Geometric Realizations of the Basic Representation of $\widehat{\mathfrak{sl}}$

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Goal

To give a geometric description of the various realizations of the basic representation of $\widehat{\mathfrak{sl}}_r$.

Outline

1. Motivation
2. Focus on the principal realization
   1. Algebraic description
   2. Cohomology and vector bundles
   3. Geometric description
3. Generalize to other realizations

Motivation

Representations of affine Lie algebras are related (in physics) to quantum states, bosons/fermions (particles that make up the universe), etc...

Mathematics (algebra)		Physics
Vector space V	$\longleftrightarrow$	Universe (quantum states)
Operators on V	$\longleftrightarrow$	Creation/annihilation of bosons and fermions
Realize affine Lie algebras using operators on V	$\longleftrightarrow$	Certain creation/ annihilation processes
Different Realizations	$\longleftrightarrow$	Different "vacuum spaces" (space where nothing exists)

Geometrizing:

"Algebra $\rightarrow$ Geometry" $\implies$ often leads to new insights.

Algebraic Description

Definition (Heisenberg algebra)

Complex Lie algebra $\mathfrak{s} = \bigoplus_{k \in \mathbb{Z} - \{0\}} \mathbb{C}\alpha(k) \oplus \mathbb{C}c$,

$$[\mathfrak{s}, c] = 0, \quad [\alpha(k), \alpha(j)] = k\delta_{k+j,0}c.$$

Action on $\mathbb{C}[x_1, x_2, \dots]$ (bosonic Fock space) via

$$\alpha(k) \mapsto \frac{\partial}{\partial x_k}, \quad \alpha(-k) \mapsto kx_k, \quad k > 0,$$

$$c \mapsto \mathrm{id}.$$

Algebraic Description

Basic representation of $\widehat{\mathfrak{g}} = \widehat{\mathfrak{gl}}_r$ or $\widehat{\mathfrak{sl}}_r$

- Recall $\widehat{\mathfrak{g}} = \left(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]\right) \oplus \mathbb{C}c$.
- Basic representation, $V_{\mathsf{basic}}(\widehat{\mathfrak{g}})$, irreducible representation characterized by the existence of a $v \in V_{\mathsf{basic}}(\widehat{\mathfrak{g}})$ such that:

$$(\mathfrak{g} \otimes \mathbb{C}[t]) \cdot v = 0, \quad \text{and} \quad c \cdot v = v.$$

- Note: $\mathfrak{s} \hookrightarrow \widehat{\mathfrak{g}}$ in various ways, call this a Heisenberg subalgebra (HSA).

Realizations of $V_{\mathsf{basic}}(\widehat{\mathfrak{g}})$

Given a HSA, as $\widehat{\mathfrak{g}}$-modules,

$$V_{\mathsf{basic}}(\widehat{\mathfrak{g}}) \cong \Omega \otimes \mathbb{C}[x_1, x_2, \dots],$$

where $\Omega = \{v \in V_{\mathsf{basic}}(\widehat{\mathfrak{g}}) \mid \alpha(k) \cdot v = 0 \text{ for all } k > 0\}$ is the vacuum space.

Algebraic Description

The problem is choice

Different choices of HSA's yield different Ω.

For $\widehat{\mathfrak{sl}}_2$:

- Principal HSA:

$$\alpha(k) \mapsto e \otimes t^{k-1} + f \otimes t^k, \quad \alpha(-k) \mapsto \frac{k}{2k-1}(e \otimes t^{-k} + f \otimes t^{1-k}),$$

$$\implies \dim \Omega = 1 \quad \text{(rest. of } V_{\text{basic}} \text{ to } \mathfrak{s} \text{ remains irred.).}$$

Note: in terms of Chevalley generators, $\alpha(1) = E_0 + E_1$ and $\alpha(-1) = F_0 + F_1$ (up to scalar mult.).

