Steinberg groups & Jordan pairs

(St. Petersburg Algebraic Groups Seminar – 2021/11/15)

Outline:

Jordan pairs

Steinberg groups for root-graded JPs

Fundamental Theorem

Example: $\mathbb{M}_{12}(\mathbb{O})$, $\mathbb{O}$ division octonions

References:

* Loos – N: Steinberg groups for Jordan pairs
  Progress in Math. 332 (2019)
* N: Steinberg groups for Jordan pairs – an introduction with open problems.
  (arXiv, home page)

Everything over k (comm assoc unital)

* **Jordan pairs** (Tsvetkov, Oct 4)

  $V = (V^+, V^-)$, V^σ, $\sigma = \pm$, k-module

  & $Q^\sigma : V^\sigma \times V^{-\sigma} \to V^\sigma$, $(x,y) \mapsto Q_x y$

  quadratic in x, linear in y

  Linearize

  $$Q_{x,z}\, y = Q_{x+z}\, y - Q_x y - Q_z y$$
  $$= D(x,y)z = \{x\,y\,z\}$$

  Identities:

  - $D(x,y)\,Q_x = Q_x\, D(y,x)$
  - $D(Q_x y, y) = D(x, Q_y x)$
  - $Q_{Q_x y} = Q_x Q_y Q_x$

  + all linearizations

* **Examples**

  - J Jordan algebra, $U_x y$ given
    (e.g. $\frac{1}{2} \in k$, $U_x y = 2x\cdot(x\cdot y) - x^2 \cdot y$)
    $V(J) = (J, J)$ Jordan pair, $Q_x y = U_x y$
  - Subpairs: $S = (S^+, S^-) \subset V = (V^+, V^-)$ JP
    s.th. $Q_x y \in S^\sigma$, $\forall (x,y) \in V^\sigma \times V^{-\sigma}$
    $\rightsquigarrow$ S Jordan pair
  - A assoc k-algebra $\rightsquigarrow$ (A, A) JP
    with $Q_x y = xyx$
    Subpairs ! e.g.
    $\mathbb{M}_{pq}(A) = (\mathrm{Mat}_{pq}(A), \mathrm{Mat}_{pq}(A))$
    $= \left(\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right)$

3-graded root systems

R root system (finite, reduced)

reflection $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$

3-grading $R = R_1 \,\dot\cup\, R_0 \,\dot\cup\, R_{-1}$ s.th.

- $R_{-1} = -R_1$
- $(R_i + R_j) \cap R \subset R_{i+j}$
- $(R_1 + R_{-1}) \cap R = R_0$

Notation (R, R_1), Fact: $\langle \alpha, \beta^\vee \rangle \in \{0,1,2\}$
$(\alpha, \beta \in R_1)$

Examples:

- R irreducible
  minuscule coweight $\longleftrightarrow$ 3-grading
  $\lambda \longmapsto R_1 = \{\alpha \in R : \lambda(\alpha) = 1\}$
  $\rightsquigarrow$ R has 3-grading $\Leftrightarrow$ $R \neq E_8, F_4, G_2$

* **Root graded Jordan pairs**

  $V = (V^+, V^-)$ JP, $Q_x y$, $Q_{x,z}\, y = D(x,y)z = \{x\,y\,z\}$
  (R, R_1) 3-graded root system
  $V = \bigoplus_{\alpha \in R_1} V_\alpha$, $V_\alpha = (V_\alpha^+, V_\alpha^-)$ s.th.

  - $Q_{V_\alpha^\sigma} V_\beta^{-\sigma} \subset V^\sigma_{2\alpha - \beta}$, $\{V_\alpha^\sigma\, V_\beta^{-\sigma}\, V_\gamma^\sigma\} \subset V^\sigma_{\alpha - \beta + \gamma}$
  - $D(V_\alpha^\sigma, V_\beta^{-\sigma}) = 0$ if $\alpha \perp \beta$

