mathfrak{F}_0$ is uniquely determined by its invariants $I_0(S)$ and $\sim_S = \approx_S$ which satisfy the restrictions listed in Lemma 12.3(e). The following definition puts this on a formal basis.

Consider a subset $I_0 \subset I$ and an equivalence relation $\sim$ on I. We say the pair $(I_0, \sim)$ is an *f-datum for* (T, I) (f as a reminder of "full") if the following conditions hold:

(i) $I_0 = \emptyset$ or I_0 is an equivalence class of $\sim$,
(ii) if $\mathrm{T} = \dot{\mathrm{A}}$ then $I_0 = \emptyset$,
(iii) if $\mathrm{T} = \mathrm{D}$ then $\mathrm{Card}\, I_0 \neq 1$.

Let $\mathbb{F}_0 = \mathbb{F}_0(\mathrm{T}, I)$ denote the set of f-data for (T, I). As mentioned before, we then have a well-defined map

$$\Upsilon \colon \mathfrak{F}_0 \to \mathbb{F}_0, \qquad S \mapsto (I_0(S), \sim_S), \tag{1}$$

and by 12.8.1 and 12.8.2 this map is $\mathrm{Sym}(I)$-equivariant, where $\mathrm{Sym}(I)$ acts on $\mathbb{F}_0$ in the obvious way. We will show in 12.17(a) that Υ is in fact a bijection. The next two results serve as a preparation for this.

Let $\sim$ be an equivalence relation on a set I and let I_0 be a saturated subset, i.e., a union (possibly empty) of equivalence classes of $\sim$. In analogy to 12.6.1 we define

$$\bar{I} := (I \setminus I_0)/\sim, \qquad \bar{I}_2 := \{J \in \bar{I} : |J| \geqslant 2\}. \tag{2}$$

We let

$$\bar{X} := \bigoplus_{J \in \bar{I}} \mathbb{R}\varepsilon_J$$

be the free vector space on the set $\bar{I}$, and define h: $X \to \bar{X}$ by

$$h(\varepsilon_i) = \left\{ \begin{matrix} 0 & \text{if } i \in I_0 \\ \varepsilon_{[i]} & \text{if } i \in I \setminus I_0 \end{matrix} \right\}, \tag{3}$$

where $[i]$ denotes the equivalence class of i.

12.15. Lemma. *The map h is surjective and has kernel*

$$\operatorname{Ker}(h) = X_{I_0,\sim} := X_{I_0} \oplus \bigoplus_{J \in \bar{I}_2} \dot{X}_J = \operatorname{span}\Big(\{\varepsilon_j : j \in I_0\} \cup \{\varepsilon_i - \varepsilon_j : i \sim j\}\Big), \tag{1}$$

where $\dot{X}_J = \dot{X} \cap X_J$. In particular, h satisfies:

$$h(\varepsilon_i - \varepsilon_j) = \left\{ \begin{matrix} 0 & \text{if } i \sim j \\ \varepsilon_{[i]} & \text{if } i \notin I_0 \ni j \\ -\varepsilon_{[j]} & \text{if } i \in I_0 \not\ni j \\ \varepsilon_{[i]} - \varepsilon_{[j]} & \text{if } i \not\sim j,\ i,j \notin I_0 \end{matrix} \right\}, \tag{2}$$

$$h(\varepsilon_i + \varepsilon_j) = \left\{ \begin{matrix} 0 & \text{if } i,j \in I_0 \\ 2\varepsilon_{[i]} & \text{if } i \sim j,\ i,j \notin I_0 \\ \varepsilon_{[i]} & \text{if } i \notin I_0 \ni j \\ \varepsilon_{[j]} & \text{if } i \in I_0 \not\ni j \\ \varepsilon_{[i]} + \varepsilon_{[j]} & \text{if } i \not\sim j,\ i,j \notin I_0 \end{matrix} \right\}. \tag{3}$$

Proof. Surjectivity of h is obvious. It is easy to see that $\dot{X}_J$ is spanned by all $\varepsilon_i - \varepsilon_j$, $i, j \in J$, which shows the inclusion from right to left in (1). Conversely, let $x = \sum_{i \in I} c_i \varepsilon_i \in \operatorname{Ker}(h)$. We rewrite x in the form

$$x = \sum_{i \in I_0} c_i \varepsilon_i + \sum_{J \in \bar{I}} \sum_{i \in J} c_i \varepsilon_i.$$

Then $0 = h(x) = \sum_{J \in \bar{I}} \big(\sum_{i \in J} c_i\big)\varepsilon_J$ shows that $\sum_{i \in J} c_i = 0$ for all $J \in \bar{I}$. Hence $c_i = 0$ if $J = \{i\}$, and if J has more than one element, i.e., $J \in \bar{I}_2$, then each $\sum_{i \in J} c_i \varepsilon_i$ is in $\dot{X}_J$. The formulas (2) and (3) follow easily from the definition of h.

12.16. Proposition. *We use the notations of* 12.14 *and let $R = \mathrm{T}_I$, where $\mathrm{T} \in \mathfrak{T}$. For $(I_0, \sim) \in \mathbb{F}_0(\mathrm{T}, I)$ define*

$$S = R_{I_0,\sim} := R \cap X_{I_0,\sim}. \tag{1}$$

Then:

(a) *S is a pure full subsystem of R, given explicitly by*

$$S = \mathrm{T}_{I_0} \cup \bigcup_{J \in \bar{I}_2} \dot{\mathrm{A}}_J, \tag{2}$$

and the linear span of S is

$$\operatorname{span}(S) = X_{I_0,\sim}. \tag{3}$$

(b) *The invariants $I_0(S)$ and $\sim_S$ of S are*

$$I_0(S) = I_0 \quad \textit{and} \quad \sim_S = \sim. \tag{4}$$

(c) *$(I_0, \sim)$ is also an f-datum for $(\mathrm{T}^\vee, I)$, cf.* 8.2, *and* $(\mathrm{T}_{I,I_0,\sim})^\vee \cong \mathrm{T}^\vee_{I,I_0,\sim}$.

Proof. (a) S is full, being the intersection of R with a subspace. We show (2): By Lemma 12.15, $S = R \cap \operatorname{Ker}(h)$. Now the inclusion from left to right in (2) follows easily from 12.14.3, 12.15.2 and 12.15.3, since any $\alpha \in R$ is either a multiple of ε_i or of the form $\pm\varepsilon_i \pm \varepsilon_j$. Conversely, $X_{I_0} \subset X_{I_0,\sim}$ by 12.15.1 so $\mathrm{T}_{I_0} = R \cap X_{I_0} \subset R \cap X_{I_0,\sim} = S$. Also, R contains all $\varepsilon_i - \varepsilon_j$, so all $\dot{\mathrm{A}}_J$, $J \in \bar{I}_2$, are contained in $R \cap X_{I_0,\sim} = S$, again by 12.15.1.

Next, we show that S is pure: If $\alpha = \varepsilon_i + \varepsilon_j \in S$ for $i \neq j$, then (2) shows that $\alpha \in \mathrm{T}_{I_0}$ and thus $i, j \in I_0$. Hence also $\varepsilon_i - \varepsilon_j \in \mathrm{T}_{I_0} \subset S$, so S is pure by 12.4(i).

The inclusion from left to right in (3) is obvious. Conversely, all $\varepsilon_i - \varepsilon_j$ (for $i, j \in J \in \bar{I}_2$) belong to S by (2) and hence to $\operatorname{span}(S)$, so it remains to show, by 12.15.1, that also all ε_i, $i \in I_0$, are in $\operatorname{span}(S)$. We may assume $R \neq \dot{\mathrm{A}}_I$, else $I_0 = \emptyset$ by 12.14(ii). If also $R \neq \mathrm{D}_I$ then ε_i or $2\varepsilon_i$ is in $\mathrm{T}_{I_0} \subset S$. If $R = \mathrm{D}_I$ then I_0 has at least two elements by (iii) of 12.14. Hence there exists $j \in I_0$, $j \neq i$, and then $\varepsilon_i \pm \varepsilon_j \in \mathrm{D}_{I_0} \subset S$, which implies $2\varepsilon_i = (\varepsilon_i + \varepsilon_j) + (\varepsilon_i - \varepsilon_j) \in \operatorname{span}(S)$.

(b) We have $i \in I_0 \iff \varepsilon_i \in \operatorname{Ker}(h)$ (by 12.14.3) $\iff \varepsilon_i \in \operatorname{span}(S)$ (by (3)) $\iff i \in I_0(S)$ (by definition of $I_0(S)$ in 12.3.3), proving the first equality of (4). Next, for $i \neq j$, we have $i \sim_S j \iff \varepsilon_i - \varepsilon_j \in S \iff$ (by (2)) $i, j \in I_0$ or $i, j \in J$ for some $J \in \bar{I}_2 \iff i \sim j$. Thus the second formula of (4) holds as well.

(c) The first claim follows from the description of $\mathrm{T}^\vee_I$ in 8.1 and the definition of f-data, the second is immediate using (2).

The following result contains the classification of the pure full subsystems of the classical root systems.

12.17. Theorem. *Let $R = \mathrm{T}_I$ where $\mathrm{T} \in \mathfrak{T}$. We use the notations introduced in* 12.7 *and* 12.14.

(a) *The map $\Upsilon\colon \mathfrak{F}_0 \to \mathbb{F}_0$ of* 12.14.1 *is bijective with inverse map $\Psi\colon \mathbb{F}_0 \to \mathfrak{F}_0$ given by $\Psi(I_0, \sim) = R_{I_0,\sim}$.*

(b) *The bijection $\Psi\colon \mathbb{F}_0 \to \mathfrak{F}_0$ of* (a) *induces a* $\operatorname{Sym}(I)$*-equivariant bijection $\mathbb{F}_0 \xrightarrow{\cong} \mathfrak{F}/N$ and hence a bijection*

$$\mathbb{F}_0/\operatorname{Sym}(I) \xrightarrow{\cong} \mathfrak{F}/G. \tag{1}$$

Proof. (a) By Prop. 12.16(b), $\Upsilon \circ \Psi$ is the identity on $\mathbb{F}_0$. Conversely, let $S \in \mathfrak{F}_0$, with invariants $(I_0(S), \sim_S) \in \mathbb{F}_0$. Then 12.11.3 and 12.16.2 say that $S = R_{I_0(S),\sim_S}$, so $\Psi \circ \Upsilon$ is the identity on $\mathfrak{F}_0$.

(b) As noted in 12.14, $\varUpsilon\colon \mathfrak{F}_0 \to \mathbb{F}_0$ is $\mathrm{Sym}(I)$-equivariant, and hence so is its inverse $\varPsi$. Combining $\varPsi$ with the $\mathrm{Sym}(I)$-equivariant bijection $\varPhi\colon \mathfrak{F}_0 \to \mathfrak{F}/N$ of 12.10.1, we obtain an $\mathrm{Sym}(I)$-equivariant bijection $\mathbb{F}_0 \xrightarrow{\cong} \mathfrak{F}/N$ and hence a bijection

$$\mathbb{F}_0/\operatorname{Sym}(I) \xrightarrow{\cong} (\mathfrak{F}/N)/\operatorname{Sym}(I) \cong (\mathfrak{F}/N)\Big/(G/N) \cong \mathfrak{F}/G.$$

12.18. *Quotients of classical root systems.* Let I be a set and J a subset of I. In addition to the root systems $\dot{\mathrm{A}}_I, \mathrm{B}_I, \mathrm{C}_I, \mathrm{BC}_I, \mathrm{D}_I$ of 8.1 we introduce the following subsets of $X = \bigoplus_{i\in I} \mathbb{R}\varepsilon_i$:

$$\mathrm{BC}_I(J) := \mathrm{B}_I \cup \{\pm 2\varepsilon_j : j \in J\}, \tag{1}$$
$$\mathrm{DC}_I(J) := \mathrm{D}_I \cup \{\pm 2\varepsilon_j : j \in J\}. \tag{2}$$

These sets are not root systems (unless J satisfies special conditions, see below), but they occur as quotients of classical root systems by full subsystems, and hence will be referred to as *quotient systems*. From the definition, it is obvious that they increase monotonically with J and that they interpolate between B_I and BC_I (resp., D_I and C_I) in the following sense:

$$\mathrm{B}_I = \mathrm{BC}_I(\emptyset) \subset \mathrm{BC}_I(J) \subset \mathrm{BC}_I(I) = \mathrm{BC}_I, \tag{3}$$
$$\mathrm{D}_I = \mathrm{DC}_I(\emptyset) \subset \mathrm{DC}_I(J) \subset \mathrm{DC}_I(I) = \mathrm{C}_I. \tag{4}$$

It is clear that $\mathrm{BC}_I(J)$ is isomorphic to $\mathrm{BC}_I(J')$ as soon as J and J' are conjugate under $\mathrm{Sym}(I)$, which is the case if and only if $\operatorname{Card} J = \operatorname{Card} J'$ *and* $\operatorname{Card}(I \setminus J) = \operatorname{Card}(I \setminus J')$ (the first condition alone is not sufficient in the infinite case). The same holds for $\mathrm{DC}_I(J)$. In case $I = \{1, \ldots, n\}$ is finite and $J = \{1, \ldots, k\}$, we will use the notations $\mathrm{BC}_n(k)$ and $\mathrm{DC}_n(k)$ instead of $\mathrm{BC}_I(J)$ and $\mathrm{DC}_I(J)$.

12.19. Proposition. (a) *Let $\mathrm{T} \in \mathfrak{T}$, and let S be a full subsystem of $R = \mathrm{T}_I$. Then R/S is either a root system $\mathrm{T}'_{I'}$ for some $\mathrm{T}' \in \mathfrak{T}$ and a suitable set I', or it is isomorphic to one of the sets $\mathrm{BC}_{I'}(J')$ or $\mathrm{DC}_{I'}(J')$ for suitable I', J'. Conversely, each such set occurs as a quotient of R by a full subsystem S.*

(b) *In more detail, let $S = R_{I_0,\sim}$ as in* 12.16 *be a pure full subsystem corresponding to the f-datum $(I_0, \sim) \in \mathbb{F}_0$. Then the quotient $\bar{R} = R/S$ is given as follows, the notations $\bar{I}$ and $\bar{I}_2$ being as in* 12.14.2*:*

$$\overline{\dot{\mathrm{A}}_I} = \dot{\mathrm{A}}_{\bar{I}}, \tag{1}$$

$$\overline{\mathrm{B}_I} = \mathrm{BC}_{\bar{I}}(\bar{I}_2), \tag{2}$$

$$\overline{\mathrm{C}_I} = \left\{ \begin{matrix} \mathrm{C}_{\bar{I}} & \textit{if } I_0 = \emptyset \\ \mathrm{BC}_{\bar{I}} & \textit{if } I_0 \neq \emptyset \end{matrix} \right\}, \tag{3}$$

$$\overline{\mathrm{BC}_I} = \mathrm{BC}_{\bar{I}}, \tag{4}$$

$$\overline{\mathrm{D}_I} = \left\{ \begin{matrix} \mathrm{DC}_{\bar{I}}(\bar{I}_2) & \textit{if } I_0 = \emptyset \\ \mathrm{BC}_{\bar{I}}(\bar{I}_2) & \textit{if } I_0 \neq \emptyset \end{matrix} \right\}. \tag{5}$$

Hence the rank of R/S is given by

$$\operatorname{rank}(R/S) = \left\{ \begin{array}{ll} \operatorname{Card} \bar{I} - 1 & \textit{if } \mathrm{T} = \dot{\mathrm{A}} \\ \operatorname{Card} \bar{I} & \textit{otherwise} \end{array} \right\}. \tag{6}$$

The quotient map $R \to \bar{R}$ may be identified with the map $h\colon X \to \bar{X}$ in case $\mathrm{T} \neq \dot{\mathrm{A}}$, and in case $\mathrm{T} = \dot{\mathrm{A}}$ with the map $h\colon \dot{X} \to \operatorname{Ker}(\bar{t})$ where $\bar{t}\colon \bar{X} \to \mathbb{R}$ is defined by $\bar{t}(\varepsilon_J) = 1$ for $J \in \bar{I}$.

Proof. By Prop. 12.10(b), any full subsystem is of the form $\sigma(S)$ for a pure S. Hence it suffices to prove (b). By Lemma 12.15 and Prop. 12.16(a), we have $\operatorname{Ker}(h) = \operatorname{span}(S)$.

Let first $\mathrm{T} \neq \dot{\mathrm{A}}$. Then $R = \mathrm{T}_I$ spans X, so we may identify the canonical map can: $X \to X/\operatorname{span}(S)$ with $h\colon X \to \bar{X}$, and correspondingly the quotient R/S with $h(R) \subset \bar{X}$. Now (2) – (5) follow easily from 12.14.3, 12.15.2 and 12.15.3.

Next, let $\mathrm{T} = \dot{\mathrm{A}}$. Then $I_0 = \emptyset$ and $\operatorname{span}(R) = \dot{X} = \operatorname{Ker}(t)$ is the kernel of the trace form t. We claim that $h(\dot{X}) = \operatorname{Ker}(\bar{t})$ for $\bar{t}$ defined as above. Indeed, $\dot{X}$ and $\operatorname{Ker}(\bar{t})$ are spanned by all $\varepsilon_i - \varepsilon_j$ $(i, j \in I)$ and $\varepsilon_J - \varepsilon_K$ $(J, K \in \bar{I})$, respectively, and by 12.15.2 and because of $I_0 = \emptyset$ we have $h(\varepsilon_i - \varepsilon_j) = \varepsilon_{[i]} - \varepsilon_{[j]}$. Thus we may identify the canonical map can: $\dot{X} \to \dot{X}/\operatorname{span}(S)$ with the map $h\colon \dot{X} \to \operatorname{Ker}(\bar{t})$ and then have (1). Finally, (6) is clear from (1) – (5).

12.20. Corollary. *Let $R = \mathrm{T}_I$ and $S \subset R$ a full subsystem. Then S is of scalar type, i.e., $S = R_0(f)$ for some linear form $f \in X^*$, if and only if $\operatorname{rank}(R/S) \leqslant \operatorname{Card}(\mathbb{R})$.*

Proof. If S is of scalar type then $\operatorname{rank}(R/S) = \operatorname{rank}(f) \leqslant \operatorname{Card}(\mathbb{R})$ by 8.11. Conversely, let $\operatorname{rank}(R/S) \leqslant \operatorname{Card}(\mathbb{R})$. Then also $\operatorname{Card}(\bar{I}) \leqslant \operatorname{Card}(\mathbb{R})$ by 12.19.6. For any automorphism u, we have S scalar if and only if $u(S)$ is so, because of the easily verified formula $R_0(f \circ u^{-1}) = u(R_0(f))$. Hence we may assume S pure by Prop. 12.10(c). It suffices to find a linear form $\bar{f}$ on $\bar{X} = X/\operatorname{span}(S)$ with the property that $\bar{f}(\bar{\alpha}) \neq 0$ for all $\bar{\alpha} \neq 0$ in R/S, and then put $f = \bar{f} \circ \mathrm{can}$. Since $\operatorname{Card}(\bar{I}) \leqslant \operatorname{Card}(\mathbb{R}) = \operatorname{Card}(\mathbb{R}_{++})$, there exists an injective map $\varphi\colon \bar{I} \to \mathbb{R}_{++}$. Now define $\bar{f}$ by $\bar{f}(\varepsilon_J) = \varphi(J)$, for all $J \in \bar{I}$. Then it follows easily from (1) – (5) of Prop. 12.19 that $\bar{f}$ has the required property.

12.21. *The quotient systems* $\mathrm{BC}_I(J)$ *and* $\mathrm{DC}_I(J)$. We finish this section by proving some structural results on the quotient systems $\mathrm{BC}_I(J)$ and $\mathrm{DC}_I(J)$.

In the sequel, Q denotes one of these two sets. We recall that the full subsets of Q are determined by the First Isomorphism Theorem 1.7(c): if $Q = R/S$ for a full subsystem S of a suitable root system R of type B or D, there is a bijection between the full subsets of Q and the full subsets (= full subsystems) of R which contain S. Similarly, by Prop. 10.19(c), there is a bijection between the parabolic subsets of Q and the parabolic subsets of R containing S. Moreover, under this bijection the positive systems of Q correspond to the parabolic subsets with symmetric part S. We will describe the parabolic subsystems of the root systems $R = \mathrm{T}_I$ in the following section 13.

For the description of the automorphisms of Q, the following concept will be useful. We let Q_r be the union of $\{0\}$ and the subset of all $\alpha \in Q^\times$ for which there exists a reflection s in the sense of 3.1 such that $s(\alpha) = -\alpha$ and $s(Q) = Q$. Recall from 3.2 that such a reflection is unique if it exists, since Q is locally finite. Local finiteness of Q can either easily be seen directly or as an application of Th. 6.4.

We first consider $\mathrm{DC}_2(1)$ which plays a special role:

12.22. LEMMA. $\mathrm{DC}_2(1) = \{0, \pm 2\varepsilon_1, \pm\varepsilon_1 \pm \varepsilon_2\}$ *is isomorphic to the root system* A_2 *via the isomorphism* $\varepsilon_1 - \varepsilon_2 \mapsto \varepsilon_1' - \varepsilon_2' \in \mathrm{A}_2$ *and* $\varepsilon_1 + \varepsilon_2 \mapsto \varepsilon_2' - \varepsilon_3' \in \mathrm{A}_2$. *Consequently,* $\mathrm{DC}_2(1)_r = \mathrm{DC}_2(1)$ *and*

$$\mathrm{Aut}(\mathrm{DC}_2(1)) \cong \mathfrak{S}_3 \times \{\pm \mathrm{Id}\}. \tag{1}$$

The proof is a simple verification which is left to the reader.

12.23. PROPOSITION. *Let* $Q = \mathrm{BC}_I(J)$ *or* $\mathrm{DC}_I(J)$, *and assume* $Q \neq \mathrm{DC}_2(1)$. *Then with the notations of* 12.1 *and* $K := I \setminus J$ *we have*

$$Q_r = (Q \cap X_J) \cup (Q \cap X_K) = \left\{ \begin{array}{ll} \mathrm{BC}_J \cup \mathrm{B}_K & \text{if } Q = \mathrm{BC}_I(J) \\ \mathrm{C}_J \cup \mathrm{D}_K & \text{if } Q = \mathrm{DC}_I(J) \end{array} \right\}. \tag{1}$$

Proof. We may assume that both J and K are nonempty, otherwise Q is one of the root systems listed in 12.18.3 and 12.18.4 and hence $Q = Q_r$. For the inclusion from right to left, note that the set $(Q \cap X_J) \cup (Q \cap X_K)$ obviously has the given description, in particular it is a direct sum of two root systems. The usual orthogonal reflection of $\alpha \in (Q \cap X_J) \cup (Q \cap X_K)$ leaves Q invariant, which, for example, follows easily from the description

$$Q = R \setminus \{\pm 2\varepsilon_k : k \in K\} \quad \text{for} \quad R = \left\{ \begin{array}{ll} \mathrm{BC}_I & \text{if } Q = \mathrm{BC}_I(J) \\ \mathrm{C}_I & \text{if } Q = \mathrm{DC}_I(J) \end{array} \right\}.$$

For the inclusion from left to right in (1), it suffices to show that $\alpha = \pm\varepsilon_j \pm \varepsilon_k$ with $j \in J$ and $k \in K$ does not belong to Q_r. Suppose to the contrary that $\alpha \in Q_r$ and denote by s the corresponding reflection. It is no restriction to assume $\alpha = \varepsilon_j + \varepsilon_k$ because the full group $\mathbf{2}^I$ of sign changes clearly acts by automorphisms of Q. We will also write $1 = j$ and $2 = k$ to simplify notation.

By 3.1.1, the reflection s has the form $s(x) = x - \langle x, l\rangle(\varepsilon_1 + \varepsilon_2)$ where $l \in X^*$ satisfies $l(\varepsilon_1 + \varepsilon_2) = 2$. It follows from this that s leaves the plane $\mathbb{R}\varepsilon_1 \oplus \mathbb{R}\varepsilon_2$ invariant. For $i \in I$ we put $a_i := \langle \varepsilon_i, l\rangle$.

We first suppose $Q = \mathrm{DC}_I(J)$ and then have $Q \cap (\mathbb{R}\varepsilon_1 \oplus \mathbb{R}\varepsilon_2) = \mathrm{DC}_2(1)$. Hence the restriction of s to $\mathrm{DC}_2(1)$ corresponds, under the isomorphism of Lemma 12.22, to the usual reflection of A_2 in the root $\varepsilon_2' - \varepsilon_3'$. A simple computation then shows that

$$s(\varepsilon_1 + \varepsilon_2) = -(\varepsilon_1 + \varepsilon_2), \qquad s(\varepsilon_1 - \varepsilon_2) = 2\varepsilon_1,$$

and hence

$$s(\varepsilon_1) = \frac{1}{2}(\varepsilon_1 - \varepsilon_2), \quad s(\varepsilon_2) = -\frac{3}{2}\varepsilon_1 - \frac{1}{2}\varepsilon_2, \qquad a_1 = \frac{1}{2}, \quad a_2 = \frac{3}{2}.$$

Now assume there exists $1 \neq j \in J$. Then $2\varepsilon_j \in Q$, hence also $s(2\varepsilon_j) = 2\varepsilon_j - 2a_j(\varepsilon_1 + \varepsilon_2) \in Q$ which is only possible if $a_j = 0$, because no element of Q

has nonzero coefficients at three ε_i's. This implies $s(\varepsilon_j + \varepsilon_2) = \varepsilon_j + s(\varepsilon_2) = \varepsilon_j - (3/2)\varepsilon_1 - (1/2)\varepsilon_2 \in Q$, contradiction. Thus we must have $J = \{1\}$, a singleton.

Next, assume that there exists $2 \neq k \in K$. Then

$$s(\varepsilon_1 + \varepsilon_k) = \frac{1}{2}(\varepsilon_1 - \varepsilon_2) + \varepsilon_k - a_k(\varepsilon_1 + \varepsilon_2) = \big(\frac{1}{2} - a_k\big)\varepsilon_1 - \big(\frac{1}{2} + a_k\big)\varepsilon_2 + \varepsilon_k \in Q$$

implies $a_k \in \{\pm(1/2)\}$. Similarly,

$$s(\varepsilon_2 + \varepsilon_k) = \varepsilon_k - \big(\frac{3}{2} + a_k\big)\varepsilon_1 - \big(\frac{1}{2} + a_k\big)\varepsilon_2 \in Q$$

shows $a_k \in \{-(1/2), -(3/2)\}$. Hence we obtain $a_k = -(1/2)$ for all $k \in K \setminus \{2\}$. But then

$$s(\varepsilon_2 - \varepsilon_k) = -\frac{3}{2}\varepsilon_1 - \frac{1}{2}\varepsilon_2 - \big(\varepsilon_k + \frac{1}{2}(\varepsilon_1 + \varepsilon_2)\big) = -2\varepsilon_1 - \varepsilon_2 - \varepsilon_k \in Q,$$

contradiction. Thus also $K = \{2\}$ is a singleton, and we are in the excluded case $Q = \mathrm{DC}_2(1)$.

We still need to consider the case $Q = \mathrm{BC}_J(I)$. Here $Q' := Q \cap (\mathbb{R}\varepsilon_1 \oplus \mathbb{R}\varepsilon_2) = \{\pm\varepsilon_1, \pm 2\varepsilon_1, \pm\varepsilon_1 \pm \varepsilon_2\}$. Since both ε_1 and $2\varepsilon_1$ lie in Q', we must have $s(\varepsilon_1) = \pm\varepsilon_1$, which yields $a_1 = 0$, $a_2 = 2$, and then the contradiction $s(\varepsilon_1 - \varepsilon_2) = \varepsilon_1 - (\varepsilon_2 - 2(\varepsilon_1 + \varepsilon_2)) = 3\varepsilon_1 + \varepsilon_2 \in Q$. This completes the proof.

12.24. Proposition. *Let $Q = \mathrm{BC}_I(J)$ or $\mathrm{DC}_I(J)$ where $\emptyset \neq J \neq I$, and suppose $Q \neq \mathrm{DC}_2(1)$. Also let $K = I \setminus J$ and embed $\mathrm{Sym}(J) \times \mathrm{Sym}(K)$ into $\mathrm{Sym}(I)$ in the natural way. Then*

$$\mathrm{Aut}(Q) = \big(\mathrm{Sym}(J) \times \mathrm{Sym}(K)\big) \ltimes \mathbf{2}^I. \tag{1}$$

Proof. From the structure of Q it follows immediately that we have the inclusion from right to left in (1). To prove the inclusion from left to right, observe that

$$\mathrm{Aut}(Q) \subset \mathrm{Aut}(Q_r). \tag{2}$$

By Prop. 12.23, Q_r is a direct sum of the two non-isomorphic root systems $Q \cap X_J$ and $Q \cap X_K$, so φ must leave $Q \cap X_J$ and $Q \cap X_K$ invariant. Also, $Q \cap X_J$ is either BC_J or C_J, and hence spans X_J. Thus φ stabilizes X_J, and $\varphi\big|X_J \in \mathrm{Aut}(Q \cap X_J) = \mathrm{Sym}(J) \ltimes \mathbf{2}^J$. We now distinguish the following cases:

Case 1: $Q = \mathrm{BC}_I(J)$. Then $Q \cap X_K = \mathrm{B}_K$ spans X_K, so $\varphi\big|X_K \in \mathrm{Aut}(\mathrm{B}_K) = \mathrm{Sym}(K) \ltimes \mathbf{2}^K$.

Case 2: $Q = \mathrm{D}_I(J)$, and $|K| \notin \{1, 4\}$. Then $Q \cap X_K = \mathrm{D}_K$ spans X_K, so again φ stabilizes X_K and $\varphi\big|X_K \in \mathrm{Aut}(\mathrm{D}_K)$ which is, because of the restriction on $|K|$, the group $\mathrm{Sym}(K) \ltimes \mathbf{2}^K$.

Case 3: $Q = \mathrm{D}_I(J)$, and $|K| = 1$. Let, say, $K = \{1\}$. Then $Q \cap X_K = \mathrm{D}_1 = \{0\}$ does not span $X_K = \mathbb{R}\varepsilon_1$. We have $|J| \geqslant 2$ since the case $Q = \mathrm{D}_2(1)$ was excluded. The restriction $\varphi|X_J = g \in \mathrm{Aut}(\mathrm{C}_J) = \mathrm{Sym}(J) \ltimes \mathbf{2}^J$ can be extended to an automorphism $\tilde{g}$ of Q by $\tilde{g}(\varepsilon_1) = \varepsilon_1$, and then $\psi := \varphi \circ \tilde{g}^{-1}$ has $\psi|X_J = \mathrm{Id}$.

Consider $\psi(\varepsilon_1)$, which must have the form $\psi(\varepsilon_1) = a_1\varepsilon_1 + \sum_{j\in J'} a_j\varepsilon_j$ where J' is a finite subset of J and the coefficients a_1 and a_j are nonzero. For every $i \in J$ we have $\psi(\varepsilon_i + \varepsilon_1) = \varepsilon_i + a_1\varepsilon_1 + \sum_{j\in J'} a_j\varepsilon_j \in Q$. Since no element of Q has nonzero coefficients at more than two ε's, this already implies $|J'| \leqslant 2$, and we also must have $a_1 \in \{\pm 1\}$. If $|J'| = 1$, say, $J' = \{2\}$, then because J contains an element different from 2, say 3, we have $\varepsilon_1 + \varepsilon_3 \in Q$ and obtain the contradiction $\psi(\varepsilon_1 + \varepsilon_3) = a_1\varepsilon_1 + a_2\varepsilon_2 + \varepsilon_3 \in Q$. If J' has two elements, say $J' = \{2, 3\}$, then $\psi(\varepsilon_1 \pm \varepsilon_2) = a_1\varepsilon_1 + (a_2 \pm 1)\varepsilon_2 + a_3\varepsilon_3 \in Q$ implies $a_2 \pm 1 = 0$, which is again impossible. Thus we must have $J' = \emptyset$, and ψ is either the identity or $\psi = \sigma_{\{1\}} \in \mathbf{2}^K$. Hence $\varphi = \psi \circ \tilde{g}$ belongs to the right hand side of (1).

Case 4: $Q = \mathrm{D}_I(J)$, and $|K| = 4$. Again φ stabilizes $X_K = \mathrm{span}(\mathrm{D}_4)$ and thus $\varphi\big|X_K \in \mathrm{Aut}(\mathrm{D}_4)$. Now $\mathrm{Aut}(\mathrm{D}_4)$ contains $A := \mathfrak{S}_4 \ltimes \mathbf{2}^4$ as a subgroup of index three. To prove φ lies in the right hand side of (1), it suffices to show that $\varphi|X_K \in A$.

Let $K = \{1, \ldots, 4\}$ and consider the root basis $\{\varepsilon_2 + \varepsilon_1, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \varepsilon_4 - \varepsilon_3\}$ of D_4. Then the diagram automorphism fixing $\varepsilon_3 - \varepsilon_2$ and mapping

$$\varepsilon_2 + \varepsilon_1 \mapsto \varepsilon_2 - \varepsilon_1, \quad \varepsilon_2 - \varepsilon_1 \mapsto \varepsilon_4 - \varepsilon_3, \quad \varepsilon_4 - \varepsilon_3 \mapsto \varepsilon_2 + \varepsilon_1 \tag{3}$$

extends to an automorphism τ of order three of D_4, and $\mathrm{Aut}(\mathrm{D}_4) = T \cdot A$ where $T = \langle\tau\rangle$ is cyclic of order three. (This is not a semidirect product because neither T nor A is normal in $\mathrm{Aut}(\mathrm{D}_4)$).

Now assume, aiming for a contradiction, that $\varphi\big|X_K \notin A$. Then $\varphi\big|X_K = \tau^n g$ where $n \in \{1, 2\}$ and $g \in A$. Extend g to an automorphism $\tilde{g}$ of Q by $\tilde{g}\big|X_J = \varphi\big|X_J$. Then $\psi := \varphi \circ \tilde{g}^{-1} \in \mathrm{Aut}(Q)$ satisfies $\psi\big|X_J = \mathrm{Id}$ and $\psi\big|X_K = \tau^n$. From (3), one computes easily that

$$\tau(\varepsilon_1) = \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4), \quad \tau^2(\varepsilon_1) = \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4).$$

Since $\varepsilon_j + \varepsilon_1 \in Q$ for any $j \in J$, we arrive at the contradiction $\psi(\varepsilon_j + \varepsilon_1) = \varepsilon_j + (1/2)(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \in Q$. This completes the proof.

Remark. From (1) and 12.23.1 and the structure of the automorphism groups of the classical root systems (cf. §9), it follows that (2) is in general not an equality.

§13. Parabolic subsets of root systems: classification

13.1. *Notations and conventions.* We classify in this section the parabolic subsets of the irreducible infinite root systems $R = \mathrm{T}_I$, $\mathrm{T} \in \mathfrak{T} = \{\dot{\mathrm{A}}, \mathrm{B}, \mathrm{C}, \mathrm{BC}, \mathrm{D}\}$, up to equivalence under the big Weyl group. For finite root systems, the description of the parabolic subsets is well known, see Lemma 11.1. As in §12 it turns out that our methods do not require I to be infinite, and we therefore let I be an arbitrary set, finite or infinite. We use the notations and conventions introduced in 12.1 and 12.7.

We follow the same procedure as in our description of full subsystems in §12: With every parabolic subset P of R we associate in 13.2 combinatorial invariants $I_\nu(P) \subset I$ (where $\nu \in \{+, -, 0, 1\}$) and $\succcurlyeq_P$, the latter being a total preorder (see B.2) on I. The condition $I_-(P) = \emptyset$ defines a subset $\mathfrak{P}_0$ of the set $\mathfrak{P}$ of all parabolic subsets of R, whose elements we call pure. The invariants $I_0(P)$ and $\succcurlyeq_P$ of a pure P suffice to describe it uniquely (Prop. 13.4), and $\mathfrak{P}_0$ is a fundamental domain for the action of N on $\mathfrak{P}$ (Prop. 13.6). As an application, we give necessary and sufficient conditions for a parabolic subset to be of scalar type (13.7). The invariants satisfy certain restrictions which, in turn, define a set $\mathbb{P}_0$ of combinatorial data, and the map $P \mapsto (I_0(P), \succcurlyeq_P)$ is a $\mathrm{Sym}(I)$-equivariant bijection $\mathfrak{P}_0 \cong \mathbb{P}_0$ which yields a bijection $\mathfrak{P}/G \cong \mathbb{P}_0/\operatorname{Sym}(I)$ (Th. 13.11).

13.2. Lemma. *With the notations and conventions of* 12.1 *and* 13.1*, let* $\mathrm{T} \in \mathfrak{T}$ *and* $R = \mathrm{T}_I$*. Let* $P \subset R$ *be a parabolic subset with symmetric part* $P_s = P \cap (-P)$*, and let* $K = \mathbb{R}_+[P]$ *be the convex cone spanned by* P*. Consider the relation* $\succcurlyeq_P$ *on* I *defined by*

$$i \succcurlyeq_P j \quad :\Longleftrightarrow \quad \varepsilon_i - \varepsilon_j \in P, \tag{1}$$

as well as the partition of I *into the following four subsets:*

$$\begin{aligned} I_0(P) &:= \{i \in I : \pm\varepsilon_i \in K\}, &\qquad I_+(P) &:= \{i \in I : \varepsilon_i \in K,\ -\varepsilon_i \notin K\},\\ I_1(P) &:= \{i \in I : \pm\varepsilon_i \notin K\}, &\qquad I_-(P) &:= \{i \in I : \varepsilon_i \notin K,\ -\varepsilon_i \in K\}. \end{aligned}$$

Then:

(a) $\succcurlyeq_P$ *is a total preorder on* I*, whose associated equivalence relation* $\sim_P$ *is given by*

$$i \sim_P j \quad \Longleftrightarrow \quad \varepsilon_i - \varepsilon_j \in P_s \quad \Longleftrightarrow \quad i \sim_{P_s} j,$$

so with the notation of 12.3.3 *we have*

$$I_0(P) = I_0(P_s).$$

(b) *The subsets* $I_\pm(P)$ *satisfy* $I_+(P) \succnsim I_-(P)$*. They are either empty or a union of equivalence classes of* $\sim_P$*.*

(c) *We have* $I_0(P) = \emptyset$ *or* $I_1(P) = \emptyset$*, hence* $I = I_+(P) \,\dot\cup\, I_\nu(P) \,\dot\cup\, I_-(P)$ *for* $\nu \in \{0, 1\}$*. Moreover,*

$$I_+(P) \gtrsim I_\nu(P) \gtrsim I_-(P). \tag{2}$$

If $I_0(P)$ is not empty then it is a full equivalence class of $\sim_P$.

(d) *The subset $I_1(P)$ satisfies the following conditions, depending on the type* T*:*

(i) *if* $\mathrm{T} = \dot{\mathrm{A}}$ *then* $I = I_1(P)$,
(ii) *if* $\mathrm{T} = \mathrm{B}, \mathrm{C}$ *or* BC *then* $I_1(P) = \emptyset$,
(iii) *if* $\mathrm{T} = \mathrm{D}$ *then* $|I_1(P)| \leqslant 1$.

(e) *Let* $\mathrm{T} \neq \dot{\mathrm{A}}$ *and let* $i, j \in I_+(P) \cup I_0(P) \cup I_1(P)$, $i \neq j$. *Then* $\varepsilon_i + \varepsilon_j \in P$, *and*

$$-(\varepsilon_i + \varepsilon_j) \in P \iff i, j \in I_0(P) \iff \pm\varepsilon_i \pm \varepsilon_j \in P. \tag{3}$$

Proof. For simpler notation, we will drop the subscript P at $\succcurlyeq$ and $\sim$ and abbreviate $I_\nu(P) = I_\nu$ for $\nu \in \{0, 1, \pm\}$.

(a) The property $i \succcurlyeq j$ or $j \succcurlyeq i$ holds because $P \cup (-P) = R$ and R contains all $\varepsilon_i - \varepsilon_j$. To prove transitivity, suppose $i \succcurlyeq j$ and $j \succcurlyeq k$. Since P is additively closed, we then have $\varepsilon_i - \varepsilon_k = (\varepsilon_i - \varepsilon_j) + (\varepsilon_j - \varepsilon_k) \in P$, or $i \succcurlyeq k$. The assertion about $\sim_P$ is immediate from $P_s = P \cap (-P)$. Finally, we have $j \in I_0(P)$ if and only if $\varepsilon_j \in K \cap (-K) = \operatorname{span}(P_s)$ (by 10.17.6), which proves $I_0(P) = I_0(P_s)$.

We will prove (b) and (c) simultaneously. First we show

$$(I_+ \cup I_0) \gtrsim (I_- \cup I_1). \tag{4}$$

Let $i \in I_+ \cup I_0$, $j \in I_- \cup I_1$ and assume $i \preccurlyeq j$, i.e., $\varepsilon_j - \varepsilon_i \in P \subset K$. Since $\varepsilon_i \in K$ by definition of I_+ and I_0, we have $\varepsilon_i + (\varepsilon_j - \varepsilon_i) = \varepsilon_j \in K$, contradicting $j \in I_- \cup I_1$. Because $\succcurlyeq$ is a total preorder, this implies $i \gtrsim j$. Similarly we prove that

$$(I_+ \cup I_1) \gtrsim (I_- \cup I_0). \tag{5}$$

Indeed, let $i \in I_+ \cup I_1$, $j \in I_- \cup I_0$ and assume $i \preccurlyeq j$. Thus $\varepsilon_j - \varepsilon_i \in P \subset K$, but also $-\varepsilon_j \in K$ by definition of I_-, I_0. Hence $(\varepsilon_j - \varepsilon_i) + (-\varepsilon_j) = -\varepsilon_i \in K$, contradicting $i \in I_+ \cup I_1$. Therefore $i \gtrsim j$.

