Corestriction (or norm) – an introduction

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Plan: $F'/F \rightsquigarrow R'/R \rightsquigarrow S'/S$

Corestriction in increasing generality:

1. F field:
   (a) F'/F finite Galois extension of fields
   (b) F'/F finite étale extension
2. Application $A_1^2 \cong D_2$
3. R commutative ring: R' finite étale or finite projective R–algebra
4. S scheme: $S' \to S$ finite locally free (or even étale) morphism

General goal of corestriction:

$$\text{Objects over } F' \text{ (or } R' \text{ or } S'\text{)} \quad \rightsquigarrow \quad \text{Objects over } F \text{ (or } R \text{ or } S\text{)}$$

Idea: Use "augmented" Galois descent (Albert, ...)

Galois descent (field case)

F'/F finite Galois extension of fields, $G = \mathrm{Gal}(F'/F)$ Galois group
ALG_F = a category of nonassociative F–"algebras" allowing base change

$$A \in \mathrm{ALG}_F \implies A' = F' \otimes_F A \in \mathrm{ALG}_{F'},$$

Examples: most types of "algebras" (vector spaces, associative, Lie, ...)

For $A' \in \mathrm{ALG}_{F'}$: $\mathrm{Aut}_F(A')$ = F–linear automorphisms of the F–algebra A' preserving the structure of $\mathrm{ALG}_{F'}$, e.g.,...

Galois descent datum: group homomorphism

$$\delta \colon G \to \mathrm{Aut}_F(A'), \quad g \mapsto \bar{g}, \qquad \bar{g}(f'a') = g(f')\,\bar{g}(a')$$

for $f' \in F'$, $a' \in A'$. Put

$$A'^G = \{a' \in A' : \bar{g}(a') = a' \ \text{ for all } g \in G\}$$

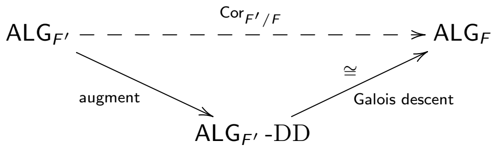
Facts:

- $A \in \mathrm{ALG}_F$, then $A' = F' \otimes_F A \in \mathrm{ALG}_{F'}$ has canonical descent datum: $G \to \mathrm{Aut}_F(F' \otimes_F A)$, $\bar{g}(f' \otimes_F a) = g(f') \otimes_F a$, and $(F' \otimes_F A)^G \cong A$
- $A' \in \mathrm{ALG}_{F'}$ with descent datum, then $A'^G \in \mathrm{ALG}_F$ and $F' \otimes_F A'^G \cong A'$.
- Equivalence of categories:

$$\mathrm{ALG}_F \xrightarrow{\sim} \mathrm{ALG}_{F'} - \mathrm{DD} = \text{objects with descent data}$$

Corestriction = augmented Galois descent

Recall: F'/F finite Galois, $G = \mathrm{Gal}(F'/F)$



Assume $\mathsf{ALG}_{F'}$ has base change + tensor products ($A', B' \in \mathsf{ALG}_{F'} \implies A' \otimes_{F'} B' \in \mathsf{ALG}_{F'}$)

Example: $\mathsf{ALG}_{F'} = F'$ vector spaces or associative F'-algebras, more ??

Given $A' \in \mathsf{ALG}_{F'}$, for $g \in \mathrm{Gal}(F'/F)$ define $A'_g \in \mathsf{ALG}_{F'}$ by

- $A' \xrightarrow{\sim} A'_g$, $a' \mapsto a'_g$, as F–algebra, i.e., $a'_g + b'_g = (a' + b')_g$, ...
- twisted F'–action: $f' \cdot a'_g = (g^{-1}(f')a')_g$, so $(f' \cdot a')_g = g(f') \cdot a'_g$.

$\bigotimes_{g \in G} A'_g$ ($\otimes = \otimes_{F'}$) has decent datum using $\delta_g : A'_h \to A'_{gh}$, $a'_h \mapsto a'_{gh}$

Define

$$\boxed{\mathrm{Cor}_{F'/F}(A') = \Big(\bigotimes_{g \in G} A'_g\Big)^G}$$

Have functor

$$\mathrm{Cor}_{F'/F} \colon \mathsf{ALG}_{F'} \to \mathsf{ALG}_F, \quad A' \mapsto \mathrm{Cor}_{F'/F}(A').$$

Example: $[F' : F] = 2$

$\mathrm{Gal}(F'/F) = \{\mathrm{Id}, \kappa\}$

A' an F'-algebra, define $A'_\kappa = \{a'_\kappa : a' \in A'\}$ and operations

$$a'_\kappa + b'_\kappa = (a' + b')_\kappa, \quad a'_\kappa b'_\kappa = (a'b')_\kappa, \quad (f'a')_\kappa = \kappa(f')a'$$