- Homogeneous HSA:

$$\alpha(k) \mapsto h \otimes t^k, \quad \alpha(-k) \mapsto \frac{1}{8}(h \otimes t^{-k}),$$

$$\implies \dim \Omega = \infty.$$

Algebraic Description

In general

HSA's of $\widehat{\mathfrak{sl}}_r$ are parametrized by **partitions** of r, i.e.

$$\underline{r} = (r_1, \ldots, r_s), \quad \text{s.t.} \quad r_1 + \cdots + r_s = r \text{ and } r_1 \geq \cdots \geq r_s.$$

Two extreme cases:

- Principal HSA $\longleftrightarrow \underline{r} = (r)$.
- Homogeneous HSA $\longleftrightarrow \underline{r} = (1, 1, \ldots, 1)$.

Algebraic Description

Definition (Fermionic Fock space)

Infinite wedge space, i.e.
$$F := \mathrm{span}_{\mathbb{C}}\{i_1 \wedge i_2 \wedge \cdots \mid i_k \in \mathbb{Z},\ i_k > i_{k+1},$$
$$i_{k+1} = i_k - 1 \text{ for } k \gg 0\}.$$

Define zero-charge subspace

$$F_0 := \mathrm{span}_{\mathbb{C}}\{i_1 \wedge i_2 \wedge \cdots \in F \mid i_k = 1 - k \text{ for } k \gg 0\}.$$

Definition (Fermions)

Operators $\psi(j)$, $\psi^*(j)$ on F: for all $j \in \mathbb{Z}$,

- $\psi(j) = $ wedge j,
- $\psi^*(j) = $ contract j.
- This defines an irreducible representation of the Clifford algebra, Cl.

Algebraic Description

How to describe different realizations?

Ex: Principal realization. Define:

$$\psi(z) := \sum_{k \in \mathbb{Z}} \psi(k) z^k, \quad \psi^*(z) := \sum_{k \in \mathbb{Z}} \psi^*(k) z^{-k}.$$

The homogeneous components of

$$:\psi(\omega^p z)\psi^*(\omega^q z): -\frac{\omega^{p-q}}{1-\omega^{p-q}}, \quad 1 \leq p, q \leq r, \ p \neq q,$$

and $\quad :\psi(z)\psi^*(z):,$

where $\omega = e^{2\pi i/r}$, span a Lie algebra of operators on F_0 isomorphic to $\widehat{\mathfrak{gl}}_r$, and $F_0 \cong V_{\text{basic}}$.

Principal HSA given by $\alpha(k) \mapsto \sum_{i \in \mathbb{Z}} :\psi(i)\psi^*(i+k):.$

Algebraic Description

How to describe different realizations? (continued)

Via the boson-fermion correspondence,

$$F_0 \cong \mathbb{C}[x_1, x_2, \dots] \cong \mathbb{C} \otimes \mathbb{C}[x_1, x_2, \dots], \quad \text{(1-dim vacuum space)},$$

as $\widehat{\mathfrak{gl}}_r$-modules. Need a slight "tweak" to get a realization for $V_{\text{basic}}(\widehat{\mathfrak{sl}}_r)$.

For other realizations... we'll see later.

Geometrizing in a nutshell

<u>Goal:</u> Find geometric analogs of algebraic objects.

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Algebra   Geometry

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Algebra		Geometry
Vector space V	$\longleftrightarrow$	(co)homology of algebraic varieties
Linear maps on V	$\longleftrightarrow$	Geometric operators on (co)homology
Realize algebras using linear maps on V	$\longleftrightarrow$	Realize Geometric versions of algebras

Equivariant Cohomology

Equivariant cohomology

- Let X be a (nice) $4n$-dim. variety,
- $T = (\mathbb{C}^*)^d$ torus acting on X.
- $H_T^*(\mathrm{pt}) = \mathbb{C}[t_1, \ldots, t_d]$,
- Localized equivariant cohomology

$$\mathcal{H}_T^*(X) = H_T^*(X) \otimes_{\mathbb{C}[t_1, \ldots, t_d]} \mathbb{C}(t_1, \ldots, t_d).$$

- $\mathcal{H}_T^*(X) \cong \mathcal{H}_T^*(X^T)$.