Let now $i \in I_0$ and $j \in I_1$. Then $i \gtrsim j$ by (4), while $j \gtrsim i$ by (5), contradiction. Thus $I_0 = \emptyset$ or $I_1 = \emptyset$, and (2) follows from (4) and (5) above. To finish the proof of (b), let $i \in I_+$ and suppose $i \sim j$ for a $j \in I$. Then $j \succcurlyeq i$ so (2) implies $j \in I_+$. Hence I_+ is either empty or a union of equivalence classes of $\sim$. The proof for I_- is analogous. Finally, because the equivalence relations $\sim$ and $\sim_{P_s}$ coincide, it follows from 12.11 that $I_0 = I_0(P_s)$ is either empty or an equivalence class of $\sim$.

(d) Let t be the trace form, given by $t(\varepsilon_i) = 1$ for all i. Since $\dot{\mathrm{A}}_I \subset \dot{X} = \operatorname{Ker}(t)$, it is clear that also $K \subset \dot{X}$ in case $R = \dot{\mathrm{A}}_I$, proving case (i). If $\mathrm{T} \in \{\mathrm{B}, \mathrm{C}, \mathrm{BC}\}$ then either ε_i or $2\varepsilon_i$ belongs to $R = P \cup (-P)$ whence $I_1(P) = \emptyset$. Finally, let $R = \mathrm{D}_I$, and assume I_1 contains more than one element, say, that $i, j \in I_1$. Since $R = P \cup (-P)$, we have $\varepsilon_i + \varepsilon_j \in P$ or $-(\varepsilon_i + \varepsilon_j) \in P$. For the same reason, $\varepsilon_i - \varepsilon_j \in P$ or $\varepsilon_j - \varepsilon_i \in P$. Possibly after exchanging i and j, we may assume $s\varepsilon_i \pm \varepsilon_j \in P$ for $s = +$ or $s = -$. But then $\frac{1}{2}\big((s\varepsilon_i + \varepsilon_j) + (s\varepsilon_i - \varepsilon_j)\big) = s\varepsilon_i \in K$ since K is convex, contradicting $i \in I_1$. This proves $|I_1| \leqslant 1$.

(e) Let $i, j \in I_+ \cup I_0$. Then $\varepsilon_i, \varepsilon_j \in K$ and so $\varepsilon_i + \varepsilon_j \in K \cap R = P$ by 10.17.3. Suppose $j \in I_1$. Then necessarily $I_0 = \emptyset$ and $i \in I_+$, whence $\varepsilon_i \in K$. If $\varepsilon_i + \varepsilon_j \notin P$ then $-(\varepsilon_i + \varepsilon_j) \in P$ and $\varepsilon_i + (-\varepsilon_i - \varepsilon_j) = -\varepsilon_j \in K$, contradiction. Therefore $\varepsilon_i + \varepsilon_j \in P$ in all cases.

Now suppose $-(\varepsilon_i + \varepsilon_j) \in P$. We may assume $i \succcurlyeq j$, so $\varepsilon_i - \varepsilon_j \in P$ and then $(-\varepsilon_i - \varepsilon_j) + (\varepsilon_i - \varepsilon_j) = -2\varepsilon_j \in K$. This implies $j \in I_- \cup I_0$ whence $j \in I_0$. Then $I_1 = \emptyset$ by (c), so $i \in I_0 \cup I_+$, i.e., $\varepsilon_i \in K$. Since $\pm\varepsilon_j \in K$, we also have $-\varepsilon_i = (-\varepsilon_i - \varepsilon_j) + \varepsilon_j \in K$, proving $i \in I_0$. We have now established "$\Longrightarrow$" of the first equivalence in (3). The remaining implications follow from 12.11.2 and $I_0 = I_0(P_s)$.

Remarks. (i) Just as the invariants $\sim_S$ and $I_0(S)$ of a subsystem S in 12.3, the subsets $I_\nu(P)$ depend not only on the root system R and the parabolic subset P, but on the triple (T, I, P). Indeed, $\dot{\mathrm{A}}_4 = \mathrm{A}_3 \cong \mathrm{D}_3$, so any parabolic subset $P \subset \dot{\mathrm{A}}_4$ has $|I_1(P)| = 4$ while $|I_1(P)| \leqslant 1$ for $P \subset \mathrm{D}_3$.

(ii) By definition of $\succcurlyeq$ we have

$$P \cap \dot{\mathrm{A}}_I = \{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\} \tag{6}$$

for any parabolic subset $P \subset R = \mathrm{T}_I$. In particular, $P = \{\varepsilon_i - \varepsilon_j : i \succcurlyeq_P j\}$ for $R = \dot{\mathrm{A}}_I$. We will see later in 13.10 that, conversely, any total preorder $\succcurlyeq$ gives rise to a parabolic subset of $\dot{\mathrm{A}}_I$ by (6). Note also that (e) describes $P \cap \{\pm(\varepsilon_i + \varepsilon_j) : i \neq j\}$ in case $I_-(P) = \emptyset$, while $P \cap \mathbb{Z}\varepsilon_i$ is determined by the subsets $I_\nu(P)$. We postpone the description of a general parabolic P until 13.12 and concentrate now on the special class of pure parabolic subsets defined below. The structure of a general parabolic subset will then be obtained by conjugation.

13.3. Definition. We let $\mathrm{T} \in \mathfrak{T}$ and keep the notations of Lemma 13.2. A parabolic subset P of $R = \mathrm{T}_I$ will be called *pure* if $I_-(P) = \emptyset$. Then I decomposes

$$I = I_0(P) \mathbin{\dot\cup} I_+(P) \mathbin{\dot\cup} I_1(P) \tag{1}$$

where, as we recall from 13.2(c), $I_0(P)$ and $I_1(P)$ cannot both be non-empty. We denote by $\mathfrak{P} = \mathfrak{P}(R)$ the set of all parabolic subsets of R and by $\mathfrak{P}_0 = \mathfrak{P}_0(\mathrm{T}, I)$ the set of pure parabolic subsets. Note that $\mathfrak{P}(\dot{\mathrm{A}}_I) = \mathfrak{P}_0(\dot{\mathrm{A}}, I)$ by 13.2(d).

Before showing that a pure parabolic subset is uniquely determined by its invariants $I_0(P)$ and $\succcurlyeq_P$, we introduce the following notation. Let $I_0 \subset I$ be any subset and let $\succcurlyeq$ be any relation on I. Then we let $R_{I_0,\succcurlyeq} = \mathrm{T}_{I,I_0,\succcurlyeq}$ denote the following subsets of $R = \mathrm{T}_I$:

$$\begin{aligned}
\dot{\mathrm{A}}_{I,I_0,\succcurlyeq} &= \dot{\mathrm{A}}_{I,\succcurlyeq} = \{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\}, && (2)\\
\mathrm{D}_{I,I_0,\succcurlyeq} &= \mathrm{D}_{I_0} \cup \{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\} \cup \{\varepsilon_i + \varepsilon_j : i \neq j\}, && (3)\\
\mathrm{B}_{I,I_0,\succcurlyeq} &= \mathrm{B}_{I_0} \cup \{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\} \cup \{\varepsilon_i + \varepsilon_j : i \neq j\} \cup \{\varepsilon_i : i \in I\}, && (4)\\
\mathrm{C}_{I,I_0,\succcurlyeq} &= \mathrm{C}_{I_0} \cup \{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\} \cup \{\varepsilon_i + \varepsilon_j : i \neq j\} \cup \{2\varepsilon_i : i \in I\}, && (5)\\
\mathrm{BC}_{I,I_0,\succcurlyeq} &= \mathrm{BC}_{I_0} \cup \{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\} \cup \{\varepsilon_i + \varepsilon_j : i \neq j\} \cup \{\varepsilon_i, 2\varepsilon_i : i \in I\}. && (6)
\end{aligned}$$

The simplified notation $\mathrm{T}_{I,\succcurlyeq}$ will be employed in case $I_0 = \emptyset$ or $\mathrm{T} = \dot{\mathrm{A}}$, since in this case the set on the right hand side of (2) obviously does not depend on I_0.

13.4. PROPOSITION. *Let $P \in \mathfrak{P}_0(\mathrm{T}, I)$ be a pure parabolic subset of $R = \mathrm{T}_I$ and let $(I_0(P), \succcurlyeq_P)$ be as in* 13.2. *Then with the definitions of* 13.3 *and* 12.16,

$$P = R_{I_0(P), \succcurlyeq_P}. \tag{1}$$

The symmetric part of P is a pure full subsystem given by

$$P_s = \mathrm{T}_{I_0(P)} \cup \bigcup_{J \in \bar{I}_2(P)} \dot{\mathrm{A}}_J = R_{I_0(P), \sim_P}, \tag{2}$$

where

$$\bar{I}(P) = (I \setminus I_0(P))/{\sim_P}, \quad \bar{I}_2(P) = \{J \in \bar{I}(P) : |J| \geqslant 2\}. \tag{3}$$

In particular, P is a positive system if and only if $I_0(P) = \emptyset$ and $\succcurlyeq_P$ is a total order. Moreover, we have:

(a) *$I_0(P)$ is either empty or $I_0(P) = \min(I/{\sim_P}, \geqslant)$ where $\geqslant$ is the total order induced on $I/{\sim_P}$ from the total preorder $\succcurlyeq_P$.*

(b) *Let $\mathrm{T} = \mathrm{D}$. Then $I_1(P) \neq \emptyset$ implies that $(I, \succcurlyeq_P)$ has a minimal element 0, and then $I_1(P) = \{0\}$ and $I_0(P) = \emptyset$.*

Remark. It will follow from 13.10.6 that the converse in (b) also holds, hence $I_1(P) \neq \emptyset \iff (I, \succcurlyeq_P)$ has a minimal element.

Proof. If $\mathrm{T} = \dot{\mathrm{A}}$ then $P = \dot{\mathrm{A}}_{I, \succcurlyeq_P}$ as we have already noted above. So we assume $\mathrm{T} \neq \dot{\mathrm{A}}$ from now on. Then

$$P \cap \mathrm{D}_I = \mathrm{D}_{I, I_0(P), \succcurlyeq_P} \tag{4}$$

follows from 13.2(e). In particular, $P = \mathrm{D}_{I, I_0(P), \succcurlyeq_P}$ if $\mathrm{T} = \mathrm{D}$. Next, let $\mathrm{T} = \mathrm{B}$. Then $I_1(P)$ is empty by 13.2(d), so $I = I_0(P) \,\dot\cup\, I_+(P)$. In particular, all $\varepsilon_i \in K$, while $-\varepsilon_j \in K \iff j \in I_0(P)$, so because of $P = K \cap R$ we have

$$\varepsilon_i \in P \quad \text{for all } i \in I, \tag{5}$$

$$-\varepsilon_j \in P \quad \iff \quad j \in I_0(P). \tag{6}$$

From (4), (5) and (6) we then obtain that $P = \mathrm{B}_{I, I_0(P), \succcurlyeq_P}$. The proof of the remaining cases $\mathrm{T} = \mathrm{C}$ and $\mathrm{T} = \mathrm{BC}$ is similar. Thus (1) holds.

Formula (2) for P_s follows easily from (1). Cor. 12.6 then shows that P_s is pure. Since P is a positive system if and only if $P_s = \{0\}$, the criterion for positivity is immediate from (2).

Finally, (a) and (b) follow from 13.2.2 and the fact that $I_0(P)$ is a full equivalence class of $\sim_P$.

13.5. LEMMA. *Let $P \subset \mathrm{T}_I$ be a parabolic subset. We use the notations introduced in* 12.7, 13.2 *and* 13.3.

(a) *For a permutation $\pi \in \mathrm{Sym}(I)$ and a sign change $\sigma = \sigma_L \in N$ we have*

$$I_\nu(\pi(P)) = \pi(I_\nu(P)) \quad \textit{for } \nu \in \{+, -, 0, 1\}, \tag{1}$$

$$\succcurlyeq_{\pi(P)} = (\pi \times \pi)(\succcurlyeq_P), \tag{2}$$

$$I_\nu(\sigma(P)) = I_\nu(P) \quad \textit{for } \nu = 0, 1, \tag{3}$$

$$I_\varepsilon(\sigma(P)) = \big(I_\varepsilon(P) \setminus L\big) \cup \big(I_{-\varepsilon}(P) \cap L\big) \quad \textit{for } \varepsilon \in \{+, -\}. \tag{4}$$

(b) *For $P \in \mathfrak{P}_0$ and a sign change $\sigma = \sigma_L \in N$, the following conditions are equivalent:*

(i) $\sigma(P) = P$,
(ii) $\sigma(P) \in \mathfrak{P}_0$,
(iii) $L \subset I_0(P) \cup I_1(P)$.

Proof. (a) This follows easily from the definitions.

(b) The implication (i) $\Longrightarrow$ (ii) is trivial. We prove (ii) $\Longrightarrow$ (iii). By (4) and because $I_-(P)$ is empty, $I_-(\sigma(P)) = \big(I_-(P) \setminus L\big) \cup \big(I_+(P) \cap L\big) = I_+(P) \cap L$. Hence $\sigma(P) \in \mathfrak{P}_0$ if and only if $I_+(P) \cap L = \emptyset$ or $L \subset I_0(P) \cup I_1(P)$, because of 13.3.1.

(iii) $\Longrightarrow$ (i): If $\mathrm{T} = \dot{\mathrm{A}}$ then $N = \{\mathrm{Id}\}$ by definition in 12.7, so we are done. We thus assume $\mathrm{T} \neq \dot{\mathrm{A}}$ and $L \neq \emptyset$. By Lemma 13.2(c), $I_0(P)$ and $I_1(P)$ cannot both be non-empty, and by 13.2(d), $I_1(P)$ has at most one element. Hence there are two possibilities:

Case 1: $L \subset I_0(P)$. Then $I_0(P) = \min(I/{\sim_P}, \geqslant)$ by Prop. 13.4(a), so $\sigma(\varepsilon_j) = -\varepsilon_j$ implies $i \succcurlyeq_P j$ for all $i \in I$. Now it follows easily from the explicit description of P in 13.4 resp. 13.3.3 – 13.3.6 that $\sigma(P) = P$.

Case 2: $L = I_1(P)$. Then $\mathrm{T} = \mathrm{D}$, $I_0(P) = \emptyset$ by Lemma 13.2(d), and Prop. 13.4(b) shows that $I_1(P) = \{0\}$ where 0 is the minimal element of $(I, \succcurlyeq_P)$. Again, it follows from the description of P in 13.3.3 that $\sigma(P) = P$.

13.6. Proposition. *Let $\mathrm{T} \in \mathfrak{T}$ and $R = \mathrm{T}_I$. We use the notations introduced in* 12.7 *and* 13.3.

(a) *The subset $\mathfrak{P}_0$ of all pure parabolic subsets is a fundamental domain for the action of N on the set $\mathfrak{P}$ of all parabolic subsets of R.*

(b) *The symmetric group* $\mathrm{Sym}(I) = G/N$ *acts naturally on $\mathfrak{P}/N$, and the map*

$$\tilde{\varPhi}\colon \mathfrak{P}_0 \hookrightarrow \mathfrak{P} \xrightarrow{\text{can}} \mathfrak{P}/N \tag{1}$$

is bijective and $\mathrm{Sym}(I)$*-equivariant.*

Proof. (a) By Lemma 13.5(b), an N-orbit intersects $\mathfrak{P}_0$ in at most one point. It remains to show that for every $P \in \mathfrak{P}$ there exists $\sigma = \sigma_L \in N$ such that $\sigma(P) \in \mathfrak{P}_0$. Let $L = I_-(P)$. Then by 13.5.4, $I_-(\sigma(P)) = \emptyset$ so $\sigma(P) \in \mathfrak{P}_0$.

(b) From 13.5.1 it is clear that $\mathfrak{P}_0$ is stable under $\mathrm{Sym}(I)$. The remainder of the proof is identical with that of Prop. 12.10(b).

13.7. *Characterization of scalar parabolic subsets.* As an application, we now give necessary and sufficient conditions for a parabolic subset P of a root system R to be of scalar type, i.e., $P = R_+(f)$ for some linear form $f \in X^*$, cf. 10.9. By 10.9.2, we may assume R irreducible. If R is finite it follows from Lemma 11.1 that P is of scalar type, so we restrict ourselves to the infinite irreducible case. By 13.6(a), any parabolic subset is of the form $\sigma(P)$ for some $\sigma \in N$ and $P \in \mathfrak{P}_0$ pure. By 10.9.1 we may assume $\sigma = \mathrm{Id}$. Let $\bar{I}(P)$ be defined as in 13.4.3, with the total order $\geqslant$ induced from $\succcurlyeq_P$. Then:

Proposition. *A pure parabolic subset $P \subset \mathrm{T}_I$ is of scalar type if and only if the totally ordered set $(\bar{I}(P), \succcurlyeq)$ embeds into $\mathbb{R}$ with its usual ordering. In particular, this is so if the rank of R/P_s is at most countable.*

An example of K.H. Hofmann [**50**, Remark II.2(c)] shows that not every totally ordered set $(A, \geqslant)$ with $\operatorname{Card} A = \operatorname{Card} \mathbb{R}$ imbeds into $\mathbb{R}$ with the standard order.

Proof. We first recall that the restriction map $X^* \to (\dot{X})^*$ is surjective with kernel $\mathbb{R}t$, so every element of $(\dot{X})^*$ is the restriction $\dot{f} = f\big|\dot{X}$ of some $f \in X^*$.

Assume $P = R_+(f)$ (resp., $P = R_+(\dot{f})$ in case $R = \dot{\mathrm{A}}_I$) is of scalar type, and define φ: $I \to \mathbb{R}$ by $\varphi(i) = f(\varepsilon_i)$. Then $i \succcurlyeq_P j \iff \varphi(i) \geqslant \varphi(j)$ is immediate from the definition of $\succcurlyeq_P$ in 13.2.1, and hence, by B.2.1, $i \sim_P j \iff \varphi(i) = \varphi(j)$. Now it is clear that φ induces a strictly increasing map $\bar{\varphi}$: $\bar{I}(P) \to \mathbb{R}$.

Conversely, let ψ: $\bar{I}(P) \to \mathbb{R}$ be strictly increasing. It is no restriction to assume that ψ takes values in $\mathbb{R}_{++}$, by composing it with the exponential function if necessary. Define $f \in X^*$ by $f(\varepsilon_i) = \psi(\varepsilon_{[i]})$ for $i \notin I_0(P)$, and $f(\varepsilon_i) = 0$ for $i \in I_0(P)$. Then it follows from the description of P in 13.4 that $P = R_+(f)$ (resp., $P = R_+(\dot{f})$ in case $R = \dot{\mathrm{A}}_I$).

The last assertion follows from 12.19.6, 13.4.2 and the following well-known lemma, see e.g. [**31**, Ch. 5, Th. 2.6]. We include a proof for the convenience of the reader.

Lemma. *A countable totally ordered set is order-isomorphic to a subset of $\mathbb{Q}$ with its usual ordering.*

Proof. We may identify the set in question with $\mathbb{N}$, equipped with a total order $\geqslant$ which, of course, need not be the standard order of $\mathbb{N}$. Define a strictly increasing map ψ: $\mathbb{N} \to \mathbb{Q}$ inductively as follows. Put $\psi(0) := 0$, and suppose ψ: $\{0, \ldots, n\} \to \mathbb{Q}$ is already defined. Since $\{0, \ldots, n+1\}$ is a finite totally ordered set, there are three possibilities for $n+1$:

(a) If $n+1 > i$ for all $i = 0, \ldots, n$ put $\psi(n+1) := \max\{\psi(0), \ldots, \psi(n)\} + 1$.

(b) If $n+1 < i$, for all $i = 0, \ldots, n$ define $\psi(n+1) := \min\{\psi(0), \ldots, \psi(n)\} - 1$.

(c) Otherwise, there exist (uniquely determined) $i, j \in \{0, \ldots, n\}$ such that $i < n+1 < j$ and no $k \in \{0, \ldots, n\}$ lies strictly between i and $n+1$ or between $n+1$ and j (i.e., i and j are the predecessor and successor of $n+1$, respectively). Then define $\psi(n+1) := \frac{1}{2}(\psi(i) + \psi(j))$.

13.8. Corollary. *Let R be a root system with the property that every irreducible component has at most countable rank. Then every positive system P of R is of scalar type.*

Proof. Immediate from 13.7, since P is a positive system if and only if $P_s = \{0\}$.

13.9. Definition. Prop. 13.4 shows how a pure parabolic subset P of $R = \mathrm{T}_I$ is determined by its invariants $I_0(P)$ and $\succcurlyeq_P$. Conversely, it is natural to ask for which $(I_0, \succcurlyeq)$ the formulas (2) – (6) of 13.3 define pure parabolic subsets of R. Necessary for this is that $(I_0, \succcurlyeq)$ satisfy the conditions listed in Lemma 13.2(b) and Prop. 13.4(a). It turns out that these conditions are also sufficient. We introduce the following terminology.

A *p-datum for* (T, I) (p standing for parabolic) is a pair $(I_0, \succcurlyeq)$, where I_0 is a subset I_0 of I and $\succcurlyeq$ is a total preorder $\succcurlyeq$ on I, with associated equivalence relation $\sim$ as in B.2, such that the following conditions hold:

(i) $I_0 = \emptyset$ or $I_0 = \min(I/{\sim})$ is the minimum of the totally ordered set $I/{\sim}$,
(ii) if $\mathrm{T} = \dot{\mathrm{A}}$ then $I_0 = \emptyset$,
(iii) if $\mathrm{T} = \mathrm{D}$ then $\operatorname{Card} I_0 \neq 1$.

Let $\mathbb{P}_0 = \mathbb{P}_0(\mathrm{T}, I)$ denote the set of p-data for (T, I). For later use we observe:

$$\textit{If } (I_0, \succcurlyeq) \in \mathbb{P}_0(\mathrm{D}, I) \textit{ and } (I, \succcurlyeq) \textit{ has a minimal element } 0 \textit{ then } I_0 = \emptyset. \tag{1}$$

Indeed, the minimum of $(I/{\sim}, \geqslant)$ is then $\{0\}$, so we must have $I_0 = \emptyset$ by (i) and (iii).

Lemma 13.2(d) and Prop. 13.4(a) say that there is a p-datum $(I_0(P), \succcurlyeq_P)$ associated to any pure parabolic subset P, i.e., there is a well-defined map

$$\tilde{\Upsilon}\colon \mathfrak{P}_0 \to \mathbb{P}_0, \quad P \mapsto \tilde{\Upsilon}(P) := (I_0(P), \succcurlyeq_P), \tag{2}$$

and by 13.5.1 and 13.5.2, this map is $\mathrm{Sym}(I)$-equivariant.

Comparing the definition of $\mathbb{P}_0$ with that of $\mathbb{F}_0$ in 12.14, we see that there is a natural map $\mathbb{P}_0 \to \mathbb{F}_0$ given by $(I_0, \succcurlyeq) \mapsto (I_0, \sim)$. Then the diagram

$$\begin{tikzcd} \mathfrak{P}_0 \arrow[r, "\tilde{\Upsilon}"] \arrow[d, "s"'] & \mathbb{P}_0 \arrow[d] \\ \mathfrak{F}_0 \arrow[r, "\Upsilon"'] & \mathbb{F}_0 \end{tikzcd} \tag{3}$$

is commutative, where s stands for the map $P \mapsto P_s$, sending P to its symmetric part. Indeed, $P_s \in \mathfrak{F}_0$ for $P \in \mathfrak{P}_0$, by Prop. 13.4. Now commutativity of (3) follows from $i \sim_P j \iff \varepsilon_i - \varepsilon_j \in P_s \iff i \sim_{P_s} j$ and $I_0(P_s) = I_0(P)$ (Lemma 13.2(a)).

13.10. Proposition. *Let* $\mathrm{T} \in \mathfrak{T}$ *and* $R = \mathrm{T}_I$. *We use the notations introduced in* 13.9. *For a p-datum* $(I_0, \succcurlyeq) \in \mathbb{P}_0(\mathrm{T}, I)$ *let* $P := R_{I_0, \succcurlyeq} \subset R$ *be defined as in* (2) – (6) *of* 13.3. *Then:*

(a) *P is a parabolic subset of R, given by*

$$P = R \cap X_{I_0, \succcurlyeq}, \tag{1}$$

where

$$X_{I_0,\succcurlyeq} = \mathbb{R}_+\big[\{\varepsilon_i : i \in I\} \cup \{-\varepsilon_j : j \in I_0\} \cup \{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\}\big] \tag{2}$$

is the cone of type B *defined by* $(I, I_0, \succcurlyeq)$ *as in* B.3.

(b) *The convex cone $\mathbb{R}_+[P]$ generated by P is as follows:*

(b_1) *If* $\mathrm{T} = \dot{\mathrm{A}}$ *then*

$$\mathbb{R}_+[P] = \dot{X}_{\succcurlyeq} = \mathbb{R}_+\big[\{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\}\big] \tag{3}$$

is the cone of type $\dot{\mathrm{A}}$ *defined by* $(I, \succcurlyeq)$ *as in* B.7.

(b$_2$) *If* $\mathrm{T} = \mathrm{D}$ *and* $(I, \succcurlyeq)$ *has a minimal element* 0 *then* $I_0 = \emptyset$ *by* 13.9.1 *and*

$$\mathbb{R}[P] = X_{\succcurlyeq,0} = \mathbb{R}_+\big[\{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\} \cup \{\varepsilon_i + \varepsilon_0 : i \neq 0\}\big] \tag{4}$$

is the cone of type D *defined by* $(I, \succcurlyeq, 0)$ *as in* B.9.

(b$_3$) *In all other cases,* $\mathbb{R}_+[P] = X_{I_0,\succcurlyeq}$ *is the cone of type* B *defined by* $(I, I_0, \succcurlyeq)$.

(c) *The invariants* $\succcurlyeq_P$ *and* $I_\nu(P)$ *of* P *are*

$$I_-(P) = \emptyset, \quad I_0(P) = I_0, \quad \succcurlyeq_P = \succcurlyeq, \tag{5}$$

$$I_1(P) = \left\{ \begin{array}{ll} I & \textit{in case } (\mathrm{b}_1) \\ \{0\} & \textit{in case } (\mathrm{b}_2) \\ \emptyset & \textit{in all other cases} \end{array} \right\}. \tag{6}$$

In particular, P *is a pure parabolic subset.*

(d) $(I_0, \succcurlyeq)$ *is also a p-datum for* $(\mathrm{T}^\vee, I)$*, and* $(\mathrm{T}_{I,I_0,\succcurlyeq})^\vee \cong \mathrm{T}^\vee_{I,I_0,\succcurlyeq}$.

Proof. (a) The inclusion $P \subset R \cap X_{I_0,\succcurlyeq}$ is evident from (2) and the description of P in (2) – (6) of 13.3. The converse follows by a straightforward application of the criteria for an element $x \in X$ to belong to $X_{I_0,\succcurlyeq}$ given in B.5(a). The details are left to the reader. It remains to show that P is parabolic. Since $X_{I_0,\succcurlyeq}$ is additively closed, being a convex cone, we have P additively closed. The condition $P \cup (-P) = R$ follows from the immediately checked fact $R \subset X_{I_0,\succcurlyeq} \cup (-X_{I_0,\succcurlyeq})$.

(b) Case (b$_1$) is evident from 13.3.2 and B.7.1. In case (b$_2$), the inclusion $\mathbb{R}_+[P] \supset X_{\succcurlyeq,0}$ is clear from 13.3.3 and B.9.1. For the reverse inclusion, it suffices to show $\varepsilon_i + \varepsilon_j \in X_{\succcurlyeq,0}$ whenever $i, j, 0$ are pairwise distinct. But because $i \succcurlyeq 0$ we have $\varepsilon_i - \varepsilon_0 \in X_{\succcurlyeq,0}$ and hence $\varepsilon_i + \varepsilon_j = (\varepsilon_i - \varepsilon_0) + (\varepsilon_j + \varepsilon_0) \in X_{\succcurlyeq,0}$. In case (b$_3$), $\mathbb{R}_+[P] = X_{I_0,\succcurlyeq}$, is clear from (4) – (6) of 13.3, provided $\mathrm{T} = \mathrm{B}, \mathrm{C}$ or BC. It thus remains to consider the case where $\mathrm{T} = \mathrm{D}$ and $(I, \succcurlyeq)$ does not have a minimal element. Let $K := \mathbb{R}_+[P]$ for short. The inclusion $K \subset X_{I_0,\succcurlyeq}$ is clear from (1). For the reverse inclusion, we must show $\varepsilon_i \in K$ and $-\varepsilon_j \in K$, for all $i \in I$ and all $j \in I_0$. Since i is not minimal, there exists $k \in I$ with $i \succcurlyeq k$ and $i \neq k$, so $\varepsilon_i \pm \varepsilon_k \in P$ and hence $\varepsilon_i = \frac{1}{2}\big((\varepsilon_i + \varepsilon_k) + (\varepsilon_i - \varepsilon_k)\big) \in K$. Also, because I_0, if not empty, has at least two elements, there exists $l \in I_0$, $l \neq j$, whence $\pm\varepsilon_j \pm \varepsilon_l \in \mathrm{D}_{I_0} \subset P$, and therefore $-\varepsilon_j = \frac{1}{2}\big((\varepsilon_l - \varepsilon_j) + (-\varepsilon_l - \varepsilon_j)\big) \in K$.

(c) For $\succcurlyeq_P = \succcurlyeq$ we must show that $\alpha := \varepsilon_k - \varepsilon_l \in P \iff k \succcurlyeq l$. Here "$\Longleftarrow$" follows from 13.3.2 – 13.3.6. The converse is clear in case $R = \dot{\mathrm{A}}_I$. In the other cases, we must have $\alpha \in \mathrm{T}_{I_0} \cup \{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\}$. If $\alpha \in \mathrm{T}_{I_0}$ then $k, l \in I_0$ so $k \sim l$ by (i) of 13.9, in particular, $k \succcurlyeq l$. If $\alpha \in \{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\}$, it is clear that we have $k \succcurlyeq l$.

Next we compute the sets $I_\nu(P)$. In case (b$_1$) we have $\mathbb{R}_+[P] \subset \dot{X} = \mathrm{Ker}(t)$, so $\pm\varepsilon_i \notin \mathbb{R}_+[P]$ for all i, showing $I_1(P) = I$ and thus the other $I_\nu(P)$ are empty. In case (b$_3$), where $K := \mathbb{R}_+[P] = X_{I_0,\succcurlyeq}$, it follows from (2) that all $\varepsilon_i \in K$ so $I_-(P) = I_1(P) = \emptyset$. Again by (2), $-\varepsilon_i \in K$ for all $j \in I_0$, so $I_0 \subset I_0(P)$. Assume there exists $k \in I_0(P) \setminus I_0$. Then $-\varepsilon_k \in K$ and $[k, \rightarrow[$ is a final segment of I not meeting I_0, so $-1 = q_{[k,\rightarrow[}(-\varepsilon_k) \geqslant 0$ by condition (iv) of Lemma B.5(a), contradiction.

It remains to deal with case (b$_2$) where $\mathbb{R}_+[P] = K_0$ is the cone of type D defined by $(I, \succcurlyeq, 0)$. Let $i \neq 0$. Then $i \succcurlyeq 0$, so $2\varepsilon_i = (\varepsilon_i + \varepsilon_0) + (\varepsilon_i - \varepsilon_0) \in K_0$. We claim that $0 \in I_1(P)$, i.e., $\pm\varepsilon_0 \notin K_0$. Indeed, $q_\pm(\varepsilon_0) = \pm(1/2)$, so neither ε_0 nor $-\varepsilon_0$ belong to K_0, by condition (iii) of Lemma B.11(a). We also have $-\varepsilon_i \notin K_0$ for all $i \neq 0$, else $-\varepsilon_i + (\varepsilon_i + \varepsilon_0) = \varepsilon_0 \in K_0$. Now it follows that the $I_\nu(P)$ are as indicated.

(d) is immediate from (1) and $(\mathrm{T}_I)^\vee \cong \mathrm{T}_I^\vee$.

We can now prove the analogue of Th. 12.17. Recall the definition of the groups N and G in 12.7.1. In particular, G induces the big Weyl group $\overline{W}(R)$ except in the finite case for $R = \mathrm{D}_n$ where $W(\mathrm{D}_n)$ has index 2 in G.

13.11. Theorem. *Let* $\mathrm{T} \in \mathfrak{T}$ *and* $R = \mathrm{T}_I$. *We use the notations introduced in* 12.7 *and* 13.9.

(a) *The map* $\tilde{\Upsilon}\colon \mathfrak{P}_0 \to \mathbb{P}_0$ *of* 13.9.2 *is a bijection, with inverse map* $\tilde{\Psi}\colon \mathbb{P}_0 \to \mathfrak{P}_0$ *given by* $\tilde{\Psi}(I_0, \succcurlyeq) = R_{I_0,\succcurlyeq}$.

(b) *The bijection* $\tilde{\Psi}\colon \mathbb{P}_0 \to \mathfrak{P}_0$ *of* (a) *composed with the bijection* $\tilde{\Phi}\colon \mathfrak{P}_0 \to \mathfrak{P}/N$ *of* 13.6.1 *is a* $\mathrm{Sym}(I)$*-equivariant bijection* $\mathbb{P}_0 \to \mathfrak{P}/N$ *which induces a bijection*

$$\mathbb{P}_0/\operatorname{Sym}(I) \xrightarrow{\cong} \mathfrak{P}/G. \tag{1}$$

Proof. The proof is analogous to that of Th. 12.17.

(a) By Prop. 13.10(c), $\tilde{\Psi}$ has values in $\mathfrak{P}_0$, and 13.10.5 says that $\tilde{\Upsilon} \circ \tilde{\Psi} = \mathrm{Id}$. It remains to show that $\tilde{\Psi} \circ \tilde{\Upsilon} = \mathrm{Id}$ which is precisely 13.4.1.

(b) Since $\tilde{\Upsilon}$ is $\mathrm{Sym}(I)$-equivariant, so is its inverse $\tilde{\Psi}$. We therefore obtain a $\mathrm{Sym}(I)$-equivariant bijection $\tilde{\Phi} \circ \tilde{\Psi}\colon \mathbb{P}_0 \xrightarrow{\cong} \mathfrak{P}/N$ and hence the bijection (1).

13.12. *Classification of parabolic subsets.* Let $R = \mathrm{T}_I$. The bijection $\mathbb{P}_0 \to \mathfrak{P}/N$ of Th. 13.11(b) provides in particular a description of all parabolic subsets (not necessarily pure) of $R = \mathrm{T}_I$. For the convenience of the reader we make this explicit here.

Given a p-datum $(I_0, \succcurlyeq) \in \mathbb{P}_0(\mathrm{T}, I)$ and a subset $I_- \subset I$ with $I_- = \emptyset$ in case $\mathrm{T} = \dot{\mathrm{A}}$, the set $R_{I,I_0,I_-,\succcurlyeq} = \sigma_{I_-}(R_{I,I_0,\succcurlyeq})$ is a parabolic subset of R and, conversely, every parabolic subset of R arises in this way for a unique p-datum $(I_0, \succcurlyeq)$ and a suitable subset I_- with $I_- = \emptyset$ in case $\mathrm{T} = \dot{\mathrm{A}}$. Indeed, let $P \subset R$ be a parabolic subset. By 13.6 there is a unique pure parabolic subset P' of R such that $P = \sigma(P')$ for some $\sigma \in N$, and by 13.11(a), we have $P' = R_{I,I_0,\succcurlyeq}$ for a unique p-datum $(I_0, \succcurlyeq)$.

The parabolic subsets $R_{I,I_0,I_-,\succcurlyeq}$ have the same description as the subsets $R_{I,I_0,\succcurlyeq}$, defined in 13.3.2 – 13.3.6, if one replaces the ε_i by

$$\varepsilon_i' = \sigma_{I_-}(\varepsilon_i) = \left\{ \begin{matrix} -\varepsilon_i & \text{if } i \in I_- \\ \varepsilon_i & \text{if } i \notin I_- \end{matrix} \right\}.$$

Hence $R_{I,I_0,I_-,\succcurlyeq}$ is given explicitly by

$$\dot{\mathrm{A}}_{I,I_0,I_-,\succcurlyeq} = \dot{\mathrm{A}}_{I,\succcurlyeq} = \{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\}, \tag{1}$$

$$\mathrm{D}_{I,I_0,I_-,\succcurlyeq} = \mathrm{D}_{I_0} \cup \{\varepsilon_i' - \varepsilon_j' : i \succcurlyeq j\} \cup \{\varepsilon_i' + \varepsilon_j' : i \neq j\}, \tag{2}$$

$$\mathrm{B}_{I,I_0,I_-,\succcurlyeq} = \mathrm{B}_{I_0} \cup \{\varepsilon_i' - \varepsilon_j' : i \succcurlyeq j\} \cup \{\varepsilon_i' + \varepsilon_j' : i \neq j\} \cup \{\varepsilon_i' : i \in I\}, \tag{3}$$

$$\mathrm{C}_{I,I_0,I_-,\succcurlyeq} = \mathrm{C}_{I_0} \cup \{\varepsilon_i' - \varepsilon_j' : i \succcurlyeq j\} \cup \{\varepsilon_i' + \varepsilon_j' : i \neq j\} \cup \{2\varepsilon_i' : i \in I\}, \tag{4}$$

$$\mathrm{BC}_{I,I_0,I_-,\succcurlyeq} = \mathrm{BC}_{I_0} \cup \{\varepsilon_i' - \varepsilon_j' : i \succcurlyeq j\} \cup \{\varepsilon_i' + \varepsilon_j' : i \neq j\} \cup \{\varepsilon_i', 2\varepsilon_i' : i \in I\}. \tag{5}$$

We note that a different description of parabolic subsets in T_I for $T \neq \mathrm{BC}$ is given in [**23**, Prop. 4].

§14. Positive systems in root systems

14.1. *Extremal rays.* In this section, we study positive systems of root systems in more detail. We also specialize the results of the previous section and thus obtain the classification of the positive systems of the infinite irreducible root systems. We first establish notation and recall some facts on extremal rays from Appendix B.

Let P be a positive system of a root system (R, X). Then P is in particular a parabolic subset with $P_s = \{0\}$ and $P_u = P^\times$, so the cones $K = \mathbb{R}_+[P]$ and $K_u = \mathbb{R}_+[P_u]$ introduced in 10.17 coincide. Also, K is proper by 10.17(d) and thus determines a partial ordering on the vector space X, compatible with the vector space structure, by

$$x \geqslant y \quad \Longleftrightarrow \quad x - y \in K, \tag{1}$$

see also 10.7. Finally, the partial orderings on $\mathbb{Z}[R] = \mathcal{Q}(R)$ induced by K and P coincide by Prop. 11.2: For $x, y \in \mathcal{Q}(R)$ we have

$$x \geqslant y \quad \Longleftrightarrow \quad x \succcurlyeq_P y. \tag{2}$$

In the sequel, we will simply write $\succcurlyeq$ instead of $\succcurlyeq_P$.

Recall from B.1 that an extremal ray of K is a half-line $\mathbb{R}_+x \subset K$ such that $x = y + z$ (where $y, z \in K$) implies $y, z \in \mathbb{R}_+x$. By B.1.1,

$$\textit{an extremal ray of } K \textit{ must be one of the generating rays } \mathbb{R}_+\alpha,\ \alpha \in P^\times. \tag{3}$$

Note also that by 3.4.2, each extremal ray $\mathbb{R}_+\gamma$ contains exactly one indivisible root, namely γ itself or $\gamma/2$, depending on whether γ is indivisible or not.

14.2. *Simple roots.* Let P be a positive system of a root system (R, X). An element $\gamma \in P^\times$ is called a *simple root* of P if it satisfies the following equivalent conditions:

(i) γ is indivisible and $\mathbb{R}_+\gamma$ is an extremal ray of $K = \mathbb{R}_+[P]$,

(ii) $\gamma \in P_{\min}$ (as in 10.11),

(iii) γ is a minimal element of $(P^\times, \geqslant)$ with respect to the partial ordering of 14.1.1.

The equivalence of these conditions will be shown below. The set of simple roots of P will be denoted by $\operatorname{simp}(P)$. We note that for a positive system P determined by a root basis B, the set of simple roots of P is precisely B, as follows easily from the properties of root bases. Hence this terminology is consistent with established usage. We also note that, by 14.1.3 and (i):

$$\textit{The extremal rays of } K \textit{ are precisely the rays } \mathbb{R}_+\gamma \textit{ where } \gamma \in \operatorname{simp}(P). \tag{1}$$

It remains to prove the equivalence of (i) – (iii).

(i) $\Longrightarrow$ (ii): By Prop. 10.11, it suffices to show that γ is not the sum of two elements of $P^\times$. If $\gamma = \alpha_1 + \alpha_2$ for $\alpha_i \in P^\times$, then by extremality, $\alpha_i \in \mathbb{R}_+\gamma$, say

$\alpha_i = c_i\gamma$. By 3.4.2 and indivisibility of γ, we have $c_i \in \{1, 2\}$. Hence $\gamma = (c_1 + c_2)\gamma$ where $c_1 + c_2 \geqslant 2$, which is impossible.

(ii) $\iff$ (iii): This is obvious from 14.1.2.