($f' \in F'$ and $a' \in A'$)
The switch map

$$\mathrm{sw}\colon A' \otimes_{F'} A'_\kappa \to A' \otimes_{F'} A'_\kappa, \quad a' \otimes b'_\kappa \mapsto b' \otimes a'_\kappa$$

is κ–linear, hence defines a Galois descent datum on $A' \otimes_{F'} A'_\kappa$. Thus

$$\mathrm{Cor}_{F'/F}(A') = \{x' \in A' \otimes_{F'} A'_\kappa : \mathrm{sw}(x') = x'\}.$$

Properties of $\mathrm{Cor}_{F'/F}$ for $[F' : F] = d$

1. V' an F'- vector space: $\mathrm{Cor}_{F'/F}(V')$ is an F-vector space with
$$\dim_F \mathrm{Cor}_{F'/F}(V') = \big(\dim_{F'} V'\big)^d,$$
e.g., $\mathrm{Cor}_{F'/F}(F') = F$

2. A' an associative (or unital) F'–algebra:
$\implies$ $\mathrm{Cor}_{F'/F}(A')$ is an associative (unital resp.) F–algebra

3. $\dim_{F'} V' < \infty$: $\mathrm{Cor}_{F'/F}\big(\mathrm{End}_{F'}(V')\big) \cong \mathrm{End}_F\big(\mathrm{Cor}_{F'/F}(V')\big)$ (as algebras)

4. Central-simple algebras:
Recall: central-simple F–algebra A = associative, unital, central, simple, and $\dim_F A < \infty$
   - A' a central-simple F'–algebra: $\mathrm{Cor}_{F'/F}(A')$ is a central-simple F–algebra
   - A' and B' central-simple F'–algebras:
$$\mathrm{Cor}_{F'/F}(A' \otimes_{F'} B') \cong \mathrm{Cor}_{F'/F}(A') \otimes_F \mathrm{Cor}_{F'/F}(B')$$
   - $\mathrm{Cor}_{F'/F}(\cdot)$ preserves Brauer equivalence, hence get group homomorphism
$$\mathrm{Cor}_{F'/F}\colon \mathrm{Br}(F') \to \mathrm{Br}(F), \quad [A'] \mapsto [\mathrm{Cor}_{F'/F}(A')]$$
   $\mathrm{Br}(\cdot)$ = Brauer group.
   - A central-simple F–algebra, so $\mathrm{Res}_{F'/F}(A) = F' \otimes_F A$ is central-simple F'–algebra, then
$$\boxed{\mathrm{Cor}_{F'/F} \circ \mathrm{Res}_{F'/F}\colon \mathrm{Br}(F) \to \mathrm{Br}(F), \quad [A] \mapsto [A]^d.}$$

Corestriction for F'/F finite étale (Riehm, Scharlau, Tignol, ...)

F field,

F' commutative associative unital F–algebra, which is finite étale, i.e.,

$F' = K_1 \times \cdots \times K_n$, K_i/F finite separable field extension

$\iff$ there exists L/F finite Galois s.th. $F' \otimes_F L \cong L \times \cdots \times L$

Modified corestriction construction:

$A' \in \mathrm{ALG}_{F'}$, $x \in X = \mathrm{Hom}_{F\text{-}\mathbf{alg}}(F', L)$,

$A'_x \in \mathrm{ALG}_L$ via $\ell a'_x = x^{-1}(\ell) \cdot a'$ $(\ell \in L,\ a' \in A')$

$$\boxed{\mathrm{Cor}_{F'/F}(A') = \left(\bigotimes_{x\in X} A'_x\right)^G}$$

compare:

Galois	**finite étale**
$F \to F' = L$	$F' \leftarrow F \to L$
$G = \mathrm{Gal}(F'/F) = X$	$X = \mathrm{Hom}_{F\text{-}\mathbf{alg}}(F', L)$, $G = \mathrm{Gal}(L/F)$
$\mathrm{Cor}_{F'/F}(A') = \left(\bigotimes_{g\in G} A'_g\right)^G$	$\mathrm{Cor}_{F'/F}(A') = \left(\bigotimes_{x\in X} A'_x\right)^G$

Facts: All properties of the Galois case generalize (mutatis mutandis)

Example

$F' = F \times \cdots \times F$ (d factors) finite étale,

$L = F,\ G = \mathrm{Gal}(L/F) = \{1\}$,

$X = \mathrm{Hom}_{F\text{-}\mathbf{alg}}(F', L) = \{\mathrm{pr}_1, \ldots, \mathrm{pr}_d\}$