Bilinear form on $\mathcal{H}_T^{2n}(X)$

- $X \xleftarrow{\ i\ } X^T \xrightarrow{\ p\ } \text{pt.}$
- For $a, b \in \mathcal{H}_T^{2n}(X)$,

$$\langle a, b \rangle_X = p_*(i_*)^{-1}(a \cup b).$$

Bilinear form on $\mathcal{H}_T^{2(n_1+n_2)}(X_1 \times X_2)$

Given $T \curvearrowright X_1, X_2$ and $a, b \in \mathcal{H}_T^{2(n_1+n_2)}(X_1 \times X_2)$,

$$\langle a, b \rangle_{X_1 \times X_2} = p_*((i_1 \times i_2)_*)^{-1}(a \cup b).$$

Vector Bundle Operator

Operator

$\alpha \in \mathcal{H}_T^{2(n_1+n_2)}(X_1 \times X_2)$ gives a map $\alpha : \mathcal{H}_T^{2n_1}(X_1) \to \mathcal{H}_T^{2n_2}(X_2)$ with structure constants

$$\langle \alpha(a), b \rangle_{X_2} = \langle a \otimes b, \alpha \rangle_{X_1 \times X_2}.$$

Operators from vector bundles

- Let $E \to X_1 \times X_2$ be a T-equivariant vector bundle.
- Then the k-th Chern class $c_k(E) \in \mathcal{H}_T^{2k}(X_1 \times X_2)$.
- For $\beta \in \mathcal{H}_T^{2(n_1+n_2-k)}(X_1 \times X_2)$,

$$\beta \cup c_k(E) : \mathcal{H}_T^{2n_1}(X_1) \to \mathcal{H}_T^{2n_2}(X_2).$$

Need:

- A variety whose cohomology corresponds to F_0.

- A vector bundle whose Chern classes give the appropriate operators:
  - Heisenberg algebra
  - $\widehat{\mathfrak{sl}}_r$
  - Clifford algebra

Picking the Right Variety (Principal Realization)

Definition (Hilbert scheme)

The Hilbert scheme of n points in the plane can be defined as

$$\mathrm{HS}(n) := \{I \trianglelefteq \mathbb{C}[x,y] \mid \dim(\mathbb{C}[x,y]/I) = n\}.$$

Note: $\dim \mathrm{HS}(n) = 4n$.

Torus action

Fix $T = \mathbb{C}^*$. Action on $\mathrm{HS}(n)$ induced by

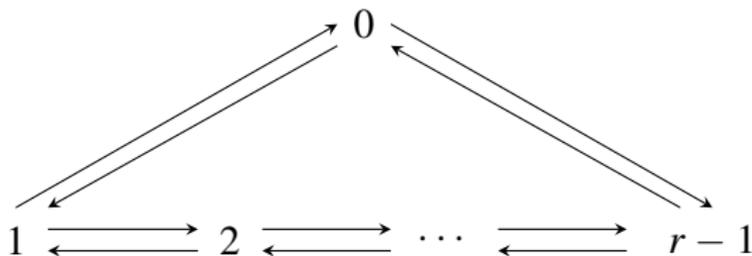$$t \cdot x = tx, \qquad \text{and} \qquad t \cdot y = t^{-1}y.$$

Fact (by a result of Nakajima, Yoshioka 2005):

$$\coprod_n \mathrm{HS}(n)^T \xleftrightarrow{\sim} \text{ basis of } F_0 \subseteq F.$$

Quiver Varieties

Definition (Quiver)

A quiver is a directed graph, i.e. $Q = (Q_0, Q_1)$, where $Q_0 = \{\text{vertices}\}$ and $Q_1 = \{\text{arrows}\}$.