(iii) $\implies$ (i): γ is indivisible, for otherwise $\gamma/2 \in P^\times$ and then $\gamma/2 \leqslant \gamma$ would show γ not minimal. It remains to prove $\mathbb{R}_+\gamma$ extremal. Let thus $\gamma = y + z$ where $y, z \in K$, and write $y = \sum_{i=1}^n c_i\alpha_i$, $z = \sum_{i=1}^n d_i\alpha_i$ where $c_i, d_i \geqslant 0$ and $\alpha_1, \dots, \alpha_n \in P$. Let R' be a finite full subsystem containing the α_i (and hence also γ) and let $P' = P \cap R'$. Then P' is a positive system of R', and in view of the one-to-one correspondence between root bases and positive systems for finite root systems (cf. 10.5), there is a unique root basis B' of R' determining P'. Since γ and $\alpha_i \in P'$, we have

$$\gamma = \sum_{\beta \in B'} n_\beta \beta, \qquad \alpha_i = \sum_{\beta \in B'} m_{i\beta}\beta,$$

with $n_\beta, m_{i\beta} \in \mathbb{N}$. Since $\gamma \neq 0$, we have $n_{\beta_1} \geqslant 1$ for some $\beta_1 \in B'$, and then $\gamma - \beta_1 = (n_{\beta_1} - 1)\beta_1 + \sum_{\beta \neq \beta_1} n_\beta \beta$ shows $\beta_1 \leqslant \gamma$ and therefore $\beta_1 = \gamma \in B'$, by minimality of γ. Thus $n_{\beta_1} = 1$ and $n_\beta = 0$ for $\beta \neq \beta_1$. By substituting we obtain

$$\beta_1 = y + z = \Big(\sum_{i=1}^n (c_i + d_i)m_{i,\beta_1}\Big) \cdot \beta_1 + \sum_{\beta \neq \beta_1} \Big(\sum_{i=1}^n (c_i + d_i)m_{i\beta}\Big)\beta$$

whence, by comparing coefficients at elements of B', $0 = \sum_i (c_i + d_i)m_{i\beta} = 0$ for $\beta \neq \beta_1$. This implies $\sum_i c_i m_{i\beta} = \sum_i d_i m_{i\beta} = 0$ for $\beta \neq \beta_1$, so $y = \Big(\sum_i c_i m_{i\beta_1}\Big)\beta_1$ is a positive multiple of $\beta_1 = \gamma$, and similarly so is z.

14.3. Proposition. *Let P be a positive system of a root system (R, X), and let $K := \mathbb{R}_+[P]$ be the associated positive cone. Furthermore, let $B = \operatorname{simp}(P)$ be the set of simple roots of P, and put $\tilde{X} := \operatorname{span}(B)$. Then B is a root basis of the full subsystem $\tilde{R} := R \cap \tilde{X}$, whose associated positive system is $\tilde{P} := P \cap \tilde{R}$. The cone $\tilde{K} := \mathbb{R}_+[\tilde{P}]$ is given by*

$$\tilde{K} = K \cap \tilde{X}. \tag{1}$$

Proof. We first prove (1). The inclusion from left to right is clear from the definitions. Conversely, let $x \in K \cap \tilde{X}$, say,

$$x = \sum_{\alpha \in E} c_\alpha \alpha = \sum_{\beta \in F} b_\beta \beta, \tag{2}$$

where $E \subset P$ and $F \subset B$ are finite, $c_\alpha > 0$, and $b_\beta \in \mathbb{R}$. Consider the finite full subsystem $R' := R \cap \operatorname{span}(E \cup F)$. Then $P' := P \cap R'$ is a positive system of R', and clearly $E \subset P'$. By the one-to-one correspondence between positive systems and root bases of finite root systems, $B' := \operatorname{simp}(P')$ is a root basis of R' with associated positive system $P' = \mathbb{N}[B'] \cap R'$. From the characterization (ii) of simple roots in 14.2 in terms of indecomposability (cf. 10.11) it is evident that a simple root of P contained in P' is a fortiori a simple root of P'. Hence $F \subset B'$, and every $\alpha \in E \subset P'$ can be written $\alpha = \sum_{\beta \in B'} n_{\alpha\beta}\beta$, where $n_{\alpha\beta} \in \mathbb{N}$. Substituting this into (2) and comparing coefficients at $\beta \in B'$ yields

$$\sum_{\alpha \in E} c_\alpha n_{\alpha\beta} = b_\beta \geqslant 0 \quad \text{for } \beta \in F, \tag{3}$$

$$\sum_{\alpha \in E} c_\alpha n_{\alpha\beta} = 0 \quad \text{for } \beta \in B' \setminus F. \tag{4}$$

From (2) and (3) we see that $x \in \mathbb{R}_+[F] \subset \mathbb{R}_+[\tilde{P}] = \tilde{K}$, proving (1). Also, (4) implies, because $c_\alpha > 0$, that $n_{\alpha\beta} = 0$ for $\alpha \in E$ and $\beta \in B' \setminus F$, so

$$\alpha = \sum_{\beta \in F} n_{\alpha\beta}\beta \in \mathbb{N}[F]. \tag{5}$$

As remarked in 10.5, $\tilde{P} = \tilde{R} \cap P$ is a positive system of $\tilde{R}$. For B to be a root basis of $\tilde{R}$ with $\tilde{P}$ as associated positive system, it suffices to show that B is linearly independent and that $\tilde{P} \subset \mathbb{N}[B]$. Indeed, from $\tilde{R} = \tilde{P} \cup (-\tilde{P})$ it then follows that every root of $\tilde{P}$ is an integer linear combination of B with coefficients of the same sign, so B is a root basis of $\tilde{R}$. We then also have $\mathbb{N}[\tilde{P}] = \mathbb{N}[B]$, so $\tilde{R} \cap \mathbb{N}[B] = (\tilde{P} \cup (-\tilde{P})) \cap \mathbb{N}[\tilde{P}] = \tilde{P} \cap \mathbb{N}[\tilde{P}]$ (by Lemma 10.10(b)) $= \tilde{P}$, showing $\tilde{P}$ is the positive system of $\tilde{R}$ defined by B.

We prove linear independence of B. Assume that $\sum_{\beta \in F} b_\beta \beta = 0$ where $F \subset B$ is finite. Then in particular, $x = 0 \in K \cap \tilde{X}$, so the proof above (specialized to the case $E = \emptyset$) shows $F \subset B'$, and since B' is linearly independent, we must have $b_\beta = 0$. Similarly, let $\alpha \in \tilde{P}$. Then in particular, $x = \alpha \in K \cap \tilde{X}$, so (specializing $E = \{\alpha\}$ above) (5) shows $\alpha \in \mathbb{N}[B]$.

As a consequence, we have the following "geometric" characterization of those positive systems which are determined by a root basis.

14.4. Corollary. *Let P be a positive system of a root system (R, X). Then the following conditions are equivalent.*

(i) *P is the positive system determined by a root basis B of R,*
(ii) *the convex cone $K = \mathbb{R}_+[P]$ is spanned by its extremal rays,*
(iii) *X is spanned by the simple roots of P.*

Proof. (i) $\Longrightarrow$ (ii): From $P \subset \mathbb{N}[B]$ we conclude $K = \mathbb{R}_+[P] = \mathbb{R}_+[B]$, and by 14.2.1, the elements of B span extremal rays of K.

(ii) $\Longrightarrow$ (iii): Again by 14.2.1, each extremal ray of K contains a simple root. Thus K is spanned by $\mathrm{simp}(P)$, and from $P \subset K$ and $X = \mathrm{span}(P)$ it follows that X is spanned by $\mathrm{simp}(P)$.

(iii) $\Longrightarrow$ (i) is a consequence of 14.3.

14.5. *Subsets of P associated to automorphisms.* Let P be a positive system of a root system (R, X). To an automorphism f of R we associate the subset

$$P_f = \{\alpha \in P^\times : f(\alpha) \in (-P)\} = P^\times \cap f^{-1}(-P) \tag{1}$$

of P. The following properties are elementary:

$$f(P_f) = -P_{f^{-1}}\,, \quad f(P \setminus P_f) = P \setminus P_{f^{-1}}\,. \tag{2}$$

By definition, $P_f = \emptyset \iff f(P) = P$, but there may of course be nontrivial automorphisms f with $f(P) = P$. However, for $f = w$ in the Weyl group $W(R)$, we have, by 15.8,

$$P_w = \emptyset \quad \iff \quad w = \mathrm{Id}. \tag{3}$$

For automorphisms f, g of R we claim

$$g^{-1}(P_{fg^{-1}}) = (P_f \setminus P_g) \,\dot\cup\, (-(P_g \setminus P_f)). \tag{4}$$

Indeed, by (2) we have

$$\begin{aligned} P_{fg^{-1}} &= \{\alpha \in P \setminus P_{g^{-1}} : g^{-1}(\alpha) \in P_f\} \,\dot\cup\, \{\alpha \in P_{g^{-1}} : -g^{-1}(\alpha) \in P \setminus P_f\} \\ &= \Big(g(P \setminus P_g) \cap g(P_f)\Big) \,\dot\cup\, \Big(g(-P_g) \cap g(-(P \setminus P_f))\Big) \\ &= g(P_f \setminus P_g) \,\dot\cup\, g(-(P_g \setminus P_f)), \end{aligned}$$

which is equivalent to (4).

14.6. Lemma. *Let $f \in \mathrm{Aut}(R)$ and suppose $\alpha \in P_f$ is a minimal element of the partially ordered set P_f with the partial order induced from $(P, \succcurlyeq)$. Then α is a simple root.*

Proof. We use the characterization 14.2(ii) of simple roots and thus have to show that α is minimal not only in $(P_f, \succcurlyeq)$ but even in all of $P^\times$. By way of contradiction, suppose that $\alpha = \beta + \gamma$ for $\beta, \gamma \in P^\times$. Then $\beta \preccurlyeq \alpha$ and also $\gamma \preccurlyeq \alpha$. From $f(\alpha) = f(\beta) + f(\gamma) \in (-P)$ it follows that, say, $f(\beta) \in (-P)$. Then $\beta \in P_f$ so $\beta = \alpha$ by minimality of α, and therefore $\gamma = 0$, contradiction.

We can now prove yet another characterization of simple roots.

14.7. Corollary. *Let P be a positive system of a root system (R, X). A root $\alpha \in P$ is simple if and only if the only roots of P mapped into $-P$ by s_α are those in $\mathbb{R}_+\alpha$, i.e., $P_\alpha := P_{s_\alpha} \subset \{\alpha, 2\alpha\}$.*

Proof. Suppose α is simple and let $\gamma \in P_\alpha$, i.e., $\beta := -s_\alpha(\gamma) = -\gamma + \langle \gamma, \alpha^\vee \rangle \alpha \in P$. Hence $\beta + \gamma = \langle \gamma, \alpha^\vee \rangle \alpha \in K = \mathbb{R}_+[P]$. Since K is a proper cone, $\langle \gamma, \alpha^\vee \rangle > 0$, and since $\mathbb{R}_+\alpha$ is an extremal ray of K by 14.2(i), we have $\gamma \in \mathbb{R}_+\alpha$, and then $\gamma \in \{\alpha, 2\alpha\}$ because α is indivisible. The converse follows from 14.6 applied to $f = s_\alpha$.

Following the usual terminology we call the s_α where $\alpha \in \mathrm{simp}(P)$ the *simple reflections*. We also recall that R_{ind} denotes the subsystem of indivisible roots of R, see 3.4.

14.8. Proposition. *Let R be a root system and P a positive system of R. For $w \in W(R)$ the following conditions are equivalent:*

(i) *P_w is finite,*
(ii) *w is a product of simple reflections,*
(iii) *$w \in W(\tilde R)$, where $\tilde R = R \cap \mathrm{span}(\mathrm{simp}(P))$ as in* 14.3.

If these conditions hold, w is a product of $|P_w \cap R_{\mathrm{ind}}|$ simple reflections, say $w = s_{\alpha_1} \cdots s_{\alpha_n}$ for $\alpha \in \mathrm{simp}(P)$, and

$$P_w \cap R_{\rm ind} = \{s_{\alpha_n} \cdots s_{\alpha_{i+1}}(\alpha_i) : 1 \leqslant i \leqslant n\}. \tag{1}$$

In particular, $P_w \subset \tilde{R}$.

Proof. We first observe that 14.5.4 implies for any $f \in \mathrm{Aut}(R)$ and $\alpha \in \mathrm{simp}(P)$

$$P_{fs_\alpha} = \left\{ \begin{array}{ll} s_\alpha(P_f \setminus P_\alpha) & \text{if } \alpha \in P_f \\ s_\alpha(P_f) \,\dot\cup\, P_\alpha & \text{if } \alpha \notin P_f \end{array} \right\}. \tag{2}$$

Suppose (i) holds. Then the partially ordered set $(P_w, \preccurlyeq)$ has a minimal element, say α, which by 14.6 is a simple root. By (2) the set P_{ws_α} has smaller cardinality. Continuing in this fashion, we find finitely many simple reflections $s_1, \ldots, s_n$ such that $P_{ws_n \cdots s_1} = \emptyset$. By 14.5.3, we then have $w = s_1 \cdots s_n \in W(\tilde{R})$, i.e., (iii). Since (ii) $\iff$ (iii) is obvious, we now suppose (ii). Observe that $|P_{fs_\alpha}| \leqslant |P_f| + 2$ for $f \in \mathrm{Aut}(R)$ by (2), so that $|P_w| < \infty$ follows by induction. This proves the equivalence of (i) – (iii).

Suppose now that $w = s_1 \cdots s_n$ is a product of simple reflections $s_i = s_{\alpha_i}$, $\alpha_i \in \mathrm{simp}(P)$. Then (1) follows by a standard reduction to the finite-dimensional theory, which is dealt with in [**12**, VI, §1.6, Cor. 2]. For the convenience of the reader we include the argument here: $\alpha \in P_w$ if and only if there exists i such that $s_{i+1} \cdots s_n(\alpha) \in P$ while $s_i s_{i+1} \cdots s_n \in -P$, i.e., $s_{i+1} \cdots s_n(\alpha) \in P_{s_i \cdots s_n} \subset \{\alpha_i, 2\alpha_i\}$. Hence $\alpha \in P_w \cap R_{\rm ind}$ if and only $s_{i+1} \cdots s_n(\alpha) = \alpha_i$ for some i, which is equivalent to (1). If $\alpha \in P_w$ is divisible then $\alpha/2 \in R_{\rm ind} \cap P_w$, hence $P_w \subset \tilde{R}$.

14.9. *Positive systems in classical root systems.* In the remainder of this section we describe in more detail the positive systems of the root systems $R = \dot{\mathrm{A}}_I, \mathrm{B}_I, \mathrm{C}_I, \mathrm{BC}_I$ and D_I for an arbitrary set I.

By Prop. 13.4, a pure parabolic subset $P = R_{I_0, \succcurlyeq}$ of a root system $R = \mathrm{T}_I$ is a positive system if and only if $I_0 = \emptyset$ and $\succcurlyeq$ is a total order on I, which we therefore write $\geqslant$. Accordingly, we specialize the notations introduced in 13.3 to this case and introduce subsets $R_\geqslant$ of R as follows:

$$\begin{aligned} \dot{\mathrm{A}}_{I,\geqslant} &= \{\varepsilon_i - \varepsilon_j : i \geqslant j\}, && (1)\\ \mathrm{D}_{I,\geqslant} &= \{\varepsilon_i - \varepsilon_j : i \geqslant j\} \cup \{\varepsilon_i + \varepsilon_j : i \neq j\}, && (2)\\ \mathrm{B}_{I,\geqslant} &= \{\varepsilon_i - \varepsilon_j : i \geqslant j\} \cup \{\varepsilon_i + \varepsilon_j : i \neq j\} \cup \{\varepsilon_i : i \in I\}, && (3)\\ \mathrm{C}_{I,\geqslant} &= \{\varepsilon_i - \varepsilon_j : i \geqslant j\} \cup \{\varepsilon_i + \varepsilon_j : i \neq j\} \cup \{2\varepsilon_i : i \in I\}, && (4)\\ \mathrm{BC}_{I,\geqslant} &= \{\varepsilon_i - \varepsilon_j : i \geqslant j\} \cup \{\varepsilon_i + \varepsilon_j : i \neq j\} \cup \{\varepsilon_i, 2\varepsilon_i : i \in I\}. && (5) \end{aligned}$$

It will also be useful to have analogous notations for the sets $\mathrm{BC}_I(J)$ and $\mathrm{DC}_I(J)$ introduced in 12.18, so we introduce the following subsets:

$$\begin{aligned} \mathrm{BC}_{I,\geqslant}(J) &= \mathrm{B}_{I,\geqslant} \cup \{2\varepsilon_j : j \in J\}, && (6)\\ \mathrm{DC}_{I,\geqslant}(J) &= \mathrm{D}_{I,\geqslant} \cup \{2\varepsilon_j : j \in J\}. && (7) \end{aligned}$$

14.10. Proposition. *Let* $P = R_{I_0, \succcurlyeq}$ *be a pure parabolic subset as defined in* 13.3.2 – 13.3.6 *of* 13.3. *With the notations introduced in* 13.4.3 *and* 14.9, *the* *positive system $\bar{P} = P/P_s$ in the quotient $\bar{R} = R/P_s$ (cf.* 10.19(c)*) has the following description, where $\geqslant$ is the total order induced by $\succcurlyeq$ on $\bar{I}$:*

$$\overline{\dot{\mathrm{A}}_{I,\succcurlyeq}} = \dot{\mathrm{A}}_{\bar{I},\geqslant}, \tag{1}$$

$$\overline{\mathrm{D}_{I,I_0,\succcurlyeq}} = \left\{ \begin{array}{ll} \mathrm{DC}_{\bar{I},\geqslant}(\bar{I}_2) & \text{if } I_0 = \emptyset \\ \mathrm{BC}_{\bar{I},\geqslant}(\bar{I}_2) & \text{if } I_0 \neq \emptyset \end{array} \right\}, \tag{2}$$

$$\overline{\mathrm{B}_{I,I_0,\succcurlyeq}} = \mathrm{BC}_{\bar{I},\geqslant}(\bar{I}_2), \tag{3}$$

$$\overline{\mathrm{C}_{I,I_0,\succcurlyeq}} = \left\{ \begin{array}{ll} \mathrm{C}_{\bar{I},\geqslant} & \text{if } I_0 = \emptyset \\ \mathrm{BC}_{\bar{I},\geqslant} & \text{if } I_0 \neq \emptyset \end{array} \right\}, \tag{4}$$

$$\overline{\mathrm{BC}_{I,I_0,\succcurlyeq}} = \mathrm{BC}_{\bar{I},\geqslant}. \tag{5}$$

Proof. By 12.19(b), the quotient map $R \to \bar{R}$ may be identified with the map $h\colon X \to \bar{X}$ in case $R \neq \dot{\mathrm{A}}_I$, and the map $h\colon \dot{X} \to \mathrm{Ker}(\bar{t})$ in case $R = \dot{\mathrm{A}}_I$. Now formulas (1) – (5) are an easy consequence of the formulas 12.14.3, 12.15.2 and 12.15.3 describing the map h and the description of P given by 13.3.2 – 13.3.6.

14.11. *Total orders and order types.* Let us recall that a totally ordered set I is *well-ordered* if every non-empty subset has a minimum.

Let $\mathrm{Ord}(I)$ be the set of total orders on a set I. Under the action of the symmetric group $\mathrm{Sym}(I)$, the set $\mathrm{Ord}(I)$ decomposes into equivalence classes, called *order types* of I. If I is finite, $\mathrm{Sym}(I)$ acts transitively on $\mathrm{Ord}(I)$ so there is only one order type. For infinite I this is no longer the case: There are infinitely many different order types. Also, unlike the finite case, the action of $\mathrm{Sym}(I)$ on $\mathrm{Ord}(I)$ is no longer free; e.g., the natural order on $\mathbb{Z}$ admits the shift $n \mapsto n+1$ as a nontrivial order automorphism. However, if we let $\mathrm{Word}(I) \subset \mathrm{Ord}(I)$ be the set of well-orderings of I, then $\mathrm{Sym}(I)$ acts freely on $\mathrm{Word}(I)$, and the set $\mathrm{Word}(I)/\mathrm{Sym}(I)$ of types of well-orderings of I is itself well-ordered by the relation $[\geqslant] \preccurlyeq [\geqslant']$ if and only if there exists an order isomorphism between $(I, \geqslant)$ and an initial segment of $(I, \geqslant')$. In fact, $\mathrm{Word}(I)/\mathrm{Sym}(I)$ may be identified with the set of ordinals of cardinality $\mathrm{Card}(I)$, and then its smallest element becomes $\mathrm{Card}(I)$, the cardinal defined by I. Here we consider ordinals as special well-ordered sets, and cardinals as the initial ordinals, see [**18**, Ch. 4,5]. Thus also for infinite I, there is a distinguished element in $\mathrm{Ord}(I)/\mathrm{Sym}(I)$, namely the class of minimal well-orderings of I.

14.12. THEOREM. *Let $R = \mathrm{T}_I$ be one of the root systems $\dot{\mathrm{A}}_I, \mathrm{B}_I, \mathrm{C}_I, \mathrm{BC}_I$ or D_I. We denote by $\mathfrak{P}^+ = \mathfrak{P}^+(R)$ the set of positive systems of R, and use the notation of* 14.9 *and* 12.7.1.

The map $\tilde{\Psi}^+\colon \mathrm{Ord}(I) \to \mathfrak{P}^+(R)$ which sends $\geqslant$ to $R_\geqslant$ is $\mathrm{Sym}(I)$*-equivariant, and induces a* $\mathrm{Sym}(I)$*-equivariant bijection*

$$\mathrm{Ord}(I) \xrightarrow{\cong} \mathfrak{P}^+(R)/N \tag{1}$$

which in turn gives rise to a bijection between the set $\mathrm{Ord}(I)/\mathrm{Sym}(I)$ of order types of I, and the set $\mathfrak{P}^+(R)/G$ of conjugacy classes of positive systems of R under the group G of automorphisms of R, defined in 12.7.1*:*

$$\operatorname{Ord}(I)/\operatorname{Sym}(I) \xrightarrow{\cong} \mathfrak{P}^+(R)/G. \tag{2}$$

We recall that $G = \overline{W}(R)$ unless $R = \mathrm{D}_n$ is a finite, in which case $[G : W(\mathrm{D}_n)] = 2$.

Proof. The set $\operatorname{Ord}(I)$ of total orders on I can be identified with a subset of the set $\mathbb{P}_0$ of p-data (cf. 13.9) by assigning to $\geqslant$ the p-datum $(\emptyset, \geqslant)$. The map $\tilde{\Psi}$ of Th. 13.11 maps $\geqslant\, = (\emptyset, \geqslant)$ to $R_{\geqslant}$. By Th. 13.11 and Prop. 13.4 we have $\tilde{\Psi}(\operatorname{Ord}(I)) = \mathfrak{P}_0^+(\mathrm{T}, I) := \mathfrak{P}_0(\mathrm{T}, I) \cap \mathfrak{P}^+(R)$, the set of pure positive systems of R. As a group of automorphisms, N maps positive systems to positive systems. Hence the bijection $\tilde{\Phi}$ of 13.6.1 maps $\mathfrak{P}_0(\mathrm{T}, I) \cap \mathfrak{P}^+(R)$ onto the set of orbits of positive systems under N. This proves (1). The second assertion is then immediate from Th. 13.11.

14.13. Corollary. *Let R be one of the root systems $\dot{\mathrm{A}}_I, \ldots, \mathrm{BC}_I$. Then the positive systems of R are of the form $\dot{\mathrm{A}}_{\geqslant}$ if $R = \dot{\mathrm{A}}_I$, and of the form $\sigma(R_{\geqslant})$ for a sign change $\sigma \in \mathbf{2}^I$ in the other cases, where $\geqslant$ is a total order on I.*

For reduced root systems, this description is due to Neeb [**50**, Prop. II.1, V.1, VI.1, VII.1]. It can also be deduced from [**23**, Prop. 3].

We now describe the simple roots of the pure positive systems $R_{\geqslant}$.

14.14. Proposition. *Let I be a totally ordered set, and let* $\operatorname{pre}(I)$ *be the set of $i \in I$ which have a successor $i+1$ as in* B.2*. Also let* 0 *denote the minimum (if present) of I. Then the set of simple roots of the pure positive system $R_{\geqslant}$ of* 14.9 *is given by*

$$\operatorname{simp}(R_{\geqslant}) = \{\varepsilon_{i+1} - \varepsilon_i : i \in \operatorname{pre}(I)\} \ \cup\ \Sigma$$

where Σ is as follows:

$$\Sigma = \left\{ \begin{array}{ll} \{\varepsilon_0\} & \textit{if } R = \mathrm{B}_I \textit{ or } \mathrm{BC}_I, \textit{ and } 0 \in I \\ \{2\varepsilon_0\} & \textit{if } R = \mathrm{C}_I \textit{ and } 0 \in I \\ \{\varepsilon_1 + \varepsilon_0\} & \textit{if } R = \mathrm{D}_I \textit{ and } 0 \in \operatorname{pre}(I) \\ \emptyset & \textit{in all other cases} \end{array} \right\}.$$

Proof. The cones $\mathbb{R}_+[P]$ spanned by $P = R_{\geqslant}$ are described in Prop. 13.10(b), and their extremal rays are given in B.6(b), B.8(b) and B.12(b). Now the result follows from condition (i) of 14.2.

14.15. Example. From the description of the simple roots given above, it is easy to see that even in root systems admitting a root basis, not every positive system is determined by a root basis. For example, the root system $R = \dot{\mathrm{A}}_{\mathbb{N}}$ admits the root basis $\{\varepsilon_{i+1} - \varepsilon_i : i \in \mathbb{N}\}$ by 6.11. On the other hand, $\dot{\mathrm{A}}_{\mathbb{N}} \cong \dot{\mathrm{A}}_{\mathbb{Q}}$ since $\mathbb{Q}$ is countable, and the natural order of $\mathbb{Q}$ defines a positive system P of R. Since no element of $\mathbb{Q}$ has a predecessor, the set of simple roots of P is empty by 14.14(d), so 14.4 shows that P is not determined by any root basis.

14.16. Proposition. *Let $R = \mathrm{T}_I$, where $|I| \geqslant 5$ for* $\mathrm{T} = \mathrm{D}$ *and $|I| \geqslant 2$ in the other cases, and let $\geqslant$ be a total order on I. We denote by* $\operatorname{Aut}_{\uparrow\downarrow}(I, \geqslant)$ *the group of monotone (i.e., increasing or decreasing) bijections of the ordered set I, and by* $\operatorname{Aut}(I, \geqslant)$ *its normal subgroup of order automorphisms (= increasing bijections).*

The stabilizer of the positive system $P = R_{\geqslant}$ in $\operatorname{Aut}(R)$*, denoted* $\operatorname{Aut}(R, P)$*, is then given by*

$$\operatorname{Aut}(R,P) = \left\{ \begin{array}{ll} \operatorname{Aut}_{\uparrow\downarrow}(I, \geqslant) & \textit{if } \mathrm{T} = \dot{\mathrm{A}} \\ \operatorname{Aut}(I, \geqslant) \times \{\mathrm{Id}, \sigma_0\} & \textit{if } \mathrm{T} = \mathrm{D} \textit{ and } 0 \in I \\ \operatorname{Aut}(I, \geqslant) & \textit{otherwise} \end{array} \right\}. \tag{1}$$

For an infinite I we have

$$\overline{W}(R) \cap \operatorname{Aut}(R,P) = \operatorname{Aut}(I, \geqslant) \times \left\{ \begin{array}{ll} \{\mathrm{Id}, \sigma_0\} & \textit{if } \mathrm{T} = \mathrm{D} \textit{ and } 0 \in I \\ \{\mathrm{Id}\} & \textit{otherwise} \end{array} \right\}. \tag{2}$$

Proof. Under our assumptions on $|I|$, it follows from 9.5 that

$$\operatorname{Aut}(R) \cong \left\{ \begin{array}{ll} \operatorname{Sym}(I) \times \{\pm \mathrm{Id}\} & \text{if } \mathrm{T} = \dot{\mathrm{A}} \\ \operatorname{Sym}(I) \ltimes \mathbf{2}^I & \text{otherwise} \end{array} \right\}. \tag{3}$$

Let $f \in \operatorname{Aut}(R,P)$ with permutation part $\pi \in \operatorname{Sym}(I)$, cf. 9.1. In case $\mathrm{T} = \dot{\mathrm{A}}$ we then have $f = \pm\pi$, and it is immediate that either $f = \pi \in \operatorname{Aut}(I, \geqslant)$ or $f = -\pi$ for a decreasing π, establishing the first case in (1). In the following we will assume $\mathrm{T} \neq \dot{\mathrm{A}}$, hence $f = \sigma\pi$ for some $\sigma \in \mathbf{2}^I$, i.e., $f(\varepsilon_i) = \sigma(i)\varepsilon_{\pi(i)}$. The cone $K = \mathbb{R}_+[P]$ is invariant under f. Also, since P is pure, we have $I = I_+(P) \mathbin{\dot\cup} I_1(P)$. From this and the definition of $I_\nu(P)$ in 13.2 it follows that π leaves $I_\nu(P)$, $\nu \in \{+, 1\}$, invariant, and that $\sigma(i) = 1$ for $i \in I_+(P)$. In case $\mathrm{T} = \mathrm{D}_I$ and $0 \in I$, we have $I_1(P) = \{0\}$ by 13.10.6. This shows $f = \pi$ or $f = \sigma_0\pi$ for some $\pi \in \operatorname{Aut}(I, \geqslant)$. In all other cases $I_1(P) = \emptyset$, hence $\sigma = \mathrm{Id}$ and $f = \pi \in \operatorname{Aut}(I, \geqslant)$. This proves the inclusions from left to right in (1). The proof of the other inclusions is straightforward and left to the reader.

Finally, (2) is an immediate consequence of (1) and the description of $\overline{W}(R)$ in 9.5.

Remarks. (a) We note that in the finite case σ_0 is an automorphism of P which induces the nontrivial automorphism of the Dynkin diagram D_n, $n \neq 4$.

(b) Any $w \in W(R)$ stabilizing P is trivial. This is well-known for a finite R and follows from 15.8 for an infinite I. The analogous result for $\overline{W}(R)$ fails: It follows from (1) that, in particular, all order automorphisms give rise to non-trivial elements in $\overline{W}(R)$ stabilizing P. For infinite I, such order automorphisms may well exist, for example, the shift $n \mapsto n + 1$ in case $I = \mathbb{Z}$ with its natural ordering.

§15. Positive linear forms and facets

15.1. *The dual cone of a parabolic subset.* Let $P \subset R$ be a parabolic subset of a root system (R, X). A linear form $f \in X^*$ is called *positive* (with respect to P) if $\langle \alpha, f \rangle \geqslant 0$ for all $\alpha \in P$. We denote by

$$D^\vee(P) = \{f \in X^* : \langle P, f \rangle \geqslant 0\}, \tag{1}$$

also called the *dual cone of* P, the set of these linear forms. Clearly, $D^\vee(P)$ is the polar set (dual cone) of the convex cone $K(P) = \mathbb{R}_+[P]$ spanned by P, cf. B.1. Hence $D^\vee(P)$ is a weak-$*$-closed convex cone which is proper since P spans X.

It is obvious that $P_1 \subset P_2$ implies $D^\vee(P_1) \supset D^\vee(P_2)$. Furthermore, denoting by Z the linear span of the symmetric part $P_s = P \cap (-P)$ of P, every $f \in D^\vee(P)$ vanishes on Z. Thus we may regard $D^\vee(P)$ as a cone in $(X/Z)^*$, namely the polar of the canonical image $\mathrm{can}(K(P))$ in X/Z. This also shows that $\mathrm{rank}(f) \leqslant \mathrm{rank}(R/P_s)$ for any $f \in D^\vee(P)$. Furthermore, we have:

$$\textit{The union of all } D^\vee(P),\ P \textit{ a positive system, is all of } X^*. \tag{2}$$

Indeed, if $f \in X^*$ then $R_+(f) = \{\alpha \in R : \langle \alpha, f \rangle \geqslant 0\}$ is parabolic by 10.8, and hence contains a positive system P by 10.14, showing $f \in D^\vee(P)$.

When R is a direct sum of root systems R_i and correspondingly $P = \bigcup P_i$, the cones $K(P)$ and $D^\vee(P)$ behave as follows:

$$K(P) = \bigoplus K(P_i), \qquad D^\vee(P) = \prod D^\vee(P_i). \tag{3}$$

By Lemma 10.18, $P^\vee$ is a parabolic subset of the coroot system $(R^\vee, X^\vee)$. It makes therefore sense to define

$$D(P) := D^\vee(P^\vee) = \{g \in (X^\vee)^* : \langle P^\vee, g \rangle \geqslant 0\}. \tag{4}$$

The natural isomorphism $(R, X) \cong (R^{\vee\vee}, X^{\vee\vee})$ of 4.9.2 then gives rise to an isomorphism

$$D(P^\vee) \cong D^\vee(P). \tag{5}$$

of cones (in the obvious meaning).

15.2. Lemma. *If R is finite and $P \subset R$ a positive system, then $D(P)$ is the closure of the Weyl chamber determined by P.*

Proof. Indeed, identifying $(X^\vee)^*$ with X, we have $D(P) = \{x \in X : \langle x, B^\vee \rangle \geqslant 0\}$ where $B = \mathrm{simp}(P)$ is the root basis associated to P, so that 15.2 follows from [**12**, V, § 1.4, Rem. 1 and VI, § 1.5, Th. 2].

Examples show (cf. 16.3) that the attempt to define analogs of the usual open Weyl chambers by replacing $\geqslant$ with $>$ in 15.1.1, may yield the empty set. We also remark that, unlike in the finite case, it is essential to consider $D^\vee(P)$ as a subset of the full dual X^* (and $D(P)$ as a subset of $(X^\vee)^*$). The intersection of $D^\vee(P)$ with the subspace $X^\vee = \operatorname{span}(R^\vee)$ of X^* may be too small or even trivial; see, again, 16.3.

Next we show that for a positive system P, the cone $D^\vee(P)$ may be characterized via the Weyl group just as in the finite case (see A.13). It is in fact easy to prove the following version for parabolic subsets. Recall that the Weyl group acts on X^* by $w(f) = f \circ w^{-1}$.

15.3. Proposition. *Let P be a parabolic subset of a root system (R, X), and consider the cones $K^\vee = \mathbb{R}_+[P^\vee] \subset X^\vee$ and $D^\vee(P) \subset X^*$ as in* 15.1.1. *Let $f \in X^*$. Then*

$$f \in D^\vee(P) \qquad \Longleftrightarrow \qquad \langle P_s, f\rangle = 0 \ \textit{ and } \ f - w(f) \in K^\vee \ \textit{ for all } w \in W(R).$$

Proof. $\Longrightarrow$: As noted in 15.1, every $f \in D^\vee(P)$ vanishes on P_s. Let $\tilde{P} \subset P$ be a positive system (cf. 10.14). Again by 15.1, we have $D^\vee(P) \subset D^\vee(\tilde{P})$, and $\tilde{P} \subset P$ implies $\tilde{K}^\vee = \mathbb{R}_+[\tilde{P}^\vee] \subset K^\vee$. Hence it suffices to prove that $f \in D^\vee(\tilde{P})$ implies $f - w(f) \in \tilde{K}^\vee$, i.e., we may replace P by $\tilde{P}$ and thus assume that P is a positive system.

Let $w = s_{\alpha_1} \cdots s_{\alpha_m}$, and choose a finite full subsystem S with linear span Y containing $\alpha_1, \ldots, \alpha_m$. By Cor. 5.8, we may identify $W(S)$ with the subgroup W_S of $W(R)$ generated by $\{s_\alpha : \alpha \in S\}$. By Th. 5.7, we have $X = Y \oplus S^\perp$ and W_S acts trivially on $S^\perp$. Passing to the dual space, and keeping in mind that $Y^* = Y^\vee$ since Y is finite-dimensional, we obtain the W_S-invariant decomposition

$$X^* = Y^\vee \oplus Y^\circ \tag{1}$$

where $Y^\circ = \{f \in X^* : f\big|Y = 0\} \cong (S^\perp)^*$ is the polar of Y, and $Y^\vee$ is identified with a subspace of $X^\vee \subset X^*$ as in 4.10. Also, W_S acts trivially on Y°. Now $P \cap S$ is a positive system in S, with dual cone $D^\vee(P \cap S) = \{g \in Y^\vee : \langle P \cap S, g\rangle \geqslant 0\}$. Let $f \in D^\vee(P)$, decomposed in $f = g + f^\circ$ according to (1). Then $\langle Y, f^\circ\rangle = 0$ implies $\langle P \cap S, g\rangle = \langle P \cap S, f\rangle \subset \langle P, f\rangle \subset \mathbb{R}_+$. Hence g belongs to $D^\vee(P \cap S)$ which by 15.2 is the closure of the Weyl chamber determined by $(P \cap S)^\vee$. From $w(f^\circ) = f^\circ$ and A.13 we then conclude $f - w(f) = g - w(g) \in \mathbb{R}_+[(P \cap S)^\vee] \subset K^\vee$.

$\Longleftarrow$: It suffices to prove $\langle \alpha, f\rangle \geqslant 0$ for all $\alpha \in P_u$. Assume to the contrary that $\langle \alpha, f\rangle < 0$. Then $f - s_\alpha(f) = \langle \alpha, f\rangle\alpha^\vee \in K^\vee$ implies $-\alpha^\vee \in K^\vee \cap R^\vee = P^\vee$ (by 10.17.3), so $\alpha^\vee \in P_s^\vee$ and therefore also $\alpha \in P_s$, contradiction.

Our next aim is to show that P is, in turn, uniquely determined by $D^\vee(P)$ as the set of those roots on which all $f \in D^\vee(P)$ take positive values (15.6.2). For this purpose, we need some auxiliary material on norms in X and X^*.

15.4. Definition. Let (R, X) be a root system. Since R spans X, every $x \in X$ is a linear combination $x = \sum_{\alpha \in R} c_\alpha \alpha$ with real coefficients c_α and only finitely many nonzero terms. Hence it makes sense to define

$$\|x\|_1 := \inf\Big\{\sum_{\alpha\in R}|c_\alpha| \;:\; x = \sum_{\alpha\in R} c_\alpha\alpha\Big\}.$$

It is easy to see that $\|\ \|_1$ is a seminorm on X, and it is in fact a norm: If $\|x\|_1 = 0$ there exists, for all $\varepsilon > 0$, a representation $x = \sum_{\alpha\in R} c_\alpha\alpha$ such that $\sum_{\alpha\in R}|c_\alpha| \leqslant \varepsilon$. Then, for all $\beta^\vee \in R^\vee$, $|\langle x,\beta^\vee\rangle| \leqslant \sum_{\alpha\in R}|c_\alpha||\langle\alpha,\beta^\vee\rangle| \leqslant 4\varepsilon$, because the Cartan numbers $\langle\alpha,\beta^\vee\rangle$ are bounded by 4. As ε was arbitrary, we obtain $\langle x, R^\vee\rangle = 0$ so $x \in R^\perp = 0$ by 3.5.3.

Clearly the 1-norm is invariant under all automorphisms of R. We note also its behavior under direct sums: If $(R,X) = \coprod_{i\in I}(R_i,X_i)$ and $x = \sum_{i\in I} x_i$ is decomposed accordingly, then

$$\|x\|_1 = \sum_{i\in I}\|x_i\|_1.$$

This follows easily from the definitions.

Next, we define, for any $f \in X^*$,

$$\|f\|_\infty := \sup\{|\langle\alpha,f\rangle| : \alpha\in R\},$$

called the *maximum norm* of f, and denote by X^*_{bd} the set of linear forms f for which $\|f\|_\infty < \infty$, also called *bounded linear forms.* Note that

$$X^\vee \subset X^*_{\mathrm{bd}} \tag{1}$$

because $R^\vee$ spans $X^\vee$ and $\|\beta^\vee\|_\infty \leqslant 4$ by the aforementioned property of Cartan numbers.

The bounded coweights introduced in 7.3 are just the coweights which are bounded in the above sense, so that

$$\mathcal{P}_{\mathrm{bd}}(R^\vee) = \mathcal{P}^\vee(R) \cap X^*_{\mathrm{bd}}.$$

We finally note that the basic coweights are bounded, in fact,

$$\|q\|_\infty \leqslant 6 \quad \text{for } q \in \mathcal{B}^\vee(R), \tag{2}$$

as follows immediately from Prop. 7.12.

15.5. Lemma. *Let $f \in X^*$. Then $f \in X^*_{\mathrm{bd}}$ if and only if $f\colon X \to \mathbb{R}$ is continuous in the* 1*-norm, and then*

$$\|f\|_\infty = \sup\{|\langle x,f\rangle| : x\in X,\ \|x\|_1 \leqslant 1\}. \tag{1}$$

Hence $(X^*_{\mathrm{bd}}, \|\ \|_\infty)$ is the topological dual of the normed vector space $(X, \|\ \|_1)$; in particular, it is a real Banach space.

Proof. From the definition of the 1-norm it is clear that $\|\alpha\|_1 \leqslant 1$ for all $\alpha \in R$. Hence continuity of f implies $f \in X^*_{\mathrm{bd}}$, and we have the inequality "$\leqslant$" in (1). Conversely, let $f \in X^*_{\mathrm{bd}}$ and let $\|x\|_1 \leqslant 1$. Then, for any $\varepsilon > 0$, there exists a representation $x = \sum_{\alpha\in R} c_\alpha\alpha$ such that $\sum_\alpha |c_\alpha| \leqslant 1+\varepsilon$, and therefore

$$|\langle x,f\rangle| \leqslant \sum_\alpha |c_\alpha||\langle\alpha,f\rangle| \leqslant \big(\sum_\alpha|c_\alpha|\big)\|f\|_\infty \leqslant (1+\varepsilon)\|f\|_\infty.$$

As ε was arbitrary, we conclude that f is continuous with respect to the 1-norm, and we have the inequality "$\geqslant$" in (1).

