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# LIE TORI

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RÉSUMÉ. On annonce ici quelques résultats concernant les tores de Lie nécessaires à la construction des algèbres de Lie affines étendues dans [15].

ABSTRACT. We announce some results on Lie tori which are used in the description of extended affine Lie algebras in the following article [15].

**0. Introduction.** Lie tori are a class of Lie algebras that arise in the construction of extended affine Lie algebras; see the following article [15]. Examples of Lie tori are the loop algebras of finite-dimensional split simple Lie algebras; more examples are given in 4 below.

An important property of a Lie torus $L$ is that $L$ is graded by a finite irreducible root system $\Delta$. Although one knows the structure of root-graded Lie algebras in general (Allison-Benkart-Gao [2], Berman-Moody [11], Benkart-Smirnov [7], Benkart-Zelmanov [8] and Neher [16]), it is non-trivial to characterize those that are Lie tori. As of now, the precise structure of a centreless Lie torus $L$ has been worked out for the case of a reduced $\Delta$ and in a special case for $\Delta = \mathrm{BC}_1$ (see 4 for a summary).

In this note we announce some results on Lie tori that are needed for the description of extended affine Lie algebras: A Lie torus $L$ is finitely generated as Lie algebra and the dimension of its homogeneous components are uniformly bounded (Theorem 5). The centroid $\mathrm{Cent}(L)$ of a centreless Lie torus $L$ is always a Laurent polynomial ring, and if $\Delta$ is not of type A then $L$ is a free $\mathrm{Cent}(L)$-module of finite rank (Theorem 7); the derivation algebra of $L$ is a semidirect product of the ideal of inner derivations and the subalgebra of centroidal derivations (Theorem 9).

Details of proofs will appear elsewhere. The author thanks Bruce Allison and Yoji Yoshii for having provided him with their preprints [5], [19] and [23].

**1. Notations and terminology.** All vector spaces and algebras considered in this note will be defined over a field $F$ of characteristic 0, except when indicated otherwise. For an abelian group $G$ and a subset $R \subset G$ we denote by $\langle R \rangle$ the subgroup generated by $R$. Root systems will always contain 0. This has some notational advantages and follows the conventions in [1]. We will call $\Delta$ a finite root system if $\Delta^\times := \Delta \setminus \{0\}$ is a root system in the sense of [12, Ch.VI, §1.1]. In particular, $\Delta$ need not be reduced. For $\alpha, \beta \in \Delta$ we denote by $\langle \alpha, \beta^\vee \rangle$ the Cartan

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integer of $\alpha, \beta$ (thus $\langle \alpha, \beta^\vee \rangle = n(\alpha, \beta)$ in the notation of [12]) and by $\mathcal{Q}(\Delta) = \langle \Delta \rangle$ the root lattice of $\Delta$. We denote by $\Delta_{\text{ind}} = \{0\} \cup \{\alpha \in \Delta^\times : \alpha/2 \notin \Delta\}$ the subsystem of indivisible roots of $\Delta$.

**2. Definition.** Let $\Delta$ be a finite irreducible root system and let $\Lambda$ be a free abelian group of finite rank. A *Lie torus of type* $(\Delta, \Lambda)$ is a Lie algebra $L$ satisfying the following axioms:

(LT1) $L$ has a $(\mathcal{Q}(\Delta) \oplus \Lambda)$-grading of the form

$$L = \bigoplus_{\alpha \in \mathcal{Q}(\Delta),\, \lambda \in \Lambda} L_\alpha^\lambda, \quad [L_\alpha^\lambda, L_\beta^\mu] \subset L_{\alpha+\beta}^{\lambda+\mu}, \quad \text{and } L_\alpha^\lambda = 0 \text{ if } \alpha \notin \Delta. \tag{2.1}$$

(LT2) For $\alpha \in \Delta^\times$ and $\lambda \in \Lambda$ we have

(i) $\dim L_\alpha^\lambda \le 1$, with $\dim L_\alpha^0 = 1$ if $\alpha \in \Delta_{\text{ind}}$,

(ii) if $\dim L_\alpha^\lambda = 1$ then there exists $(e_\alpha^\lambda, f_\alpha^\lambda) \in L_\alpha^\lambda \times L_{-\alpha}^{-\lambda}$ such that $h_\alpha^\lambda = [e_\alpha^\lambda, f_\alpha^\lambda] \in L_0^0$ acts on $x_\beta^\mu \in L_\beta^\mu$ ($\beta \in \Delta, \mu \in \Lambda$) by $[h_\alpha^\lambda, x_\beta^\mu] = \langle \beta, \alpha^\vee \rangle x_\beta^\mu$.

(LT3) For $\lambda \in \Lambda$ we have $L_0^\lambda = \sum_{\alpha \in \Delta^\times, \mu \in \Lambda} [L_\alpha^\mu, L_{-\alpha}^{\lambda-\mu}]$.