$A' \in \mathrm{ALG}_{F'}$: $A' = A_1 \times \cdots \times A_d$ with $A_i \in \mathrm{ALG}_F$,

$A'_{\mathrm{pr}_i} = (A_1 \times \cdots \times A_d)_{\mathrm{pr}_i} = A_i$,

$$\boxed{\mathrm{Cor}_{F'/F}(A_1 \times \cdots \times A_d) = A_1 \otimes_F \cdots \otimes_F A_d}$$

The exceptional isomorphism $\mathrm{A}_1 \times \mathrm{A}_1 \cong \mathrm{D}_2$, i.e., $\cdot\ \ \cdot \cong\ :$

adjoint algebraic groups: $\mathrm{PGL}_2(K) \times \mathrm{PGL}_2(K) \cong \mathrm{PGO}^+(K) \quad (K = \bar{K})$

Magic of stacks: The following 4 categories are equivalent (fix field F):

1. $\mathfrak{A}_1^2$ – objects are $(F'/F, Q')$ where F'/F finite étale, $[F' : F] = 2$,
   i.e., F'/F Galois or $F' = F \times F$, Q' quaternion F'–algebra
   morphisms = F–algebra isomorphisms
2. $\mathfrak{A}_1^{2,\mathrm{gr}}$ – objects are semisimple adjoint algebraic groups G over F "of type A_1^2",
   i.e., $G_{\bar{F}} \cong \mathrm{PGL}_2(\bar{F}) \times \mathrm{PGL}_2(\bar{F})$, $\bar{F}$ an algebraic closure of F
   morphisms = F-isomorphisms
3. $\mathfrak{D}_2$ – objects (A, σ, f) where A/F central-simple F–algebra, $\dim_F A = 16$,
   σ orthogonal involution, f semitrace
   morphisms = F–algebra isomorphisms preserving σ, f
4. $\mathfrak{D}_2^{\mathrm{gr}}$ – objects are semisimple adjoint algebraic groups over F "of type D_2"

Concrete equivalence (Book of Involutions)

$$\boxed{\mathfrak{A}_1^2 \xrightarrow{\sim} \mathfrak{D}_2, \quad (F'/F, Q') \mapsto (\mathrm{Cor}_{F'/F}(Q'), \mathrm{Cor}_{F'/F}(\iota), \ldots)}$$

(ι = canonical involution of Q')

Question: $F'/F \quad \rightsquigarrow \quad R'/R \quad \rightsquigarrow \quad S'/S$?

Example $F = \mathbb{R}$

$\mathfrak{A}_1^2$: objects $(F'/\mathbb{R}, Q')$ étale, $[F' : \mathbb{R}] = 2$, Q'/F' quaternion
$F' = \mathbb{C}$ or $F' = \mathbb{R} \times \mathbb{R}$
$F' = \mathbb{R} \times \mathbb{R} \implies Q' = Q_1 \times Q_2$, Q_i quaternion $\mathbb{R}$–algebras

$\mathfrak{D}_2$: objects (A, σ), A central-simple $\mathbb{R}$–algebra, σ orthogonal involution.

$$\begin{array}{ccc}
\mathfrak{A}_1^2 & \xrightarrow{\mathrm{Cor}_{F'/\mathbb{R}}} & \mathfrak{D}_2 \\
\big(\mathbb{C}/\mathbb{R}, \mathrm{Mat}_2(\mathbb{C}) = \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)\big) & \mapsto & \mathrm{End}_{\mathbb{R}}(\mathbb{R}^4) \cong \mathrm{Mat}_4(\mathbb{R}) \\
\big(\mathbb{R} \times \mathbb{R}, \mathrm{Mat}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{R})\big) & \mapsto & \mathrm{Mat}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathrm{Mat}_2(\mathbb{R}) \cong \mathrm{Mat}_4(\mathbb{R}) \\
(\mathbb{R} \times \mathbb{R}, \mathbb{H} \times \mathbb{H}) & \mapsto & \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathrm{Mat}_4(\mathbb{R}) \\
\big(\mathbb{R} \times \mathbb{R}, \mathrm{Mat}_2(\mathbb{R}) \times \mathbb{H})\big) & \mapsto & \mathrm{Mat}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{H} \cong \mathrm{Mat}_2(\mathbb{H})
\end{array}$$

