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EXTENDED AFFINE LIE ALGEBRAS

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RÉSUMÉ. On décrit une construction qui permet de construire tous les algèbres de Lie affines étendues à partir des tores de Lie.

ABSTRACT. We present a construction of all extended affine Lie algebras in terms of Lie tori.

0. Introduction. Extended affine Lie algebras are a class of complex Lie algebras that includes finite-dimensional simple Lie algebras, affine Lie algebras and toroidal Lie algebras. They are closely related to Saito's elliptic Lie algebras [13]. Originally proposed by the physicists Høegh-Krohn and B. Torrésani [8] under the name irreducible quasi-simple Lie algebras, extended affine Lie algebras have been put on a sound mathematical footing in the AMS-memoirs [1] by Allison, Azam, Berman, Gao and Pianzola. In particular, one can find there a detailed study of the root systems appearing in extended affine Lie algebras. The structure and representation theory of various classes of these Lie algebras has since been investigated in many papers. In this note we will describe the structure of extended affine Lie algebras in general.

Referring the reader to the main body of this note for precise definitions, we will only give a rough sketch of the relevant structures in this introduction. Two important properties of an extended affine Lie algebra are the existence of an invariant nondegenerate form and a finite-dimensional self-centralizing ad-diagonalizable subalgebra H. Thus E has a root space decomposition $E = \bigoplus E_\xi$ and a root system R, consisting of those $\xi \in H^*$ with $E_\xi \neq 0$. The form on E gives rise to a partition $R = R^0 \cup R^\times$ into isotropic roots R^0 and non-isotropic roots $R^\times$, generalizing the decomposition into imaginary and real roots in the affine case. Let E_c be the ideal generated by $\{E_\xi : \xi \in R^\times\}$, called the core of E. One assumes that E can be recovered from its core E_c in the sense that the kernel of the natural representation $E \to \operatorname{Der} E_c : x \mapsto \operatorname{ad} x|E_c$ lies in E_c. The core E_c may have a non-trivial centre, and it turns out to be easier to describe its central quotient $L = E_c/Z(E_c)$, where $Z(E_c)$ denotes the centre of E_c. The situation can thus be summarized by the following diagram

$$\begin{array}{ccc} E_c & \longrightarrow & E \\ \downarrow & & \\ L & & \end{array}$$

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familiar from the affine case where E_c is the derived algebra and L a loop algebra. In general, the Lie algebras L appearing in (0.1) can be characterized without any reference to extended affine Lie algebras: they are Lie tori as defined in Yoshii's recent preprints [15] and [14] or in the preceding article [11]. Moreover, it is shown in [14] that all centreless Lie tori appear as the "bottom algebra" in a diagram (0.1). The canonical approach to untangling the structure of an extended affine Lie algebra E is therefore to describe (I) the centreless Lie tori L and (II) how to get from L to E.

Some results on (I) have been announced in the preceding article [11]. In this note we announce a solution of (II) in general (Theorem 6 and Theorem 8). Our construction, given in **5**, describes all extended affine Lie algebras with a given centreless core. It resembles the construction of affine Lie algebras and gives a new interpretation to certain subalgebras appearing in the previously known solution for the case $\Delta = \mathrm{A}_n$, $n \geq 2$. They are described here as subalgebras of skew centroidal derivations.

While the work on Lie tori can be done for Lie algebras over fields of characteristic 0, one has up to now only considered complex extended affine Lie algebras since one of their defining axioms is a topological (discreteness) condition. To remedy this discrepancy, we are proposing here a new definition of an extended affine Lie algebra over an arbitrary field F of characteristic 0. Roughly speaking, we are allowing more possibilities for the subalgebra $H \subset E$. In case $F = \mathbb{C}$ the algebras satisfying the old axiom system are recovered as the discrete extended affine Lie algebras in our sense (Theorem 8).

We continue with the terminology and notation of the preceding article [11].

1. A preliminary setting. Let E be a Lie algebra satisfying the following two axioms (EA1) and (EA2):

(EA1) E has a nondegenerate invariant symmetric bilinear form $(\ |\ )$.

(EA2) E contains a nontrivial finite-dimensional self-centralizing and ad-diagonalizable subalgebra H.