  Example

  - $e = (e^+, e^-) \in V = (V^+, V^-)$ idempotent: $e^\sigma = Q_{e^\sigma} e^{-\sigma} = e^\sigma$
    (e.g. $V = (J, J)$ J JA, $e = e^2$ JA idempotent
    $\rightsquigarrow$ (e, e) idempotent in $V(J) = (J, J)$)
    $V = V_2(e) \oplus V_1(e) \oplus V_0(e)$ Peirce decomposition
    $V_2(e) = (Q_{e^+} V^-, Q_{e^-} V^+)$, $V_1(e) = \dots$
    is a C_2-grading:
    $C_2 = \{ \varepsilon_i \pm \varepsilon_j : 0 \le i, j \le 1 \} \setminus \{0\}$
    3-graded by $(C_2)_1 = \{2\varepsilon_0, \varepsilon_0 + \varepsilon_1, 2\varepsilon_1\}$
    $V^\sigma_{\varepsilon_i + \varepsilon_j} = V^\sigma_{i+j}(e)$; $V_2^\sigma(e) = V^\sigma_{2\varepsilon_1}$, $V_1^\sigma(e) = V^\sigma_{\varepsilon_0 + \varepsilon_1}$

  Idempotent root gradings:

  $V = \bigoplus_{\beta \in R_1} V_\beta$ root graded JP by (R, R_1)
  is idempotent if $\exists\, \Delta \subset R_1$ & $(e_\alpha)_{\alpha \in \Delta}$
  family of idempotents e_α ($\rightsquigarrow$ Peirce dec)
  s.th. $V_\beta = \bigcap_{\alpha \in \Delta} V_{\langle \beta, \alpha^\vee \rangle}(e_\alpha)$, $\forall \beta \in R_1$
  (taken componentwise,
  recall $\langle \beta, \alpha^\vee \rangle \in \{0, 1, 2\}$)

  - Prime example:
    V JP, $e \in V$ idempotent,
    $V = V_2(e) \oplus V_1(e) \oplus V_0(e)$ Peirce dec
    $(C_2, C_{2,1})$, $(C_2)_1 = \{2\varepsilon_1, \varepsilon_1 + \varepsilon_0, 2\varepsilon_0\}$
    $\Delta = \{2\varepsilon_1\} \subset (C_2)_1$
    $\langle \beta, 2\varepsilon_1^\vee \rangle = 0, 1, 2$
  - Basic example:
    $V = M_{pq}(A) = \left(\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right)$
    E_{ij} matrix units, $n = p + q$
    $V^+ = \bigoplus_{1 \le i \le p < j \le n} A E_{ij}$
    $R = A_{n-1} = \{ \varepsilon_i - \varepsilon_j : 1 \le i \ne j \le n \}$
    $R_1 = \{ \varepsilon_i - \varepsilon_j : 1 \le i \le p < j \le n \}$
    $\rightsquigarrow$ (R, R_1) 3-graded root system
    V is (R, R_1)-graded.
    $e_{ij} = (E_{ij}, E_{ji})$ idempotent of V
    V is idempotent root graded, $\Delta = R_1$

Steinberg groups for root graded JPs

Given root graded JP $V = \bigoplus_{\alpha \in R_1} V_\alpha$,
abbreviate $\mathcal{R} = (V_\alpha)_{\alpha \in R_1}$

Steinberg group $\mathrm{St}(V, \mathcal{R})$ is presented

- generators $x_+(u)$, $x_-(v)$, $(u,v) \in V$
  for $(u,v) \in V_\alpha^+ \times V_\beta^-$ define $b(u,v)$ by
  $$x_+(u)\, x_-(v) = x_-(v + Q_v u)\, b(u,v)\, x_+(u + Q_u v)$$
- relations
  (St1) $x_\sigma(u + u') = x_\sigma(u)\, x_\sigma(u')$, $\sigma = \pm$
  (St2) $[x_+(u), x_-(v)] = 1$ if $(u,v) \in V_\alpha^+ \times V_\beta^-$, $\alpha \perp \beta$
  (St3) $[b(u,v), x_+(z)] = x_+(-\{u\,v\,z\} + Q_u Q_v z)$
  $[b(u,v)^{-1}, x_-(y)] = x_-(-\{v\,u\,y\} + Q_v Q_u z)$
  if $(u,v) \in V_\alpha^+ \times V_\beta^-$, $\alpha \neq \beta$, all $(z,y) \in V$

Example $V = \mathbb{M}_{pq}(A) = \left(\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right)$,
$n = p + q$, $R = A_{n-1}$, $R_1 = \{ \varepsilon_i - \varepsilon_j : 1 \le i \le p < j \le n \}$
$V_\alpha = (A E_{ij}, A E_{ji})$, $\alpha = \varepsilon_i - \varepsilon_j$,
$\rightsquigarrow$ $\mathrm{St}(V, \mathcal{R}) = \mathrm{St}_n(A)$ linear Steinberg

Theorem (Loos – N)