15.6. Proposition. *Let P be a parabolic subset of a root system (R, X), with symmetric part $P_s = P \cap (-P)$ and dual cone $D^\vee(P)$, and let $D^\vee_{\rm bd}(P) = D^\vee(P) \cap X^*_{\rm bd}$. Also let $\overline{K}^{\rm nor}$ be the closure of $K = \mathbb{R}_+[P]$ in the topology defined by the 1-norm. Then, with the notations of* 10.8,

$$\overline{K}^{\rm nor} = \{x \in X : \langle x, D^\vee_{\rm bd}(P)\rangle \geqslant 0\}, \tag{1}$$

$$P = \bigcap_{f \in D^\vee(P)} R_+(f) = \bigcap_{f \in D^\vee_{\rm bd}(P)} R_+(f) = \overline{K}^{\rm nor} \cap R, \tag{2}$$

$$P_s = \bigcap_{f \in D^\vee(P)} R_0(f) = \bigcap_{f \in D^\vee_{\rm bd}(P)} R_0(f). \tag{3}$$

Remark. Using the classification of parabolic subsets (§13), we will show in 16.12 that in fact $K = \overline{K}^{\rm nor}$.

Proof. Since $D^\vee_{\rm bd}(P)$ is the polar set of K in $X^*_{\rm bd}$, (1) follows from [**11**, II, §6.3, Cor. 3(ii) of Th. 1].

In (2), the inclusions from left to right are obvious or follow from (1). It remains to show $\overline{K}^{\rm nor} \cap R \subset P$. Suppose that there exists $\alpha \in \overline{K}^{\rm nor} \cap R$ but $\alpha \notin P$. Then $-\alpha \in P$ since P is parabolic. As $\alpha \in \overline{K}^{\rm nor}$, there exists $x \in K$ such that $\|x - \alpha\|_1 \leqslant 1/7$. Write $x = \sum_{i=1}^n c_i\alpha_i$ where $c_i > 0$ and $\alpha_i \in P$, and choose a full finite subsystem F of R containing $\{\alpha, \alpha_1, \dots, \alpha_n\}$. Then $P \cap F$ is parabolic in F. By Lemma 11.1(ii) there exist basic coweights $q_1, \dots, q_k$ of F such that

$$P \cap F = \{\beta \in F : \langle \beta, q_1\rangle \geqslant 0, \dots, \langle \beta, q_k\rangle \geqslant 0\}. \tag{4}$$

Since $-\alpha \in P \cap F$ but $\alpha \notin P \cap F$, one of the $\langle \alpha, q_i\rangle$ must be negative (and integral), say, $\langle \alpha, q_1\rangle \leqslant -1$. Hence

$$\langle x - \alpha, q_1\rangle = \langle x, q_1\rangle - \langle \alpha, q_1\rangle \geqslant \langle x, q_1\rangle + 1 \geqslant 1,$$

where we used $\langle x, q_1\rangle \geqslant 0$ which follows from (4), $\alpha_i \in P \cap F$ and $c_i > 0$. On the other hand, q_1 extends to a basic coweight q of R by 7.13(a) and $\|q\|_\infty \leqslant 6$ by 15.4.2. Hence $\langle x - \alpha, q_1\rangle = \langle x - \alpha, q\rangle \leqslant \|x - \alpha\|_1 \cdot \|q\|_\infty$ (by 15.5) $\leqslant 6/7$, contradiction.

15.7. *Facets.* Let (R, X) be a root system, and let $\mathfrak{H}$ be the set of hyperplanes

$$H_\alpha = \{f \in X^* : \langle \alpha, f\rangle = 0\} \tag{1}$$

where $\alpha \in R^\times$. As for a finite R we have:

If $P_1 \neq P_2$ are parabolic subsets then there exists a hyperplane $H_\alpha \in \mathfrak{H}$ such that $D^\vee(P_1)$ and $D^\vee(P_2)$ are on opposite sides of H_α. (2)

For the proof we may assume that $P_1 \setminus P_2 \neq \emptyset$ and choose an $\alpha \in P_1 \setminus P_2$. Then $-\alpha \in P_2$, so $\langle \alpha, f\rangle \geqslant 0$ for all $f \in D^\vee(P_1)$ while $\langle \alpha, g\rangle \leqslant 0$ for all $g \in D^\vee(P_2)$.

Following [**12**, V, §1.2], we define the *facets* as the equivalence classes of linear forms on X with respect to the relation

$$f \sim g \quad \iff \quad \begin{cases} \text{for all } H \in \mathfrak{H}\text{, either } f \in H \text{ and } g \in H\text{, or} \\ f \text{ and } g \text{ lie strictly on the same side of } H. \end{cases} \tag{3}$$

We denote by $\mathbf{\Phi}(R)$ or simply $\mathbf{\Phi}$ the set of facets of R.

Recall from 10.9 that every $f \in X^*$ defines the scalar parabolic subset $R_+(f) = \{\alpha \in R : f(\alpha) \geqslant 0\}$. From the definitions it is immediate that

$$f \sim g \quad \Longleftrightarrow \quad R_+(f) = R_+(g). \tag{4}$$

Hence it makes sense to define $R_+(F) := R_+(f)$, for a facet F and an element $f \in F$. Then:

$$\begin{array}{l}\textit{The map } F \mapsto R_+(F) \textit{ is a bijection between } \mathbf{\Phi} \\ \textit{and the set of scalar parabolic subsets of } R.\end{array} \tag{5}$$

If $F \in \mathbf{\Phi}$ and $P = R_+(F)$, then it is clear from the definitions and 10.8.2 that

$$f \in F \quad \Longleftrightarrow \quad \langle \alpha, f\rangle = 0 \text{ for all } \alpha \in P_s \text{ and } \langle \alpha, f\rangle > 0 \text{ for all } \alpha \in P_u. \tag{6}$$

Thus F is an intersection of a number of hyperplanes and open half spaces of $\mathfrak{H}$, which shows that F is a convex cone in X^*, not containing 0 unless $F = \{0\}$.

We define a partial order on $\mathbf{\Phi}$ by

$$F' \preccurlyeq F \quad \Longleftrightarrow \quad R_+(F') \supset R_+(F). \tag{7}$$

Clearly, $\{0\} \in \mathbf{\Phi}$ is the minimum of the partially ordered set $\mathbf{\Phi}$. In general, $\mathbf{\Phi}$ does not have maximal elements, unlike the finite case, where the open Weyl chambers are the maximal elements of $\mathbf{\Phi}$. The minimal elements of $\mathbf{\Phi} \setminus \{\{0\}\}$ ("atomic facets") will be determined in 16.14.

Recall the action of the automorphism group $\mathrm{Aut}(R)$ on X^* given by $w(f) = f \circ w^{-1}$. As already pointed out in 10.9.1, we have

$$w(R_+(f)) = R_+(w(f)) \quad \text{for } w \in \mathrm{Aut}(R), \tag{8}$$

which shows that the action of $\mathrm{Aut}(R)$ is compatible with the equivalence relation defining the facets. Hence $\mathrm{Aut}(R)$ acts on $\mathbf{\Phi}$, and

$$R_+(wF) = wR_+(F), \tag{9}$$

i.e., the bijection $F \mapsto R_+(F)$ is $\mathrm{Aut}(R)$-equivariant.

15.8. Proposition. *Let P be a parabolic subset of a root system (R, X). Then the following conditions on an element w of the Weyl group $W(R)$ are equivalent:*

(i) $w \in W(P_s)$ *(identified with a subgroup of $W(R)$ as in 5.8),*
(ii) $w \big| D^\vee(P) = \mathrm{Id}$,
(iii) *w stabilizes $D^\vee(P)$,*
(iv) *w stabilizes P.*

In particular, if P is a positive system and thus $P_s = \{0\}$, the stabilizers of P and $D^\vee(P)$ in $W(R)$ are trivial.

Proof. (i) $\Longrightarrow$ (ii): It suffices to prove this for $w = s_\alpha$ where $\alpha \in P_s$. Let $f \in D^\vee(P)$. Then $s_\alpha(f) = f - \langle \alpha, f\rangle\alpha^\vee = f$, since f vanishes on P_s, as observed in 15.1.

(ii) $\Longrightarrow$ (iii): Obvious.

(iii) $\Longrightarrow$ (iv): This follows from 15.6.2 and 15.7.8.

(iv) $\Longrightarrow$ (i): Since w is a finite product of reflections in roots, there exists a finite full subsystem R' such that $w \in W(R')$. Now $P' = P \cap R'$ is a parabolic subset of R', and clearly $w(P') = P'$. Let $F' \subset (\mathrm{span}(R'))^*$ be the facet defined by P' (recall from 15.7 that P' is of scalar type and thus $P' = R'(F')$ for a unique facet F' of R'). Then $wF' = F'$ follows from 15.7.9. Hence, by [**12**, V, §3.3 Prop. 1], w is a product of reflections s_α for which $F' \subset H_\alpha$. But by 15.7.6 this means $\alpha \in P'_s$. Hence $w \in W(P'_s) \subset W(P_s)$.

Remarks. (a) As we have seen in 14.16 this result is no longer true for the big Weyl group.

(b) Since every full subsystem S is the symmetric part of a parabolic subset by 10.8(b), the corollary shows that the subgroups $W(S)$ of the Weyl group are precisely the stabilizers of parabolic subsets of R. This justifies the terminology "parabolic subgroups" introduced in 5.8.

15.9. PROPOSITION. *Let (R, X) be a root system and let P be a parabolic subset of R.*

(a) *The dual cone $D^\vee(P)$ is a union of facets, i.e., $D^\vee(P)$ is saturated with respect to the equivalence relation* 15.7.3.

(b) *The closure of a facet F in the weak-$*$-topology is*

$$\overline{F} = \bigcup\{F' : F' \preccurlyeq F\} = D^\vee(R_+(F)). \tag{1}$$

In particular, $F' \preccurlyeq F$ if and only if $F' \subset \overline{F}$.

(c) *$D^\vee(P)$ is the closure of a facet $F \iff P = R_+(F)$ is of scalar type.*

Proof. (a) From the definitions, we have

$$f \in D^\vee(P) \quad \iff \quad \langle P, f\rangle \geqslant 0 \quad \iff P \subset R_+(f). \tag{2}$$

Hence 15.7.4 shows that $f \in D^\vee(P)$ and $g \sim f$ imply $g \in D^\vee(P)$.

(b) The second equality of (1) follows from (2) applied to $P := R_+(F)$. In particular, since $P = R_+(f)$ for all $f \in F$ by 15.7.4, we have $F \subset D^\vee(P)$, and because $D^\vee(P)$ is weak-$*$-closed, also $\overline{F} \subset D^\vee(P)$. For the reverse inclusion, let $f \in F$ and $f' \in D^\vee(P)$. Then for all $0 < t \in \mathbb{R}$, we have $f + tf' \in F$. Indeed, since f and f' take nonnegative values on P, we have $P \subset R_+(f'+tf)$. Assume that this is a proper inclusion. Then there exists $\alpha \in R_+(f' + tf) \setminus P$, so $\langle\alpha, f\rangle < 0$. This implies $-\alpha \in P \subset R_+(f')$, so $\langle\alpha, f'\rangle \leqslant 0$, and therefore $\langle\alpha, f'+tf\rangle < 0$, contradicting $\alpha \in R_+(f' + tf)$. Thus we have $P = R_+(F) = R_+(f' + tf)$ or $f' + tf \in F$, as asserted. Now $\lim_{t\downarrow 0}(f' + tf) = f'$ in the weak-$*$-topology, so $f' \in \overline{F}$.

(c) The implication from right to left is clear from (b). For the reverse, assume $D^\vee(P) = \overline{F}$ and use 15.6.2 and again (b):

$$P = \bigcap_{f\in D^\vee(P)} R_+(f) = \bigcap_{f\in\overline{F}} R_+(f) = \bigcap_{F'\preccurlyeq F} R_+(F') = R_+(F),$$

because of 15.7.7.

We can now prove the exact analogue of [**12**, V, §3.3, Prop. 1] in our setting.

15.10. PROPOSITION. *Let (R, X) be a root system, let $F \in \mathbf{\Phi}$ be a facet with closure $\overline{F}$ in the weak-$*$-topology, and let $w \in W(R)$. Also let $P = R_+(F)$ be the parabolic subset determined by F. Then the following conditions are equivalent.*

(i) $wF \cap F \neq \emptyset$,
(ii) $wF = F$,
(iii) $w\overline{F} = \overline{F}$,
(iv) *w fixes at least one $f \in F$,*
(v) *w fixes every $f \in F$,*
(vi) *w fixes every $f \in \overline{F}$,*
(vii) $w \in W(P_s)$.

Proof. The implications (vi) $\Longrightarrow$ (v) $\Longrightarrow$ (iv) are trivial, and (iv) $\Longrightarrow$ (ii) $\iff$ (i) is clear from the fact that $W(R)$ acts on $\mathbf{\Phi}$. Since the natural action of $\mathrm{GL}(X)$ on X^* is continuous with respect to the weak-$*$-topology, we have (ii) $\Longrightarrow$ (iii). The remaining implications, namely (iii) $\Longrightarrow$ (vi) $\iff$ (vii), follow from 15.9.1, 15.7.6 and Prop. 15.8.

15.11. LEMMA. *Let P_1, P_2 be parabolic subsets of a root system (R, X) such that $P_1 \cap P_2$ is parabolic, and $P_2 = wP_1$ for some $w \in W(R)$. Then $P_1 = P_2$ and $w \in W((P_1)_s)$.*

Proof. By 10.14(a) there exists a positive system $P \subset P_1 \cap P_2$. Let S be a finite full subsystem of R such that $w \in W_S \cong W(S)$ as in 5.8, and let $\mathfrak{T}$ be the set of finite full subsystems T of R with $S \subset T$. Then $R = \bigcup \mathfrak{T}$, and $wT = T$ for all $T \in \mathfrak{T}$. Also, $P \cap T$ is a positive system in T and $P_i \cap T$ are parabolic subsets satisfying $P \cap T \subset (P_1 \cap T) \cap (P_2 \cap T)$ and $w(P_1 \cap T) = P_2 \cap T$. By [**12**, VI, §1.7, Cor. of Prop. 21], we have $P_1 \cap T = P_2 \cap T$, so $P_1 = P_2$ since R is the union of the T's. The last statement follows from 15.8.

15.12. PROPOSITION. *Let P be a parabolic subset of a root system (R, X), let $f_1, f_2 \in D^\vee(P)$, and suppose that $w(f_1) = f_2$ for some $w \in W(R)$. Then $f_1 = f_2$. Hence $D^\vee(P)$ is a fundamental domain for the action of $W(R)$ on the set $U^\vee = \bigcup_{w \in W(R)} w(D^\vee(P))$.*

Proof. Let F_i be the facet containing f_i, and $P_i = R_+(F_i)$. From $f_i \in D^\vee(P)$ we conclude $P \subset P_1 \cap P_2$ by 15.9.2, and hence $P_1 \cap P_2$ is again parabolic. Since w permutes facets we have $wF_1 = F_2$, and hence $wP_1 = P_2$ by 15.7.9. From Lemma 15.11 we obtain $P_1 = P_2$, and hence $F_1 = F_2 = wF_1$. Now $w(f_1) = f_1$ follows from Prop. 15.10.

§16. Dominant and fundamental weights

16.1. Definition. Let (R, X) be a root system, $P \subset R$ a parabolic subset, and $D^\vee(P) \subset X^*$ (resp. $D(P) \subset X^{\vee *}$) the dual cone of P (resp. $P^\vee$) as in 15.1.1. A coweight $q \in \mathcal{P}^\vee(R) \subset X^*$ is called *dominant with respect to* P if it belongs to $D^\vee(P)$. Thus $q \in X^*$ is dominant if and only if $\langle P, q\rangle \subset \mathbb{N}$. Analogously, the *dominant weights with respect to* P are the elements of $\mathcal{P}(R) \cap D(P)$. A (co)weight is called *fundamental with respect to* P if it is both dominant and basic (cf. 7.10). Explicitly, this means that a linear form $f \in X^*$ is a fundamental coweight of P if and only if

(i) $\langle P, f\rangle \subset \mathbb{N}$,
(ii) $1 \in \langle R, f\rangle$ and $R_0(f) = \{\alpha \in R : f(\alpha) = 0\}$ spans the hyperplane $\operatorname{Ker} f$.

An analogous characterization holds for fundamental weights, after replacing R and P by $R^\vee$ and $P^\vee$. We denote by

$$\mathcal{D}(P) := D(P) \cap \mathcal{P}(R) \quad \text{and} \quad \mathcal{D}^\vee(P) := D^\vee(P) \cap \mathcal{P}^\vee(R)$$

the sets of dominant weights and coweights, and by

$$\mathcal{F}(P) = D(P) \cap \mathcal{B}(R) \quad \text{and} \qquad \mathcal{F}^\vee(P) = D^\vee(P) \cap \mathcal{B}^\vee(R)$$

the sets of fundamental weights and coweights of P. Here $\mathcal{B}(R)$ and $\mathcal{B}^\vee(R)$ denote the sets of basic weights and coweights as in 7.10.

As in 15.1.1, we have canonical identifications $\mathcal{D}(P^\vee) = \mathcal{D}^\vee(P)$ and $\mathcal{F}(P^\vee) = \mathcal{F}^\vee(P)$, i.e., passing from R and P to $R^\vee$ and $P^\vee$ switches weights and coweights. For notational convenience, we will usually deal with coweights, although the dominant and fundamental weights are probably more important in applications to the representation theory of Lie algebras and groups.

If R is the direct sum of root systems R_i and correspondingly $P = \bigcup P_i$, then

$$\mathcal{D}^\vee(P) = \prod \mathcal{D}^\vee(P_i), \qquad \mathcal{F}^\vee(P) = \bigcup \mathcal{F}^\vee(P_i), \tag{1}$$

with analogous formulas for $\mathcal{D}(P)$ and $\mathcal{F}(P)$. This follows easily from 7.10.5 and 15.1.3. In case R is finite and P is a positive system, the fundamental weights defined here are the usual ones, as will follow from Prop. 16.2 below. Note that, by 15.4.2,

$$\textit{fundamental weights and coweights are bounded}, \tag{2}$$

but they need not be finite. We remark that Neeb [**50**] introduced fundamental weights in an ad hoc manner, with a different definition for each type of infinite irreducible root system.

16.2. Proposition. *Let (R, X) be a finite root system, let P be a parabolic subset of R and let P be described in terms of a root basis B and a decomposition $B = B_s \dot\cup B_u$ as in* 11.1. *Also let q_β ($\beta \in B$) be the dual basic coweights of B as in* 7.10.3. *Then*

$$D^\vee(P) = \mathbb{R}_+\big[\{q_\beta : \beta \in B_u\}\big], \tag{1}$$
$$\mathcal{D}^\vee(P) = \mathbb{N}\big[\{q_\beta : \beta \in B_u\}\big], \tag{2}$$
$$\mathcal{F}^\vee(P) = \{q_\beta : \beta \in B_u\}. \tag{3}$$

In particular, if P is a positive system and therefore $B_u = B$, then the dominant (fundamental) coweights of P defined in 16.1 *are precisely the dominant (fundamental) weights of $P^\vee$ as in* [**12**, VI, §1.10].

Remark. With P as above, the set B_u depends on the choice of a root basis $B \subset P$, unless, of course, P happens to be a positive system where B is uniquely determined by P. For example, in $R = \mathrm{B}_2 = \{0, \pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_1 \pm \varepsilon_2\}$ the parabolic subset $P = R_+(t) = \{0, \pm(\varepsilon_1 - \varepsilon_2), \varepsilon_1, \varepsilon_2, \varepsilon_1 + \varepsilon_2\}$ determined by the trace form t contains two root bases, $B = \{\varepsilon_1, \varepsilon_2 - \varepsilon_1\}$ and $B' = \{\varepsilon_2, \varepsilon_1 - \varepsilon_2\}$ for which $B_u = \{\varepsilon_1\} \neq B'_u = \{\varepsilon_2\}$. Nevertheless, the set of linear forms $\{q_\beta : \beta \in B_u\}$ depends only on P, as (3) shows. Also, if $Z = \operatorname{span}(P_s)$ then the set $\operatorname{can}(B_u) \subset X/Z$ depends only on P because it is just the basis dual to the basis $\{q_\beta : \beta \in B_u\}$ of $(X/Z)^*$.

Proof. By 11.1.1 we have $\alpha \in P$ if and only if $q_\beta(\alpha) \geqslant 0$ for all $\beta \in B_u$. Hence the q_β ($\beta \in B_u$) are dominant and then also fundamental because they are basic, so we have the inclusions from right to left in (1) – (3). Conversely, let $f \in D^\vee(P)$. As $\{q_\beta : \beta \in B\}$ is a vector space basis of X^*, we can write $f = \sum_{\beta\in B} c_\beta q_\beta$ with real coefficients c_β. Then $0 \leqslant f(\beta) = c_\beta$ for all $\beta \in B_u = B \cap P_u$. Also, $B_s = B \cap P_s$, so $\pm\beta \in P$ for all $\beta \in B_s$, which implies $c_\beta = 0$ for $\beta \in B_s$. Thus $f = \sum_{\beta\in B_u} c_\beta q_\beta$ is a positive linear combination of $\{q_\beta : \beta \in B_u\}$. This proves (1), and (2) follows immediately because $c_\beta = f(\beta) \in \mathbb{Z}$ when f is a coweight.

It remains to prove the inclusion from left to right in (3). Let $f \in \mathcal{F}^\vee(P)$. By (2), $f = \sum_{\beta\in B_u} n_\beta \cdot q_\beta$ with $n_\beta \in \mathbb{N}$. Since every $\alpha \in R$ is a linear combination of B with coefficients of the same sign, we have $f(\alpha) = 0$ if and only if α is a linear combination of $B_0(f) := \{\beta \in B : f(\beta) = 0\}$. Hence $R_0(f)$ has $B_0(f)$ as root basis, and thus f has rank 1 if and only if $B \setminus B_0(f) = \{\beta\}$ consists of one element. It follows that $f = n_\beta q_\beta$, and $n_\beta = 1$ because f is indivisible.

16.3. *Dominant and fundamental coweights of* $\dot{\mathrm{A}}_I$. We now determine explicitly the positive linear forms and the dominant and fundamental coweights of the parabolic subsets P of the classical root systems $R = \mathrm{T}_I$, $\mathrm{T} \in \mathfrak{T} = \{\dot{\mathrm{A}}, \ldots, \mathrm{BC}\}$. The case of a finite I is of course well-known. It is included here since our methods do not depend on the cardinality of I. Our results, together with the corresponding ones for weights, are summarized in 16.6. By Prop. 13.6, we may assume P to be a pure parabolic subset. Then by Th. 13.11, $P = R_{I_0,\succcurlyeq}$ is one of the parabolic subsets defined in 13.3, where $(I_0, \succcurlyeq)$ is a p-datum for (T, I) (cf. 13.9). Therefore, the cone $\mathbb{R}_+[P]$ spanned by P is given by Prop. 13.10(b); in particular, it is one of the cones studied in Appendix B.

Throughout, we use the notation of 12.1, so $X = \bigoplus_{i\in I}\mathbb{R}\varepsilon_i$ and $\dot X$ is the hyperplane of those elements $x = \sum x_i\varepsilon_i$ for which the trace $t(x) = q_I(x) = \sum x_i = 0$. For $f \in X^*$ we let $\dot f$ denote the restriction of f to $\dot X$.

Let $P = \dot{\mathrm{A}}_{I,\succcurlyeq} = \{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\}$ where $\succcurlyeq$ is a total preorder, as in 13.3.2. We use the notation $\mathfrak{E}$ of B.2 for the set of final segments of $(I, \succcurlyeq)$ and $\dot{\mathfrak{E}}$ for the set of proper final segments. Then by Prop. 13.10(b), Case (b$_1$), $\mathbb{R}_+[P] = \dot K$ is the cone of type $\dot{\mathrm{A}}$ determined by $\succcurlyeq$, so by B.7.2, $\dot f \in D^\vee(P) = \dot K^\circ$ if and only if the map $i \mapsto f(\varepsilon_i)$, $I \to \mathbb{R}$, is increasing. Hence $\dot f$ is a dominant coweight if and only if $\dot f(\varepsilon_i - \varepsilon_j) = f(\varepsilon_i) - f(\varepsilon_j) \in \mathbb{N}$, for all $i \succcurlyeq j$. By 8.12(a), the basic coweights of $\dot{\mathrm{A}}_I$ are the linear forms $\dot q_J$ where $\emptyset \neq J \subsetneqq I$. Now the map $i \mapsto f(\varepsilon_i) = \chi_J(i)$ is increasing if and only if $J = \Sigma$ is a final segment of $(I, \succcurlyeq)$ so we have:

$$\mathcal{F}^\vee(\dot{\mathrm{A}}_{I,\succcurlyeq}) = \{\dot q_\Sigma : \Sigma \in \dot{\mathfrak{E}}\}. \tag{1}$$

(Recall that final segments, see B.2, are not empty by definition). It is also immediately seen that $\dot q_\Sigma \neq \dot q_{\Sigma'}$ for different Σ, Σ' in $\dot{\mathfrak{E}}$, so (1) actually establishes a bijection between $\dot{\mathfrak{E}}$ and $\mathcal{F}^\vee(\dot{\mathrm{A}}_{I,\succcurlyeq})$. Comparing this with B.8(a), we see that the extremal rays of $D^\vee(P)$ are spanned by $\mathcal{F}^\vee(P)$.

Let us now consider the case where P is a positive system and hence $\succcurlyeq$ a total order, written $\geqslant$. We determine the intersection of $D^\vee(P)$ with the space $\dot X^\vee \subset \dot X^*$ which we may identify with the set of those $\dot f$ for which only finitely many $f(\varepsilon_i)$ are non-zero and which satisfy $\sum_{i\in I} f(\varepsilon_i) = 0$. Assuming $\dot f \neq 0$, let $\{i_{-m} < \cdots < i_{-1} < i_1 < \cdots < i_n\}$ be the set of those $i \in I$ with $f(\varepsilon_i) \neq 0$, where moreover $f(\varepsilon_{i_{-m}}) \leqslant \cdots \leqslant f(\varepsilon_{i_{-1}}) < 0 < f(\varepsilon_{i_1}) \leqslant \cdots \leqslant f(\varepsilon_{i_n})$. Since the map $i \mapsto f(\varepsilon_i)$ is increasing, we must have $i_{-m} = \min(I)$, $i_n = \max(I)$, and the order on I must be of type

$$I = \big(i_{-m} < \cdots < i_{-1} < I_0 < i_1 < \cdots < i_n\big),$$

where $I_0 = \{i \in I : f(\varepsilon_i) = 0\}$. Thus in general, we will have $D^\vee(P) \cap \dot X^\vee = \{0\}$, illustrating the fact that it is important to consider $D^\vee(P)$ in the full dual X^* of X and not just in $X^\vee = \mathrm{span}(R^\vee)$.

16.4. *Dominant and fundamental coweights of* B_I, C_I *and* BC_I. Let R be one of these root systems, and let $P = R_{I_0,\succcurlyeq}$ be a pure parabolic subset as in (4) – (6) of 13.3 where $(I_0, \succcurlyeq)$ is a p-datum, so I_0 is either empty or the minimum of the totally ordered set $I/{\sim}$. By Case (b$_3$) of Prop. 13.10(b), $\mathbb{R}_+[P] = K$ is the cone of type B determined by $(I_0, \succcurlyeq)$. By B.3.3, $f \in D^\vee(P) = K^\circ$ if and only if the map $i \mapsto f(\varepsilon_i)$ is non-negative, increasing, and vanishes on I_0. In case $R = \mathrm{B}_I$ or BC_I, the condition for f to be a dominant coweight is that in addition all $f(\varepsilon_i) \in \mathbb{N}$, while for $R = \mathrm{C}_I$, the $f(\varepsilon_i)$ must either be all integers or all half-integers. In particular, $I_0 \neq \emptyset$ implies that all $f(\varepsilon_i) \in \mathbb{N}$. We now determine the fundamental coweights.

(a) Let $R = \mathrm{B}_I$ or BC_I. By 8.12, the basic coweights of R are the linear forms q_J^σ where $\emptyset \neq J \subset I$ and $\sigma \in \mathbf{2}^I$. By definition, $q_J^\sigma(\varepsilon_i) = \sigma(i)\chi_J(i)$. Since $q_J^\sigma = q_J^\tau$ as long as $\sigma(i) = \tau(i)$ for all $i \in J$, it is no restriction to assume $\sigma(i) = 1$ for $i \in I \setminus J$. It is then easy to see that the conditions for q_J^σ to belong to $D^\vee(P)$ are $I_0 \cap J = \emptyset$, $\sigma = \mathrm{Id}$, and $J = \Sigma$ a final segment. Thus we have

$$\mathcal{F}^\vee(\mathrm{B}_{I,I_0,\succcurlyeq}) = \mathcal{F}^\vee(\mathrm{BC}_{I,I_0,\succcurlyeq}) = \{q_\Sigma : \Sigma \in \mathfrak{E},\ I_0 \cap \Sigma = \emptyset\}, \tag{1}$$

and the map $\Sigma \mapsto q_\Sigma$ is a bijection between the set of final segments not meeting I_0 and the fundamental coweights.

(b) Let $R = \mathrm{C}_I$. Then by 8.12, the basic coweights are $q_I^\sigma/2$ and all q_J^σ, where $\sigma \in \mathbf{2}^I$ and $\emptyset \neq J \subsetneqq I$. If $I_0 \neq \emptyset$, no coweight of type $q_I^\sigma/2$ can be in $D^\vee(P)$. A similar argument as before then shows

$$\mathcal{F}^\vee(\mathrm{C}_{I,I_0,\succcurlyeq}) = \left\{ \begin{array}{ll} \{q_+\} \cup \{q_\Sigma : \Sigma \in \dot{\mathfrak{E}}\} & \text{if } I_0 = \emptyset \\ \{q_\Sigma : \Sigma \in \dot{\mathfrak{E}},\ I_0 \cap \Sigma = \emptyset\} & \text{if } I_0 \neq \emptyset \end{array} \right\}, \tag{2}$$

where $q_+ = (1/2)q_I$. From (1) and (2) and B.6(a), we see that, as in 16.3, the extremal rays of $D^\vee(P)$ are spanned by $\mathcal{F}^\vee(P)$. Again, these formulas establish bijections between $\mathcal{F}^\vee(P)$ and suitably defined sets of final segments.

16.5. *Dominant and fundamental coweights of* D_I. Let $R = \mathrm{D}_I$, and let $P = \mathrm{D}_{I,I_0,\succcurlyeq}$ be a pure parabolic subset as in 13.3.3. Thus I_0 has cardinality $\neq 1$, and if $(I, \succcurlyeq)$ has a minimal element 0 then $I_0 = \emptyset$, by 13.9.1. Accordingly, there are two subcases:

(a) *$(I, \succcurlyeq)$ has no minimal element.* Then by Prop. 13.10(b), Case (b$_3$), $\mathbb{R}_+[P] = K$ is the cone of type B determined by $(I_0, \succcurlyeq)$. Thus by B.3.3, $D^\vee(P)$ consists again of all linear forms f for which the map $i \mapsto f(\varepsilon_i)$ is non-negative, increasing, and vanishes on I_0. The condition for f to be a dominant coweight is that, in addition, the $f(\varepsilon_i)$ are either all integers or all half-integers.

From 8.12 it follows that the basic coweights are the q_J^σ for some non-empty subset J with $|I \setminus J| \geqslant 2$, and $q_I^\sigma/2$, where $\sigma \in \mathbf{2}^I$. As before, this implies that the fundamental coweights are the q_Σ where Σ is a final segment not meeting I_0 and with $|I \setminus \Sigma| \geqslant 2$, as well as $q_I/2$, provided that $I_0 = \emptyset$. However, the condition $|I \setminus \Sigma| \geqslant 2$ is now automatic for a final segment $\Sigma \neq I$, because I has no minimal element: If $I \setminus \Sigma = \{i_0\}$ a singleton then necessarily i_0 must be the minimal element of I which is not present. This shows:

If $(I, \succcurlyeq)$ has no minimal element then

$$\mathcal{F}^\vee(\mathrm{D}_{I,I_0\succcurlyeq}) = \mathcal{F}^\vee(\mathrm{C}_{I,I_0,\succcurlyeq}) \textit{ as in } 16.4.2. \tag{1}$$

(b) *$(I, \succcurlyeq)$ has a minimal element* 0. Then $I_0 = \emptyset$, and by Prop. 13.10(b), Case (b$_2$), $\mathbb{R}_+[P] = K_0$ is the cone of type D determined by $(I, \succcurlyeq, 0)$. By B.9, a linear form $f \in X^*$ belongs to $D^\vee(P) = K_0^\circ$ if and only if the map $i \mapsto f(\varepsilon_i)$ is increasing, and $f(\varepsilon_i) \geqslant -f(\varepsilon_0)$, for all $i \neq 0$. The dominant coweights are then characterized by the additional condition that the $f(\varepsilon_i)$ are all integers or all half-integers. We claim that the fundamental coweights of P are precisely the q_Σ with $\Sigma \in \ddot{\mathfrak{E}}$, i.e., Σ is a final segment with $|I \setminus \Sigma| \geqslant 2$, and the "spin coweights" $q_\pm$, as in B.9. That these linear forms are indeed fundamental is easily verified. Conversely, let f be fundamental. By 8.12, either $f = q_J^\sigma$ where $J \neq \emptyset$ and $|I \setminus J| \geqslant 2$, or $f = q_I^\sigma/2$, for some $\sigma \in \mathbf{2}^I$. In the first case, the conditions describing $D^\vee(P)$ show we must have $\sigma(j) = j$ for $j \in J$, and thus $q_J^\sigma = q_J$ where in addition $J = \Sigma$ must be a final segment. In the second case, they imply that either $\sigma = \mathrm{Id}$ or $\sigma = \sigma_0$, so $f = q_+$ or $f = q_-$. Thus we have shown:

If $(I, \succcurlyeq)$ *has a minimal element then*

$$\mathcal{F}^\vee(\mathrm{D}_{I,\succcurlyeq}) = \{q_\pm\} \,\dot\cup\, \{q_\Sigma : \Sigma \in \ddot{\mathfrak{E}}\}. \tag{2}$$

A comparison with B.12(a) shows that again the extremal rays of $D^\vee(P)$ are spanned by $\mathcal{F}^\vee(P)$.

16.6. *Summary.* We summarize our results on the fundamental coweights $\mathcal{F}^\vee(P)$ established in 16.3 – 16.5 in the following table. For convenience, we also list the fundamental weights $\mathcal{F}(P)$, obtained from $\mathcal{F}^\vee(P)$ by passing to $R^\vee$ and $P^\vee$. Throughout, $P = \mathrm{T}_{I,I_0,\succcurlyeq}$ is a pure parabolic subset in $R = \mathrm{T}_I$ and Σ is a final segment of $(I, \succcurlyeq)$. Also, the notation $0 \in I$ or $0 \notin I$ refers to the case where $(I, \succcurlyeq)$ has or does not have a minimal element 0. For the definition of the weights $\dot p_\Sigma$, p_Σ and coweights $\dot q_\Sigma$ and q_Σ see 8.9. The coweights $q_\pm$ are defined in B.9.2, and the $p_\pm$ are defined analogously, with ε_i replaced by e_i, cf. 8.1.6.

P (parabolic)	$\mathcal{F}(P)$	$\mathcal{F}^\vee(P)$
$\dot{\mathrm{A}}_{I,\succcurlyeq}$	$\dot p_\Sigma,\ \Sigma \neq I$	$\dot q_\Sigma,\ \Sigma \neq I$
$\mathrm{B}_{I,I_0,\succcurlyeq}$	$p_\Sigma,\ \Sigma \neq I;\ p_+ \quad$ if $I_0 = \emptyset$ $p_\Sigma,\ \Sigma \cap I_0 = \emptyset \quad$ if $I_0 \neq \emptyset$	$q_\Sigma,\ \Sigma \cap I_0 = \emptyset$
$\mathrm{C}_{I,I_0,\succcurlyeq}$	$p_\Sigma,\ \Sigma \cap I_0 = \emptyset$	$q_\Sigma,\ \Sigma \neq I;\ q_+ \quad$ if $I_0 = \emptyset$ $q_\Sigma,\ \Sigma \cap I_0 = \emptyset \quad$ if $I_0 \neq \emptyset$
$\mathrm{D}_{I,I_0,\succcurlyeq},\ 0 \notin I$	$p_\Sigma,\ \Sigma \neq I;\ p_+ \quad$ if $I_0 = \emptyset$ $p_\Sigma,\ \Sigma \cap I_0 = \emptyset \quad$ if $I_0 \neq \emptyset$	$q_\Sigma,\ \Sigma \neq I;\ q_+ \quad$ if $I_0 = \emptyset$ $q_\Sigma,\ \Sigma \cap I_0 = \emptyset \quad$ if $I_0 \neq \emptyset$
$\mathrm{D}_{I,\succcurlyeq},\ 0 \in I$	$p_\Sigma,\ \ \lvert I \setminus \Sigma\rvert \geqslant 2;\ \ p_\pm$	$q_\Sigma,\ \ \lvert I \setminus \Sigma\rvert \geqslant 2;\ \ q_\pm$
$\mathrm{BC}_{I,I_0,\succcurlyeq}$	$p_\Sigma,\ \Sigma \cap I_0 = \emptyset$	$q_\Sigma,\ \Sigma \cap I_0 = \emptyset$

In case P is a positive system the table above specializes as follows.

P (positive system)	$\mathcal{F}(P)$	$\mathcal{F}^\vee(P)$
$\dot{\mathrm{A}}_{I,\geqslant}$	$\dot p_\Sigma,\ \Sigma \neq I$	$\dot q_\Sigma,\ \Sigma \neq I$
$\mathrm{B}_{I,\geqslant}$	$p_\Sigma,\ \Sigma \neq I;\ p_+$	q_Σ
$\mathrm{C}_{I,\geqslant}$	p_Σ	$q_\Sigma,\ \Sigma \neq I;\ q_+$
$\mathrm{D}_{I,\geqslant},\ 0 \notin I$	$p_\Sigma,\ \Sigma \neq I;\ \ p_+$	$q_\Sigma,\ \Sigma \neq I;\ \ q_+$
$\mathrm{D}_{I,\geqslant},\ 0 \in I$	$p_\Sigma,\ \ \lvert I \setminus \Sigma\rvert \geqslant 2;\ \ p_\pm$	$q_\Sigma,\ \ \lvert I \setminus \Sigma\rvert \geqslant 2;\ \ q_\pm$
$\mathrm{BC}_{I,\geqslant}$	p_Σ	q_Σ

As a first application, we show:

16.7. Proposition. *Let P be a positive system of a root system (R, X). Then every coroot is an integer linear combination of $\mathcal{F}^\vee(P)$.*

Proof. After decomposing R into irreducible components we may assume R irreducible. If R is finite then P determines a root basis B, and $\mathcal{F}^\vee(P) = \{q_\beta : \beta \in B\}$ (by 16.2.3) is a $\mathbb{Z}$-basis of the coweight lattice $\mathcal{P}^\vee(R)$. Now $R^\vee \subset \mathcal{Q}^\vee(R) \subset \mathcal{P}^\vee(R)$ by 7.3, so we are done.

If $R = \mathrm{T}_I$ is infinite, we may assume $P = R_\geqslant$ is a pure positive system. Note that, for all $i \in I$,

$$e_i = q_{\{i\}} = q_{[i,\rightarrow[} - q_{]i,\rightarrow[} \tag{1}$$

where e_i is defined in 8.1.6, $[i,\rightarrow[$ is the principal final segment determined by i, and $]i,\rightarrow[\; = \{j \in I : j > i\}$ is a final segment or empty. Moreover, by 8.1, the coroots of T_I are given by $(\varepsilon_i - \varepsilon_j)^\vee = \dot{e}_i - \dot{e}_j$ in case $\dot{\mathrm{A}}_I$, and $(\varepsilon_i \pm \varepsilon_j)^\vee = e_i \pm e_j$, $\varepsilon_i^\vee = 2e_i$, $(2\varepsilon_i)^\vee = e_i$ in the other cases. Now the assertion follows easily from (1) and the structure of $\mathcal{F}^\vee(P)$ in the table above, using the fact that $q_I = 2q_+$ and $q_{\{0\}} = q_+ - q_-$, in case $P = \mathrm{D}_{I,\geqslant}$ and $0 \in I$.

For the case of classical root systems the following lemma is obvious from the discussion above. It is interesting that one can give a short classification-free proof for root systems in general.

16.8. Lemma. *Let $P \subset R$ be parabolic and let $f \in \mathcal{F}^\vee(P)$ be a fundamental coweight. Then $\mathbb{R}_+ f$ is an extremal ray of $D^\vee(P)$.*

Proof. Suppose $f = f_1 + f_2$ with $f_i \in D^\vee(P)$. Then for all $\alpha \in P \cap R_0(f)$, $0 = f(\alpha) = f_1(\alpha) + f_2(\alpha)$ and $f_i(\alpha) \geqslant 0$ implies $f_i(\alpha) = 0$. Hence $P \cap R_0(f) \subset R_0(f_i)$, and since $R = P \cup (-P)$, we see that $R_0(f) \subset R_0(f_i)$. A fundamental coweight is basic, in particular, it is of rank 1. Hence $R_0(f)$ spans a hyperplane, and therefore f_i is a multiple of f, as desired.