(LT4) $\Lambda = \langle \{\lambda \in \Lambda : L_\alpha^\lambda \neq 0 \text{ for some } \alpha \in \Delta\} \rangle$.

The rank of $\Lambda$ is called the *nullity* of $L$. If $(\Delta, \Lambda)$ is not important or clear from the context, we will simply call $L$ a *Lie torus*. Similarly, a Lie torus of type $\Delta$ and nullity $n$ is a Lie torus of type $(\Delta, \Lambda)$ for some $\Lambda$ of rank $n$.

Examples of Lie tori will be given in 4 below. It will emerge that Lie tori can be constructed using certain $\Lambda$-graded algebras, like Jordan, alternative or structurable algebras, which have been called Jordan tori, alternative tori or structurable tori respectively. This, together with the fact that toroidal Lie algebras are one of the main examples of Lie tori, is the justification for the name "Lie torus".

It is natural to consider Lie tori for more general groups $\Lambda$ and with less restrictive conditions as (LT2i); see [21], [22] and [23] for some work in this direction. However, the results stated below require the axioms above.

**3. Some properties of Lie tori.** Let $L$ be a Lie torus of type $(\Delta, \Lambda)$. Then $L$ has a $\Lambda$-grading

$$L = \bigoplus_{\lambda \in \Lambda} L^\lambda, \quad L^\lambda := \bigoplus_{\alpha \in \Delta} L_\alpha^\lambda \tag{3.1}$$

as well as a $\mathcal{Q}(\Delta)$-grading

$$L = \bigoplus_{\alpha \in \Delta} L_\alpha, \quad L_\alpha := \bigoplus_{\lambda \in \Lambda} L_\alpha^\lambda. \tag{3.2}$$

The subalgebra $\mathfrak{g}$ of $L^0$ generated by $\{L^0_\alpha : \alpha \in \Delta^\times\}$ is a finite-dimensional split simple Lie algebra of type $\Delta_{\text{ind}}$ with splitting Cartan subalgebra

$$\mathfrak{h} = \sum_{\alpha \in \Delta^\times} [L^0_\alpha, L^0_{-\alpha}]. \tag{3.3}$$

With respect to $\mathfrak{g}$, $\mathfrak{h}$ and the decomposition (3.2), $L$ is a $\Delta$-graded Lie algebra; see [11], [8] or [16] for the case of a reduced $\Delta$ and [2], [7] for the case $\Delta = \text{BC}$. It is then easily seen that our definition of a Lie torus is equivalent to the one given in [23] and [19]. A Lie torus is called *centreless* if its centre $Z(L)$ vanishes.

Let $C \subset Z(L) = \bigoplus_{\lambda\in\Lambda} (Z(L) \cap L^\lambda)$ be a $\Lambda$-graded subspace of $Z(L)$. Then $L/C$ is canonically a Lie torus of type $(\Delta, \Lambda)$. In particular, $L/Z(L)$ is a centreless Lie torus. Conversely, the universal central extension of a Lie torus (more generally, any $\Lambda$-cover of $L$ in the sense of [17, 1.15]) is again a Lie torus.

**4. Examples.** (a) Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra of type $\Delta$, and let $F[t_1^{\pm1}, \dots, t_n^{\pm1}]$ be the ring of Laurent polynomials in $n$ variables. Then $\mathfrak{g} \otimes F[t_1^{\pm1}, \dots, t_n^{\pm1}]$ is a centreless Lie torus of type $\Delta$ and nullity $n$. Hence, by **3**, its universal central extension, i.e., the associated *toroidal Lie algebra* [14], is also a Lie torus of type $\Delta$ and nullity $n$. Conversely, by [9, Theorem 1.37], every Lie torus of type $\Delta = \text{D}_l$, $l \geq 4$, or $\Delta = \text{E}_l$, $l = 6, 7, 8$ and nullity $n$ is a central extension of $\mathfrak{g} \otimes F[t_1^{\pm1}, \dots, t_n^{\pm1}]$.

(b) The special case $n = 1$ and $F = \mathbb{C}$ of example (a) is worth pointing out. Then the *loop algebra* $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t^{\pm1}]$ and its universal central extension $\hat{L}(\mathfrak{g})$ are Lie tori of nullity 1. More generally, it follows from the proof of [3, Theorem 1.19] that the complex Lie tori of nullity 1 are precisely the derived affine Lie algebras and their central quotients.