By (EA2), E has a root space decomposition $E = \bigoplus_{\xi \in H^*} E_\xi$ with $E_0 = H$, where, as usual, $E_\xi = \{e \in E : [h, e] = \xi(h)e \text{ for all } h \in H\}$. The invariance of $(\ |\ )$ implies that $(E_\xi | E_\zeta) = 0$ for $\xi + \zeta \neq 0$. It follows that $(\ |\ )$ restricted to $H \times H$ is nondegenerate. We can therefore transfer the restricted form $(\ |\ )|H \times H$ to a nondegenerate symmetric bilinear form on H^* by setting $(\xi|\zeta) = (t_\xi | t_\zeta)$ where $t_\xi \in H$ is defined by $(t_\xi | h) = \xi(h)$ for all $h \in H$. We define

$$R = \{\xi \in H^* : E_\xi \neq 0\} \quad \text{(root system of } E\text{)},$$

$$R^0 = \{\xi \in R : (\xi|\xi) = 0\} \quad \text{(isotropic roots)},$$

$$R^\times = \{\xi \in R : (\xi|\xi) \neq 0\} \quad \text{(nonisotropic roots)}.$$

The subalgebra E_c of E, generated by $\{E_\xi : (\xi|\xi) \neq 0\}$ is called the *core of* E. It is in fact an ideal if E is an extended affine Lie algebra as defined below.

2. Definition. An *extended affine Lie algebra of nullity* n, or extended affine Lie algebra for short, is a Lie algebra E satisfying (EA1), (EA2) of **1** and, in addition, the following axioms (EA3)–(EA6):

(EA3) For $\xi \in R^\times$ and $x_\xi \in E_\xi$, the endomorphism $\operatorname{ad} x_\xi \in \operatorname{End}_F L$ is locally nilpotent.
(EA4) $R^\times$ is *irreducible*, i.e., $R^\times = R_1 \cup R_2$ and $(R_1|R_2) = 0$ implies $R_1 = \varnothing$ or $R_2 = \varnothing$.
(EA5) E is *tame* in the sense that $\{e \in E : [e, E_c] = 0\} \subset E_c$.
(EA6) $\Lambda := \langle R^0 \rangle \subset V$ is a free abelian group of rank n.

It is appropriate to immediately point out that this definition for $F = \mathbb{C}$ is more general than the usual definition of an extended affine Lie algebra ([1] or **7** below). The relation between the two definitions is discussed in **7** and **8**. As we will see, there is a close connection between extended affine Lie algebras and Lie tori. The following proposition is the first step in this direction. It can be proven using the techniques of [1, Ch. I] and [11, Thm. 5].

3. Proposition. *Let E be an extended affine Lie algebra with root system R, and put $\Lambda = \langle R^0 \rangle$.*

(a) *There exists a finite irreducible root system Δ, an imbedding $\Delta_{\text{ind}} \hookrightarrow R$ and a family $(\Lambda_\alpha : \alpha \in \Delta) \subset \Lambda$ such that*

$$V = \operatorname{span}_{\mathbb{Q}}(\Delta) \oplus \operatorname{span}_{\mathbb{Q}}(R^0) \quad \textit{and} \quad R = \bigcup_{\alpha \in \Delta} (\alpha \oplus \Lambda_\alpha).$$

The subspaces $(E_c)^\lambda_\alpha = E_c \cap E_{\alpha \oplus \lambda}$ give E_c the structure of a Lie torus of type (Δ, Λ).

(b) *The root spaces E_ξ of E have uniformly bounded finite dimension.*

We note that the family $(\Lambda_\alpha : \alpha \in \Delta^\times)$ is a "reduced root system of type Δ extended by Λ" in the terminology of [15]. Theorem 2.4 of that paper gives the structure of this family, generalizing [1, II, Thm. 2.37].

4. Skew centroidal derivations. Let L be a centreless Lie torus. It follows from [14, Thm. 2.2 and Thm. 7.1] that L has a non-zero invariant (necessarily) symmetric bilinear form $(\ |\ )$, which is Λ-graded in the sense that $(L^\lambda | L^\mu) = 0$ if $\lambda + \mu \neq 0$. Moreover, any such form is unique up to a non-zero scalar, and is nondegenerate since L is centreless. In the following, we fix such a form $(\ |\ )$.

Recall the definition of the centroidal derivations in [11, §8]. Let $\operatorname{SCDer}_F L$ be the subalgebra of $\operatorname{CDer}_F L$ consisting of skew derivations with respect to the form $(\ |\ )$. Then $\operatorname{SCDer}_F L = \bigoplus_{\mu \in \Gamma} (\operatorname{SCDer}_F L)^\mu$ is Γ-graded with 0-component $\mathcal{D}$. One can show that $\operatorname{SCDer}_F L$ is the semidirect product of the subalgebra $\mathcal{D}$ of degree derivations and the ideal $\mathcal{D}' = \bigoplus_{0 \neq \mu \in \Gamma} (\operatorname{SCDer}_F L)^\mu$.