16.9. Theorem. *Let P be a parabolic subset of a root system (R, X), let $C = \mathbb{R}_+[P]$ be the convex cone generated by P, with polar $C^\circ = D^\vee(P)$ as in* 15.1.1*, and let $\mathcal{F}^\vee(P)$ be the set of fundamental coweights of P.*

(a) *For an element $x \in X$, the following conditions are equivalent:*

(i) *$x \in C$,*
(ii) *$x \in C^{\circ\circ}$, i.e., $f(x) \geqslant 0$ for all $f \in D^\vee(P)$,*
(iii) *$f(x) \geqslant 0$ for all $f \in \mathcal{F}^\vee(P)$.*

(b) *The extremal rays of $D^\vee(P)$ are precisely the rays spanned by the fundamental coweights of P, i.e., the map $\mathcal{F}^\vee(P) \to \operatorname{extr}(D^\vee(P))$, $q \mapsto \mathbb{R}_+ q$, is bijective.*

Proof. Since the set of extremal rays of a direct product of cones is the union of the sets of extremal rays of its factors, it follows from 16.1.1 and 15.1.3 that it suffices to prove the theorem in case R is irreducible.

Let first R be finite. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) of (a) are obvious. For (iii) $\Longrightarrow$ (i), we describe P in terms of a root basis B and a subset B_u as in 11.1. Write $x = \sum_{\beta \in B} c_\beta \beta$ with $c_\beta \in \mathbb{R}$. By 16.2.3, $q_\beta \in \mathcal{F}^\vee(P)$ for $\beta \in B_u$, so (iii) says $c_\beta = q_\beta(x) \geqslant 0$ for $\beta \in B_u$. As $B \setminus B_u = B_s \subset P_s$ by 11.1 and therefore $\pm\beta \in P$ for all $\beta \in B_s$, it follows that x is a positive linear combination of P, so $x \in K$. This proves (a). By 16.2.1, $D^\vee(P)$ is the simplicial cone spanned by the linear forms q_β ($\beta \in B_u$) which are, by 16.2.3, precisely the fundamental coweights. Hence, by 16.8 and B.1.1, the extremal rays of $D^\vee(P)$ are spanned precisely by $\mathcal{F}^\vee(P)$.

Now let R be infinite and irreducible, so $R = \mathrm{T}_I$ for an infinite set I and $\mathrm{T} \in \mathfrak{T} = \{\dot{\mathrm{A}}, \ldots, \mathrm{BC}\}$. By the results of §13, we may assume P pure.

We first show (b). By Lemma 16.8, a fundamental coweight spans an extremal ray of $D^\vee(P)$. The converse follows from the case-by-case discussion in 16.3 – 16.5. This completes the proof of (b). Now (a) follows from the corresponding statements of B.5, B.7, and B.11.

16.10. Corollary. *A parabolic subset P of a root system R is determined by its fundamental coweights:*

$$P = \bigcap_{f \in \mathcal{F}^\vee(P)} R_+(f). \tag{1}$$

This follows immediately from Th. 16.9(a) and the fact that $P = K \cap R$ by 10.17.3.

16.11. Corollary. *With the notations of Th.* 16.9*, the convex subcone of $D^\vee(P)$ spanned by $\mathcal{F}^\vee(P)$ is weak-$*$-dense in $D^\vee(P)$.*

This is an immediate consequence of 16.9(b) and Cor. B.14.

16.12. Corollary. *With the notations of* 16.9*, the cone $\mathbb{R}_+[P]$ spanned by a parabolic subset P of a root system R is closed in the norm topology of X.*

Proof. As remarked in 16.1.2, fundamental coweights are bounded and hence, by 15.5, norm-continuous. Now the corollary follows from condition (iii) of Th. 16.9(a).

16.13. Proposition. (a) *For a parabolic subset P of a root system (R, X), the following conditions are equivalent:*

(i) *P is maximal among the proper parabolic subsets of R,*
(ii) *$P = R_+(f)$ where $f \in \mathcal{B}^\vee(R)$ is a basic coweight,*
(iii) *$P = R_+(f)$ where f is a linear form of rank* 1,
(iv) *P_s has corank* 1,
(v) *P_s is maximal among the proper full subsets of R.*

(b) *Every parabolic subset of a root system R is the intersection of the proper maximal parabolic subsets containing it.*

Proof. (a) (i) $\Longrightarrow$ (ii): Since $P \neq R$, we have $\mathcal{F}^\vee(P) \neq \emptyset$ by 16.10.1. Let $f \in \mathcal{F}^\vee(P)$. Then $R_+(f)$ is a proper parabolic subset containing P, so $P = R_+(f)$ by maximality of P.

The implication (ii) $\Longrightarrow$ (iii) is obvious from the definition of basic coweights, and (iii) $\Longrightarrow$ (iv) follows from the fact that the symmetric part of $R_+(f)$ is $R_0(f)$, see 10.8.2. Condition (iv) says that P_s spans a hyperplane. Since P_s is a full subset of R, this easily implies (v).

It remains to prove (v) $\Longrightarrow$ (i). Let P' be a proper parabolic subset with $P \subset P'$. Then $P_s \subset P'_s$, so by maximality of P_s, either $P_s = P'_s$ or $P'_s = R$. The second possibility is excluded because P' is proper. Now assume that there exists

$\alpha \in P' \setminus P$. Then $-\alpha \in P \subset P'$ so $\alpha \in P'_s = P_s \subset P$, contradiction. Thus $P = P'$, proving P maximal.

(b) This follows immediately from (a) and 16.10.1.

16.14. Corollary. *Let (R, X) be a root system. For a facet F of R (see* 15.7*), the following conditions are equivalent:*

(i) *F is minimal among the facets different from $\{0\}$ with respect to the partial order $\preccurlyeq$ of* 15.7.7*,*
(ii) *$F = \mathbb{R}_{++}f$ where f is a basic coweight of R,*
(iii) *F is an open half-line.*

Proof. (i) $\Longrightarrow$ (ii): Let $P = R_+(F)$ be the scalar parabolic subset determined by F. Then $P \neq R$ because $F \neq \{0\}$. By Prop. 16.13, there exists a parabolic subset $P' = R_+(f') \supset P$, where $f' \in \mathcal{B}^\vee(R)$ is a basic coweight. Let F' be the facet determined by f'. Then $P' \supset P$ implies $F' \preccurlyeq F$ by 15.7.7, and $F' \neq \{0\}$ because $f' \neq 0$. Hence $F' = F$ by minimality of F. It remains to show $F = \mathbb{R}_{++}f'$. Thus let $f \in F$. Then $R_+(f) = R_+(f')$ and hence $R_0(f) = R_0(f')$. Since $R_0(f')$ spans a hyperplane, it follows that $f = cf'$ for some $c \neq 0$, and $c > 0$ follows from $R_+(f) = R_+(f')$.

The implication (ii) $\Longrightarrow$ (iii) is trivial. We prove (iii) $\Longrightarrow$ (i). If F is an open half-line, the weak-$*$-closure of F is $\{0\} \,\dot\cup\, F$. Hence Prop. 15.9(b) shows that the only facet $F' \preccurlyeq F$ and different from F is $\{0\}$, so F is minimal.

By Cor. 16.11, a dominant coweight $f \in \mathcal{D}^\vee(P)$ (P parabolic) is the limit, in the weak-$*$-topology, of a net (g_λ) where each g_λ is a finite linear combination with non-negative coefficients of fundamental coweights. Our next aim is to derive a more precise series representation of f, similarly to B.15. We begin with the following result on the restriction of fundamental coweights to suitable finite subsystems.

16.15. Proposition. *Let P be a parabolic subset of a root system (R, X), and let $F \subset R$ and $\mathcal{E} \subset \mathcal{F}^\vee(P)$ be finite subsets. Then there exists a finite full subsystem (R', X') of (R, X) such that, letting $P' := P \cap R'$,*

(i) *$F \subset R'$,*
(ii) *for every $q \in \mathcal{E}$, the restriction* $\operatorname{res}(q) := q\big|X'$ *belongs to $\mathcal{F}^\vee(P')$, and*
(iii) $\operatorname{res}\colon \mathcal{E} \to \mathcal{F}^\vee(P')$ *is injective.*

Proof. Using 16.1.1 it is easily seen that we may assume R irreducible. Since the case of a finite R is trivial, it remains to consider $R = \mathrm{T}_I$ for $\mathrm{T} \in \mathfrak{T} = \{\dot{\mathrm{A}}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{BC}\}$ and I infinite. Then $F \subset \mathrm{T}_J$ for a suitable finite $J \subset I$. If $\mathcal{E} = \emptyset$, the assertions (ii) and (iii) are trivially satisfied while (i) follows from local finiteness of R. We thus always assume $\mathcal{E}$ non-empty.

Our claim is invariant under automorphisms, so we can assume that P is pure and hence of the form $P = \mathrm{T}_{I, I_0, \succcurlyeq}$ for a p-datum $(I_0, \succcurlyeq) \in \mathbb{P}_0(\mathrm{T}, I)$ as in 13.9. We will find a suitable finite $I' \subset I$ such that, with the notation of 12.1, $(R', X') := (\mathrm{T}_{I'}, X_{I'})$ satisfies (i) – (iii).

Let $I' \subset I$ be a non-empty subset, put $I'_0 := I_0 \cap I'$ and let $\succcurlyeq'$ denote the restriction of $\succcurlyeq$ to I'. We also assume that $|I'| \geqslant 2$ and $|I'_0| \neq 1$ in case $\mathrm{T} = \mathrm{D}$. Then it follows immediately from the definition in 13.9 that $(I'_0, \succcurlyeq') \in \mathbb{P}_0(\mathrm{T}, I')$ is a p-datum for (T, I'). Also, from the formulas in 13.3 defining $\mathrm{T}_{I, I_0, \succcurlyeq}$, we see that

$$P' = P \cap R' = \mathrm{T}_{I,I_0,\succcurlyeq} \cap \mathrm{T}_{I'} = \mathrm{T}_{I,I_0',\succcurlyeq'}$$

is the pure parabolic subset defined by the p-datum $(I_0', \succcurlyeq')$. Hence the results of 16.6 describing the structure of the fundamental coweights apply to P as well as to P'. This will make it easy to verify that condition (ii) holds. Once this is established, condition (iii) just means that $|\mathcal{E}| = |\operatorname{res}(\mathcal{E})|$.

By 16.3 – 16.5, a fundamental coweight q of P is one of the following: If $\mathrm{T} = \dot{\mathrm{A}}$ then $q = \dot{q}_\Sigma$ for a proper final segment Σ of I, while in the other cases, q is of the form q_Σ where Σ is a final segment, possibly satisfying certain restrictions, or $q = q_\pm$ is one of the spin coweights where $q_+ = (1/2)q_I$, and q_- is only defined in case I has a minimal element 0.

With $(I', I_0', \succcurlyeq')$ as above, let Σ' be a final segment of I', and define linear forms $q'_{\Sigma'}$, $q'_\pm$ on X' and $\dot{q}'_{\Sigma'}$ on $\dot{X}' = \dot{X} \cap X'$ in analogy to q_Σ, $q_\pm$ and $\dot{q}_\Sigma$. Then the fundamental coweights of P behave as follows upon restriction to X' resp. $\dot{X}'$:

$$\operatorname{res}(q_\Sigma) = q'_{\Sigma\cap I'}, \qquad \operatorname{res}(\dot{q}_\Sigma) = \dot{q}'_{\Sigma\cap I'}, \tag{1}$$

$$\operatorname{res}(q_+) = q'_+, \qquad \operatorname{res}(q_-) = q'_- \quad \text{in case } 0 \in I'. \tag{2}$$

Note here that $\Sigma \cap I'$ is either empty or a final segment of I' with respect to $\succcurlyeq'$. We now discuss the possibilities for T and show that in each case, conditions (i) — (iii) can be met by a judicious choice of I'.

(a) $\mathrm{T} = \dot{\mathrm{A}}$: By 16.3.1 we have $\mathcal{E} = \{\dot{q}_{\Sigma_1}, \ldots, \dot{q}_{\Sigma_n}\}$ where the Σ_ν are proper final segments and $n = |\mathcal{E}| \geqslant 1$. Since the set of final segments of I is totally ordered by inclusion, we can assume the Σ_ν strictly descending: $I \supsetneqq \Sigma_1 \supsetneqq \cdots \supsetneqq \Sigma_n$. Let $\Sigma_0 := I$, and choose $i_\nu \in I$ such that

$$i_\nu \in \Sigma_\nu \setminus \Sigma_{\nu+1} \quad \text{for } 0 \leqslant \nu < n, \quad i_n \in \Sigma_n. \tag{3}$$

Now define $I' := J \cup \{i_0, \ldots, i_n\}$. Then (3) ensures that the $\Sigma'_\nu := \Sigma_\nu \cap I'$ $(\nu = 1, \ldots, n)$ are proper final segments of I' in strictly descending order: $I' \supsetneqq \Sigma'_1 \supsetneqq \cdots \supsetneqq \Sigma'_n$; in particular, they are pairwise different. Hence (1) and 16.3.1 applied to $R' = \dot{\mathrm{A}}_{I'}$ show that $\operatorname{res}(\mathcal{E}) = \{\dot{q}'_{\Sigma'_1}, \ldots \dot{q}'_{\Sigma'_n}\} \subset \mathcal{F}^\vee(P')$ has cardinality n.

(b) $\mathrm{T} = \mathrm{B}$ or $\mathrm{T} = \mathrm{BC}$: By 16.4.1, $\mathcal{E} = \{q_{\Sigma_1}, \ldots, q_{\Sigma_n}\}$ where $n = |\mathcal{E}|$ and the Σ_ν are final segments with $\Sigma_\nu \cap I_0 = \emptyset$. As before, we may assume them in strictly descending order. Choose $i_\nu \in I$ satisfying

$$i_\nu \in \Sigma_\nu \setminus \Sigma_{\nu+1} \quad \text{for } 1 \leqslant \nu < n, \quad i_n \in \Sigma_n, \tag{4}$$

and define $I' := J \cup \{i_1, \ldots, i_n\}$. Then the $\Sigma'_\nu = \Sigma_\nu \cap I'$ are final segments of I' not meeting I_0', and they are again in strictly descending order. By (1) and 16.4.1 applied to R', we have $\operatorname{res}(\mathcal{E}) = \{q'_{\Sigma'_1}, \ldots q'_{\Sigma'_n}\} \subset \mathcal{F}^\vee(P')$ and $|\operatorname{res}(\mathcal{E})| = n = |\mathcal{E}|$, as desired.

(c) $\mathrm{T} = \mathrm{C}$: First let $I_0 \neq \emptyset$. Then by 16.4.2, $\mathcal{E}$ has the form discussed in (b) above, and the same method proves our assertion.

Now let $I_0 = \emptyset$. Then the spin coweight $q_+ = (1/2)q_I$ is in $\mathcal{F}^\vee(P)$. We make the trivial but useful remark that we may always enlarge $\mathcal{E}$ (as long as it stays finite).

Hence it is no restriction to assume $q_+ \in \mathcal{E}$, and then $\mathcal{E} = \{q_+\} \,\dot\cup\, \{q_{\Sigma_1}, \dots, q_{\Sigma_n}\}$ where the $\Sigma_\nu \in \dot{\mathfrak{E}}$ are proper final segments in strictly descending order, and $|\mathcal{E}| = n+1$, so $n = 0$ is possible. Let $\Sigma_0 := I$ and choose $i_0, \dots, i_n$ as in (3). Let $I' := J \cup \{i_0, \dots, i_n\}$, and define again $\Sigma'_\nu := \Sigma_\nu \cap I'$. Then (1) and (2) together with 16.4.2 show $\mathrm{res}(\mathcal{E}) = \{q'_+\} \,\dot\cup\, \{q'_{\Sigma'_1}, \dots q'_{\Sigma'_n}\} \subset \mathcal{F}^\vee(P')$ and $|\,\mathrm{res}(\mathcal{E})| = n+1 = |\mathcal{E}|$.

(d) $\mathrm{T} = \mathrm{D}$, where $(I, \succcurlyeq)$ has no minimal element. First, assume again that $I_0 \neq \emptyset$. By 16.5.1 we have $\mathcal{E}$ as in the first part of (c), and pick elements $i_1, \dots, i_n$ as in (4). Also pick two elements $i_0, i'_0 \in I_0$. This is possible by condition (iii) of 13.9. Put $I' := J \cup \{i_0, i'_0, i_1, \dots, i_n\}$. Then $|I'_0| \geqslant 2$, so $(I'_0, \succcurlyeq')$ is a p-datum for (D, I'). As before, it is easily checked that $\mathrm{res}(\mathcal{E}) \subset \mathcal{F}^\vee(P')$ and $|\,\mathrm{res}(\mathcal{E})| = |\mathcal{E}|$.

Next, let $I_0 = \emptyset$. Then we may assume $\mathcal{E}$ as in the second part of (c) above and pick $i_0, \dots, i_n$ in the same way. Because I has no minimal element, there exists a further element $i'_0 \preccurlyeq i_0$, $i'_0 \neq i_0$. Put $I' := J \cup \{i'_0, i_0, i_1, \dots, i_n\}$. Then $|I'| \geqslant 2$, so $(I'_0 = \emptyset, \succcurlyeq')$ is a p-datum for (D, I'), and again one shows that $\mathrm{res}(\mathcal{E}) \subset \mathcal{F}^\vee(P')$ has the same cardinality as $\mathcal{E}$.

(e) $\mathrm{T} = \mathrm{D}$ and $0 \in I$: Recall from 13.9.1 that $I_0 = \emptyset$ in this case, so that any $I' \subset I$ with at least two elements gives rise to a p-datum in $\mathbb{P}_0(\mathrm{D}, I')$. By 16.5.2 we have $q_\pm \in \mathcal{F}^\vee(P)$, so by the remark made earlier, there is no harm in assuming $q_\pm \in \mathcal{E}$. Then $\mathcal{E} = \{q_+, q_-\} \cup \{q_{\Sigma_1}, \dots, q_{\Sigma_n}\}$ where $|\mathcal{E}| = n+2 \geqslant 2$, and the Σ_ν form a strictly descending sequence of final segments with $|I \setminus \Sigma_\nu| \geqslant 2$. Let $\Sigma_0 := I \setminus \{0\}$, choose i_ν as in (3) and define $I' := J \cup \{0, i_0, i_1, \dots, i_n\}$. Then $0 \in I'$ and $|I'| \geqslant 2$, the $\Sigma'_\nu = \Sigma_\nu \cap I'$ are final segments in strictly descending order for $(I', \succcurlyeq')$, and they satisfy $|I' \setminus \Sigma'_\nu| \geqslant 2$ for $\nu = 1, \dots, n$. Now (1) and (2) (and of course 16.5.2 applied to P') show that $\mathrm{res}(\mathcal{E}) = \{q'_+, q'_-\} \,\dot\cup\, \{q'_{\Sigma'_1}, \dots q'_{\Sigma'_n}\} \subset \mathcal{F}^\vee(P')$ and $|\,\mathrm{res}(\mathcal{E})| = n+2 = |\mathcal{E}|$. This completes the proof.

16.16. Corollary. *Let P be a parabolic subset of a root system (R, X).*

(a) *The set $\mathcal{F}^\vee(P)$ of fundamental coweights is linearly independent.*

(b) *Let $\mathcal{E} \subset \mathcal{F}^\vee(P)$ be a finite subset, and fix an element $q' \in \mathcal{E}$. Then there exists $\beta \in P$ with*

$$\langle \beta, q \rangle = \left\{ \begin{matrix} 1 & \text{for } q = q' \\ 0 & \text{for all } q \in \mathcal{E},\ q \neq q' \end{matrix} \right\}.$$

If $\alpha \in P$ is a root with $\langle \alpha, q' \rangle > 0$ then β can be chosen in such a way that in addition $\beta \preccurlyeq_P \alpha$ with respect to the partial preorder defined by P (cf. 10.7 and 11.2), i.e., $\alpha - \beta \in \mathbb{N}[P]$.

Proof. It is easily seen that (b) implies (a). To prove (b), we apply Prop. 16.15 with $F = \{\alpha\}$. This reduces us to the case of a finite R. Then, with the notations of 16.2, $q' = q_\beta$ for a unique $\beta \in B_u$. Writing $\alpha = \sum_{\gamma \in B} n_\gamma \gamma$, we have $\langle \alpha, q' \rangle = n_\beta \geqslant 1$, which implies that all $n_\gamma \in \mathbb{N}$ since B is a root basis. Hence $\alpha - \beta = (n_\beta - 1)\beta + \sum_{\gamma \neq \beta} n_\gamma \gamma \in \mathbb{N}[P]$, so β has the required properties.

16.17. Theorem. *Let (R, X) be a root system and let $P \subset R$ be a parabolic subset, with set of fundamental coweights $\mathcal{F}^\vee(P)$. Suppose $f \in X^*$ has a representation as a weak-$*$-convergent series*

$$f = \sum_{q \in \mathcal{F}^\vee(P)} c_q \cdot q \tag{1}$$

with real coefficients c_q. Then the c_q belong to the closure of $f(P)$ in $\mathbb{R}$. They are uniquely determined by f, and satisfy

$$f \in D^\vee(P) \quad \iff \quad \text{all } c_q \in \mathbb{R}_+, \tag{2}$$
$$f \in \mathcal{P}^\vee(R) \quad \iff \quad \text{all } c_q \in \mathbb{Z}, \tag{3}$$
$$f \in \mathcal{D}^\vee(P) \quad \iff \quad \text{all } c_q \in \mathbb{N}. \tag{4}$$

Remark. Convergence of (1) means of course convergence of the net $f_{\mathcal{E}} := \sum_{q \in \mathcal{E}} c_q \cdot q$ in the weak-$*$-topology of X^*, where $\mathcal{E}$ runs over the directed set of finite subsets of $\mathcal{F}^\vee(P)$. Since convergence in the weak-$*$-topology is pointwise convergence, (1) says that for every $x \in X$ the family $(c_q \cdot \langle x, q\rangle)_{q \in \mathcal{F}^\vee(P)}$ of real numbers is summable with sum $f(x)$.

Proof. Let q' be a fundamental coweight. We must show that, for every positive ε, there exists $\beta \in P$ such that $|f(\beta) - c_{q'}| < \varepsilon$. Choose any root $\alpha \in P$ with $\langle \alpha, q'\rangle > 0$. Evaluating (1) on α yields,

$$f(\alpha) = \sum_{q \in \mathcal{F}^\vee(P)} c_q \cdot \langle \alpha, q\rangle.$$

We note that $\langle \alpha, q\rangle \in \mathbb{N}$ by (i) of 16.1. It is well known that a summable family of real numbers is absolutely summable [**9**, IV, §7.2, Th. 3]. Hence there exists a finite subset $\mathcal{E} \subset \mathcal{F}^\vee(P)$ such that, with $\mathcal{C} := \mathcal{F}^\vee(P) \setminus \mathcal{E}$,

$$\sum_{q \in \mathcal{C}} |c_q| \cdot \langle \alpha, q\rangle < \varepsilon. \tag{5}$$

Since we may always enlarge $\mathcal{E}$ (and correspondingly diminish $\mathcal{C}$) without disturbing the estimate (5), it is no restriction to assume $q' \in \mathcal{E}$. Then $\mathcal{E}$, q', and α satisfy the hypotheses of Corollary 16.16(b), so we can find $\beta \in P$ such that $\beta \preccurlyeq_P \alpha$ and $\langle \beta, q\rangle = \delta_{qq'}$ for all $q \in \mathcal{E}$. By evaluating (1) on β we obtain

$$f(\beta) = \sum_{q \in \mathcal{E}} c_q \cdot \langle \beta, q\rangle + \sum_{q \in \mathcal{C}} c_q \cdot \langle \beta, q\rangle = c_{q'} + \sum_{q \in \mathcal{C}} c_q \cdot \langle \beta, q\rangle. \tag{6}$$

From $\beta \preccurlyeq_P \alpha$ we conclude $0 \leqslant \langle \beta, q\rangle \leqslant \langle \alpha, q\rangle$ for all $q \in \mathcal{F}^\vee(P)$, so (5) and (6) yield

$$|f(\beta) - c_{q'}| = \Big|\sum_{q \in \mathcal{C}} c_q \cdot \langle \beta, q\rangle\Big| \leqslant \sum_{q \in \mathcal{C}} |c_q| \cdot \langle \beta, q\rangle \leqslant \sum_{q \in \mathcal{C}} |c_q| \cdot \langle \alpha, q\rangle < \varepsilon,$$

as desired.

In particular, $f = 0$ implies that all $c_q = 0$, from which uniqueness of the coefficients follows easily. Also, the equivalences (2) – (4) now follow easily from the definitions in 16.1: $f \in D^\vee(P) \iff f(P) \subset \mathbb{R}_+$, so $f \in D^\vee(P)$ implies all $c_q \in \overline{f(P)} \subset \mathbb{R}_+$. Conversely, if all $c_q \geqslant 0$ then $f(\alpha) = \sum c_q \langle \alpha, q\rangle \geqslant 0$ for all $\alpha \in P$, since $\langle \alpha, q\rangle \in \mathbb{N}$, showing $f \in D(P)$. The proof of (3) and (4) is the same, with $\mathbb{R}_+$ replaced by $\mathbb{Z}$ and $\mathbb{N}$, respectively.

16.18. Theorem. *Let P be a parabolic subset of a root system R. Then every dominant coweight f of P is a weak-$*$-convergent series*

$$f = \sum_{q \in \mathcal{F}^\vee(P)} n_q \cdot q \tag{1}$$

with uniquely determined coefficients $n_q \in \mathbb{N}$. If R is irreducible there are at most countably many $n_q \neq 0$, and f is bounded if and only if only finitely many $n_q \neq 0$.

Proof. Uniqueness of the coefficients n_q and the fact that $n_q \in \mathbb{N}$ follows from Th. 16.17. By 16.1.1 it suffices to prove existence of a representation as in (1) for an irreducible R. In the finite case, the result follows from Prop. 16.2. For an infinite R we can assume that P is a pure parabolic subset of $R = \mathrm{T}_I$ and is therefore of the form $P = \mathrm{T}_{I,I_0,\succcurlyeq}$ for a suitable p-datum $(I_0, \succcurlyeq)$. Then by Prop. 13.10, the cone $C = \mathbb{R}_+[P]$ spanned by P is one of the cones of type $\dot{\mathrm{A}}$, B, D treated in App. B. By Th. 16.9(b), each extremal ray of C contains exactly one fundamental coweight. The discreteness condition B.15.1 of Th. B.15 is clearly satisfied for a dominant coweight because $f(\varepsilon_i - \varepsilon_k) \in \mathbb{Z}$ for all $i, k \in I$. Hence the existence of the representation (1) and the remaining statements all follow from Th. B.15.

16.19. Corollary. *Let P be a positive system of a root system R and put $U^\vee = \bigcup_{w \in W(R)} w \cdot D^\vee(P)$ as in 15.12. Then every $g \in \mathcal{P}^\vee(R) \cap U^\vee = \bigcup_{w \in W(R)} w \cdot \mathcal{D}^\vee(P)$ is a weak-$*$-convergent sum*

$$g = \sum_{q \in \mathcal{F}^\vee(P)} m_q \cdot q \tag{1}$$

with uniquely determined coefficients $m_q \in \mathbb{Z}$. If R is irreducible then at most countably many $m_q \neq 0$, only finitely many are negative, and g is bounded if and only if only finitely many m_q are $\neq 0$.

Proof. Let $g = w(f)$ where $f \in \mathcal{D}^\vee(P)$. Then $f - w(f) \in \mathcal{Q}(R^\vee) \cap K^\vee$ (by 7.4.3 and 15.3) $= \mathbb{N}[P^\vee]$ (by 11.2.1 applied to $P^\vee$) $\subset \mathbb{Z}[\mathcal{F}^\vee(P)]$ (by 16.7). Thus $g = f + f'$ where f' is a finite integral linear combination of fundamental coweights. Now the corollary follows easily from 16.17 and 16.18.

§17. Gradings of root systems

17.1. Definition. Let (R, X) be a root system and A an abelian group, written additively. An A-*grading* of R is a family $(R_a)_{a \in A}$ of subsets of R such that

$$R = \dot{\bigcup_{a \in A}} R_a \quad \text{and} \quad R \cap (R_a + R_b) \subset R_{a+b} \tag{1}$$

holds for all $a, b \in A$.

Let $\mathcal{Q}(R)$ be the root lattice of R. By Lemma 7.9 any homomorphism $g\colon \mathcal{Q}(R) \to A$ defines an A-grading of R by

$$R_a = R_a(g) := \{\alpha \in R : g(\alpha) = a\} = g^{-1}(a) \cap R, \tag{2}$$

and, conversely, every A-grading of R arises in this way. Therefore, we will often identify an A-grading with the *associated homomorphism* g, and refer to a graded root system as to (R, g). As a consequence of (2),

$$0 \in R_0 \quad \text{and} \quad R_{-a} = -R_a \tag{3}$$

holds for all $a \in A$, cf. 7.9.4.

Let B be an integral basis of R. Since B is in particular a basis of the free abelian group $\mathcal{Q}(R)$, a grading homomorphism g is uniquely determined by $g\big|B$, and in this way $\mathrm{Hom}(\mathcal{Q}(R), A) \cong A^B$, the group of functions from B to A. This remark is useful in the case of $\mathbb{Z}$-gradings of finite root systems, see 17.5.

A *morphism* between A-graded root systems $(R_a)_{a \in A}$ and $({R'}_a)_{a \in A}$ is a morphism $f\colon (R, X) \to (R', X')$ in the category **RS** respecting the grading, i.e., a linear map $f\colon X \to X'$ with $f(R_a) \subset R'_a$ for all $a \in A$. This is equivalent to the condition $g = g' \circ (f\big|\mathcal{Q}(R))$ for the associated grading homomorphisms g and g'. In particular, an *isomorphism* is a vector space isomorphism $f\colon X \to X'$ satisfying $f(R_a) = R'_a$ for all $a \in A$. Embeddings of graded root systems are defined analogously. Note that $-\mathrm{Id}$ is an isomorphism between a grading and its *opposite*, given by $R_a^{\mathrm{op}} = R_{-a}$, whose associated homomorphism is $-g$. If $f\colon (S, Y) \to (R, X)$ is a morphism of root systems and R is A-graded, then $S_a := S \cap f^{-1}(R_a)$ defines an A-grading of S, called the *induced grading*. Its associated homomorphism is of course just $g \circ (f\big|\mathcal{Q}(S))$. This applies in particular to the case where f is the inclusion of a subsystem S of R.

The *support* of a grading g is

$$\mathrm{supp}(g) = \{a \in A : R_a = R \cap g^{-1}(a) \neq \emptyset\}.$$

Because $0 \in R_0$ and $R = -R$, we always have $0 \in \mathrm{supp}(g)$ and $\mathrm{supp}(g) = -\mathrm{supp}(g)$. A grading is called *trivial* if $\mathrm{supp}(g) = \{0\}$, i.e., $R = R_0$ or $g = 0$.

17.2. *Effective gradings.* Let $(R_a)_{a\in A}$ be an A-graded root system, with associated grading homomorphism g. It follows immediately from the definitions that

$$R_0 \textit{ is an additively closed subsystem of } R. \tag{1}$$

In general, R_0 is not full. For example, consider $R = \mathrm{BC}_1 = \{0, \pm\alpha, \pm 2\alpha\}$ with the natural $\mathbb{Z}/2\mathbb{Z}$-grading given by $R_0 = \{0, \pm 2\alpha\}$ and $R_1 = \{\pm\alpha\}$.

By 11.5, it makes sense to define a grading to be *effective* if R_0 is an effective subsystem. The equivalent conditions (i) – (iii) of 11.5 may then be augmented as follows:

(iv) every $\gamma \in R_0^\times$ is of the form $\gamma = \alpha - \beta$ where $\alpha, \beta \in R_a$ for some $0 \neq a \in A$,
(v) the induced grading on all connected components of R is nontrivial.

Indeed, (iv) is clearly equivalent to the condition (iii) in 11.5. To see that also (v) characterizes effectiveness of R_0, note that the induced grading on a connected component C of R is trivial $\iff g\big|C = 0 \iff C \subset R_0$. Thus (v) is equivalent to condition (ii) of 11.5.

17.3. Lemma. *Let $(R_a)_{a\in A}$ be an A-grading of a root system (R, X), with associated homomorphism g.*

(a) *For any subset $\Sigma \subset R \setminus R_0$, the induced grading of the subsystems $R' = R \cap \mathrm{span}(\Sigma)$ and $R'' = R \cap \mathbb{Z}[\Sigma]$ is effective.*

(b) *If g is effective then so is the induced grading on the subsystem R_{ind} of indivisible roots. Conversely, any effective A-grading of R_{ind} extends to an effective A-grading of R.*

Proof. (a) Both R' and R'' are root systems in the subspace $X' = \mathrm{span}(\Sigma)$. As $R'_0 = R' \cap R_0$, we have $\Sigma \subset R' \setminus R'_0$, so the latter spans X' and thus R' is effectively graded. The proof for R'' is identical.

(b) By 8.5, R_{ind} is irreducible for an irreducible R, and the converse holds trivially. Hence we may assume that R and R_{ind} are irreducible. Then it suffices to observe that a grading is nontrivial on R if and only if it is nontrivial on R_{ind}, because $\mathcal{Q}(R) = \mathcal{Q}(R_{\mathrm{ind}})$.

17.4. *$\mathbb{Z}$-gradings and coweights.* We now specialize to the case $A = \mathbb{Z}$. By Th. 7.5(c), applied to the coroot system, we have $\mathrm{Hom}(\mathcal{Q}(R), \mathbb{Z}) \cong \mathcal{P}^\vee(R)$, the group of coweights of R (see 7.1.1). Thus $\mathbb{Z}$-gradings $(R_i)_{i\in\mathbb{Z}}$ may naturally be identified with coweights q via

$$R_i = R_i(q) = \{\alpha \in R : \langle \alpha, q\rangle = i\}, \tag{1}$$

see also Cor. 7.9. Note that

$$\gcd(\mathrm{supp}(q)) = 1 \quad \iff \quad q \text{ is indivisible.} \tag{2}$$

By 7.12 the gradings given by basic coweights q all satisfy $R_1(q) \neq \emptyset$ and $\mathrm{supp}(q) \subset \{-6, \dots, 6\}$, and even $\mathrm{supp}(q) \subset \{-2, \dots, 2\}$ for non-exceptional irreducible root systems, by 8.12.

For a $\mathbb{Z}$-grading $(R_i)_{i\in\mathbb{Z}}$ we put

$$R_+ = R_+(q) = \dot{\bigcup_{i \geqslant 0}} R_i \quad \text{and} \quad R_{++} = \dot{\bigcup_{i>0}} R_i = R_+ \setminus R_0. \tag{3}$$

Then by Lemma 10.8(a), R_+ is a parabolic subset of R with unipotent part R_{++} and symmetric part R_0, and R_0 is a full subsystem of R. We also note that 15.8 shows, for $w \in W(R)$:

$$w(R_+) = R_+ \iff w \in W(R_0) \iff w(R_i) = R_i \text{ for all } i \in \mathbb{Z}. \tag{4}$$

(For the implication $\Longrightarrow$ in the second equivalence, it suffices to observe that $s_\alpha(q) = q - \langle q, \alpha \rangle \alpha^\vee = q$ for all $\alpha \in R_0(q)$.)

The following lemma gives an explicit description of the $\mathbb{Z}$-gradings of finite root systems. Note that the set of isomorphism classes of $\mathbb{Z}$-gradings of R may be identified with the orbit space $\mathcal{P}^\vee(R)/\operatorname{Aut}(R)$, the automorphism group acting naturally on $\mathcal{P}^\vee(R)$ on the right by composition.

17.5. Lemma. *Let R be a finite root system and let B be a root basis of R. For an element $\mathbf{n} = (n_\beta)_{\beta \in B}$ of $\mathbb{N}^B$ let $g(\mathbf{n}) \in \mathcal{P}^\vee(R)$ be the unique coweight such that $\langle \beta, g(\mathbf{n}) \rangle = n_\beta$ for all $\beta \in B$. Let $\Delta = \operatorname{Dyn}(B)$ the Dynkin diagram of B and $\operatorname{Aut}(\Delta)$ its group of automorphisms, and observe that $\operatorname{Aut}(\Delta)$ acts naturally on the right on $\mathbb{N}^B$ by composition. Then the map $\mathbf{n} \mapsto g(\mathbf{n})$ induces a bijection*

$$\mathbb{N}^B \big/ \operatorname{Aut}(\Delta) \xrightarrow{\cong} \mathcal{P}^\vee(R)/\operatorname{Aut}(R). \tag{1}$$

Remark. An element $\mathbf{n}$ of $\mathbb{N}^B$ may be visualized as a *weighted Dynkin diagram*, by attaching to each vertex β of $\operatorname{Dyn}(B)$ the value n_β. Then (1) reduces the classification of $\mathbb{Z}$-gradings of finite root systems to the determination of the orbits of the group of diagram automorphisms on the set of weighted Dynkin diagrams.

Proof. B is in particular a basis of the free abelian group $\mathcal{Q}(R)$. Hence the restriction map res: $\mathcal{P}^\vee(R) \cong \operatorname{Hom}(\mathcal{Q}(R), \mathbb{Z}) \to \mathbb{Z}^B$, $\operatorname{res}(q) = q\big|B$, is bijective. Let P be the positive system determined by B and let $\mathcal{D}^\vee := \mathcal{D}^\vee(P)$ be the set of dominant coweights, cf. 16.1. As $P = R \cap \mathbb{N}[B]$, we have an induced bijection $\operatorname{res}'\colon \mathcal{D}^\vee \to \mathbb{N}^B$, inverse to the map $\mathbf{n} \mapsto g(\mathbf{n})$ in the statement of the lemma. Let H be the stabilizer of B (equivalently, of P) in $\operatorname{Aut}(R)$. Then $H \cong \operatorname{Aut}(\Delta)$ by Cor. 6.10. Also, H and $\operatorname{Aut}(\Delta)$ act naturally on the right by composition of maps on $\mathcal{D}^\vee$ and $\mathbb{N}^B$, respectively. The bijection $\mathcal{D}^\vee \to \mathbb{N}^B$ being clearly equivariant, we obtain a bijection

$$\mathcal{D}^\vee/H \xrightarrow{\cong} \mathbb{N}^B \big/ \operatorname{Aut}(\Delta). \tag{2}$$

(We note that (2) is also valid for an infinite R.) Next, let $W = W(R)$ be the Weyl group, acting on $\mathcal{P}^\vee(R)$ on the right. We claim that the map

$$\mathcal{D}^\vee \to \mathcal{P}^\vee(R)/W \tag{3}$$

induced from the inclusion $\mathcal{D}^\vee \subset \mathcal{P}^\vee(R)$, is bijective. Indeed, injectivity follows from Prop. 15.12 (and does not require R to be finite). To show that the map is surjective, let $q \in \mathcal{P}^\vee(R) \subset X^*$ be a coweight. By 15.1.2, X^* is the union of the cones $D^\vee(P')$ where P' runs over the positive systems of R. Since positive systems in finite root systems are conjugate under $W(R)$, there exists $w \in W(R)$ such that $q \circ w \in \mathcal{D}^\vee$.

By simple transitivity of W on the set of root bases (A.9), we have $\mathrm{Aut}(R) = W \cdot H$ (semidirect product), in particular, $H \cong \mathrm{Aut}(R)/W$, and it is evident that the bijection (3) is equivariant with respect to the action of H and of $\mathrm{Aut}(R)/W$, respectively. Hence there is an induced bijection

$$\mathcal{D}^\vee/H \xrightarrow{\cong} (\mathcal{P}^\vee(R)/W)/(\mathrm{Aut}(R)/W) \xrightarrow{\cong} \mathcal{P}^\vee(R)/\mathrm{Aut}(R). \tag{4}$$

Now the lemma follows by combining (2) and (4).

In the remainder of this section we consider two special types of effective $\mathbb{Z}$-gradings.

17.6. 3-*gradings.* An effective $\mathbb{Z}$-grading $(R_i)_{i\in\mathbb{Z}}$ with support $\{0, \pm 1\}$ is called a 3-*grading*, see [**57**]. In other words, a 3-grading of a root system R is a partition $R = R_1 \,\dot\cup\, R_0 \,\dot\cup\, R_{-1}$ satisfying

(i) $(R_i + R_j) \cap R \subset R_{i+j}$ with the understanding that $R_k = \emptyset$ for $k \notin \{\pm 1, 0\}$,

(ii) $(R_1 - R_1) \cap R = R_0$.

Equivalently, a $\mathbb{Z}$-grading is a 3-grading if and only if the induced grading on every irreducible component of R has support $\{0, \pm 1\}$.

A 3-grading of a root system R is uniquely determined by the subset R_1 since, by 17.1.3, $R_{-1} = -R_1$ and then $R_0 = R \setminus (R_1 \cup R_{-1})$. We will therefore denote 3-gradings of R by (R, R_1). We will say that R *is* 3-*graded* if a 3-grading of R has been specified. Morphisms and embeddings between 3-graded root systems are morphisms respectively embeddings in the category of $\mathbb{Z}$-graded root systems (17.1).

By specializing the bijection between $\mathbb{Z}$-gradings and coweights described in 17.1 we obtain a *bijection between* 3-*gradings and minuscule coweights*, as defined in 7.14.

It is not our goal here to present all the known results concerning 3-graded root systems. Rather, we limit ourselves to the classification, announced in [**57**], and the following characterization of the parabolic subsets determined by 3-gradings, which is a corollary of the presentation of $\mathcal{Q}(R)$ given in Prop. 11.12. More results will be presented in the following section §18.

17.7. Proposition. *For an effective parabolic subset P of a root system R, recall the subsets $P_{\min}$ and $P_{\max}$ introduced in* 10.11. *Then the following conditions are equivalent:*

(i) $P_u = P_{\min}$,

(ii) $P_u = P_{\max}$,

(iii) *there exists a minuscule coweight q such that $P = R_+(q) = R_0(q) \,\dot\cup\, R_1(q)$, i.e., P_u is the* 1-*part of a* 3-*grading of R.*

Proof. The equivalence of (i) and (ii) was shown in 10.12.