Generalizations of corestriction

New terminology: Corestriction becomes norm

- Grothendieck [EGA II] (1962):
  $S' \to S$ finite locally free morphism of schemes,
  e.g. $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ for R' finite projective as R–module
  Norm $\mathsf{N}_{S'/S}$ for invertible $\mathcal{O}_{S'}$–modules, via Zariski "gluing",
  $\mathsf{N}_{S'/S} \colon \mathrm{Pic}(S') \to \mathrm{Pic}(S), [L'] \mapsto [\mathsf{N}_{S'/S}(L')]$ group homomorphism

R base ring (commutative associative unital),
R' commutative associative unital R–algebra, finite projective as R–module
Norm functor $\mathsf{N}_{R'/R} \colon R'\text{-}\mathbf{mod} \to R\text{-}\mathbf{mod}$

- Knus-Ojanguren 1975:
  R'/R finite étale, $\mathsf{N}_{R'/R}$ via flat descent instead of Galois descent

- Ferrand 1998:
  $\mathsf{N}_{R'/R}$ via universal property

- Rost 2003 (preprint):
  $\mathsf{N}_{R'/R}$ via symmetric tensors,
  (multiplicative transfer for Witt-Grothendieck groups)

Properties of $\mathsf{N}_{R'/R}$: R'-**mod** $\to$ R-**mod**

R' an R–algebra finite projective as R–module

- $\mathsf{N}_{R'/R}(M')$ of R'–module M' is "polynomial", i.e., T an associative commutative R–algebra, $T' = R' \otimes_R T$ finite projective T–algebra, $M' \otimes_R T$ is T'–module,

$$\mathsf{N}_{R'/R}(M') \otimes_R T \cong \mathsf{N}_{T'/T}(M' \otimes_R T)$$

- $R' = R \times \cdots \times R$ (d factors), so $M' = M_1 \times \cdots \times M_d$ ($M_i \in R$-**mod**)

$$\mathsf{N}_{R'/R}(M_1 \times \cdots \times M_d) = M_1 \otimes_R \cdots \otimes_R M_d$$

- A' nonassociative R'–algebra $\implies$ $\mathsf{N}_{R'/R}(A')$ canonically nonassociative R–algebra s. th. A' associative (unital resp.) $\implies$ $\mathsf{N}_{R'/R}(A')$ associative (unital resp.)
- R'/R finite étale
  - M' finite projective $\implies$ $\mathsf{N}_{R'/R}(M')$ finite projective
  - A' Azumaya R'-algebra $\implies$ $\mathsf{N}_{R'/R}(A')$ Azumaya R–algebra
  - $\mathsf{N}_{R'/R}$: $\mathrm{Br}(R') \to \mathrm{Br}(R)$, $[A'] \to [\mathsf{N}_{R'/R}(A')]$ group homomorphism s. th. for $[R' : R] = d$ and A Azumaya R–algebra

$$[\mathsf{N}_{R'/R}(R' \otimes_R A)] = [A \otimes_R \cdots \otimes_R A] = [A]^d \in \mathrm{Br}(R)$$

Norm for $S' \to S$ finite locally free

$\mathfrak{Sch}_S$ big fppf site of schemes over a scheme S,
$\mathcal{O}$ structure sheaf on $\mathfrak{Sch}$, given by $X(\in \mathfrak{Sch}_S) \mapsto \mathcal{O}_X(X)$,
$\mathfrak{QCoh}(S)$ category of quasi-coherent $\mathcal{O}$–modules over $\mathfrak{Sch}_S$

Theorem (Gille-N-Ruether, arXiv 2024)

Assume $S' \to S$ finite locally free morphism of schemes.

- *There exists a norm functor $\mathsf{N}_{S'/S} \colon \mathfrak{QCoh}(S') \to \mathfrak{QCoh}(S)$ such that for all affine $U \to S$ and all $M' \in \mathfrak{QCoh}(S')$*

  $$\mathsf{N}_{S'/S}(M')(U) = \mathsf{N}_{\mathcal{O}(S' \times_S U)/\mathcal{O}(U)} \left(M'(S' \times_S U)\right)$$

  where N on the right-hand side is the Ferrand-norm
  $\mathcal{O}(S' \times_S U)$ is a finite projective $\mathcal{O}(U)$–algebra, $M'(S' \times_S U)$ is an $\mathcal{O}(S' \times_S U)$–module
- *The norm functor preserves the properties of $\mathsf{N}_{R'/R}$, R' finite projective or finite étale*

Corollary

The previously defined categories $\mathfrak{A}_1^2$, $\mathfrak{A}_1^{2,\mathrm{gr}}$, $\mathfrak{D}_2$ and $\mathfrak{D}_2^{\mathrm{gr}}$ have natural generalizations to schemes, and they are all equivalent. The equivalence $\mathfrak{A}_1^2 \xrightarrow{\sim} \mathfrak{D}_2$ is induced by the norm functor above.