Fix Q: $Q_0 = \mathbb{Z}_r$, $Q_1 = \{k \to k+1\}_{k \in \mathbb{Z}_r} \cup \{k \to k-1\}_{k \in \mathbb{Z}_r}$

Quiver Varieties

Definition (Nakajima quiver variety)

- $\mathbb{Z}_r$-graded vector space V, $\mathbf{v} = (\dim V_k)_{k \in \mathbb{Z}_r}$, $|\mathbf{v}| = \sum_k \mathbf{v}_k$.
- Let $G_\mathbf{v} := \prod_{k \in \mathbb{Z}_r} \mathrm{GL}(V_k)$.

Define $M :=$ variety whose points consist of

- $i \in V_0$,
- linear maps $C_k^\pm : V_k \to V_{k\pm 1}$ for all $k \in \mathbb{Z}_r$, such that

  1. $V_{k-1} \; \underset{C_k^-}{\overset{C_{k-1}^+}{\rightleftarrows}} \; V_k \; \underset{C_{k+1}^-}{\overset{C_k^+}{\rightleftarrows}} \; V_{k+1}$ commutes,

  2. i generates V under application of $C_k^\pm$.

The Nakajima quiver variety is

$$QV(\mathbf{v}) = M /\!\!/ G_\mathbf{v}.$$

Quiver Varieties

Theorem (Nakajima/Barth)

The Hilbert scheme is isomorphic to the quiver variety with 1 vertex. That is, for

$$Q : \quad \bullet$$

we have $\mathrm{HS}(n) \cong \mathrm{QV}(n)$.

Observation

- $\mathbb{Z}_r \hookrightarrow T$.
- Thus, $\mathbb{Z}_r \curvearrowright \mathrm{HS}(n)$.
- Fact: $\mathrm{HS}(n)^{\mathbb{Z}_r} \cong \coprod_{|\mathbf{v}|=n} \mathrm{QV}(\mathbf{v})$, ($r$ vertices).
- $\mathrm{QV}(\mathbf{v})$ inherits a T-action.

Idea

$$\mathcal{H}_T^*(-) \quad \leftarrow \text{Define operators}$$

$$\text{Heisenberg} \atop \text{algebra} \curvearrowright \text{HS}(n) \quad \supseteq \quad \text{HS}(n)^{\mathbb{Z}_r} \quad \supseteq \quad \text{HS}(n)^T \quad \curvearrowleft \quad {\text{Clifford} \atop \text{algebra}}$$

$$\cong$$

$$\coprod_{\mathbf{v}} \text{QV}(\mathbf{v}) \quad \curvearrowleft \quad \widehat{\mathfrak{sl}}_r$$
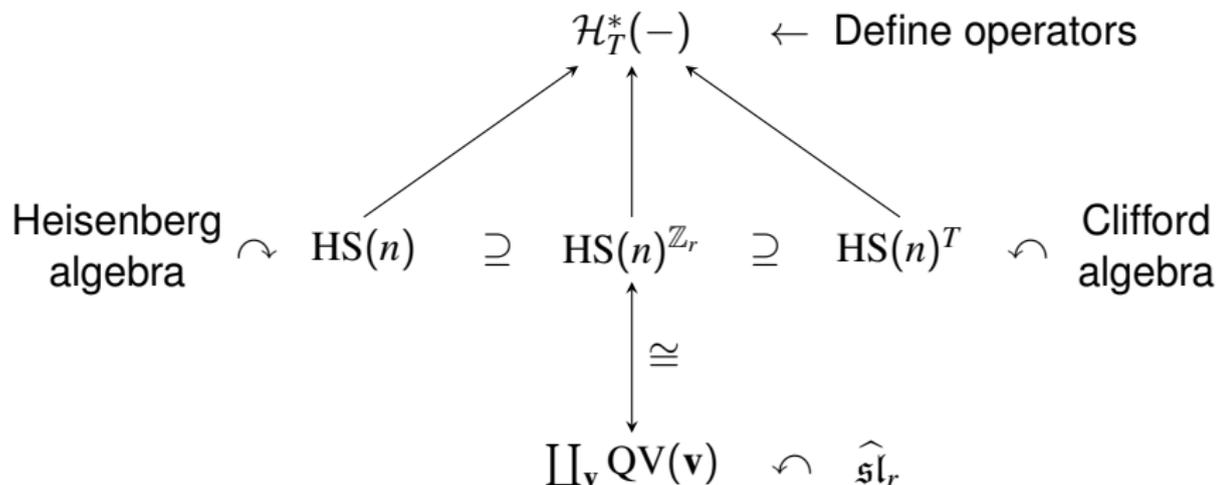
Vector bundles on $QV(\mathbf{v})$