(ii) $\Longrightarrow$ (iii): Define $q'(x_\alpha) = 1$ for $\alpha \in P_u$. Since $P_u = P_{\max}$, $\alpha, \beta \in P_u$ implies $\alpha + \beta \notin P_u$, so the relation 11.12.1 is empty, and relation 11.12.2 is trivially compatible with q'. Hence Prop. 11.12 shows that there exists a unique homomorphism $q \colon \mathcal{Q}(R) \to \mathbb{Z}$ of abelian groups (i.e., a coweight) extending q', and it clearly takes the values $-1, 0, 1$ on $-P_u, P_s, P_u$.

(iii) $\Longrightarrow$ (i): Assuming $\alpha \in P_u \setminus P_{\min}$, we have $\alpha = \beta + \gamma$ decomposable for some $\beta, \gamma \in P_u$, and thus $1 = q(\alpha) = q(\beta) + q(\gamma) = 1 + 1 = 2$, contradiction.

17.8. *Classification of* 3*-gradings of the classical root systems.* The minuscule coweights of the infinite irreducible root systems, or more generally the classical root systems of types A, …, D, have been determined in 8.12. We can therefore use these results to classify the 3-gradings of these root systems. We will use the notation of 8.1. The name for the 3-grading (R, R_1) has been chosen in such a way that it coincides with the name of the Jordan pair covered by a grid with associated 3-graded root system (R, R_1) [**55**, **60**]. Graphs of the small rank examples are given in 18.2. Let us also recall from Th. 12.13 that the subsystems R_0 of a 3-graded root system (R, R_1) are precisely the maximal proper closed subsystems of R that are full.

Type $\dot{\mathrm{A}}_I$, $|I| \geqslant 1$: Any $\emptyset \neq J \subsetneqq I$ defines a 3-grading, denoted $\dot{\mathrm{A}}_I^J$ and called a *rectangular grading*, by

$$(\dot{\mathrm{A}}_I^J)_1 = \{\varepsilon_j - \varepsilon_k : j \in J, k \notin J\},$$
$$(\dot{\mathrm{A}}_I^J)_0 = \{\varepsilon_i - \varepsilon_j : \text{ both } i, j \in J \text{ or both } i, j \notin J\} \cong \dot{\mathrm{A}}_J \times \dot{\mathrm{A}}_{I \setminus J}.$$

Every 3-grading of $\dot{\mathrm{A}}_I$ is of type $\dot{\mathrm{A}}_I^J$ for a suitable J. By 9.5 an automorphism φ of $\dot{\mathrm{A}}_I$ has the form $\varphi = \pi$ or $\varphi = -\pi$ for some permutation $\pi \in \mathrm{Sym}(I)$. For such a map we have, respectively, $\varphi(\dot{\mathrm{A}}_I^J) = \dot{\mathrm{A}}_I^{\pi(J)}$ or $\varphi(\dot{\mathrm{A}}_I^J) = \dot{\mathrm{A}}_I^{I \setminus \pi(J)} = (\dot{\mathrm{A}}_I^{\pi(J)})^{\mathrm{op}}$. Hence $\dot{\mathrm{A}}_I^J \cong \dot{\mathrm{A}}_I^{J'}$ if and only if there exists $\pi \in \mathrm{Sym}(I)$ such that $\pi(J) = J'$ or $\pi(J) = I \setminus J'$. For example, up to isomorphism, we can always assume $|J| \leqslant |I \setminus J|$.

A 3-grading $\dot{\mathrm{A}}_I^J$ with $|J| = 1$ is called a *collinear grading* of $\dot{\mathrm{A}}_I$ and denoted by $\dot{\mathrm{A}}_I^{\mathrm{coll}}$. For a collinear grading, any two roots in $(\dot{\mathrm{A}}_I^{\mathrm{coll}})_1$ are pairwise collinear in the sense of 11.16. Any two collinear gradings of $\dot{\mathrm{A}}_I$ are isomorphic. Clearly A_1 and A_2 admit only the collinear gradings $\mathrm{A}_1^{\mathrm{coll}}$ and $\mathrm{A}_2^{\mathrm{coll}}$. We note that $|(\mathrm{A}_1^{\mathrm{coll}})_i| = 1$ for $i = \pm 1, 0$.

Type B_I, $|I| \geqslant 2$: To any sign $s = \pm$ and fixed $i_0 \in I$ we associate a 3-grading, denoted $\mathrm{B}_I^{si_0}$ and called an *odd quadratic form grading of* B_I, by

$$(\mathrm{B}_I^{si_0})_1 = \{s\varepsilon_{i_0}\} \cup \{s\varepsilon_{i_0} \pm \varepsilon_i : i_0 \neq i \in I\},$$
$$(\mathrm{B}_I^{si_0})_0 = \mathrm{B}_{I \setminus \{i_0\}}.$$

Every 3-grading of B_I is of this type. It easily follows from 9.5 that any two 3-gradings of B_I are conjugate by a Weyl group element. For easier notation we will abbreviate $\mathrm{B}_I^{\mathrm{qf}} = \mathrm{B}_I^{+i_0}$. We have $(\mathrm{B}_I^{si_0})^{\mathrm{op}} = \mathrm{B}_I^{-si_0}$.

Type C_I, $|I| \geqslant 3$: Any sign distribution $\sigma \in \mathbf{2}^I$ gives rise to a 3-grading of C_I, denoted C_I^{σ} and called a *hermitian grading.* It is defined by

$$(\mathrm{C}_I^{\sigma})_1 = \{\sigma(i)\varepsilon_i + \sigma(j)\varepsilon_j : i, j \in I\},$$
$$(\mathrm{C}_I^{\sigma})_0 = \{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j : i, j \in I\} \cong \dot{\mathrm{A}}_I.$$

Every 3-grading of C_I is of this type, and any two 3-gradings of C_I are conjugate by an element in the big Weyl group $\overline{W}(\mathrm{C}_I)$. We have $(\mathrm{C}_I^{\sigma})^{\mathrm{op}} = \mathrm{C}_I^{-\sigma}$. The case where all signs $\sigma(i) = 1$ will be abbreviated by $\mathrm{C}_I^{\mathrm{her}}$.

Type D_I, $|I| \geqslant 4$: There are two types of 3-gradings in this case, both of them arising as induced gradings on the subsystem D_I of C_I and B_I, respectively.

First, for any sign distribution $\sigma \in \mathbf{2}^I$ we have a 3-grading, denoted D_I^σ and called an *alternating grading*. It is defined by

$$\begin{aligned}(\mathrm{D}_I^\sigma)_1 &= \{\sigma(i)\varepsilon_i + \sigma(j)\varepsilon_j : i, j \in I, i \neq j\},\\ (\mathrm{D}_I^\sigma)_0 &= \{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j : i, j \in I\} \cong \dot{\mathrm{A}}_I,\end{aligned}$$

and is of course nothing but the 3-grading induced by the hermitian grading C_I^σ on the subsystem D_I. We have $(\mathrm{D}_I^\sigma)^{\mathrm{op}} = \mathrm{D}_I^{-\sigma}$. The special case where all $\sigma(i) = 1$ will be denoted $\mathrm{D}_I^{\mathrm{alt}}$.

Second, to any sign $s = \pm$ and fixed $i_0 \in I$ we associate a 3-grading, denoted $\mathrm{D}_I^{si_0}$ and called an *even quadratic form grading*, for which

$$\begin{aligned}(\mathrm{D}_I^{si_0})_1 &= \{s\varepsilon_{i_0} \pm \varepsilon_i : i_0 \neq i \in I\},\\ (\mathrm{D}_I^{si_0})_0 &= \mathrm{D}_{I \setminus \{i_0\}}.\end{aligned}$$

This is the 3-grading induced by $\mathrm{B}_I^{si_0}$ on the subsystem D_I. As in type B we abbreviate $\mathrm{D}_I^{\mathrm{qf}} = \mathrm{D}_I^{+i_0}$. We have $(\mathrm{D}_I^{si_0})^{\mathrm{op}} = \mathrm{D}_I^{-si_0}$.

Every 3-grading of D_I is of type D_I^σ or D^{si_0} for suitable choices of σ, s and $i_0 \in I$. For $|I| > 4$ there are exactly two isomorphism classes under the big Weyl group $\overline{W}(\mathrm{D}_I)$, namely $\mathrm{D}_I^{\mathrm{alt}}$ and $\mathrm{D}_I^{\mathrm{qf}}$ for a fixed i_0. That these two types are not isomorphic for $|I| > 4$ is immediate by considering the 0-part of the two gradings. For $|I| = 4$, $\mathrm{D}_4^{\mathrm{qf}}$ and $\mathrm{D}_4^{\mathrm{alt}}$ are conjugate by a diagram automorphism.

Type BC_I: These root systems do not have minuscule coweights and therefore no 3-gradings.

Taking into account the well-known low rank isomorphisms 8.2.1 we have

$$\mathrm{A}_1^{\mathrm{coll}} \cong \mathrm{B}_1^{\mathrm{qf}} \cong \mathrm{C}_1^{\mathrm{her}}, \quad \mathrm{B}_2^{\mathrm{qf}} \cong \mathrm{C}_2^{\mathrm{her}}, \quad \mathrm{A}_3^{\mathrm{coll}} \cong \mathrm{D}_3^{\mathrm{alt}} \quad \text{and} \quad \mathrm{A}_3^2 \cong \mathrm{D}_3^{\mathrm{qf}}, \tag{1}$$

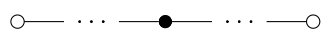
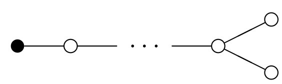
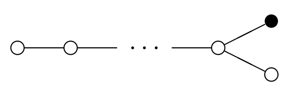
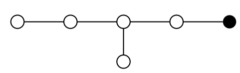
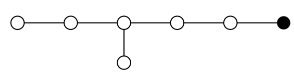
where we used the abbreviation $\mathrm{A}_n^p = \dot{\mathrm{A}}_I^J$ for $|J| = p$, $I = \{0, 1, \ldots, n\}$, and $\mathrm{T}_n = \mathrm{T}_I$ for $\mathrm{T} = \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ and $|I| = n$.

A different method of classifying 3-graded root systems can easily be derived from [**57**]. A description of the 0-part of these 3-gradings was also given by Neeb and Stumme in [**54**, Prop. VII.2] and of the 1-part by Neeb in [**51**, IV.5] (without proof). The cases $\mathrm{B}_I^{-i_0}$ and $\mathrm{D}_I^{-i_0}$ seem to be missing in Neeb's description.

17.9. *Classification of* 3*-gradings of finite root systems.* Since 17.8 does not cover the exceptional root systems, we shortly review the classification of 3-gradings of a finite irreducible root system R, based on the well-known description of minuscule weights [**14**, VIII, §7.3].

A minuscule coweight is always basic. By 7.10.4, applied to the coroot system, it is fundamental with respect to some root basis B of R, and therefore of the form q_β for some $\beta \in B$. Not all q_β are minuscule coweights. Indeed, since the highest root with respect to B lies in R_1, its β-coefficient must be 1. It is easily seen that this condition is not only necessary but also sufficient for defining a minuscule coweight.

The coefficients of the highest root with respect to a root basis can be found in the tables of [**12**]. This readily gives a list of all minuscule coweights of finite root systems. The isomorphism classes of 3-gradings are then obtained by applying 17.5. In the following table the simple root β determining the 3-grading is marked with a $\bullet$.

Type	Dynkin diagram	Name
A_n^p $\left(1 \leqslant p \leqslant [\frac{n+1}{2}]\right)$		rectangular
$\mathrm{B}_n^{\mathrm{qf}}$		odd quadratic form
$\mathrm{C}_n^{\mathrm{her}}$		hermitian
$\mathrm{D}_n^{\mathrm{qf}}$		even quadratic form
$\mathrm{D}_n^{\mathrm{alt}}$		alternating
$\mathrm{E}_6^{\mathrm{bi}}$		bi-Cayley
$\mathrm{E}_7^{\mathrm{alb}}$		Albert

In type A_n, every simple root gives rise to a 3-grading. The restrictions on p come from the diagram automorphism $\mathrm{A}_n^p \cong \mathrm{A}_n^{n+1-p}$. Similarly, both roots at the right end of the Dynkin diagram of D_n give rise to a 3-grading, but are conjugate by a diagram automorphism. The same holds for the two outer roots in E_6. For D_4 both types, $\mathrm{D}_n^{\mathrm{qf}}$ and $\mathrm{D}_n^{\mathrm{alt}}$, are conjugate by a diagram automorphism.

The names for the two exceptional 3-gradings are again taken from the names of the corresponding Jordan pairs. It is easily seen from [**12**, Planches V, VI] that the 1-parts of the Bi-Cayley and Albert grading have 16 and 27 roots, respectively.

The root systems $\mathrm{BC}_n, \mathrm{E}_8, \mathrm{F}_4$ and G_2 do not have 3-gradings.

17.10. 5-*gradings.* In analogy to 3-gradings, we define a 5-*grading* of a root system R as a $\mathbb{Z}$-grading with the property that on every irreducible component of R the induced grading has support $\{0, \pm 1, \pm 2\}$. In particular, a 5-grading is effective, but not every effective $\mathbb{Z}$-grading with support $\{0, \pm 1, \pm 2\}$ is a 5-grading in our sense.

A classification of 5-gradings could be obtained along the lines of 17.8 and 17.9. However, we limit ourselves to the following example showing that all root systems have a 5-grading, unless they have an irreducible component of type A_1. In particular, every irreducible root system possesses either a 3- or a 5-grading.

We may assume that R is irreducible and fix a long root $\alpha \in R$. Then it follows from A.2 that the $\mathbb{Z}$-grading induced by the coweight $\alpha^\vee$ is $R = R_{-2} \mathbin{\dot\cup} R_{-1} \mathbin{\dot\cup} R_0 \mathbin{\dot\cup}$

$R_1 \dot\cup R_2$ where

$$R_i = \{\beta \in R : \langle \beta, \alpha^\vee \rangle = i\}. \tag{1}$$

By 5.6 two $\mathbb{Z}$-gradings of type (1), induced by two different long roots, are isomorphic. Since $\pm\alpha \in R_{\pm 2}$ and $0 \in R_0$, (1) is a 5-grading if and only if $R_{\pm 1} \neq \emptyset$. It is immediate from the classification of root systems that this is always the case unless $R = \mathrm{A}_1$. The coweights corresponding to these special 5-gradings are called *quasi-minuscule* in [**41**] and *distinguished* in [**40**].

The parabolic subsets corresponding to 5-gradings have a characterization that is analogous to the one given in 17.7 for the 3-graded case.

17.11. Proposition. *For an effective parabolic subset P of a root system R, the following conditions are equivalent:*

(i) $P_u = P_{\min} \dot\cup P_{\max}$,

(ii) $P = R_+$ *is the parabolic subset of a* 5*-grading of* R,

(iii) $P = R_+(q) = R_0 \dot\cup R_1 \dot\cup R_2$ *for some coweight* q *where the* $R_i := R_i(q)$ *satisfy* $(R_1 + R_1) \cap R = R_2$ *and* $R_1 = (R_2 - R_1) \cap R$,

If these conditions are satisfied then $P_{\min} = R_1$ *and* $P_{\max} = R_2$.

Proof. (i) $\Longrightarrow$ (ii): Let C be a connected component of R. From 10.11 it is easily seen that $P_{\min} \cap C = (P \cap C)_{\min}$ and $P_{\max} \cap C = (P \cap C)_{\max}$. We thus may assume R connected. Since $P_u \neq \emptyset$, it follows from 17.7 and our assumption (i) that $P_{\min} \neq \emptyset \neq P_{\max}$. Define $q'(\alpha) = 2$ for $\alpha \in P_{\max}$ and $q'(\beta) = 1$ for $\beta \in P_{\min}$. We show that q' extends to a coweight $q\colon \mathfrak{Q}(R) \to \mathbb{Z}$ by showing compatibility with the defining relations for $\mathfrak{Q}(R)$ given in Prop. 11.12. For 11.12.1, let $\alpha, \beta, \alpha+\beta \in P_u$. By our assumption (i), necessarily $\alpha + \beta \in P_u \setminus P_{\min} = P_{\max}$, and $\alpha, \beta \in P_u \setminus P_{\max} = P_{\min}$, so $q'(x_{\alpha+\beta}) = 2 = 1 + 1 = q'(x_\alpha) + q'(x_\beta)$. For 11.12.2, it suffices to show that $\mu = \alpha - \beta \in P_s$, for suitable $\alpha, \beta \in P_u$, implies α and β are both in $P_{\max}$ or both in $P_{\min}$. Assume to the contrary that $\alpha \in P_{\max}$, $\beta \in P_{\min}$. Since $\alpha \in P_{\max} = P_u \setminus P_{\min}$ is decomposable, there exist $\gamma, \delta \in P_u$ with $\alpha = \gamma + \delta$. Then $0 = \mu - \gamma - \delta + \beta$, and the triple $(\gamma, \delta, -\mu)$ satisfies the hypotheses of Lemma 11.10. Hence either $\gamma - \mu$ or $\delta - \mu$ belongs to $R^\times$. In the first case, $\gamma - \mu \in (P_u + P_s) \cap R \subset P_u$, and we have $\beta - \delta = \gamma - \mu \in P_u$, contradicting the fact that $\beta \in P_{\min}$. In the second case, $\beta - \gamma = \delta - \mu \in (P_u + P_s) \cap R \subset P_u$, and this again contradicts $\beta \in P_{\min}$. Thus q is now a well-defined coweight, and we have $R_1(q) = P_{\min} \neq \emptyset$ and $R_2(q) = P_{\max} \neq \emptyset$. Since R is irreducible, q defines a 5-grading of R.

(ii) $\Longrightarrow$ (iii): It is again no restriction to assume R irreducible. Then P is connected by 11.9. We fix an invariant inner product and note that

$$(R_2 | R_1) \geqslant 0, \tag{1}$$

because of A.3 and $R_3 = \emptyset$.

Let $\alpha_2 \in R_2$. The condition $\alpha_2 \in R_1 + R_1$ is equivalent to $\alpha_2 - \beta_1 \in R$ (and hence in R_1 since we have a $\mathbb{Z}$-grading) for some $\beta_1 \in R_1$. Assume this not to be the case. Then $(\alpha_2 | R_1) = 0$ follows from A.3 and (1). Now pick an element $\beta_1 \in R_1$. By connectedness of P there exists $\gamma \in P$ such that $\alpha_2 \not\perp \gamma \not\perp \beta_1$, and by our assumption on α_2, we have $\gamma = \gamma_2 \in R_2$. Furthermore, $(\gamma_2 | \beta_1) > 0$, whence $\gamma_2 - \beta_1 \in R_1$, and then $(\gamma_2 - \beta_1 | \alpha_2) = (\gamma_2 | \alpha_2) \neq 0$, contradiction.

We show similarly that every $\alpha_1 \in R_1$ can be written in the form $\alpha_1 = \beta_2 - \beta_1$ for some $\beta_2 \in R_2$ and $\beta_1 \in R_1$. By A.3 this is certainly true if there exists $\beta_2 \in R_2$ with $(\alpha_1|\beta_2) > 0$ or if there exists $\beta_1 \in R_1$ with $(\alpha_1|\beta_1) < 0$. So assume that neither condition is satisfied. Then we have $(\alpha_1|R_1) \geqslant 0$ and $(\alpha_1|R_2) = 0$ by (1). Pick an element $\beta_2 \in R_2$ and choose a connecting chain $\alpha_1 \not\perp \gamma \not\perp \beta_2$, where $\gamma \in P$. Then necessarily $\gamma = \gamma_1 \in R_1$, and hence $(\beta_2|\gamma_1) > 0$ and $(\alpha_1|\gamma_1) > 0$. It follows that $\beta_2 - \gamma_1 \in R_1$, and $(\beta_2 - \gamma_1|\alpha_1) = 0 - (\gamma_1|\alpha_1) < 0$, contradiction.

(iii) $\Longrightarrow$ (i): An easy argument using the coweight q shows that $R_1 \subset P_{\min}$ and $R_2 \subset P_{\max}$. Because of $P_u = R_1 \dot\cup R_2$, the reverse inclusions are equivalent to

$$R_2 \cap P_{\min} = \emptyset \quad \text{and} \quad R_1 \cap P_{\max} = \emptyset.$$

Assuming $\alpha \in P_{\min} \cap R_2$, we have $\alpha = \beta + \gamma$ for $\beta, \gamma \in R_1$. This contradicts the fact that α is indecomposable. Similarly, assume $\alpha \in R_1 \cap P_{\max}$. Then we can write $\alpha = \beta - \gamma$ for $\beta \in R_2$, $\gamma \in R_1$, and so $\alpha + \gamma = \beta \in P_u$, contradicting $\alpha \in P_{\max}$.

§18. Elementary relations and graphs in 3-graded root systems

18.1. *Elementary relations and graphs.* Recall from 11.16 that for two roots α, β in a root system R we have defined the following relations:

$$
\begin{aligned}
\alpha \top \beta \quad (\alpha \text{ collinear to } \beta) \quad &\iff \quad \langle \alpha, \beta^\vee \rangle = 1 = \langle \beta, \alpha^\vee \rangle \,, \\
\alpha \vdash \beta \quad (\alpha \text{ governs } \beta) \quad &\iff \quad \langle \alpha, \beta^\vee \rangle = 1,\ \langle \beta, \alpha^\vee \rangle = 2 \,.
\end{aligned}
$$

As we will see in 18.5.2, the elementary relations $\perp$, $\top$ and $\vdash$ describe all possibilities between two roots in the 1-part of a 3-graded root system. Up to signs, they describe the relations between linearly independent roots in an irreducible root system $R \neq \mathrm{G}_2$. Indeed, two linearly independent $\alpha, \beta \in R$ with $\langle \alpha, \beta^\vee \rangle \geqslant 0$ either satisfy an elementary relation or $\angle(\alpha, \beta) = \pi/6$. In the latter case, by 4.5, α and β span an irreducible component of type G_2.

Elementary relations appearing in a sequence have the obvious meaning. We have

$$\alpha \vdash \beta \vdash \gamma \quad \Longrightarrow \quad R \text{ is not reduced.} \tag{1}$$

Indeed, the assumption implies $(\alpha|\alpha) = 2(\beta|\beta) = 4(\gamma|\gamma)$ for any invariant inner product, and so R is not reduced by 4.4.

It is sometimes helpful to visualize elementary relations among elements of a family of roots in the form of a partially directed graph whose vertices are the members of the family and whose edges are determined by the rules

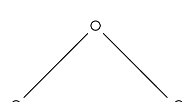
$$
\begin{array}{lll}
\alpha \perp \beta : & \alpha \qquad \beta & \text{(no edge)}, \\
\alpha \top \beta : & \alpha \longrightarrow\!\!\!\!\!\!- \;\beta & \\
\alpha \vdash \beta : & \alpha \longleftarrow \beta & (\text{or } \beta \longrightarrow \alpha).
\end{array}
$$

As a mnemonic, note that the transition from $\vdash$ to $\longleftarrow$ is obtained by bending over the $\mid$ in $\vdash$ to form an arrow. Since collinear roots α and β have the same length and α is shorter than β in case $\alpha \vdash \beta$, these definitions are consistent with the usual ones for Dynkin diagrams.

18.2. *Graphs of* 3*-gradings of small rank.* The classification of the 3-gradings (R, R_1) of the classical finite root systems R is described in 17.9. For small ranks the graphs of R_1 are as follows.

Type A_n: The graph of $\mathrm{A}_1^{\mathrm{coll}}$ consists only of one vertex, and that of $\mathrm{A}_2^{\mathrm{coll}}$ is $\circ$ — $\circ$. In general, the graph of $\mathrm{A}_n^{\mathrm{coll}}$ is the complete (undirected) graph on n vertices, so

$\mathrm{A}_3^{\mathrm{coll}}$:



is a triangle and $\mathrm{A}_4^{\mathrm{coll}}$ is the graph of a tetrahedron. The root system A_3 admits a second 3-grading A_3^2 with 1-part $\{\varepsilon_j - \varepsilon_k : 0 \leqslant j \leqslant 1 < k \leqslant 3\}$. The corresponding graph is a quadrangle (see 18.3):

$$\dot{\mathrm{A}}_I^J = \mathrm{A}_3^2 : \qquad \begin{tikzcd} \varepsilon_1 - \varepsilon_2 \arrow[r, dash] \arrow[d, dash] & \varepsilon_1 - \varepsilon_3 \arrow[d, dash] \\ \varepsilon_0 - \varepsilon_2 \arrow[r, dash] & \varepsilon_0 - \varepsilon_3 \end{tikzcd}$$

corresponding to $J = \{0, 1\} \subset I = \{0, 1, 2, 3\}$.

Type B_I: We suppose $0 = i_0 \in I$. The odd quadratic form grading $\mathrm{B}_I^{\mathrm{qf}}$ is then defined by $(\mathrm{B}_I^{\mathrm{qf}})_1 = \{\varepsilon_0\} \cup \{\varepsilon_0 \pm \varepsilon_i : i \in I \setminus \{0\}\}$. The graphs of $\mathrm{B}_2^{\mathrm{qf}}$ and $\mathrm{B}_3^{\mathrm{qf}}$ are

$$\mathrm{B}_2^{\mathrm{qf}} : \quad \varepsilon_0 - \varepsilon_1 \longrightarrow \varepsilon_0 \longleftarrow \varepsilon_0 + \varepsilon_1 \,, \qquad \mathrm{B}_3^{\mathrm{qf}} : \quad \begin{tikzcd} \varepsilon_0 + \varepsilon_2 \arrow[rr, dash] \arrow[dd, dash] \arrow[dr] & & \varepsilon_0 + \varepsilon_1 \arrow[dd, dash] \arrow[dl] \\ & \varepsilon_0 & \\ \varepsilon_0 - \varepsilon_1 \arrow[rr, dash] \arrow[ur] & & \varepsilon_0 - \varepsilon_2 \arrow[ul] \end{tikzcd}$$

Type C_I: Recall that the hermitian grading $\mathrm{C}_I^{\mathrm{her}}$ is given by $(\mathrm{C}_I^{\mathrm{her}})_1 = \{\varepsilon_i + \varepsilon_j : i, j \in I\}$. For $|I| = 1, 2$ we have the following graphs:

$$\mathrm{C}_2^{\mathrm{her}} : \quad 2\varepsilon_0 \longrightarrow \varepsilon_0 + \varepsilon_1 \longleftarrow 2\varepsilon_1 \,, \qquad \mathrm{C}_3^{\mathrm{her}} : \quad \begin{tikzcd} & & 2\varepsilon_1 \arrow[dl] \arrow[dr] & & \\ & \varepsilon_1 + \varepsilon_2 \arrow[rr, dash] \arrow[dr, dash] & & \varepsilon_1 + \varepsilon_3 \arrow[dl, dash] & \\ 2\varepsilon_2 \arrow[ur] \arrow[rr] & & \varepsilon_2 + \varepsilon_3 & & 2\varepsilon_3 \arrow[ll] \arrow[ul] \end{tikzcd}$$

Type D_I: The graph of the isomorphic 3-gradings $\mathrm{D}_4^{\mathrm{alt}}$ and $\mathrm{D}_4^{\mathrm{qf}}$ is the graph of an octahedron. In general, the graph of $\mathrm{D}_n^{\mathrm{alt}}$ is the graph of 2-element subsets of an n-element set, with (undirected) edges between non-disjoint subsets.

It is useful to give some of these low-rank 3-gradings a special name.

18.3. DEFINITION. We refer to the following families of roots $\alpha_i \in R$ as the *elementary configurations.* We call

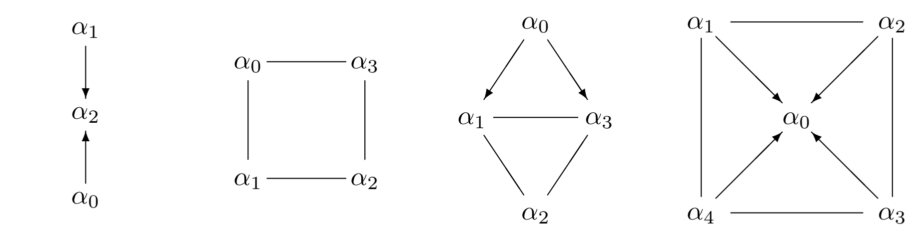
(i) $(\alpha_0; \alpha_1, \alpha_2)$ a *triangle* or a *double arrow* if $\alpha_0 \vdash \alpha_1 \perp \alpha_2 \dashv \alpha_0$,
(ii) $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ a *quadrangle* if $\alpha_i \top \alpha_{i+1} \perp \alpha_{i+3}$ for indices mod 4,
(iii) $(\alpha_0; \alpha_1, \alpha_2, \alpha_3)$ a *diamond* if $\alpha_0 \dashv \alpha_1 \top \alpha_2 \perp \alpha_0 \dashv \alpha_3 \top \alpha_2$ and $\alpha_1 \top \alpha_3$.

In addition to these elementary configurations we call

(iv) $(\alpha_0; \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ a *pyramid* if $\alpha_0 \vdash \alpha_i$ for $1 \leqslant i \leqslant 4$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a quadrangle.

The names "collinear" and "governing" come from the theory of grids in Jordan triple systems [**44**, **56**] where orthogonal, collinear and governing tripotents have a well-defined meaning. There is a close connection between grids and 3-graded root systems, as defined in 17.6: it is shown in [**58**] that for every grid $\mathcal{G}$ in a Jordan triple system there exists a 3-graded root system (R, R_1) and a bijection $R_1 \to \mathcal{G}$, $\alpha \mapsto g_\alpha$, which preserves the elementary relations in R_1 and in $\mathcal{G}$, i.e., two roots $\alpha, \beta \in R_1$ are orthogonal roots if and only if g_α, g_β are orthogonal tripotents, and analogously for collinear and governing. This connection to Jordan theory also explains the names for the elementary configurations "triangle", "quadrangle" and "diamond" which are established terminologies in Jordan theory. From the point of view of their graphical representation, it is more natural to call a "triangle" a double arrow, and we will therefore use both names interchangeably.

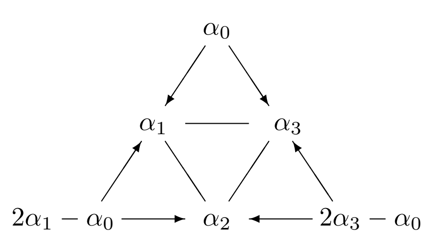
The graphs corresponding to these configurations are



double arrow quadrangle diamond pyramid

Hence a double arrow, quadrangle and pyramid have the same graph as the 1-part of the 3-gradings $\mathrm{B}_2^{\mathrm{qf}}$, A_3^2 and $\mathrm{B}_3^{\mathrm{qf}}$, respectively. They generate a (not necessarily closed) subsystem S which has an induced 3-grading such that $(S, S_1) \cong \mathrm{B}_2^{\mathrm{qf}}$, A_3^2 and $\mathrm{B}_3^{\mathrm{qf}}$. Indeed, let $E \subset R$ be one of the three elementary configurations, and let (T, T_1) be one of the 3-graded root systems $\mathrm{B}_2^{\mathrm{qf}}$, A_3^2 and $\mathrm{B}_3^{\mathrm{qf}}$ such that E and T_1 have the same graph. Thus, there exists a bijection $f \colon T_1 \to E$ with the property that f composed with the injection $E \to R$ satisfies the condition 11.7.1. Hence f extends to an embedding $f \colon T \to R$. Let $S = f(T)$ and $S_1 = f(T_1) = E$. Then S is a subsystem isomorphic to T via f, whence (S, S_1) is isomorphic to (T, T_1).

The subfamily $(2\varepsilon_0; \varepsilon_0 + \varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_0 + \varepsilon_2)$ of $\mathrm{C}_3^{\mathrm{her}}$ is a diamond. As in this example, any diamond $(\alpha_0; \alpha_1, \alpha_2, \alpha_3)$ can be completed to a subfamily with the same graph as the 1-part of $\mathrm{C}_3^{\mathrm{her}}$:



The elementary relations satisfied by the enlarged family can easily be checked using 18.4.3 below. The same argument as above shows that any diamond generates a 3-graded subsystem S such that $(S, S_1) \cong \mathrm{C}_3^{\mathrm{her}}$.

In any elementary configuration the last root can be "generated" from the previous ones. The following lemma makes this more precise.

18.4. LEMMA. *Let α_i be roots in a root system R.*

(a) *If $\alpha_0 \vdash \alpha_1$ then there exists a unique α_2 in R such that $(\alpha_0; \alpha_1, \alpha_2)$ is a triangle, namely $\alpha_2 = -s_{\alpha_0}(\alpha_1) = 2\alpha_0 - \alpha_1$. In particular, for any triangle $(\alpha_0; \alpha_1, \alpha_2)$ we have*

$$2\alpha_0 = \alpha_1 + \alpha_2 \quad \text{and} \quad \alpha_0^\vee = \alpha_1^\vee + \alpha_2^\vee. \tag{1}$$

(b) *If $\alpha_0 \top \alpha_1 \top \alpha_2 \perp \alpha_0$ then there exists a unique $\alpha_3 \in R$ such that $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ is a quadrangle, namely $\alpha_3 = s_{\alpha_0 - \alpha_1}(\alpha_2) = \alpha_0 - \alpha_1 + \alpha_2$. In particular, for each quadrangle $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ we have*

$$\alpha_0 + \alpha_2 = \alpha_1 + \alpha_3 \quad \text{and} \quad \alpha_0^\vee + \alpha_2^\vee = \alpha_1^\vee + \alpha_3^\vee. \tag{2}$$

(c) *If $\alpha_0 \dashv \alpha_1 \top \alpha_2 \perp \alpha_0$ then there exists a unique $\alpha_3 \in R$ such that $(\alpha_0; \alpha_1, \alpha_2, \alpha_3)$ is a diamond, namely $\alpha_3 = s_{\alpha_0 - \alpha_1}(\alpha_2) = \alpha_0 - \alpha_1 + \alpha_2$. Similarly, if $\alpha_0 \dashv \alpha_1 \top \alpha_3 \vdash \alpha_0$ then there exists a unique $\alpha_2 \in R$ such that $(\alpha_0; \alpha_1, \alpha_2, \alpha_3)$ is a diamond, namely $\alpha_2 = s_{\alpha_0 - \alpha_1}(\alpha_3) = \alpha_1 - \alpha_0 + \alpha_3$. In particular, for any diamond $(\alpha_0; \alpha_1, \alpha_2, \alpha_3)$ we have*

$$\alpha_0 + \alpha_2 = \alpha_1 + \alpha_3 \quad \text{and} \quad 2\alpha_0^\vee + \alpha_2^\vee = \alpha_1^\vee + \alpha_3^\vee. \tag{3}$$

Proof. (a) It is straightforward to check that $-s_{\alpha_0}(\alpha_1) = 2\alpha_0 - \alpha_1$ and that $(\alpha_0; \alpha_1, 2\alpha_0 - \alpha_1)$ is a triangle. For an arbitrary triangle $(\alpha_0; \alpha_1, \alpha_2)$ one verifies that $(\alpha_2|\alpha_2) = (\alpha_2|2\alpha_0 - \alpha_1) = (2\alpha_0 - \alpha_1|2\alpha_0 - \alpha_1)$ for any invariant inner product $(\ |\ )$. Therefore $\alpha_2 = 2\alpha_0 - \alpha_1$ by the criterion

$$x = y \quad \Longleftrightarrow \quad (x|x) = (x|y) = (y|y),$$

an immediate consequence of the Cauchy-Schwarz inequality. The second equation in (1) then follows from the general formula 4.8.2. The claims in (b) and (c) are proven in the same way.

The lemma has the following interpretation in terms of graphs. An arrow $\alpha \longrightarrow \beta$ generates a unique double arrow $(\beta; \alpha, \gamma)$:

$$\alpha \longrightarrow \beta \qquad \hookrightarrow \qquad \alpha \longrightarrow \beta \longleftarrow \gamma, \qquad \gamma = 2\beta - \alpha. \tag{4}$$

A "hook" generates a unique quadrangle by completing the missing corner:

$$\begin{array}{ccc} \alpha & & \\ \Big| & & \\ \beta & \text{———} & \gamma \end{array} \qquad \hookrightarrow \qquad \begin{array}{ccc} \alpha & \text{———} & \delta \\ \Big| & & \Big| \\ \beta & \text{———} & \gamma \end{array}, \qquad \delta = \alpha - \beta + \gamma. \tag{5}$$

Finally, diamonds are created in two ways:

$$
\begin{tikzcd}
& \alpha \arrow[dl] & & & \alpha \arrow[dl] \arrow[dr] & & & & \alpha \arrow[dl] \arrow[dr] & \\
\beta \arrow[dr, dash] & & \hookrightarrow & \beta \arrow[rr, dash] \arrow[dr, dash] & & \delta \arrow[dl, dash] & \hookleftarrow & \beta \arrow[rr, dash] & & \delta \\
& \gamma & & & \gamma & & & & &
\end{tikzcd}
\tag{6}
$$

by completing the missing vertex δ resp. γ from the equation $\alpha + \gamma = \beta + \delta$.

We now turn to 3-graded root systems (R, R_1).

18.5. LEMMA. *Let (R, R_1) be a 3-graded root system.*

(a) *For $\alpha, \beta \in R_1$ we have*

$$
\begin{aligned}
&\langle \alpha, \beta^\vee \rangle \in \{0, 1, 2\}, \quad \textit{hence} && (1)\\
&\alpha \perp \beta \ \textit{ or } \ \alpha \top \beta \ \textit{ or } \ \alpha \vdash \beta \ \textit{ or } \ \alpha \dashv \beta \ \textit{ or } \ \alpha = \beta\,, && (2)\\
&\alpha - \beta \in R_0 \quad \iff \quad \langle \alpha, \beta^\vee \rangle > 0. && (3)
\end{aligned}
$$

Therefore the 0-part R_0 has the description

$$R_0 = \{\alpha - \beta : \alpha, \beta \in R_1,\ \langle \alpha, \beta^\vee \rangle > 0\}. \tag{4}$$

Every root $\mu \in R_0^\times$ has a standard representation of the form

$$\mu = \alpha - \beta = s_\beta(\alpha) \quad \textit{with } \alpha, \beta \in R_1 \textit{ and } \langle \alpha, \beta^\vee \rangle = 1. \tag{5}$$

The coroot is given by $\mu^\vee = \alpha - \langle \beta, \alpha^\vee \rangle \beta^\vee$.

(b) *R is reduced.*

(c) *In obvious notation,*

$$|\langle R_0, R_1^\vee \rangle| \leqslant 1 \quad \textit{and} \quad |\langle R, R^\vee \rangle| \leqslant 2. \tag{6}$$

(d) *Let $E \subset R$ be an elementary configuration. If all elements of E, possibly with one exception, lie in R_1, then in fact $E \subset R_1$.*

Proof. (a) If $\langle \alpha, \beta^\vee \rangle < 0$ then $\alpha + \beta \in R$ by A.3, but $\alpha + \beta \in R_2 = \emptyset$ by 17.6(i). The assumption $\langle \alpha, \beta^\vee \rangle \geqslant 3$ leads to the contradiction $s_\beta(\alpha) \in R_{1-\langle \alpha, \beta^\vee \rangle} = \emptyset$. Therefore $\langle \alpha, \beta^\vee \rangle \in \{0, 1, 2\}$ which implies (2).

For the proof of (3) suppose that $\alpha \perp \beta$. Then $s_\beta(\alpha - \beta) = \alpha + \beta$ shows $\alpha - \beta \notin R$. This proves the implication from left to right. Conversely, if $\langle \alpha, \beta^\vee \rangle > 0$ then $\alpha = \beta$ or $\alpha - \beta \in R^\times$ by A.3 and hence $\alpha - \beta \in R_0$. The description of R_0 in (4) is immediate from (3). Regarding the standard representation, see 11.14 and note that R_1 does not contain weakly orthogonal roots.

(b) We may assume that R is irreducible. By 8.5 the set of indivisible roots $R_{\rm ind}$ is a reduced subsystem of R. It is immediate that $R_1 \cup R_{-1} \subset R_{\rm ind}$. Now let $\alpha - \beta \in R_0$ where $\alpha, \beta \in R_1 \subset R_{\rm ind}$. Then $s_\beta(\alpha) = \alpha - \langle \alpha, \beta^\vee \rangle \beta \in R_{\rm ind}$ and $\langle \alpha, \beta^\vee \rangle > 0$ by (b). By A.5, the set $\{j \in \mathbb{Z} : \alpha + j\beta \in R_{\rm ind}\}$ is an interval in $\mathbb{Z}$. Therefore in particular $\alpha - \beta \in R_{\rm ind}$, proving that also $R_0 \subset R_{\rm ind}$.

(c) For the proof of $|\langle R_0, R_1^\vee \rangle| \leqslant 1$ it suffices to exclude the case $\langle \mu, \beta^\vee \rangle \geqslant 2$ for $\beta \in R_1$, $\mu = \alpha_1 - \alpha_2 \in R_0$ and $\alpha_i \in R_1$ with $\langle \alpha_1, \alpha_2^\vee \rangle > 0$. Since then $2 \leqslant \langle \alpha_1, \beta^\vee \rangle - \langle \alpha_2, \beta^\vee \rangle$, it follows from (2) that $\langle \alpha_1, \beta^\vee \rangle = 2$ and $\langle \alpha_2, \beta^\vee \rangle = 0$, whence $\alpha_1 \dashv \beta$, $2\beta - \alpha_1 \in R_1$ and $\langle 2\beta - \alpha_1, \alpha_2 \rangle = -\langle \alpha_1, \alpha_2^\vee \rangle < 0$ which contradicts (2).