(c) Let $\mathbf{q} = (q_{ij}) \in \text{M}_n(F)$ be a $(n\times n)$-matrix over $F$ satisfying $q_{ii} = 1 = q_{ij}q_{ji}$ for $1 \leq i, j \leq n$, and let $F_\mathbf{q}$ be the associated *quantum torus*, which, by definition, is the unital associative algebra with $2n$ generators $t_1^{\pm1}, \dots, t_n^{\pm1}$ and defining relations $t_i t_i^{-1} = 1 = t_i^{-1} t_i$ and $t_i t_j = q_{ij} t_j t_i$ for $1 \leq i, j \leq n$. Denote by $[F_\mathbf{q}, F_\mathbf{q}]$ the span of all commutators $[a, b] = ab - ba$ with $a, b \in F_\mathbf{q}$. Then $\mathfrak{sl}_{l+1}(F_\mathbf{q}) = \{x \in \text{M}_{l+1}(F_\mathbf{q}) : \text{tr}(x) \in [F_\mathbf{q}, F_\mathbf{q}]\}$ is a Lie torus of type $\text{A}_l$, $l \geq 1$, and nullity $n$. It is shown in [9, Theorem 2.65] that every Lie torus of type $\text{A}_l$, $l \geq 3$, and nullity $n$ is a central extension of $\mathfrak{sl}_{l+1}(F_\mathbf{q})$ for some quantum torus $F_\mathbf{q}$.

(d) Lie tori of type $\text{A}_2$ are classified in [9] and [10]. The centreless Lie tori of type $\text{A}_1$ are precisely the Tits-Kantor-Koecher algebras of the so-called Jordan tori, classified in [20]. A description of the centreless Lie tori is given in [4] for $\Delta$ of type $\text{B}_l, \text{C}_l$, $\text{F}_4$, $\text{G}_2$ and, under additional assumptions, in [5] for $\Delta = \text{BC}_1$.

**5. Theorem.** *Let $L$ be a Lie torus of type $(\Delta, \Lambda)$.*

(a) *$L$ is finitely generated as Lie algebra, and has uniformly bounded dimension with respect to the $(\mathcal{Q}(\Delta) \oplus \Lambda)$-grading* (2.1), *i.e., there exists a $M \in \mathbb{N}$ such that* $\dim_F L_\alpha^\lambda \le M$ *for all $\alpha \in \Delta$ and $\lambda \in \Lambda$.*

(b) *The Lie algebra* $\operatorname{Der}_F L$ *of $F$-linear derivations of $L$ is $(\mathcal{Q}(\Delta) \oplus \Lambda)$-graded:*

$$\operatorname{Der}_F L = \bigoplus_{\alpha \in \Delta,\, \lambda \in \Lambda} (\operatorname{Der}_F L)_\alpha^\lambda\,, \tag{5.1}$$

*where* $(\operatorname{Der}_F L)_\alpha^\lambda$ *consists of those derivations mapping $L_\beta^\mu$ to $L_{\alpha+\beta}^{\lambda+\mu}$. Moreover,* $\operatorname{Der}_F L$ *has uniformly bounded dimension with respect to the $(\mathcal{Q}(\Delta) \oplus \Lambda)$-grading* (5.1).

**6.** Let $L$ be a Lie torus of type $(\Delta, \Lambda)$. Recall that the *centroid* of $L$, denoted $\operatorname{Cent}(L)$, is the set of all $\chi \in \operatorname{End}_F L$ satisfying $[\chi, \operatorname{ad} x] = 0$ for all $x \in L$. Since $L$ is perfect, $\operatorname{Cent}(L)$ is a unital associative commutative algebra, and one can thus consider $L$ as a module or as a Lie algebra over $\operatorname{Cent}(L)$. Since $L$ is $\Delta$-graded, a $\chi \in \operatorname{Cent}(L)$ leaves every root space $L_\alpha$ invariant. Moreover, $\chi$ is uniquely determined by $\chi | L_\alpha$ for a short root $\alpha$. It follows that $\operatorname{Cent}(L)$ is $\Lambda$-graded, $\operatorname{Cent}(L) = \bigoplus_{\lambda \in \Lambda} \operatorname{Cent}(L)^\lambda$ with $\dim_F \operatorname{Cent}(L)^\lambda \le 1$, where $\operatorname{Cent}(L)^\lambda$ consists of endomorphisms of degree $\lambda$ with respect to the $\Lambda$-grading (3.1) of $L$. We put $\Gamma = \{\lambda \in \Lambda : \operatorname{Cent}(L)^\lambda \ne 0\}$. The following result justifies to call $\Gamma$ the *centroid grading group*.