Tautological vector bundles

$$V_k \times_{G_\mathbf{v}} M \to QV(\mathbf{v}) \quad \text{and} \quad \mathbb{C} \times QV(\mathbf{v}) \to QV(\mathbf{v})$$

Denote by $\mathcal{V}_k$ ($k \in Q_0$) and $\mathcal{W}$, respectively.

Note: T-equivariant w.r.t. trivial action on V_k and $\mathbb{C}$.

On the product $QV(\mathbf{v}^1) \times QV(\mathbf{v}^2)$

Have Hom-bundles:

$$\operatorname{Hom}(\mathcal{V}_k^1, \mathcal{V}_j^2), \quad \operatorname{Hom}(\mathcal{W}^1, \mathcal{V}_0^2), \quad \operatorname{Hom}(\mathcal{V}_0^1, \mathcal{W}^2).$$

Picking the Right Vector Bundle (Principal Realization)

On the product $QV(\mathbf{v}^1) \times QV(\mathbf{v}^2)$

$$\mathcal{E} := \bigoplus_{k \in Q_0} \mathrm{Hom}(\mathcal{V}_k^1, \mathcal{V}_k^2), \quad \mathcal{E}^{\pm} := \bigoplus_{k \in Q_0} \mathrm{Hom}(\mathcal{V}_k^1, \mathcal{V}_{k\pm1}^2).$$

Complex of vector bundles

$$\mathcal{E} \overset{\sigma}{\hookrightarrow} \begin{matrix} t\mathcal{E}^+ \oplus \mathrm{Hom}(\mathcal{W}, \mathcal{V}_0^2) \\ \oplus \\ t^{-1}\mathcal{E}^- \oplus \mathrm{Hom}(\mathcal{V}_0^1, \mathcal{W}) \end{matrix} \overset{\tau}{\twoheadrightarrow} \mathcal{E}$$

(Similar to construction of Nakajima's Hecke correspondence)

Theorem ($\sim$ Licata, Savage 2009)

$\ker \tau / \operatorname{im} \sigma \to QV(\mathbf{v}^1) \times QV(\mathbf{v}^2)$ *is a vector bundle.*

Denote this vector bundle by $\mathcal{K}(QV(\mathbf{v}^1), QV(\mathbf{v}^2))$.

Picking the Right Vector Bundle (Principal Realization)

Observations

- $\mathcal{K}(\mathrm{HS}(n_1), \mathrm{HS}(n_2))$ is a vector bundle on $\mathrm{HS}(n_1) \times \mathrm{HS}(n_2)$.

- $\mathcal{K}(\mathrm{HS}(n_1), \mathrm{HS}(n_2))^{\mathbb{Z}_r}$ is a v.b. on $\mathrm{HS}(n_1)^{\mathbb{Z}_r} \times \mathrm{HS}(n_2)^{\mathbb{Z}_r}$.