Proposition 3 associates a centreless Lie torus to every extended affine Lie algebra E, namely $L = E_c/Z(E_c)$. We will now describe a construction which, conversely, associates an extended affine Lie algebra to any centreless Lie torus.

5. Construction. Let L be a centreless Lie torus of type (Δ, Λ) and nullity n, and let $(\ |\ )$ be a nondegenerate invariant Λ-graded symmetric bilinear form on L. We denote by Γ the centroid grading group ([11, §6]).

The second ingredient of our construction is a Γ-graded subalgebra of $\mathrm{SCDer}_F L$,

$$D = \bigoplus_{\gamma\in\Gamma} D_\gamma, \quad D_\gamma \subset (\mathrm{SCDer}_F L)^\gamma,$$

which has the property that D_0 induces the Λ-grading [11, (3.1)] of L, i.e., $L^\lambda = \{x \in L : \partial_\theta(x) = \theta(\lambda)x \text{ for all } \partial_\theta \in D_0\}$. Equivalently, the canonical evaluation map

$$\mathrm{ev}\colon \Lambda \to D_0^* : \lambda \mapsto \mathrm{ev}(\lambda), \quad \text{where } \big(\mathrm{ev}(\lambda)\big)(\partial_\theta) = \theta(\lambda), \tag{5.1}$$

is injective. Let $D^{\mathrm{gr}*} = \bigoplus_{\gamma\in\Gamma} D_\gamma^*$ be the graded dual space of D. Thus, $f \in D_\gamma^*$ is extended to a linear form on D by $f|D_\delta = 0$ for $\delta \neq \gamma$. We consider $D^{\mathrm{gr}*}$ as a Γ-graded vector space with γ-component $(D^{\mathrm{gr}*})_\gamma = D_{-\gamma}^*$. It is easily seen that $\sigma_D\colon L\times L \to D^{\mathrm{gr}*}$, $\sigma_D(x,y)(d) = (dx|y)$ is a 2-cocycle for L with values in the trivial L-module $D^{\mathrm{gr}*}$ which respects the gradings of L and $D^{\mathrm{gr}*}$.

The third ingredient of our construction is a 2-cocycle $\tau\colon D\times D \to D^{\mathrm{gr}*}$ of D with values in $D^{\mathrm{gr}*}$, considered a D-module via the contragredient action $d\cdot f$, which is graded and invariant, i.e.,

$$\tau(D_\gamma, D_\delta) \subset (D^{\mathrm{gr}*})_{\gamma+\delta} = D_{-\gamma-\delta}^* \quad \text{and} \quad \tau(d_1,d_2)(d_3) = \tau(d_2,d_3)(d_1) \tag{5.2}$$

for $d_1, d_2, d_3 \in D$. Moreover, we suppose that

$$\tau(D_0, D) = 0. \tag{5.3}$$

Let $D' = \bigoplus_{0\neq\gamma\in\Gamma} D_\gamma$, an ideal of D. Because of condition (5.3), the map $\tau \mapsto \tau|D'\times D'$ is a bijection between the 2-cocycles of D satisfying (5.2) and (5.3) and the 2-cocycles of D' with values in the D'-module $D^{\mathrm{gr}*}$ satisfying (5.2).

Finally, for L, D and τ as above we define

$$E = E(L, D, \tau) = L \oplus D^{\mathrm{gr}*} \oplus D.$$

Then E is a Lie algebra with respect to the product ($x_i \in L, f_i \in D^{\mathrm{gr}*}, d_i \in D$)

$$\begin{aligned}[x_1 \oplus f_1 \oplus d_1\,,\, x_2 \oplus f_2 \oplus d_2] &= \big([x_1,x_2] + d_1(x_2) - d_2(x_1)\big) \\ &\quad \oplus \big(\sigma_D(x_1,x_2) + d_1\cdot f_2 - d_2\cdot f_1 + \tau(d_1,d_2)\big) \oplus [d_1,d_2].\end{aligned}$$

Moreover, E has a nondegenerate invariant form $(\ |\ )$ given by

$$(x_1 \oplus f_1 \oplus d_1 | x_2 \oplus f_2 \oplus d_2) = (x_1|x_2) + f_1(d_2) + f_2(d_1).$$

Let $H = \mathfrak{h} \oplus D_0^* \oplus D_0$ for $\mathfrak{h}$ as in [11, (3.3)]. We identify $\Lambda = \mathrm{ev}(\Lambda) \subset D_0^*$ and view $\Lambda \subset H^*$ by letting $\lambda \in \Lambda$ act by 0 on $\mathfrak{h} \oplus D_0^*$. Similarly, any $\alpha \in \Delta \subset \mathfrak{h}^*$ gives rise to a linear form on H by putting $\alpha | D_0^* \oplus D_0 = 0$. With these identifications, H becomes a self-centralizing, ad-diagonalizable subalgebra of E whose root spaces are

$$E_{\alpha \oplus \lambda} = \begin{cases} L_\alpha^\lambda & ; \alpha \neq 0, \\ L_0^\lambda \oplus D_{-\lambda}^* \oplus D_\lambda & ; \alpha = 0. \end{cases}$$

It is then easy to verify part (a) of the following theorem.