$V = \bigoplus_{\alpha \in R_1} V_\alpha$ root graded JP,
$\mathcal{R} = (V_\alpha)_{\alpha \in R_1}$ idempotent wrt $(e_\alpha)_{\alpha \in \Delta}$,
Δ "big enough" (depending on R, eg $\Delta = R_1$)
rank $R \geq 5$
$\Rightarrow$ $\mathrm{St}(V, \mathcal{R})$ is centrally closed, ie

$$E \underset{\exists !}{\overset{\text{cent ext}}{\rightleftarrows}} \mathrm{St}(V, \mathcal{R})$$

Remark

V JP $\rightsquigarrow$ $\mathrm{PE}(V)$ projective elementary group,
subgroup of $\mathrm{Aut}(\mathrm{TKK}(V))$
$\mathrm{St}(V, \mathcal{R}) \overset{\pi}{\twoheadrightarrow} \mathrm{PE}(V)$
$\rightsquigarrow$ $\mathrm{St}(V, \mathcal{R})$ is uce of $\mathrm{PE}(V)$,
if π is a central extension (eg $|R| = \infty$)

$\boxed{\text{Problem:}}$ When is π a central extension ?

Remark

$V = \bigoplus_{\alpha \in R_1} V_\alpha$, (R, R_1) irreducible
Thm holds for $R = A_4, B_4, C_4$, $\cancel{D_4}$
Steinberg (Yale, 1981):
exceptions if rank R "low", $V_\alpha \cong F$ field

$\boxed{\text{Conjecture}}$ $V = \bigoplus_\alpha V_\alpha$, V_α division pair,
R irred, $2 \le \operatorname{rank} R \le 3$ or $R = D_4$
$\Rightarrow$ $\mathrm{St}(V, \mathcal{R})$ is centrally closed
(even uce of $\mathrm{PE}(V)$)
except in the known exceptions

Friday, November 5, 2021 11:45 AM

Example $\mathbb{M}_{12}(C)$

C unital alternative algebra

$(x,y,z) = (xy)z - x(yz)$ alternating

$V = \mathbb{M}_{12}(C) = (\mathrm{Mat}_{12}(C), \mathrm{Mat}_{21}(C))$ is JP

$Q_x y = x(yx)$, $Q_y x = (yx)y$ for $(x,y) \in V$

idempotent $e = ((10), \binom{1}{0}) \in V$

Peirce decomp $V = V_2(e) \oplus V_1(e) =: V_2 \oplus V_1$

$V_2 = V_2(e) = ((C0), \binom{C}{0})$, $V_1 = V_1(e) = ((0C), \binom{0}{C})$

idempotent root grading, type A_2

Steinberg group $\mathrm{St}(V,R)$

- generators $x_+(u), x_-(v)$, $(u,v) \in V$

  define $x_+(u)x_-(v) = x_-(v)\,b(u,v)\,x_+(u)$

- relations

  * $x_\sigma(w+w') = x_\sigma(w)\,x_\sigma(w')$

  * $[b(u,v), x_+(z)] = x_+(-\{uvz\})$

    $[b(u,v)^{-1}, x_-(y)] = x_-(-\{vuy\})$

    $(u,v) \in V_i^+ \times V_j^-$, $i \neq j$, all $(z,y) \in V$

Steinberg: $C = F$ field

$\mathrm{St}(V,R)$ centrally closed $\Leftrightarrow F \neq \mathbb{F}_2, \mathbb{F}_4$

Theorem (N)

C alternative division algebra, $C \neq \mathbb{F}_2, \mathbb{F}_4$

$\Rightarrow$ $\mathrm{St}(V,R)$ centrally closed (and a uce of $\mathrm{PE}(V)$)

Remarks

- V_i $(i=1,2)$ division JP

  $\Leftrightarrow$ C alternative division

  $\Leftrightarrow$ (Bruck-Kleinfeld Skornyakov) Either C assoc division or C octonion division algebra

- C octonion algebra / K = centre of C

  $\Rightarrow$ $\mathrm{PE}(V)$ = octonion (= Cayley) plane

  $= G(K)$, G ss alg K-gp, type E_6

  Fact: C either division algebra, $G = {}^1E_{6,2}^{28}$

  or C split: A_2-grading $\rightsquigarrow$ E_6-grading

  $\mathrm{St}(V,R)$ is centrally closed (Steinberg)

  G is split E_6; Tsvetkov (Oct 4)

THANK YOU