- $\mathrm{HS}(n_1)^{\mathbb{Z}_r} \times \mathrm{HS}(n_2)^{\mathbb{Z}_r} \cong \coprod_{\mathbf{v}^1, \mathbf{v}^2} \mathrm{QV}(\mathbf{v}^1) \times \mathrm{QV}(\mathbf{v}^2)$.

Theorem (L.)

$$\mathcal{K}(\mathrm{HS}(n_1), \mathrm{HS}(n_2))^{\mathbb{Z}_r} \cong \coprod_{\mathbf{v}^1, \mathbf{v}^2} \mathcal{K}(\mathrm{QV}(\mathbf{v}^1), \mathrm{QV}(\mathbf{v}^2)).$$

Observation

Gives a geometric interpretation of $\alpha(1) = \sum_k E_k$.

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Algebraically   **Geometrically**

$\alpha(k)$ changes energy $\boldsymbol{\alpha}(k)$ in terms of
$n \mapsto n - k$ $\implies$ $\mathcal{K}(\mathrm{HS}(n), \mathrm{HS}(n - k))$

E_k, F_k change weight $\mathbf{E}_k, \mathbf{F}_k$ in terms of
$\mathbf{v} \mapsto \mathbf{v} \mp 1_k$ $\implies$ $\mathcal{K}(\mathrm{QV}(\mathbf{v}), \mathrm{QV}(\mathbf{v} \mp 1_k))$

Principal HSA in $\widehat{\mathfrak{sl}}_r$ $\implies$ "Geometric" HSA in $\widehat{\mathfrak{sl}}_r$.

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Putting it all together (Principal Realization)

Definition (Geometric operators)

$$\boldsymbol{\alpha}(k), \mathbf{E}_k, \mathbf{F}_k : \bigoplus_n \mathcal{H}_T^{2n}(\mathrm{HS}(n)) \to \bigoplus_n \mathcal{H}_T^{2n}(\mathrm{HS}(n)),$$

restricted to $\mathcal{H}_T^{2n}(\mathrm{HS}(n))$,

$$\boldsymbol{\alpha}(k) := \beta \cup c_{\mathrm{tnv}}(\mathcal{K}(\mathrm{HS}(n), \mathrm{HS}(n-k))),$$

$$\mathbf{E}_k, \mathbf{F}_k := \gamma \cup c_{\mathrm{tnv}}(\mathcal{K}(\mathrm{QV}(\mathbf{v}), \mathrm{QV}(\mathbf{v} \mp 1_k))), \quad (|\mathbf{v}| = n).$$

Theorem (Licata, Savage, L.)

- *The $\mathbf{E}_k$ and $\mathbf{F}_k$ satisfy the Kac-Moody relations for $\widehat{\mathfrak{sl}}_r$.*
- *The $\boldsymbol{\alpha}(k)$ satisfy the Heisenberg relations.*
- *Yields a* *geometric* *version of the principal realization.*

Other Realizations (Algebraically)

For a partition $\underline{r} = (r_1, \ldots, r_s)$,

Divide an $r \times r$ matrix into s^2 blocks of size $r_i \times r_j$:

$$
\begin{pmatrix}
\boxed{r_1 \times r_1} & \boxed{r_1 \times r_2} & \ldots & \boxed{r_1 \times r_s} \\
\boxed{r_2 \times r_1} & \boxed{r_2 \times r_2} & \ldots & \boxed{r_2 \times r_s} \\
\vdots & \vdots & & \vdots \\
\boxed{r_s \times r_1} & \boxed{r_s \times r_2} & \ldots & \boxed{r_s \times r_s}
\end{pmatrix}
$$

Other Realizations (Algebraically)

Diagonal blocks:

- Correspond to $\widehat{\mathfrak{gl}}_{r_i}$.