For the second part of (c) we may assume that R is irreducible. By (b) it then suffices to exclude $\langle \gamma, \delta^\vee \rangle = 3$ for $\gamma, \delta \in R$. It is well-known that such a configuration γ, δ spans a subsystem of type G_2. Therefore $R = \mathrm{G}_2$ by irreducibility and 4.5. But G_2 does not have a minuscule coweight by 17.9. (An elementary proof that $\langle \gamma, \delta^\vee \rangle = 3$ is impossible for roots γ, δ in a 3-graded root system (R, R_1), goes as follows. Since $s_\delta(\gamma) = \gamma - 3\delta \in R$ we must have $\delta \in R_0$. Also, $\langle \gamma, (\gamma - 3\delta)^\vee \rangle = \langle \gamma, s_\delta(\gamma)^\vee \rangle = \langle s_\delta(\gamma), \gamma^\vee \rangle = 2 - \langle \gamma, \delta^\vee \rangle \langle \delta, \gamma^\vee \rangle = -1$ implies $\gamma + (\gamma - 3\delta) = 2\gamma - 3\delta \in R$ and then $\gamma \in R_0$. Write $\delta = \alpha_1 - \alpha_2$ with $\alpha_i \in R_1$. Since there are only two different root lengths in R, we must have $\alpha_1 \top \alpha_2$ by what we have already shown. But then $3 = \langle \gamma, \alpha_1^\vee \rangle - \langle \gamma, \alpha_2^\vee \rangle$ which contradicts $|\langle \gamma, \alpha_i^\vee \rangle| \leqslant 1$.)

(d) follows by applying the minuscule coweight describing the 3-grading to the formulas 18.4.1, 18.4.2 and 18.4.3.

Next we describe the possible relations between three roots in R_1.

18.6. LEMMA. *Let (R, R_1) be a 3-graded root system, and let $\alpha, \beta, \gamma \in R_1$ be distinct roots. Then:*

(a) $\alpha \dashv \beta \dashv \gamma$ *is impossible.*

(b) *If $\alpha \dashv \beta$ then*

(i) $\alpha \dashv \beta \perp \gamma \implies \alpha \perp \gamma$,
(ii) $\alpha \dashv \beta \top \gamma \implies \alpha \dashv \gamma$ *or* $\alpha \perp \gamma$,
(iii) $\alpha \dashv \beta \vdash \gamma \implies \alpha \perp \gamma$ *or* $\alpha \top \gamma$.

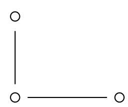
(c) *If $\alpha \vdash \beta$ then*

(i) $\alpha \vdash \beta \top \gamma \implies \alpha \vdash \gamma$,
(ii) $\alpha \vdash \beta \dashv \gamma \implies \alpha \top \gamma$.

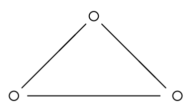
Proof. Since R is reduced by Lemma 18.5(b), (a) follows from 18.1.1. If $\alpha \dashv \beta \perp \gamma$ we have $2\beta - \alpha \in R_1$ and hence $0 \leqslant \langle 2\beta - \alpha, \gamma^\vee \rangle = -\langle \alpha, \gamma^\vee \rangle \leqslant 0$. In case (b.ii) we have $(\alpha|\alpha) > (\beta|\beta) = (\gamma|\gamma)$ for any invariant inner product, hence the claim follows from 18.5.2 and length considerations. The same argument can be used in (b.iii) where we have $(\alpha|\alpha) = (\gamma|\gamma)$. Similarly, in (c.i) we have $(\alpha|\alpha) < (\gamma|\gamma)$ whence $\alpha \vdash \gamma$ since $\alpha \perp \gamma$ contradicts (b.i) after switching α and β. (c.ii) can be proven in the same way.

It is now straightforward to write down all possible elementary relations between three given roots $\alpha, \beta, \gamma \in R_1$. Taking order into account, there are 29 cases which, in the equivalent setting of cogs in Jordan triple systems, are enumerated in [**56**, I, 3.5]. For most purposes one can assume that $\{\alpha, \beta, \gamma\}$ is connected, equivalently, that the corresponding subgraph is connected. The following classification is then easily obtained from the lemma above.

18.7. *Connected subgraphs with three vertices.* The possible connected subgraphs on 3 elements of R_1 are the following six graphs:

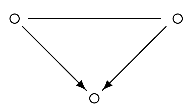


(generates a quadrangle, 18.4.5) (1)

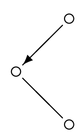


(collinear family) (2)

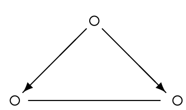
(double arrow, 18.3) (3)



(generates a pyramid, 18.3) (4)



(generates a diamond, 18.4.6) (5)



(generates a diamond, 18.4.6) (6)

We have seen in 11.9 that a 3-graded root system (R, R_1) is irreducible if and only if $R_{++} = R_1$ is connected, and in this case two roots in R_1 are connected by a chain of length at most 2. As a consequence of the classification above we can now determine precisely the possible chains connecting two orthogonal roots in R_1.

18.8. Corollary. *Let (R, R_1) be an irreducible 3-graded root system and let $\alpha, \gamma \in R_1$ be orthogonal roots. Then there exists $\beta \in R_1$ such that, possibly after switching α and γ, one of the following three cases holds:*

(i) (18.7.1) $\alpha \top \beta \top \gamma$: $\overset{\alpha}{\circ} \text{———} \overset{\beta}{\circ} \text{———} \overset{\gamma}{\circ}$

(ii) (18.7.3) $\alpha \dashv \beta \vdash \gamma$: $\overset{\alpha}{\circ} \longrightarrow \overset{\beta}{\circ} \longleftarrow \overset{\gamma}{\circ}$

(iii) (18.7.5) $\alpha \dashv \beta \top \gamma$: $\overset{\alpha}{\circ} \longrightarrow \overset{\beta}{\circ} \text{———} \overset{\gamma}{\circ}$

Moreover, $\tilde{\beta} = \alpha - \beta + \gamma \in R_1$ and $(\alpha, \tilde{\beta}, \gamma)$ is also a connecting chain.

In [**72**, sect. 6] Tits has classified all possible configurations of four roots with sum zero such that no two of these have sum zero. For the further development it will be crucial to know precisely all possibilities in case these four roots belong to R_1.

18.9. Proposition. *Let* (R, R_1) *be a* 3*-graded root system and assume that* $\alpha, \beta, \gamma \in R_1$ *satisfy* $\alpha \neq \beta \neq \gamma$*. Then the following assertions* (a), (b) *and* (c) *are equivalent:*

(a) $\alpha - \beta + \gamma \in R_1$.

(b) $\alpha \not\perp \beta \not\perp \gamma$ *and one of the following holds:*

- (i) $\alpha \perp \gamma$*, or*
- (ii) $\alpha \top \gamma$ *and* $\langle \alpha, \beta^\vee \rangle = 1 = \langle \gamma, \beta^\vee \rangle$ *, or*
- (iii) $\alpha = \gamma \vdash \beta$.

(c) *there exists* $\delta \in R_1$ *such that exactly one of the following holds:*

- (i) $(\alpha; \beta, \delta)$ *is a triangle and* $\alpha = \gamma$*,*
- (ii) $(\beta; \alpha, \gamma)$ *is a triangle and* $\beta = \delta$*,*
- (iii) $(\alpha, \beta, \gamma, \delta)$ *is a quadrangle,*
- (iv) $(\alpha; \beta, \gamma, \delta)$ *or a cyclic permutation of these four roots is a diamond.*

In all cases the root δ *is unique, namely* $\delta = \alpha - \beta + \gamma$*.*

Proof. (a) $\Longrightarrow$ (b): We put $\delta = \alpha - \beta + \gamma \in R_1$ and note $\alpha \neq \delta \neq \gamma$. From 18.5.1 we have the following inequalities:

$$\langle \delta, \alpha^\vee \rangle = 2 - \langle \beta, \alpha^\vee \rangle + \langle \gamma, \alpha^\vee \rangle \leqslant 2, \quad \text{hence} \quad 0 \leqslant \langle \gamma, \alpha^\vee \rangle \leqslant \langle \beta, \alpha^\vee \rangle, \tag{1}$$

$$\langle \delta, \beta^\vee \rangle = \langle \alpha, \beta^\vee \rangle - 2 + \langle \gamma, \beta^\vee \rangle \geqslant 0, \quad \text{hence} \quad \langle \alpha, \beta^\vee \rangle + \langle \gamma, \beta^\vee \rangle \geqslant 2, \tag{2}$$

$$\langle \delta, \gamma^\vee \rangle = \langle \alpha, \gamma^\vee \rangle - \langle \beta, \gamma^\vee \rangle + 2 \leqslant 2, \quad \text{hence} \quad 0 \leqslant \langle \alpha, \gamma^\vee \rangle \leqslant \langle \beta, \gamma^\vee \rangle. \tag{3}$$

We will first show $\alpha \not\perp \beta$. Assume to the contrary that $\alpha \perp \beta$. Then (1) implies $\alpha \perp \gamma$ and $\langle \delta, \alpha^\vee \rangle = 2$ so $\alpha \vdash \delta$ since $\alpha \neq \delta$. Similarly, (2) gives $\delta \perp \beta \vdash \gamma$. Hence $\delta \perp \gamma$ by 18.6(b.i). But then $(\delta - \alpha) \perp (\gamma - \beta) = (\delta - \alpha)$ yields the contradiction $\delta = \alpha$. Therefore $\langle \alpha, \beta^\vee \rangle > 0$ and by symmetry also $\langle \beta, \gamma^\vee \rangle > 0$.

For the remaining statements of (b) we can assume $\alpha \not\perp \gamma$. Suppose $\alpha \vdash \gamma$. Then $\beta \dashv \alpha$ follows from (1), and since $\beta \not\perp \gamma$ we obtain $\beta \top \gamma$ from 18.6(b.iii). But then (3) yields the contradiction $\delta \dashv \gamma \dashv \alpha$. Thus the possibility $\alpha \vdash \gamma$ does not occur, and by symmetry neither does $\alpha \dashv \gamma$. This leaves us with the possibilities $\alpha = \gamma$ and $\alpha \top \gamma$. In the first case, $\alpha \vdash \beta$ is immediate from (1). Suppose therefore that $\alpha \top \gamma$. The additional assumption $\langle \alpha, \beta^\vee \rangle = 2$ leads to $\langle \delta, \alpha^\vee \rangle = 2$ and hence to the contradiction $\delta \dashv \alpha \dashv \beta$. Therefore $\langle \alpha, \beta^\vee \rangle = 1$ and, by symmetry, then also $\langle \gamma, \beta^\vee \rangle = 1$ follows.

(b) $\Longrightarrow$ (c): We will use 18.4 to determine the elementary configuration generated by α, β and γ. First assume $\alpha \perp \gamma$. Because of 18.6 we then obtain the following cases: $\alpha \top \beta \top \gamma \perp \alpha$ leading to (iii), $\alpha \dashv \beta \vdash \gamma$ leading to (ii), $\alpha \top \beta \vdash \gamma$ and $\alpha \dashv \beta \top \gamma$ leading to (iv). If $\alpha \top \gamma$ we obtain the remaining two cases in (iv), and if $\alpha = \gamma$ we have (i).

(c) $\Longrightarrow$ (a): This follows from 18.4.

18.10. Corollary. *Let (R, R_1) and (S, S_1) be 3-graded root systems in X and Y respectively, and let $f\colon X \to Y$ be a linear map satisfying $f(R_1) \subset S_1$. Then the following assertions are equivalent:*

(i) *f is a morphism of 3-graded root systems, i.e., $f(R_i) \subset S_i$, $i = 0, \pm 1$,*
(ii) *$\langle \alpha, \beta^\vee \rangle > 0$ implies $\langle f(\alpha), f(\beta)^\vee \rangle > 0$ for all $\alpha, \beta \in R_1$,*
(iii) *$\langle \alpha, \beta^\vee \rangle \leqslant \langle f(\alpha), f(\beta)^\vee \rangle$ for all $\alpha, \beta \in R_1$.*

We recall from 11.7 that embeddings of 3-graded root systems can be characterized as maps $f\colon R_1 \to S$ satisfying *equality* in (iii) above.

Proof. (i) $\Longleftrightarrow$ (ii): Under our assumptions, f is a morphism of 3-graded root systems if and only if $f(R_0) \subset S_0$. In view of 18.5.3 this condition is equivalent to (ii).

(ii) $\Longrightarrow$ (iii): Because of 18.5.1 and the assumption (ii) it is enough to show $\langle \alpha, \beta^\vee \rangle = 2$ implies $\langle f(\alpha), f(\beta) \rangle = 2$. We may assume $\alpha \neq \beta$, hence $\alpha \dashv \beta$ and therefore $2\beta - \alpha \in R_1$ by Lemma 18.4(a) and Lemma 18.5(d). Applying f gives $2f(\beta) - f(\alpha) = f(\beta) - f(\alpha) + f(\beta) \in S_1$. By Prop. 18.9 we then either have $f(\alpha) = f(\beta)$ or $f(\alpha) \vdash f(\beta)$, hence in both cases $\langle f(\alpha), f(\beta)^\vee \rangle = 2$.

The implication (iii) $\Longrightarrow$ (ii) is obvious.

Examples. (i) For any (R, R_1) there exists a unique morphism $(R, R_1) \to \mathrm{A}_1^{\mathrm{coll}}$.

(ii) If $\alpha \in R_1$ is fixed, there exists a unique morphism $f_\alpha\colon (R, R_1) \to \mathrm{C}_2^{\mathrm{her}}$ with the property

$$R \cap f_\alpha^{-1}(\varepsilon_i + \varepsilon_j) = \{\beta \in R_1 : \langle \beta, \alpha^\vee \rangle = i + j\} \quad (i, j \in \{0, 1\}). \tag{1}$$

Indeed, let q be the minuscule coweight defining the 3-grading as in 17.6. Then

$$f_\alpha(x) = 2q(x)\varepsilon_0 + \langle x, \alpha^\vee \rangle (\varepsilon_1 - \varepsilon_0)$$

satisfies $f(R_i) \subset (\mathrm{C}_2^{\mathrm{her}})_i$ in view of 18.5.1 and 18.5.6.

(iii) Consider the 3-grading $\dot{\mathrm{A}}_I^J$ given by the partition $I = J \mathbin{\dot\cup} J'$ of the index set I, where J' is a second copy of J. Then there is a morphism $f\colon \dot{\mathrm{A}}_I^J \to \mathrm{C}_J^{\mathrm{her}}$ given by $f(\varepsilon_j) = \varepsilon_j$, $f(\varepsilon_{j'}) = -\varepsilon_j$ for all $j \in J$. If J has two elements, this collapses two opposite corners of a quadrangle to the middle of a double arrow, the other two corners becoming the starting points of the arrows. The reader may find it instructive to draw the corresponding picture for the morphism $\mathrm{A}_5^3 \to \mathrm{C}_3^{\mathrm{her}}$ where the graph of A_5^3 is already rather involved with nine vertices.

(iv) Let $I = \bigcup_{j \in J} I_j$ be a partition of I indexed by J. Then there is a morphism $f\colon \mathrm{C}_I^{\mathrm{her}} \to \mathrm{C}_J^{\mathrm{her}}$ with $f^{-1}(\varepsilon_j) = \{\varepsilon_i : i \in I_j\}$.

(v) Let $\dot{\mathrm{A}}_I^{\mathrm{coll}}$ be the collinear grading corresponding to the partition $I = \{0\} \mathbin{\dot\cup} J$. Then there is a morphism $f\colon \dot{\mathrm{A}}_I^{\mathrm{coll}} \to \mathrm{D}_I^{\mathrm{alt}} \subset \mathrm{C}_I^{\mathrm{her}}$ given by $f(\varepsilon_0) = \varepsilon_0$, $f(\varepsilon_j) = -\varepsilon_j$ for $j \in J$.

We finally specialize the presentations of $\mathcal{Q}(R)$ and $W(R)$ given in 11.12 and 11.17 to the situation where $P = R_0 \mathbin{\dot\cup} R_1$ is the parabolic subset given by a 3-grading.

18.11. COROLLARY. *For a 3-graded root system (R, R_1) the group $\mathcal{Q}(R)$ is isomorphic to the abelian group presented by generators x_α, $\alpha \in R_1$, and relations*

(i) $2x_\alpha = x_\beta + x_\gamma$ *for all triangles* $(\alpha; \beta, \gamma) \subset R_1$,
(ii) $x_\alpha + x_\gamma = x_\beta + x_\delta$ *for all quadrangles* $(\alpha, \beta, \gamma, \delta) \subset R_1$ *and all diamonds* $(\alpha; \beta, \gamma, \delta) \subset R_1$.

Proof. The relation 11.12.1 is vacuous since $(R_1 + R_1) \cap R = \emptyset$. So we only need to evaluate the relation 11.12.2 which is $x_\alpha + x_\gamma = x_\beta + x_\delta$ for all families $(\alpha, \beta, \gamma, \delta) \subset R_1$ satisfying $\alpha - \beta = \delta - \gamma \in R_0$, i.e., $\alpha - \beta + \gamma = \delta$. But those quadruples have been characterized in 18.9, with the result that 11.12.2 is equivalent to (i) and (ii) above.

18.12. COROLLARY. *Let (R, R_1) be a 3-graded root system. Then the Weyl group $W(R)$ is presented by generators t_α, $\alpha \in R_1$, and the following relations where always $\alpha, \beta, \gamma, \delta \in R_1$:*

$$t_\alpha^2 = 1, \tag{1}$$

$$t_\alpha t_\beta t_\alpha = \left\{ \begin{array}{ll} t_\beta & \textit{if } \alpha \perp \beta \\ t_\beta t_\alpha t_\beta & \textit{if } \alpha \top \beta \\ t_{2\alpha - \beta} & \textit{if } \alpha \vdash \beta \end{array} \right\}, \tag{2}$$

$$t_\beta t_\alpha t_\beta = t_\gamma t_\delta t_\gamma \quad \textit{if } (\alpha, \beta, \gamma, \delta) \textit{ is a quadrangle}$$
$$\textit{or } (\beta; \gamma, \delta, \alpha) \textit{ is a diamond,} \tag{3}$$

$$t_\beta \cdot t_\alpha t_\gamma t_\alpha = t_\alpha t_\gamma t_\alpha \cdot t_\beta \quad \textit{if } \alpha \top \gamma \textit{ and } \langle \beta, \alpha^\vee \rangle = 1 = \langle \beta, \gamma^\vee \rangle. \tag{4}$$

Proof. We apply 11.17 to the effective parabolic subset $P = R_0 \cup R_1$ with unipotent part $P_u = R_1$, and evaluate the relations (S1) – (S6) in our situation.

Since R is reduced and R_1 does not contain weakly orthogonal roots, the relations (S1) and (S4) are vacuous here.

To specialize the relation (S2), let $\alpha, \beta \in R_1$. Then $s_\alpha \beta \in R_1 \cup R_{-1}$ if and only if $\langle \beta, \alpha^\vee \rangle \in \{2, 0\}$ if and only if $\alpha = \beta$, $\alpha \vdash \beta$ or $\alpha \perp \beta$, and in these three cases the relation $t_\alpha t_\beta t_\alpha = t_{\pm s_\alpha \beta}$ becomes $t_\alpha^2 = 1$, $t_\alpha t_\beta t_\alpha = t_{2\alpha - \beta}$ and $t_\alpha t_\beta t_\alpha = t_\beta$ respectively. The first of these is (1), the remaining two together with (S3) yield (2).

We next evaluate (S5). Suppose $\mu = \alpha - \beta = \delta - \gamma \in R_0$ has two distinct standard representations. Since they are both of type I, the relation (S5) becomes $t_\beta t_\alpha t_\beta = t_\gamma t_\delta t_\gamma$. On the other hand, $\alpha \neq \beta \neq \gamma$ and $\alpha - \beta + \gamma = \delta \in R_1$ so that 18.9 applies. However, since $\langle \alpha, \beta^\vee \rangle = 1 = \langle \delta, \gamma^\vee \rangle$, among the cases in 18.9(c) only the following actually occur: (i) $(\alpha, \beta, \gamma, \delta)$ is a quadrangle, (ii) $(\beta; \gamma, \delta, \alpha)$ is a diamond, or (iii) $(\gamma; \delta, \alpha, \beta)$ is a diamond. Both cases (ii) and (iii) lead to the second possibility in (3).

Finally, the condition $s_\gamma(\beta) \in R_0$ of (S6) forces $\langle \beta, \gamma^\vee \rangle = 1$ and we are therefore left with the two possibilities $\alpha \top \beta \top \gamma$ and $\alpha \dashv \beta \vdash \gamma$, i.e., $\langle \beta, \alpha^\vee \rangle = 1 = \langle \beta, \gamma^\vee \rangle$. Thus (S6) becomes (4) above.

Example. Let $R = \dot{\mathrm{A}}_n = \mathrm{A}_{n-1}$ with the collinear grading $\mathrm{A}_{n-1}^{\mathrm{coll}}$ for which $R_1 = \{\varepsilon_1 - \varepsilon_i : 2 \leqslant i \leqslant n\}$. Since R_1 is a collinear family, the presentation above specializes to the following: $W(R) \cong \mathfrak{S}_n$ is generated by

$$h_i := h_{\varepsilon_1 - \varepsilon_i}, \quad 2 \leqslant i \leqslant n,$$

subject to the relations

$$\begin{aligned} &h_i^2 = 1 && (2 \leqslant i \leqslant n), \\ &(h_i h_j h_i)^3 = 1 && (2 \leqslant i < j \leqslant n), \\ &(h_i h_j h_k h_j)^2 = 1 && (i \neq j \neq k). \end{aligned}$$

This presentation of $\mathfrak{S}_n$ can already be found in Burnside's classical treatise [**15**, Note C].

Appendix A: Some standard results on finite root systems

For the convenience of the reader we list here some results on finite root systems from [**12**, VI, § 1]. We list only those results which are used frequently. In particular, it is not our intention to provide a summary of all properties of finite root systems, as given in [**12**, Résumé].

We use the notations and terminology introduced in the text. While these are quite similar to [**12**], there is an important difference inasmuch as our root systems contain 0 while the root systems in [**12**] do not. Some of the results below are stated and proven in [**12**, VI, §1] only for nonzero roots, but can easily be extended to our setting, which actually simplifies some statements. This straightforward exercise is left to the reader. Throughout, (R, X) is a finite root system and $R^\times = \{\alpha \in R : \alpha \neq 0\}$.

A.1. [**12**, VI, §1.1, Prop. 3 and Prop. 7] For $x, y \in X$ let

$$B(x,y) = \sum_{\alpha \in R^\times} \langle x, \alpha^\vee \rangle \langle y, \alpha^\vee \rangle \,.$$

Then B is an invariant inner product on X. If R is irreducible then any invariant inner product on X is a positive multiple of B. In the sequel, $(\ |\ )$ will always denote an arbitrary invariant inner product. We abbreviate $\|x\|^2 = (x, x)$ for $x \in X$.

A.2. *Relations between two roots* [**12**, VI, §1.3]. For $\alpha, \beta \in R^\times$ with $\|\alpha\|^2 \leqslant \|\beta\|^2$ there are exactly the following possibilities:

Case	$\langle \alpha, \beta^\vee \rangle$	$\langle \beta, \alpha^\vee \rangle$	$\angle(\alpha, \beta)$	$(\|\alpha\|^2 : \|\beta\|^2)$	order of $s_\alpha s_\beta$
1	0	0	$\pi/2$	indeterminate	2
2	1	1	$\pi/3$	$(1:1)$	3
3	-1	-1	$2\pi/3$	$(1:1)$	3
4	1	2	$\pi/4$	$(1:2)$	4
5	-1	-2	$3\pi/4$	$(1:2)$	4
6	1	3	$\pi/6$	$(1:3)$	6
7	-1	-3	$5\pi/6$	$(1:3)$	6
8	2	2	0	$(1:1)$	1
9	-2	-2	π	$(1:1)$	1
10	1	4	0	$(1:4)$	1
11	-1	-4	π	$(1:4)$	1

Obviously, in the cases 8 – 11 we have $\beta = s\alpha$ for $s = \pm 1, \pm 2$. Moreover [**12**, VI, §1.3, Prop. 8], if α and β are linearly independent and $\|\alpha\|^2 \leqslant \|\beta\|^2$ then $\langle \alpha, \beta^\vee \rangle \in \{0, \pm 1\}$.

A.3. [**12**, VI, §1.3, Th. 1 and Cor.] Let $\alpha, \beta \in R$.

(a) If $\langle \alpha, \beta^\vee \rangle > 0$ or, equivalently, if $(\alpha|\beta) > 0$ then $\alpha - \beta \in R$.

(b) If $\langle \alpha, \beta^\vee \rangle < 0$ or, equivalently, if $(\alpha|\beta) < 0$ then $\alpha + \beta \in R$.

(c) If $\alpha + \beta \notin R$ and $\alpha - \beta \notin R$ then $(\alpha|\beta) = 0$.

A.4. Lemma. *Let $\alpha_1, \alpha_2, \alpha_3 \in R^\times$ with $\alpha_1 + \alpha_2 + \alpha_3 = 0$, and let*

$$n = \max \Big\{ \frac{\|\alpha_i\|^2}{\|\alpha_j\|^2} : i, j = 1, 2, 3 \Big\}.$$

Then:

(a) *$n \in \{1, 2, 3, 4\}$, with $n = 4$ if and only if the α_i are multiples of each other.*

(b) *Either all three roots have the same length (the case $n = 1$), or two of them have the same length and the third one is longer. If, say, $\|\alpha_1\| = \|\alpha_2\| \leqslant \|\alpha_3\|$ then*

$$s_{\alpha_3}(\alpha_1) = -\alpha_2, \tag{1}$$

$$\alpha_1^\vee + \alpha_2^\vee + n\alpha_3^\vee = 0. \tag{2}$$

(c) *The Cartan numbers $\langle \alpha_i, \alpha_j^\vee \rangle$ for $i \neq j$ are determined by the following rules (where all three roots are considered short in case $n = 1$):*

$$\langle \text{short}, \text{short}^\vee \rangle = n - 2, \quad \langle \text{short}, \text{long}^\vee \rangle = -1, \quad \langle \text{long}, \text{short}^\vee \rangle = -n. \tag{3}$$

Proof. After renumbering, we may assume $\|\alpha_3\| \geqslant \|\alpha_1\|$ and $\|\alpha_3\| \geqslant \|\alpha_2\|$. First note that $(\alpha_3|\alpha_i) < 0$ for $i = 1, 2$. Indeed, assuming $(\alpha_3|\alpha_i) \geqslant 0$, let $j = 3 - i$. Then we would obtain $\|\alpha_j\|^2 = (\alpha_i + \alpha_3|\alpha_i + \alpha_3) = \|\alpha_i\|^2 + \|\alpha_3\|^2 + 2(\alpha_i|\alpha_3) > \|\alpha_3\|^2$, contradicting $\|\alpha_j\| \leqslant \|\alpha_3\|$. Hence the pair $(\alpha_i, \alpha_3) = (\alpha, \beta)$ is one of the cases $3, 5, 7, 9, 11$ of table A.2, and case 9 is impossible because there $\alpha = -\beta$. Now (a), $\langle \alpha_i, \alpha_3^\vee \rangle = -1$ and $\langle \alpha_3, \alpha_i^\vee \rangle = -n$ follow from A.2. Hence $s_{\alpha_3}(\alpha_i) = \alpha_i + \alpha_3 = -\alpha_j$ and therefore $\|\alpha_i\| = \|\alpha_j\|$, proving (b). Finally, $\langle \alpha_1, \alpha_2^\vee \rangle = \langle \alpha_2, \alpha_1^\vee \rangle = -\langle \alpha_1 + \alpha_3, \alpha_1^\vee \rangle = -2 + n$.

A.5. [**12**, VI, §1.3, Prop. 9 and Cor.] Let $\alpha \in R^\times$. Then for any $\beta \in R$ there exist $p, q \in \mathbb{N}$ such that

$$R \cap (\beta + \mathbb{Z}\alpha) = \{\beta + j\alpha : -q \leqslant j \leqslant p\} \quad \text{and} \quad q - p = \langle \beta, \alpha^\vee \rangle.$$

The set $R \cap (\beta + \mathbb{Z}\alpha)$, called the *$\alpha$-string through β*, is invariant under the reflection s_α. For $\gamma = \beta - q\alpha$ we have

$$-\langle \gamma, \alpha^\vee \rangle = p + q \leqslant 4.$$

In [**12**] this is only proven for linearly independent roots α, β in which case $p + q \leqslant 3$. However, the case where β and α are linearly dependent follows easily from A.2.

A.6. [**12**, VI, §1.4, Prop. 12] Let R be irreducible and reduced. Then

$$\left\{\frac{\|\alpha\|^2}{\|\beta\|^2} : \alpha, \beta \in R^\times\right\} \subset \{1, 2, \frac{1}{2}, 3, \frac{1}{3}\}$$

and

$$\operatorname{Card}\{\|\alpha\|^2 : \alpha \in R^\times\} \leqslant 2.$$

A.7. [**12**, V, §1.4, Prop. 13] Let R be an irreducible non-reduced root system. Suppose that $(\ |\ )$ is normalized such that $\min\{\|\alpha\|^2 : \alpha \in R^\times\} = 1$, and put $R_i = \{\alpha \in R : \|\alpha\|^2 = i\}$.

(i) The set R_{ind} of indivisible roots is an irreducible reduced root system in X satisfying $R_{\text{ind}} = \{0\} \cup R_1 \cup R_2$.

(ii) $R = \{0\} \cup R_1 \cup R_2 \cup R_4$ and $R_4 = 2R_1$. Two roots in R_1 are either proportional or orthogonal. If (R, X) has rank $\geqslant$2 then $R_2 \neq \emptyset$.

A.8. [**12**, VI, §1.4 Prop. 14] Let R be an irreducible reduced root system with two root lengths. Assume that the set R_{sh} of shorts roots has the property that two roots in R_{sh} are either proportional or orthogonal. Then $R' = R \cup 2R_{\text{sh}}$ is an irreducible non-reduced root system whose set of indivisible roots is R.

A.9. [**12**, VI, §1.5, Th. 2, §1.6, Th. 3 and §1.7, Cor. 3 of Prop. 20] Root bases exist. The Weyl group $W(R)$ operates simply transitively on the set of root bases of R. If B is a root basis of R, then $(W(R), \{s_\beta : \beta \in B\})$ is a Coxeter system, i.e., $W(R)$ is presented by generators $\{s_\alpha : \alpha \in B\}$ and relations $(s_\alpha s_\beta)^{m_{\alpha\beta}} = 1$, where $m_{\alpha\beta}$ is the order of $s_\alpha s_\beta$ in $W(R)$.

A.10. [**12**, VI, §1.5, Prop. 15] Let B be a root basis of R. For every root $\alpha \in R^\times$ there exists $w \in W(R)$ such that $w(\alpha) \in B$ or that $w(\alpha/2) \in B$.

A.11. [**12**, VI, §1.5 Cor. of Prop. 15] Let (R, X) and (R', X') be reduced root systems with root bases B and B' respectively. Suppose that $f\colon B \to B'$ is a bijective map preserving the Cartan integers $\langle \alpha, \beta^\vee \rangle$ for $\alpha, \beta \in B$. Then f extends to an isomorphism $f\colon (R, X) \to (R', X')$ of root systems.

A.12. [**12**, VI, §1.7, Prop. 24] Every root basis of a full subsystem of R is contained in a root basis of R.

A.13. [**12**, VI, §1.6, Prop. 18] Let B be a root basis of R and let $\overline{C} = \{x \in X : (x|\beta) \geqslant 0 \text{ for all } \beta \in B\}$ be the closed Weyl chamber corresponding to B. Then $x \in X$ lies in $\overline{C}$ if and only if $x - w(x) \in \mathbb{R}_+[B]$ if and only if $(x - w(x)|y) \geqslant 0$ for all $w \in W(R)$ and $y \in \overline{C}$.

We note that the equivalence of the two conditions arises from [**12**, VI, §1.5, Th. 2(vi)].

A.14. [**12**, VI, §1.6, Prop. 19] Let $\alpha = \beta_1 + \cdots + \beta_n$ be a sum of positive roots with respect to some root basis. Then there exists a permutation $\pi \in \mathfrak{S}_n$ such that $\beta_{\pi(1)} + \cdots + \beta_{\pi(i)}$ is a root for every i, $1 \leqslant i \leqslant n$.

A.15. [**12**, VI, §1.6, Cor. 2 of Prop. 19] Let $\varGamma$ be an abelian group and let φ: $R \to \varGamma$ be a map satisfying $\varphi(\alpha + \beta) = \varphi(\alpha) + \varphi(\beta)$ whenever α, β and $\alpha + \beta$ belong to R. Then φ extends to a unique group homomorphism $\mathcal{Q}(R) \to \varGamma$.

Remark. The present formulation is simpler than Bourbaki's, because $0 \in R$ in our setup. To see the equivalence to Bourbaki's, observe that for $\alpha = 0$ we have $\varphi(0) = \varphi(0+0) = \varphi(0) + \varphi(0)$ whence $\varphi(0) = 0$, and therefore also $0 = \varphi(\beta - \beta) = \varphi(\beta) + \varphi(-\beta)$, so $\varphi(-\beta) = -\varphi(\beta)$.

A.16. [**12**, VI, §1.7, Prop. 20] A subset P of R is parabolic if and only if there exists a root basis B of R and a subset $\varSigma$ of B such that

$$\alpha \in P \quad \iff \quad \alpha = \sum_{\beta \in B} n_\beta \beta, \qquad \text{where } n_\beta \geqslant 0 \text{ for } \beta \in B \setminus \varSigma.$$

A.17. [**12**, VI, §1.1, Prop. 1] The $\mathbb{Q}$-span $X_{\mathbb{Q}}$ of R is a rational form of X, i.e., $X_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \cong X$.

Appendix B: Cones defined by totally preordered sets

B.1. *Generalities on convex cones.* We refer to [**11**, II, §2.4] for terminology on cones in a real vector space Y. All cones considered here are convex and contain the origin. A cone C is called *proper* if $C \cap (-C) = \{0\}$.

We let Y^* denote the full algebraic dual of Y, and consider the vector spaces Y and Y^* in separating duality in the sense of [**11**, II, §6]. The vector space Y^* is endowed with the weak-$*$-topology $\sigma(Y^*, Y)$, i.e., the weakest topology making all evaluations $f \mapsto f(y)$ (for $y \in Y$) continuous. The *polar* of C,

$$C^\circ := \{f \in Y^* : f(y) \geqslant 0 \text{ for all } y \in C\},$$

is a weak-$*$-closed cone in Y^*. Note that $\operatorname{span} C = Y$ implies that C° is a proper cone. All $f \in C^\circ$ vanish on $Z = C \cap (-C)$, the largest subspace of Y contained in C, so C° can be considered in a natural way as a cone in $(Y/Z)^*$. The *double polar*

$$C^{\circ\circ} := \{y \in Y : f(y) \geqslant 0 \text{ for all } f \in C^\circ\}$$

is a cone in Y and obviously $C \subset C^{\circ\circ}$.

The cone C determines a partial preorder of Y, compatible with the vector space structure, by $x \geqslant y \iff x - y \in C$. Recall that an *extremal ray* of C [**11**, II, §7.2] is a half-line $\mathbb{R}_+ x \subset C$ such that $0 \leqslant y \leqslant x$ implies $y \in \mathbb{R}_+ x$; equivalently, $x = y + z$ (where $y, z \in C$) implies $y, z \in \mathbb{R}_+ x$. It is easy to see that only proper cones can have extremal rays. We denote by $\operatorname{extr}(C)$ the set of extremal rays of a proper cone C.

Suppose that C is given as the convex hull of a set of half-lines, say $C = \mathbb{R}_+[S]$, for some subset S of $Y^\times$, as will be the case for the cones considered below. If $0 \neq x = \sum c_i s_i \in C$ (with $s_i \in S$ and positive coefficients c_i) spans an extremal ray then $0 \leqslant c_i s_i \leqslant x$ for each i, whence all s_i are positive multiples of each other, and $x \in \mathbb{R}_+ s_i$. In particular:

An extremal ray of $\mathbb{R}_+[S]$ must be one of the generating rays $\mathbb{R}_+ s$, $s \in S$. (1)

B.2. *Total preorders.* Let I be a set. By a *total preorder on I* we mean a transitive relation $\succcurlyeq$ on I satisfying $i \succcurlyeq j$ or $j \succcurlyeq i$, for all $i, j \in I$. Note that any total preorder is reflexive. It is easily seen that

$$i \sim j \quad :\iff \quad i \succcurlyeq j \text{ and } j \succcurlyeq i. \tag{1}$$

is an equivalence relation on I, and $\succcurlyeq$ induces a total order $\geqslant$ on the set of equivalence classes $I/{\sim}$ by

$$[i] \geqslant [j] \quad :\iff \quad i \succcurlyeq j. \tag{2}$$

Clearly, $\succcurlyeq$ itself is a total order if and only if $\sim$ is equality. Conversely, every total preorder on I is obtained in this way from an equivalence relation and a total order on the set of equivalence classes. We use the symbol $i \succnsim j$ or $j \precnsim i$ for $i \succcurlyeq j$ and $i \not\sim j$, i.e., $[i] > [j]$. We will also use this symbol for subsets A, B of I where $A \succnsim B$ means $a \succnsim b$ for all $a \in A$ and $b \in B$. If $B = \{b\}$ we will simply write $A \succnsim b$. Analogous conventions apply to $\precnsim$.

A totally preordered set $(I, \succcurlyeq)$ may or may not contain a minimal element, i.e., an element 0 such that there is no $i \in I$ with $0 \succcurlyeq i$ and $0 \neq i$. If it does then 0 is unique, and we write $0 := \min(I, \succcurlyeq)$, called the *minimum* of I. In this case $\{0\}$ is an equivalence class of $\sim$, and $i \succcurlyeq 0$ for all $i \in I$. On the other hand, the set $M := \{m \in I : i \succcurlyeq m \text{ for all } i \in I\}$ may be empty or may contain more than one element. In fact, $M \neq \emptyset$ if and only if the totally ordered set $(I/\sim, \geqslant)$ has a minimum, in which case $M = \min(I/\sim, \geqslant)$ is a full equivalence class of $\sim$. Moreover, I contains a minimal element if and only if $|M| = 1$, in which case $M = \{0\}$.

A non-empty subset $\Sigma \subset I$ is said to be a *final (initial) segment* if $j \in \Sigma$ and $i \succcurlyeq j$ ($j \succcurlyeq i$) imply $i \in \Sigma$, i.e., if its characteristic function $\chi_\Sigma \colon I \to \{0, 1\} \subset \mathbb{R}$ is increasing (decreasing). Note that a final or initial segment is saturated with respect to the equivalence relation $\sim$. We denote by $\mathfrak{E}$ the set of final segments of $(I, \succcurlyeq)$, and let

$$\dot{\mathfrak{E}} = \{\Sigma \in \mathfrak{E} : \Sigma \neq I\}, \quad \ddot{\mathfrak{E}} = \{\Sigma \in \mathfrak{E} : |I \setminus \Sigma| \geqslant 2\}.$$

It is easily seen that $\mathfrak{E}$ is totally ordered by inclusion.

For an element $i \in I$, the *principal final segment defined by* i is denoted by

$$[i, \to[\ := \{j \in I : j \succcurlyeq i\}.$$

Suppose now that $(I, \geqslant)$ is a totally ordered set. As above we use the symbol $i > j$ for $i \geqslant j$ and $i \neq j$, and we write $0 := \min(I)$ for the (necessarily unique) minimum of I, provided it exists.

An element $i \in I$ is said to be a *predecessor* if the open interval $\{j \in I : j > i\}$ has a minimum, then called the *successor* of i. We denote by $\operatorname{pre}(I)$ the set of elements of I which are predecessors, and by $i + 1 = \min\{j \in I : j > i\}$ the successor of $i \in \operatorname{pre}(I)$. In particular, the successor of 0, if present, will be denoted by 1. Note that in a well-ordered set, every element different from $\max(I)$, the maximum of I (if present), is a predecessor.

B.3. *Cones of type* B. Let I be a set and let $X = \bigoplus_{i \in I} \mathbb{R}\varepsilon_i \cong \mathbb{R}^{(I)}$ be the free vector space on I. For any subset $\Sigma \subset I$ we let $q_\Sigma \in X^*$ denote the linear form defined by

$$q_\Sigma(\varepsilon_i) = \chi_\Sigma(i) = \left\{ \begin{matrix} 0 & \text{if } i \notin \Sigma \\ 1 & \text{if } i \in \Sigma \end{matrix} \right\}. \tag{1}$$

see also 8.9. We keep the notations of B.2 and let $I_0 \subset I$ be either empty or an initial segment. The cone

$$K := X_{I_0, \succcurlyeq} := \mathbb{R}_+\big[\{\varepsilon_i : i \in I\} \cup \{-\varepsilon_j : j \in I_0\} \cup \{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\}\big] \tag{2}$$

in X will be called the *cone of type* B *defined by* $(I, I_0, \succcurlyeq)$. In general, this is not a proper cone, see B.5(b) below for the description of $K \cap (-K)$. For example, a parabolic subset $\mathrm{T}_{I, I_0, \succcurlyeq}$ in a root system of type T_I, $\mathrm{T} = \mathrm{B}, \mathrm{C}$ or BC, spans such a cone, see 13.3 and Prop. 13.10(b).

A linear form $f \in X^*$ belongs to K° if and only if f is non-negative on the generators of K. Hence (2) shows that

$$f \in K^\circ \quad \iff \quad \begin{array}{l}\text{the map } i \mapsto f(\varepsilon_i),\ I \to \mathbb{R}, \text{ is increasing,} \\ \text{non-negative, and vanishes on } I_0.\end{array} \tag{3}$$

In particular, all q_Σ, where $\Sigma \subset I$ is a final segment not meeting I_0, belong to K°. We define the subspace $Z \subset X$ by

$$Z := \operatorname{span}\big(\{\varepsilon_j : j \in I_0\} \cup \{\varepsilon_i - \varepsilon_j : i \sim j\}\big). \tag{4}$$

Clearly, $Z \subset K \cap (-K)$, and therefore all $f \in K^\circ$ vanish on Z. Also, $X = \operatorname{span} K$ is obvious from (2), so K° is a proper cone in X^*.