6. Theorem. (a) *The algebra $E(L, D, \tau)$ constructed in* **5** *above is an extended affine Lie algebra of nullity n with respect to the form $(\ |\ )$ and the subalgebra H.*

(b) *Conversely, let E be an extended affine Lie algebra of nullity n and let $L = E_c/Z(E_c)$, which by Proposition* 3 *is a centreless Lie torus of nullity n. Then there exists a unique subalgebra $D \subset \mathrm{SCDer}_F L$ inducing the Λ-grading of L and a 2-cocycle $\tau \colon D \times D \to D^{\mathrm{gr}*}$ satisfying* (5.2) *and* (5.3) *such that $E \cong E(L, D, \tau)$.*

For the proof of part (b) we note that tameness of E and Proposition 3(b) imply that E can be described in terms of a Γ-graded subalgebra $D \subset \mathrm{Der}_F L$ with $D \cap \mathrm{IDer}_F L = 0$ and a 2-cocycle τ. Because of [11, Thm. 9] one can take $D \subset \mathrm{SCDer}_F L$.

Remarks. (a) The construction of the Lie algebra $E(L, D, \tau)$ makes sense in the more general setting where L is just a Lie algebra with a nondegenerate invariant form and D is a subalgebra of skew-symmetric derivations of L. For finite-dimensional algebras the Lie algebra $E(L, D, 0)$, called a *double extension*, has been used to classify finite-dimensional solvable Lie algebras admitting a nondegenerate invariant form ([8], [9, Ex. 2.10, 2.11], [10]). The more general construction with a possibly non-zero τ appears in [6, §3]. In the setting of discrete extended affine Lie algebras of type $\Delta = \mathrm{A}_l$, $l \geq 2$, the construction $E(L, D, \tau)$ appears in [4] and [5].

(b) Since $\mathrm{SCDer}_F L$ induces the Λ-grading of L, $D = \mathrm{SCDer}_F L$ is the maximal choice for D. In this case the subalgebra $L \oplus (\mathrm{SCDer}_F L)^*$ of E is the universal central extension of L. As pointed out in [4, Remark 3.71(b)], there do indeed exist non-trivial 2-cocycles τ in this case, which first have appeared in the context of toroidal Lie algebras ([12], see also [3, (2.11), (2.12)]).

7. Definition. Let E be a Lie algebra over $F = \mathbb{C}$. We call E a *discrete extended affine Lie algebra* if E satisfies the axioms (EA1)–(EA5) and the following axiom

(DE) R is a discrete subset of H^*.

In view of [2, Lemma 3.62] a discrete extended affine Lie algebra as defined above is the same as a tame extended affine Lie algebra in the sense of [1]. We have included tameness in our definition, i.e., the axiom (EA5), since our results apply to tame extended affine Lie algebras only. Moreover, as an example in [4, §3] shows, there is little hope to get a precise description of extended affine Lie algebras that are not tame. Besides the mentioned example in [4], all of the known constructions yield tame extended affine Lie algebras.

8. Theorem. *Let $F = \mathbb{C}$.* (a) *Let L be a centreless Lie torus and let D be a graded subalgebra of* $\mathrm{SCDer}_{\mathbb{C}}L$ *such that the evaluation map* $\mathrm{ev}\colon \Lambda \to D_0^*$ *of* (5.1) *is not only injective but has also discrete image. Then, with τ as in the construction* **5**, *the Lie algebra $E(L, D, \tau)$ is a discrete extended affine Lie algebra.*

(b) *Conversely, every discrete extended affine Lie algebra arises from the construction described in* (a).

We note that $\mathrm{ev}\colon \Lambda \to \mathfrak{D}^*$ is always a discrete imbedding. Hence any complex Lie torus gives rise to a discrete extended affine Lie algebra ([14, Cor. 7.3]). A construction of discrete imbeddings in terms of the maximal choice $D = \mathfrak{D}$ is given in [7, §2] (although not in this language).

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