- Take s copies of the previous construction $\implies$
  "s-coloured" versions of our previous algebras and Fock spaces:

$$\bigoplus_{i=1}^{s} \mathfrak{s}_i, \quad \mathbb{C}[x_1, x_2, \dots]^{\otimes s}, \quad \bigoplus_{i=1}^{s} \mathrm{Cl}_i, \quad F^{\otimes s}.$$

Off-diagonal blocks:

- "Mixed" vertex operators. Operators on (i, j)-th block given in terms of $\psi_i(z)$ and $\psi_j^*(z)$.

Other Realizations (Geometrically)

Need an "s-coloured version" of the Hilbert scheme.

Definition (Moduli space $\mathcal{M}(s,n)$)

Let $\mathcal{M}(s,n)$ be the moduli space of framed torsion-free sheaves on $\mathbb{P}^2$ with rank s and second Chern class n.

Note: $\mathcal{M}(1,n) \cong \mathrm{HS}(n)$. Thus, $\mathcal{M}(s,n)$ is a "higher rank" generalization of the Hilbert scheme.

Torus action

$\mathcal{M}(s,n)$ comes equipped with a natural $T = (\mathbb{C}^*)^{s+1}$ action.

Other Realizations (Geometrically)

Need a geometric interpretation of "dividing into blocks".

$\mathbb{C}^*$-action

Define an embedding $\mathbb{C}^* \hookrightarrow T$ by $z \mapsto (1, z, z^2, \ldots, z^{s-1}, 1)$.

$$\implies \mathbb{C}^* \text{ acts on } \mathcal{M}(s, n).$$

Theorem (Nakajima, 2001)

$$\mathcal{M}(s, n)^{\mathbb{C}^*} \cong \coprod_{\sum n_i = n} \mathrm{HS}(n_1) \times \cdots \times \mathrm{HS}(n_s).$$

Notation

For $\mathbf{n} \in \mathbb{N}^s$, let

$$\mathrm{HS}(\mathbf{n}) = \mathrm{HS}(\mathbf{n}_1) \times \cdots \times \mathrm{HS}(\mathbf{n}_s).$$

Other Realizations (Geometrically)

Need s-coloured versions of operators.

s-coloured vector bundles

- Similar to the principal case, we have a vector bundle

$$\mathcal{K}(\mathcal{M}(s, n), \mathcal{M}(s, m)) \text{ on } \mathcal{M}(s, n) \times \mathcal{M}(s, m).$$

- $\mathcal{K}(\mathcal{M}(s, n), \mathcal{M}(s, m))^{\mathbb{C}^*}$ is a v.b. on $\mathbb{C}^*$-fixed points.

- Can show

$$\mathcal{K}(\mathbf{n}, \mathbf{m}) := \mathcal{K}(\mathcal{M}(s, |\mathbf{n}|), \mathcal{M}(s, |\mathbf{m}|))^{\mathbb{C}^*}|_{\mathrm{HS}(\mathbf{n}) \times \mathrm{HS}(\mathbf{m})},$$

  is a v.b. on $\mathrm{HS}(\mathbf{n}) \times \mathrm{HS}(\mathbf{m})$.

s-coloured Geometric Operators

Algebraically	Geometrically
$\alpha_\ell(k)$ changes energy $n \mapsto n - k$ on ℓ-th colour	$\Longrightarrow$	$\boldsymbol{\alpha}_\ell(k)$ in terms of $\mathcal{K}(\mathbf{n}, \mathbf{n} - k1_\ell)$

Definition (s-coloured geometric Heisenberg operators)

$$\boldsymbol{\alpha}_\ell(k) : \bigoplus_{\mathbf{n}} \mathcal{H}_T^{2|\mathbf{n}|}(\mathrm{HS}(\mathbf{n})) \to \bigoplus_{\mathbf{n}} \mathcal{H}_T^{2|\mathbf{n}|}(\mathrm{HS}(\mathbf{n})),$$

restricted to $\mathcal{H}_T^{2|\mathbf{n}|}(\mathrm{HS}(\mathbf{n}))$,

$$\boldsymbol{\alpha}_\ell(k) := \beta \cup c_{\mathrm{tnv}}(\mathcal{K}(\mathbf{n}, \mathbf{n} - k1_\ell)).$$

Theorem (L.)