B.4. Lemma. (a) *For every $x \in X$ there exist representations*

$$x = z + \xi_1 \varepsilon_{i_1} + \sum_{\nu=2}^{n} \xi_\nu (\varepsilon_{i_\nu} - \varepsilon_{i_{\nu-1}}) \tag{1}$$

where $z \in Z$, $\xi_\nu \in \mathbb{R}$ for $1 \leqslant \nu \leqslant n$, and $I_0 \precnsim i_1 \precnsim \cdots \precnsim i_n$.

(b) *Let $\Sigma \subset I$ be a final segment not meeting I_0, and let x be written as in* (1). *Then*

$$q_\Sigma(x) = \left\{ \begin{array}{ll} \xi_1 & \text{if } i_1 \in \Sigma \\ \xi_\nu & \text{if } i_{\nu-1} \notin \Sigma,\ i_\nu \in \Sigma\ (\nu = 2, \dots, n) \\ 0 & \text{if } i_n \notin \Sigma \end{array} \right\}. \tag{2}$$

In particular,

$$\xi_\nu = q_{[i_\nu, \to[}(x) \quad (\nu = 1, \dots, n). \tag{3}$$

Proof. (a) Write $x = \sum_{i \in F} c_i \varepsilon_i$ for some finite subset F of I, with coefficients $c_i \in \mathbb{R}$. Let $F_0 = F \cap I_0$, and decompose $F \setminus F_0 = F_1 \dot\cup \cdots \dot\cup F_n$ (where possibly $n = 0$) into equivalence classes with respect to $\sim$. As $\succcurlyeq$ is a total preorder, we may assume $F_1 \precnsim \cdots \precnsim F_n$. Then $F_\nu \succnsim F_0$ for $\nu \geqslant 1$ because I_0 is either empty or an initial segment. Since the F_ν are not empty for $\nu \geqslant 1$, we can choose elements $i_\nu \in F_\nu$. Put $d_\nu := \sum_{i \in F_\nu} c_i$ and $y := \sum_{\nu=1}^n d_\nu \varepsilon_{i_\nu}$. Then $x = \sum_{j \in F_0} c_j \varepsilon_j + \sum_{\nu=1}^n \sum_{i \in F_\nu} c_i \varepsilon_i$ and hence

$$z := x - y = \sum_{j \in F_0} c_j \varepsilon_j + \sum_{\nu=1}^{n} \sum_{i \in F_\nu} c_i (\varepsilon_i - \varepsilon_{i_\nu}) \in Z$$

because $F_0 \subset I_0$ and $i \sim i_\nu$ for $i \in F_\nu$. Moreover, by partial summation, $y = \xi_1 \varepsilon_{i_1} + \sum_{\nu=2}^n \xi_\nu (\varepsilon_{i_\nu} - \varepsilon_{i_{\nu-1}})$ where $\xi_\nu = \sum_{\lambda=\nu}^n d_\lambda$. This shows that x has a representation of the form (1).

(b) As noted in B.3, $q_\Sigma \in K^\circ$ and hence q_Σ vanishes on Z. Now (2) follows easily from (1) and the fact that Σ is a final segment, and (3) is a special case of (2).

B.5. Lemma. *Let $K \subset X$ be the cone of type* B *defined by $(I, I_0, \succcurlyeq)$.*

(a) *For an element $x \in X$ the following conditions are equivalent:*

(i) $x \in K$,

(ii) $x \in K^{\circ\circ}$,

(iii) $q_\Sigma(x) \geqslant 0$ *for all final segments* Σ *of* I *not meeting* I_0,
(iv) $q_\Sigma(x) \geqslant 0$ *for all principal final segments* Σ *of* I *not meeting* I_0,
(v) $\xi_\nu \geqslant 0$ *for every representation of* x *in the form* B.4.1.

(b) $K \cap (-K) = Z$. *In particular,* K *is a proper cone if and only if* I_0 *is empty and* $\succcurlyeq$ *is a total order.*

Proof. (a) The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are obvious or follow from the fact that $q_\Sigma \in K^\circ$. The implication (iv) $\Longrightarrow$ (v) follows from B.4.3 and (v) $\Longrightarrow$ (i) from the definition of K, $Z \subset K$ and the fact that every $x \in X$ has a representation of the form B.4.1.

(b) As noted before, $Z \subset K \cap (-K)$. Conversely, if $x \in K \cap (-K)$ then (v) shows that all ξ_ν vanish, so $x = z \in Z$ by B.4.1.

B.6. Proposition. *Let* K *be the cone of type* B *defined by* $(I, I_0, \succcurlyeq)$, *and let* $K^\circ = \{f \in X^* : f(K) \geqslant 0\}$ *be its polar.*

(a) *The extremal rays of* K° *are precisely the rays spanned by the linear forms* q_Σ, *where* $\Sigma \subset I$ *is a final segment not meeting* I_0.

(b) *Let* K *be a proper cone, so* $I_0 = \emptyset$ *and* $\succcurlyeq$ *is a total order on* I *which we denote by* $\geqslant$. *Also let* $\mathrm{pre}(I)$ *be the set of elements* $j \in I$ *which are predecessors, and thus have successor* $j+1 = \min\{i \in I : i > j\}$, *cf.* B.2. *Then the extremal rays of* K *are spanned by the* $\varepsilon_{j+1} - \varepsilon_j$ *where* $j \in \mathrm{pre}(I)$, *and by* ε_0, *where* 0 *(if present) is the minimum of the totally ordered set* $(I, \geqslant)$.

Remark. In general I need not contain a minimum or elements which have a successor, so it may well happen that K has no extremal rays.

Proof. (a) Let Σ be as indicated and suppose $0 \leqslant f \leqslant q_\Sigma$ for some $f \in X^*$. Then $0 \leqslant f(\varepsilon_j) \leqslant q_\Sigma(\varepsilon_j) = 0$ for all $j \in I \setminus \Sigma$, and $0 \leqslant f(\varepsilon_i - \varepsilon_j) \leqslant q_\Sigma(\varepsilon_i - \varepsilon_j) = 0$ for all $i \succcurlyeq j$ whenever both $i, j \in \Sigma$. Hence $f(\varepsilon_i) = c$ for all $i \in \Sigma$ so $f = cq_\Sigma$. Also, $c \geqslant 0$ because $0 \leqslant f(\varepsilon_i) \leqslant q_\Sigma(\varepsilon_i) = 1$ for $i \in \Sigma$. Conversely, let $f \in K^\circ$ span an extremal ray, and put $a_i = f(\varepsilon_i)$. Since f vanishes on $K \cap (-K) = Z$, we have $f(\varepsilon_j) = 0$ for $j \in I_0$ and $f(\varepsilon_i) = f(\varepsilon_j)$ for $i \sim j$. Now assume f is not a positive multiple of some q_Σ where Σ is a final segment not meeting I_0. Then there exist $i_1 \precnsim i_2$ in $I \setminus I_0$ such that $0 < a_{i_1} < a_{i_2}$. Define $g \in X^*$ by

$$g(\varepsilon_i) = \left\{ \begin{array}{ll} a_i & \text{if } i \preccurlyeq i_1 \\ a_{i_1} & \text{if } i \succnsim i_1 \end{array} \right\}. \tag{1}$$

Then it is immediate that $0 \leqslant g \leqslant f$, and $g(\varepsilon_{i_1}) = a_{i_1} \neq 0$, so $g \neq 0$. By extremality, $g = cf$ with $c \neq 0$, which leads to a contradiction when evaluated on $\varepsilon_{i_2} - \varepsilon_{i_1}$.

(b) As observed in B.1.1, an extremal ray of K must be spanned by one of the generators ε_i and $\varepsilon_i - \varepsilon_j$, $i > j$. Suppose i is not the minimum of I and choose $j < i$. Then $\varepsilon_i = \varepsilon_j + (\varepsilon_i - \varepsilon_j)$ shows that $\mathbb{R}_+\varepsilon_i$ is not an extremal ray. On the other hand, let 0 be the minimum of I, and suppose $0 \leqslant x \leqslant \varepsilon_0$ for the partial order induced on X by K. Then $0 \leqslant q_\Sigma(x) \leqslant q_\Sigma(\varepsilon_0) = 0$ for all final segments Σ not containing 0, i.e., $\Sigma \neq I$. This easily implies $x = c\varepsilon_0$ for some $0 \leqslant c \leqslant 1$.

Next, let $i > j$, and assume i is not the successor of j. Then there exists k such that $i > k > j$, and hence $\varepsilon_i - \varepsilon_j = (\varepsilon_i - \varepsilon_k) + (\varepsilon_k - \varepsilon_j)$ shows that

$\varepsilon_i - \varepsilon_j$ does not generate an extremal ray. On the other hand, let $i = j+1$ be the successor of j, and suppose $0 \leqslant x \leqslant \varepsilon_{j+1} - \varepsilon_j$. By condition (iii) of B.5(a), this means $0 \leqslant q_\Sigma(x) \leqslant q_\Sigma(\varepsilon_{j+1} - \varepsilon_j)$ for all final segments Σ of I not meeting I_0. Now $q_\Sigma(\varepsilon_{j+1} - \varepsilon_j) \neq 0$ if and only if $\Sigma = [j+1, \rightarrow[$, and hence also $q_\Sigma(x) \neq 0$ if and only if $\Sigma = [j+1, \rightarrow[$. We may assume $x \neq 0$. Write $x = \xi_1 \varepsilon_{i_1} + \sum_{\nu=2}^n \xi_\nu(\varepsilon_{i_\nu} - \varepsilon_{i_{\nu-1}})$ with $i_1 < \cdots < i_n$ as in B.4.1. Then $\xi_\nu = q_{[i_\nu, \rightarrow[}(x)$ by B.4.3, so we either have $x = \xi_1 \varepsilon_{i_1}$ and $j+1 = i_1$, or $x = \xi_\nu(\varepsilon_{i_\nu} - \varepsilon_{i_{\nu-1}})$ for some $\nu \geqslant 2$, and $j+1 = i_\nu$. In the first case, also $q_{[j,\rightarrow[}(x) = \xi_1 \neq 0$, contradiction. Thus we are in the second case, and it remains to show that $i_{\nu-1} = j$. Assume to the contrary that $i_{\nu-1} < j < i_\nu = j+1$. Then $q_{[j,\rightarrow[}(x) = \xi_\nu \neq 0$, which is impossible. Hence $i_{\nu-1} = j$ and $x = \xi_\nu(\varepsilon_{j+1} - \varepsilon_j)$, as asserted.

B.7. *Cones of type* $\dot{\mathrm{A}}$. Let again $(I, \succcurlyeq)$ be a totally preordered set and X the free vector space over I. We keep the notations of B.2 and B.3 and let $t = q_I$, the trace form of X, cf 8.9. Put $\dot{X} = \mathrm{Ker}(t)$ and define

$$\dot{K} := \dot{X}_\succcurlyeq := \mathbb{R}_+\big[\{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\}\big], \tag{1}$$

called the *cone of type* $\dot{\mathrm{A}}$ *defined by* $(I, \succcurlyeq)$. It is easy to see that $\dot{X}$ is spanned by all differences $\varepsilon_i - \varepsilon_j$. Since either $i \succcurlyeq j$ or $j \succcurlyeq i$ holds, we see that $\dot{K}$ spans $\dot{X}$. The cones $\dot{K}$ are the cones spanned by parabolic subsets $\dot{\mathrm{A}}_{I,\succcurlyeq}$ in the root system $\dot{\mathrm{A}}_I$, see 13.10(b).

The restriction of a linear form $f \in X^*$ to $\dot{X}$ is denoted by $\dot{f}$. Then the map $f \mapsto \dot{f}$, $X^* \to (\dot{X})^*$, is surjective with kernel $\mathbb{R}t$. In particular, for a subset Σ of I we have $0 = \dot{q}_I = \dot{q}_\Sigma + \dot{q}_{I\setminus\Sigma}$, so $-\dot{q}_\Sigma = \dot{q}_{I\setminus\Sigma}$. Note that the polar of $\dot{K}$ is described by

$$\dot{f} \in \dot{K}^\circ \quad \iff \quad \text{the map } i \mapsto f(\varepsilon_i),\ I \to \mathbb{R}, \text{ is increasing.} \tag{2}$$

In particular, all $\dot{q}_\Sigma$, where Σ is a final segment, belong to $\dot{K}^\circ$. We put

$$\dot{Z} = \mathrm{span}\{\varepsilon_i - \varepsilon_j : i \sim j\}$$

and note that $\dot{Z} \subset \dot{K} \cap (-\dot{K})$.

Specializing Lemma B.4 to the case $I_0 = \emptyset$ and noting that $\xi_1 = t(x)$, we see that every $x \in \dot{X}$ has a representation

$$x = z + \sum_{\nu=2}^n \xi_\nu(\varepsilon_{i_\nu} - \varepsilon_{i_{\nu-1}}) \tag{3}$$

where $z \in \dot{Z}$ and $i_1 \precnsim \cdots \precnsim i_n$, whence $\dot{K} = K \cap \dot{X}$. Moreover,

$$q_\Sigma(x) = \left\{ \begin{matrix} \xi_\nu & \text{if } i_{\nu-1} \notin \Sigma \ni i_\nu \ (\nu = 2, \ldots, n) \\ 0 & \text{otherwise} \end{matrix} \right\},$$

for all final segments Σ. Also, one shows as in the proof of Lemma B.5 that the following conditions are equivalent for $x \in \dot{X}$:

(i) $x \in \dot{K}$,
(ii) $x \in \dot{K}^{\circ\circ}$,
(iii) $\dot{q}_\Sigma(x) \geqslant 0$ for all final segments Σ of I,
(iv) $\dot{q}_\Sigma(x) \geqslant 0$ for all principal final segments Σ of I,
(v) $\xi_\nu \geqslant 0$ for every representation of x in the form (3).

Furthermore, $\dot{K} \cap (-\dot{K}) = \dot{Z}$; in particular, $\dot{K}$ is a proper cone if and only if $\succcurlyeq$ is a total order.

The analogue of Prop. B.6 for cones of type $\dot{\mathrm{A}}$ is now:

B.8. PROPOSITION. *Let $\dot K \subset \dot X$ be the cone of type $\dot{\mathrm{A}}$ defined by $(I, \succcurlyeq)$.*

(a) *The extremal rays of $\dot K^\circ$ are precisely the rays spanned by the linear forms $\dot q_\Sigma$ where $\Sigma \in \dot{\mathfrak{E}}$ is a final segment $\neq I$.*

(b) *Let $\dot K$ be proper, so $\succcurlyeq$ is a total order on I, denoted $\geqslant$. Then the extremal rays of $\dot K$ are spanned by the vectors $\varepsilon_{j+1} - \varepsilon_j$ where $j \in \mathrm{pre}(I)$ has successor $j+1$.*

The proof is similar to that of Prop. B.6. The details are left to the reader.

B.9. *Cones of type* D. We now introduce a variation of the cones of type B considered in B.3. Let $(I, \succcurlyeq)$ be a set with at least 2 elements and a total preorder $\succcurlyeq$, and denote again by X the free vector space over I. We assume that I has a minimal element 0, necessarily unique, see B.2. The *cone of type* D *defined by* $(I, \succcurlyeq, 0)$ is

$$K_0 := X_{\succcurlyeq,0} := \mathbb{R}_+\big[\{\varepsilon_i + \varepsilon_0 : i \neq 0\} \cup \{\varepsilon_i - \varepsilon_j : i \succcurlyeq j\}\big]. \tag{1}$$

From $|I| \geqslant 2$ it follows easily that $\operatorname{span} K_0 = X$. By Prop. 13.10(b) the cones of type D are precisely the cones spanned by parabolic subsets $\mathrm{D}_{I,\succcurlyeq}$ where $(I, \succcurlyeq)$ has a minimal element 0.

For any subset $\Sigma \subset I$ let q_Σ be defined as in B.3.1. We also introduce linear forms $q_\pm$ by

$$q_\pm(\varepsilon_0) = \pm\frac{1}{2}, \qquad q_\pm(\varepsilon_i) = \frac{1}{2} \quad \text{for } i \neq 0. \tag{2}$$

Thus we have

$$q_+ = \frac{1}{2} q_I, \quad q_+ + q_- = q_{I\setminus\{0\}}, \quad q_+ - q_- = q_{\{0\}}. \tag{3}$$

A linear form $f \in X^*$ belongs to K_0° if and only if f is non-negative on the generators of K_0; i.e.,

$$f \in K_0^\circ \quad \iff \quad i \mapsto f(\varepsilon_i) \text{ is increasing, and } f(\varepsilon_i) \geqslant -f(\varepsilon_0) \text{ for all } i \neq 0. \tag{4}$$

In particular, $q_\Sigma \in K_0^\circ$ for all final segments Σ of I, and also $q_\pm \in K_0^\circ$. Note that the linear map σ_0 of X defined by

$$\sigma_0(\varepsilon_i) = \left\{ \begin{matrix} -\varepsilon_0 & \text{if } i = 0 \\ \varepsilon_i & \text{if } i \neq 0 \end{matrix} \right\}$$

is an automorphism of K_0 satisfying

$$q_- = q_+ \circ \sigma_0. \tag{5}$$

We finally define the subspace

$$Z = \operatorname{span}\{\varepsilon_i - \varepsilon_j : i \sim j\} \tag{6}$$

in analogy to B.3.4 and remark that again $Z \subset K_0 \cap (-K_0)$. The counterpart of Lemma B.4 is now:

B.10. LEMMA. *We keep the assumptions and notations introduced in* B.9.

(a) *Every $x \in X$ has a representation*

$$x = z + \xi_+(\varepsilon_{i_1} + \varepsilon_0) + \xi_-(\varepsilon_{i_1} - \varepsilon_0) + \sum_{\nu=2}^{n} \xi_\nu(\varepsilon_{i_\nu} - \varepsilon_{i_{\nu-1}}) \tag{1}$$

where $z \in Z$, $\xi_\pm, \xi_\nu \in \mathbb{R}$ and $0 \precnsim i_1 \precnsim \cdots \precnsim i_n$.

(b) *Let $\Sigma \subset I$ be a final segment, and let x be written as in* (1). *Then*

$$q_\pm(x) = \xi_\pm, \tag{2}$$

$$q_\Sigma(x) = \left\{ \begin{array}{ll} 2\xi_+ & \textit{if } 0 \in \Sigma \\ \xi_+ + \xi_- & \textit{if } 0 \notin \Sigma \ni i_1 \\ \xi_\nu & \textit{if } i_{\nu-1} \notin \Sigma \ni i_\nu \ (\nu = 2, \dots, n) \\ 0 & \textit{if } i_n \notin \Sigma \end{array} \right\}. \tag{3}$$

In particular,

$$\xi_\nu = q_{[i_\nu, \to[}(x) \quad \textit{for } \nu = 2, \dots, n. \tag{4}$$

Proof. (a) We apply Lemma B.4(a) with $I_0 = \{0\}$. Denoting by Z_B the space defined in B.3.4, we have $Z_B = \mathbb{R}\varepsilon_0 \oplus Z$ where Z is as in B.9.6. Hence every $x \in X$ has a representation

$$x = z + \xi_0\varepsilon_0 + \xi_1\varepsilon_{i_1} + \sum_{\nu=2}^{n} \xi_\nu(\varepsilon_{i_\nu} - \varepsilon_{i_{\nu-1}})$$

where $z \in Z$, $\xi_\nu \in \mathbb{R}$ and $0 \precnsim i_1 \precnsim \cdots \precnsim i_n$. Now (1) follows from $\xi_0\varepsilon_0 + \xi_1\varepsilon_{i_1} = \xi_+(\varepsilon_{i_1} + \varepsilon_0) + \xi_-(\varepsilon_{i_1} - \varepsilon_0)$ for appropriate $\xi_\pm \in \mathbb{R}$.

(b) This follows easily from the fact that Σ is a final segment and the definition of q_Σ and $q_\pm$.

B.11. LEMMA. *Let K_0 be the cone of type* D *defined by $(I, \succcurlyeq, 0)$. We keep the notations and assumptions of* B.9 *and use the notation $\ddot{\mathfrak{E}}$ of* B.2 *for the set of final segments Σ with $|I \setminus \Sigma| \geqslant 2$.*

(a) *For an element $x \in X$ the following conditions are equivalent:*

- (i) $x \in K_0$,
- (ii) $x \in K_0^{\circ\circ}$,
- (iii) $q_\pm(x) \geqslant 0$ *and* $q_\Sigma(x) \geqslant 0$ *for all* $\Sigma \in \ddot{\mathfrak{E}}$,
- (iv) $q_\pm(x) \geqslant 0$ *and* $q_\Sigma(x) \geqslant 0$ *for all principal final segments* $\Sigma \in \ddot{\mathfrak{E}}$,
- (v) $\xi_\pm \geqslant 0$ *and* $\xi_\nu \geqslant 0$ *for every representation of x in the form* B.10.1.

(b) $K_0 \cap (-K_0) = Z$ *as in* B.9.6. *In particular, K_0 is a proper cone if and only if $\succcurlyeq$ is a total order.*

This is an easy consequence of Lemma B.10, and is proven in the same way as Lemma B.5. The details are left to the reader.

B.12. PROPOSITION. *Let K_0 be the cone of type* D *defined by $(I, \succcurlyeq, 0)$ and let K_0° be its polar.*

(a) *The extremal rays of K_0° are precisely the rays spanned by $q_\pm$ and by the q_Σ where $\Sigma \in \ddot{\mathfrak{E}}$.*

(b) *Let K be a proper cone, so $\succcurlyeq$ is a total order, written $\geqslant$. Then the extremal rays of K_0 are spanned by the $\varepsilon_{j+1} - \varepsilon_j$ where $j \in \mathrm{pre}(I)$ has successor $j+1$, and by $\varepsilon_1 + \varepsilon_0$, if 0 has successor* 1.

Proof. (a) We first show that $q_\pm$ and the indicated q_Σ are extremal. Assume $f \in X^*$ and $0 \leqslant f \leqslant q_+$. Then in particular $0 \leqslant f(\varepsilon_i - \varepsilon_j) \leqslant q_+(\varepsilon_i - \varepsilon_j) = 0$ for all $i \succcurlyeq j$, so $f(\varepsilon_i) = c$ for all $i \in I$, and thus $f = 2cq_+$. Also, $0 \leqslant f(\varepsilon_i + \varepsilon_0) = 2c$ for all $i \neq 0$, so $c \geqslant 0$. Thus q_+ spans an extremal ray of K_0°. Since σ_0 is an automorphism of K_0 with $q_+ \circ \sigma_0 = q_-$ by B.9.5, it follows that also q_- spans an extremal ray of K_0. Next, let $\Sigma \in \ddot{\mathfrak{E}}$, and let again $f \in X^*$ with $0 \leqslant f \leqslant q_\Sigma$. Then for all $i \succcurlyeq j$, we have $0 \leqslant f(\varepsilon_i - \varepsilon_j) \leqslant q_\Sigma(\varepsilon_i - \varepsilon_j) = 0$ if both $i, j \in \Sigma$ or both $i, j \notin \Sigma$. This shows that there exist $b, c \in \mathbb{R}$ such that $f(\varepsilon_i) = \left\{\begin{matrix} b & \text{if } i \notin \Sigma \\ c & \text{if } i \in \Sigma \end{matrix}\right\}$. Since $I \setminus \Sigma$ has at least two elements and Σ is a final segment, there exists an element $j \in I$ with $j \neq 0$ and $j \notin \Sigma$. Then $\varepsilon_j + \varepsilon_0 \in K_0$ and hence $0 \leqslant f(\varepsilon_j + \varepsilon_0) = 2b \leqslant q_\Sigma(\varepsilon_j + \varepsilon_0) = 0$, so $b = 0$ and $f = cq_\Sigma$. Moreover, for $i \in \Sigma$ we have $\varepsilon_i + \varepsilon_0 \in K_0$ and hence $0 \leqslant f(\varepsilon_i + \varepsilon_0) = c$. Thus q_Σ spans an extremal ray of K_0°.

Conversely, let $\mathbb{R}_+ f$ be an extremal ray of K_0°. Then f can take at most two values on the basis $\{\varepsilon_i : i \in I\}$ of X. Indeed, let $a_i := f(\varepsilon_i)$, and assume that there exist $i_0 \precnsim i_1 \precnsim i_2$ such that $a_{i_0} < a_{i_1} < a_{i_2}$. Define g by B.6.1. Then it is easily verified that $0 \leqslant g \leqslant f$, and $g \neq 0$ because $g(\varepsilon_{i_1} - \varepsilon_{i_0}) = a_{i_1} - a_{i_0} > 0$. By extremality, $g = cf$ for some $c \neq 0$, which leads to a contradiction when evaluated on $\varepsilon_{i_2} - \varepsilon_{i_1}$. Now we distinguish the following cases:

Case 1: $a_i = c$ for all $i \in I$: Then $f = 2cq_+$.

Case 2: $\{a_i : i \in I\} = \{c_0, c_1\}$ where $c_0 < c_1$.

Subcase 2.1: $c_0 < 0$: Then by B.9, $c_0 = f(\varepsilon_0)$ and $f(\varepsilon_i) \geqslant -f(\varepsilon_0) = -c_0$ for all $i \neq 0$, and therefore $f(\varepsilon_i) = c_1$. Hence $f = (c_1 + c_0)q_+ + (c_1 - c_0)q_-$. As $c_1 + c_0 \geqslant 0$ and $c_1 - c_0 \geqslant -c_0 > 0$, it follows from extremality that $c_1 + c_0 = 0$, so $f = (c_1 - c_0)q_-$.

Subcase 2.2: $c_0 \geqslant 0$: Then $f = c_0 q_{I \setminus \Sigma} + c_1 q_\Sigma = 2c_0 q_+ + (c_1 - c_0)q_\Sigma$, where $\Sigma := \{i \in I : f(\varepsilon_i) = c_1\}$. The map $i \mapsto f(\varepsilon_i)$ is increasing by B.9 so Σ is a final segment. Since f spans an extremal ray and $c_0 \geqslant 0$, $c_1 - c_0 > 0$ we must have $c_0 = 0$, so $f = c_1 q_\Sigma$. Furthermore, $I \setminus \Sigma$ has at least 2 elements, otherwise we would have $\Sigma = I \setminus \{0\}$, but $q_{I \setminus \{0\}} = q_+ + q_-$ (by B.9) is not extremal.

(b) By B.1.1, an extremal ray of K_0 must be spanned by one of the generators $\varepsilon_i + \varepsilon_0$, $i > 0$, and $\varepsilon_i - \varepsilon_j$, $i > j$. Observe that all q_Σ (where $\Sigma \subset I$ is a final segment) take non-negative values on K_0. Hence the proof of Prop. B.6(b) can be copied and shows that $\varepsilon_i - \varepsilon_j$ is extremal if and only if $i = j+1$ is the successor of j. Also, $\varepsilon_i + \varepsilon_0$ is not extremal unless $i = 1$ is the successor of 0, because $0 < j < i$ implies $\varepsilon_i + \varepsilon_0 = (\varepsilon_i - \varepsilon_j) + (\varepsilon_j + \varepsilon_0)$. On the other hand, if 0 has successor 1 then $\varepsilon_1 + \varepsilon_0$ does span an extremal ray: Suppose $0 \leqslant x \leqslant \varepsilon_1 + \varepsilon_0$ and write $x = \xi_+(\varepsilon_{i_1} + \varepsilon_0) + \xi_-(\varepsilon_{i_1} - \varepsilon_0) + \sum_{\nu=2}^n \xi_\nu(\varepsilon_{i_\nu} - \varepsilon_{i_{\nu-1}})$ as in B.10.1, where $0 < i_1 < \cdots < i_n$. Then by B.10(b) and condition (iii) of B.11(a), we have

$$\begin{aligned}
&0 \leqslant q_+(x) = \xi_+ \leqslant q_+(\varepsilon_1 + \varepsilon_0) = 1,\\
&0 \leqslant q_-(x) = \xi_- \leqslant q_-(\varepsilon_1 + \varepsilon_0) = 0,\\
&0 \leqslant q_{[i_\nu,\to[}(x) = \xi_\nu \leqslant q_{[i_\nu,\to[}(\varepsilon_1 + \varepsilon_0) = 0 \quad \text{for } \nu \geqslant 2,
\end{aligned}$$

because $1 \leqslant i_1 < i_\nu$ for $\nu \geqslant 2$. Hence $x = \xi_+(\varepsilon_1 + \varepsilon_0)$, as desired.

B.13. Lemma. *Let S be a subset of the polar C° of a cone $C \subset Y$, and suppose $C = \{y \in Y : f(y) \geqslant 0$ for all $f \in S\}$. Then $\mathbb{R}_+[S]$, the convex subcone of C° generated by S, is weak-$*$-dense in C°.*

Proof. Let $M = \mathbb{R}_+[S]$. Then clearly $C = \{y \in Y : f(y) \geqslant 0 \text{ for all } f \in M\} = M^\circ$, and hence $M^{\circ\circ} = C^\circ$. On the other hand, the Bipolar Theorem [**11**, Chap. II, §6.3, Th. 1], applied to the pair of vector spaces (Y^*, Y), shows that $M^{\circ\circ}$ is the weak-$*$-closure of M. Thus M is weak-$*$-dense in C°.

B.14. Corollary. *Let C be one of the cones $K, \dot{K}, K_0$ of types* B, Ȧ, D, *respectively. Then the convex hull of the union of all extremal rays of C° is weak-$*$-dense in C°.*

Proof. This follows from B.13, the description of $\operatorname{extr}(C^\circ)$ given in B.6(a), B.8(a), B.12(a), and the description of $C^{\circ\circ} = C$ given in B.5(a), B.7, and B.11(a).

By this corollary, an element $f \in C^\circ$ is the limit, in the weak-$*$-topology, of a net (g_λ) where each g_λ is a convex linear combination of elements in extremal rays of C°. Under a suitable discreteness condition, the following more precise result is possible:

B.15. Theorem. *Let $C \subset Y$ be one of the cones $\dot{K} \subset \dot{X}$ or $K, K_0 \subset X$ of types* Ȧ, B, D, *respectively. Let $f \in C^\circ \subset Y^*$ be a linear form with the property that for some $k \in I$,*

$$\Delta_k := \{f(\varepsilon_i - \varepsilon_k) : i \in I\} \quad \textit{is a discrete subset of } \mathbb{R}. \tag{1}$$

Then f has a representation as a weak-$*$-convergent series

$$f = \sum_{\varrho \in \operatorname{extr}(C^\circ)} f_\varrho \tag{2}$$

where $f_\varrho \in \varrho$, and $f_\varrho \neq 0$ for at most countably many ϱ. Moreover, Δ_k is bounded if and only if $f_\varrho \neq 0$ for only finitely many ϱ.

Remarks. (a) Convergence of (2) means convergence of the net $g_F := \sum_{\varrho \in F} f_\varrho$ in the weak-$*$-topology of X^*, where F runs over the directed set of finite subsets of $\operatorname{extr}(C^\circ)$. By definition, the net (g_F) converges in the weak-$*$-topology if and only if the net $(g_F(y))$ of real numbers converges for every $y \in Y$. For this to be the case, it is sufficient (and necessary) that $(g_F(y))$ converges for all y in a spanning set of Y.

(b) Condition (1) makes sense because the $\varepsilon_i - \varepsilon_j$ belong to Y in any case. Moreover, if it holds for one $k \in I$ then it holds for all $l \in I$, for $\Delta_l = \Delta_k + f(\varepsilon_l - \varepsilon_k)$. On the other hand, (1) does not imply that the set $\{f(\varepsilon_i - \varepsilon_k) : i, k \in I\}$ is discrete. For example, let $I = \mathbb{N}$ with its natural order, let $I_0 = \emptyset$ and define f by $f(\varepsilon_{2n}) = n+1$, $f(\varepsilon_{2n+1}) = n+1+\dfrac{1}{n+1}$. Then $f \in K^\circ$ satisfies (1), but 0 is an accumulation point of $\{f(\varepsilon_i - \varepsilon_k) : i,k \in I\}$.

Proof. We begin with some general definitions. Let $f \in X^*$ be any linear form with the property that the map $i \mapsto f(\varepsilon_i)$ is increasing, and let $V = \{f(\varepsilon_i) : i \in I\}$ be the set of values of f on the basis $\varepsilon_i, i \in I$. For $v \in V$, define

$$\Sigma_v := \{i \in I : f(\varepsilon_i) \geqslant v\}.$$

Then it is immediate that $\Sigma_v \in \mathfrak{E}$ is a final segment. Also, if v is not the minimum of V (with its order induced from $\mathbb{R}$) then $\Sigma_v \in \dot{\mathfrak{E}}$ is a proper final segment, because there exists some $j \in I$ with $f(\varepsilon_j) < v$ and thus $j \notin \Sigma_v$. On the other hand, if V has minimum m then $\Sigma_m = I$. Also, one sees immediately that $v < w$ for $v, w \in V$ implies $\Sigma_v \supsetneqq \Sigma_w$, so the map $v \mapsto \Sigma_v$ from V to $\mathfrak{E}$ is strictly decreasing, in particular, it is injective. We put

$$V' := \left\{ \begin{array}{ll} V & \text{if } V \text{ has no minimum} \\ V \setminus \{m\} & \text{if } V \text{ has minimum } m \end{array} \right\}.$$

If an element $v \in V'$ has a predecessor in V, we denote it by $'v$. This is in particular the case if V is a discrete subset of $\mathbb{R}$. — We now discuss each type of cone separately.

(a) $C = \dot{K} \subset \dot{X}$. Then $f = \dot{g}$ is the restriction of some linear form $g \in X^*$, unique modulo $\mathbb{R}t$. With a slight change of notation, we write $\dot{f}$ instead of f and f instead of g. Then (1) shows that $V = f(\varepsilon_k) + \Delta_k$ is discrete in $\mathbb{R}$. By B.8(a), the extremal rays of $\dot{K}$ are in bijection with $\dot{\mathfrak{E}}$ via the map $\Sigma \mapsto \mathbb{R}_+ \dot{q}_\Sigma$. Also, $v - {}'v > 0$ is clear from the definitions. The desired representation of $\dot{f}$ is then

$$\dot{f} = \sum_{v \in V'} (v - {}'v) \cdot \dot{q}_{\Sigma_v}. \tag{3}$$

Note first that $\dot{X}$ is spanned by all $\varepsilon_i - \varepsilon_j$, $i \succcurlyeq j$, because $\succcurlyeq$ is a total preorder, so one of $i \succcurlyeq j$ and $j \succcurlyeq i$ always holds. To prove (3), it therefore suffices to show that for every pair (i,j), $i \succcurlyeq j$, the family of real numbers $\big((v - {}'v) q_{\Sigma_v}(\varepsilon_i - \varepsilon_j)\big)_{v \in V'}$ is summable with sum $f(\varepsilon_i - \varepsilon_j)$. Now by definition of Σ_v,

$$q_{\Sigma_v}(\varepsilon_i - \varepsilon_j) = \left\{ \begin{array}{ll} 1 & \text{if } f(\varepsilon_i) \geqslant v > f(\varepsilon_j) \\ 0 & \text{otherwise} \end{array} \right\}.$$

Since V is discrete in $\mathbb{R}$, the set $\{v \in V' : f(\varepsilon_i) \geqslant v > f(\varepsilon_j)\}$ is finite, say $\{v_1, \ldots, v_n\}$ where $v_1 < \cdots < v_n$, and $v_n = f(\varepsilon_i)$. Put $v_0 := f(\varepsilon_j)$. Then ${}'v_k = v_{k-1}$ for $1 \leqslant k \leqslant n$, and the right hand side of (3), evaluated on $\varepsilon_i - \varepsilon_j$, is

$$\begin{aligned} \sum_{v \in V'} (v - {}'v) q_{\Sigma_v}(\varepsilon_i - \varepsilon_j) &= \sum_{k=1}^{n} (v_k - v_{k-1}) q_{\Sigma_{v_k}}(\varepsilon_i - \varepsilon_j) \\ &= \sum_{k=1}^{n} (v_k - v_{k-1}) = v_n - v_0 = f(\varepsilon_i - \varepsilon_j), \end{aligned}$$

as asserted.

(b) Let $C = K$ or K_0 be of type B or D and again $f \in C^\circ$. By B.3.3 and B.9.4, the map $i \mapsto f(\varepsilon_i)$ is increasing and bounded below. As in (a) we see that V is discrete in $\mathbb{R}$, so now V has a minimum m, say, $m = f(\varepsilon_{i_0})$ for some $i_0 \in I$. We claim that f has the series representation

$$f = mq_I + \sum_{v \in V'} (v - {}'v) q_{\Sigma_v}. \tag{4}$$

Indeed, restriction to $\dot{X}$ is weak-$*$-continuous and maps C° to $\dot{K}^\circ$ with kernel $\mathbb{R}q_I$. Hence by what we proved in (a), $\dot{f}$ has the representation (3). For $v \in V'$ we have $q_{\Sigma_v}(\varepsilon_{i_0}) = 0$, so the right hand side of (4), evaluated on ε_{i_0}, yields $mq_I(\varepsilon_{i_0}) = m = f(\varepsilon_{i_0})$. As $X = \dot{X} \oplus \mathbb{R}\varepsilon_{i_0}$, we see that (4) holds.

(c) Let $C = K$ be of type B. Then by B.6(a), $\operatorname{extr}(K^\circ)$ is in bijection, via $\Sigma \mapsto \mathbb{R}_+ q_\Sigma$, with the set of those final segments Σ which satisfy $I_0 \cap \Sigma = \emptyset$. On the other hand, $V \subset \mathbb{R}_+$ and $f(\varepsilon_i) = 0$ for all $i \in I_0$, by B.3.3. Therefore $\Sigma_v \cap I_0 = \emptyset$ for $v \in V'$, and moreover, $m = 0$ in case $I_0 \neq \emptyset$. Hence (4) is already the asserted representation of f in the form (2).

(d) Finally, let $C = K_0$ be of type D. Here (1) and $0 \preccurlyeq i$ for all $i \in I$ implies $m = f(\varepsilon_0)$. Note that m may be negative, but $2f(\varepsilon_i) = f(\varepsilon_i + \varepsilon_0) + f(\varepsilon_i - \varepsilon_0) \geqslant 0$ for $i \neq 0$, because $\varepsilon_i \pm \varepsilon_0 \in K_0$. Hence $m' := \min(V') \geqslant 0$, $m < m'$ and $m' + m \geqslant 0$. By B.12(a), $\operatorname{extr}(K_0^\circ)$ is in bijection with $\{q_+, q_-\} \cup \{q_\Sigma : \Sigma \in \ddot{\mathfrak{E}}\}$. To obtain a representation of f in the form (2), we distinguish two cases:

Case 1: $\Sigma_{m'} \in \ddot{\mathfrak{E}}$, i.e., $|I \setminus \Sigma_{m'}| \geqslant 2$. Then also $|I \setminus \Sigma_v| \geqslant 2$ for all $v \in V'$, because $v \geqslant m'$ and thus $\Sigma_v \subset \Sigma_{m'}$. Hence q_I and the q_{Σ_v} occurring in (4) span extremal rays of K_0°. Moreover, $m \geqslant 0$: Choose an element $i \neq 0$ in $I \setminus \Sigma_{m'}$. Then $f(\varepsilon_0) = f(\varepsilon_i) \geqslant 0$ as remarked above. Thus (4) is indeed a representation of the required form.

Case 2: $|I \setminus \Sigma_{m'}| = 1$, so $\Sigma_{m'} = I \setminus \{0\}$. Let $V'' := V' \setminus \{m'\}$. Then $|I \setminus \Sigma_v| \geqslant 2$ for all $v \in V''$, so the corresponding q_{Σ_v} span extremal rays of K_0°. The predecessor of $v = m'$ is m. Hence we can rewrite (4), using B.9.3, in the form

$$\begin{aligned}
f &= mq_I + (m' - m) q_{I \setminus \{0\}} + \sum_{v \in V''} (v - {}'v) q_{\Sigma_v} \\
&= (2m) q_+ + (m' - m)(q_+ + q_-) + \sum_{v \in V''} (v - {}'v) q_{\Sigma_v} \\
&= (m' + m) q_+ + (m' - m) q_- + \sum_{v \in V''} (v - {}'v) q_{\Sigma_v}.
\end{aligned} \tag{5}$$

This shows that f has a representation of the required form.

Since V as a discrete subset of $\mathbb{R}$ is at most countable, formulas (3) – (5) show that at most countably many terms in (2) are different from zero, and that boundedness of Δ_k (and hence of V) implies finiteness of V and hence of the sum (2). The converse is also clear from (3) – (5). This completes the proof.

Remark. If f admits a representation of the type (2), then the f_ϱ are in fact uniquely determined. This could be proved directly, but follows more easily from the uniqueness statement in Theorem 16.17, and the fact that the cones considered here all occur as $\mathbb{R}_+[P]$ for a suitable parabolic subset P of one of the classical root systems T_I.

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Index of notations

We mostly follow the conventions and notations of Bourbaki; in particular:

$X \subset Y$	X is a subset of Y
$X \subsetneqq Y$	X is a proper subset of Y
$\mathbb{Z}$	rational integers
$\mathbb{N} = \mathbb{Z}_+$	natural numbers including 0
$\mathbb{N}_+ = \mathbb{Z}_{++}$	positive natural numbers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{R}_+$	nonnegative real numbers
$\mathbb{R}_{++}$	positive real numbers
$\mathrm{Card}(X),\ \lvert X\rvert$	cardinality of set X

Specific notations ocurring in the text are listed in the following table.

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