**7. Theorem.** *Let $L$ be a centreless Lie torus of type $(\Delta, \Lambda)$.*

(a) *$\Gamma$ is a subgroup of $\Lambda$, and* $\operatorname{Cent}(L)$ *is isomorphic to the group ring* $F[\Gamma]$, *hence to a Laurent polynomial ring in several variables.*

(b) *$L$ is a free* $\operatorname{Cent}(L)$*-module. If $\Delta \ne \mathrm{A}_l$, then $L$ has finite rank as* $\operatorname{Cent}(L)$*-module.*

**Remarks.** (a) Let $L = sl_{l+1}(F_{\mathbf{q}})$ as in Example 4.(c). In this case, $\operatorname{Cent}(L) = Z(F_{\mathbf{q}})\mathrm{Id}$, where $Z(F_{\mathbf{q}})$ denotes the centre of $F_{\mathbf{q}}$, and $L$ has finite rank over $\operatorname{Cent}(L)$ if and only if $F_{\mathbf{q}}$ has finite rank over $Z(F_{\mathbf{q}})$, equivalently $[\Lambda : \Gamma] < \infty$. Using the description of $Z(F_{\mathbf{q}})$ given in [9, 2.44], it is easy to construct examples for which $\operatorname{rank}(\Gamma)$ takes on every value between 0 and $n$. In particular, $L$ is in general not a finitely generated $\operatorname{Cent}(L)$-module.

(b) Let $L$ be a centreless Lie torus. Then $\operatorname{Cent}(L)$ is an integral domain, acting without torsion on $L$. Let $K$ be the quotient field of $\operatorname{Cent}(L)$, and let

$$\tilde{L} = L \otimes_{\operatorname{Cent}(L)} K \tag{7.1}$$

be its *central closure*, where in this tensor product $L$ is considered as Lie algebra over $\operatorname{Cent}(L)$. Then $L$ imbeds into $\tilde{L}$ and is a $\operatorname{Cent}(L)$-form of $\tilde{L}$. If $L$ has finite rank over $\operatorname{Cent}(L)$, $\tilde{L}$ is a simple finite-dimensional Lie algebra over $K$.

**8. Centroidal derivations.** Let $L$ be a centreless Lie torus of type $(\Delta, \Lambda)$, nullity $n$ and centroidal grading group $\Gamma$. Recall the $\Lambda$-grading (3.1) of $L$. Any $\vartheta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, F)$ induces a so-called *degree derivation* $\partial_\vartheta$ of $L$, defined by $\partial_\vartheta(x^\lambda) = \vartheta(\lambda)x^\lambda$ for $x^\lambda \in L^\lambda$. We put $\mathfrak{D} = \{\partial_\vartheta : \vartheta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, F)\}$ and note that $\vartheta \mapsto \partial_\vartheta$ is an isomorphism from $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, F)$ onto $\mathfrak{D}$, hence $\mathfrak{D} \cong F^n$. Moreover, $\mathfrak{D}$ induces the $\Lambda$-grading of $L$, i.e.,

$$L^\lambda = \{x \in L : \partial_\vartheta(x) = \vartheta(\lambda)x \text{ for all } \partial_\vartheta \in \mathfrak{D}\}. \tag{8.1}$$

If $\chi \in \mathrm{Cent}(L)$ then $\chi\partial \in \mathrm{Der}_F L$ for any $\partial \in \mathrm{Der}_F L$. It follows that $\mathrm{CDer}_F L = \mathrm{Cent}(L)\mathfrak{D} = \bigoplus_{\mu\in\Gamma} \mathrm{Cent}(L)^\mu\mathfrak{D}$ is a $\Gamma$-graded subalgebra of $\mathrm{Der}_F L$, called the algebra of *centroidal derivations of $L$*. It is a generalized Witt algebra in the sense of [18, 1.9].

**9. Theorem.** *Let $L$ be a centreless Lie torus. Denote by* $\mathrm{IDer}L$ *the ideal of inner derivations of $L$. Then*

$$\mathrm{Der}_F L = \mathrm{IDer}L \rtimes \mathrm{CDer}_F L \quad \textit{(semidirect product).} \tag{9.1}$$

In case $L$ has finite rank as $\mathrm{Cent}(L)$-module, this result can be proven by using that its central closure $\tilde{L}$, see (7.1), is a finite-dimensional simple Lie $K$-algebra and hence all $K$-linear derivations are inner. In the remaining case, where $L$ is a $\mathrm{Cent}(L)$-module of infinite rank, $\Delta$ is of type A by Theorem 7. Then the result follows from [9, 2.17, 2.53], [10, Theorem 1.40] and [18, Theorem 4.11]. For $\Delta$ of type B or D the splitting (9.1) has also been proven in [13, Corollary 4.9 and Corollary 4.10] using different methods. We note that the decomposition (9.1) is not the one proven in [6, Theorem 3.12] for arbitrary $\Delta$-graded Lie algebras: the subalgebra $\mathrm{Der}_*(\mathfrak{a}, \mathfrak{S})$ of [6] contains $\mathrm{CDer}_F L$ but has in general a non-zero intersection with $\mathrm{IDer}L$.

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