The $\boldsymbol{\alpha}_\ell(k)$ satisfy the s-coloured Heisenberg relations.

s-coloured Geometric Operators

s-coloured Chevalley operators

- $R := \mathrm{lcm}(r_1, \ldots, r_s)$, (sometimes $\times 2$).
- Can embed $\mathbb{Z}_R \hookrightarrow T$ such that

$$\mathrm{HS}(\mathbf{n})^{\mathbb{Z}_R} = \mathrm{HS}(\mathbf{n}_1)^{\mathbb{Z}_{r_1}} \times \cdots \times \mathrm{HS}(\mathbf{n}_s)^{\mathbb{Z}_{r_s}}$$

$$\cong \coprod_{|\mathbf{v}^\ell| = \mathbf{n}_\ell} \underbrace{\mathrm{QV}(\mathbf{v}^1) \times \cdots \times \mathrm{QV}(\mathbf{v}^s)}_{=: \, \mathrm{QV}(\mathbf{v}^1, \ldots, \mathbf{v}^s)}.$$

- Can show

$$\mathcal{K}(\mathbf{v}^1, \mathbf{u}^1, \ldots, \mathbf{v}^s, \mathbf{u}^s) := \mathcal{K}(\mathbf{n}, \mathbf{m})^{\mathbb{Z}_R}|_{\mathrm{QV}(\mathbf{v}^1, \ldots, \mathbf{v}^s) \times \mathrm{QV}(\mathbf{u}^1, \ldots, \mathbf{u}^s)},$$

is a v.b. on $\mathrm{QV}(\mathbf{v}^1, \ldots, \mathbf{v}^s) \times \mathrm{QV}(\mathbf{u}^1, \ldots, \mathbf{u}^s)$.

s-coloured Geometric Operators

Algebraically

E_k^ℓ, F_k^ℓ change weight
$$\mathbf{v} \mapsto \mathbf{v} \mp 1_k$$
on ℓ-th colour

$\implies$

Geometrically

$\mathbf{E}_k^\ell, \mathbf{F}_k^\ell$ in terms of
$\mathcal{K}(\mathbf{v}^1, \mathbf{v}^1, \ldots, \mathbf{v}^\ell, \mathbf{v}^\ell \mp 1_k, \ldots, \mathbf{v}^s, \mathbf{v}^s)$

Definition (s-coloured geometric Chevalley operators)

$$\mathbf{E}_k^\ell, \mathbf{F}_k^\ell : \bigoplus_{\mathbf{n}} \mathcal{H}_T^{2|\mathbf{n}|}(\mathrm{HS}(\mathbf{n})) \to \bigoplus_{\mathbf{n}} \mathcal{H}_T^{2|\mathbf{n}|}(\mathrm{HS}(\mathbf{n})),$$

restricted to $\mathcal{H}_T^{2|\mathbf{n}|}(\mathrm{HS}(\mathbf{n}))$,

$$\mathbf{E}_k^\ell, \mathbf{F}_k^\ell := \gamma \cup c_{\mathrm{tnv}}(\mathcal{K}(\mathbf{v}^1, \mathbf{v}^1, \ldots, \mathbf{v}^\ell, \mathbf{v}^\ell \mp 1_k, \ldots, \mathbf{v}^s, \mathbf{v}^s)).$$

Theorem (L.)

The $\mathbf{E}_k^\ell$ and $\mathbf{F}_k^\ell$ satisfy the Kac-Moody relations for $\bigoplus_\ell \widehat{\mathfrak{sl}}_{r_\ell}$.

We have the diagonal block operators.

Still need the off-diagonal block operators.

The end.