Admissible Nilpotent Coadjoint Orbits of Exceptional Real and p-adic Lie Groups

Preliminary Report

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The Orbit Method conjectures a deep relationship between

$$\left\{\begin{array}{c}\text{equivalence classes of}\\ \text{irreducible unitary}\\ \text{representations of G}\end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{c}\text{admissible}\\ \text{coadjoint}\\ \text{orbits of G}\end{array}\right\}$$

Origins: Quantization of classical mechanical systems

Established for:

- nilpotent Lie groups over $\mathbb{R}$ (Kirillov, 1962)
- nilpotent Lie groups over p-adic fields (C.C. Moore, 1965)
- real solvable type I Lie groups (Auslander-Kostant, 1971)
- compact Lie groups

For **reductive** Lie groups over $\mathbb{R}$:

- hyperbolic orbits $\to$ parabolic induction
- elliptic orbits $\to$ cohomological induction
- nilpotent orbits $\to$ no general construction yet known

Admissible Orbits [Duflo, 1980 for real Lie groups]:

- $f \in \mathfrak{g}^*$
- G^f = the centralizer of f in G
- $\mathfrak{g}^f$ = the centralizer of f in $\mathfrak{g}$
- Then $\mathfrak{g}/\mathfrak{g}^f$ has a symplectic structure, preserved by G^f, given by
$$\omega_f(X,Y) = f([X,Y]).$$
- This gives a map $G^f \to Sp(\mathfrak{g}/\mathfrak{g}^f)$.

Consider the pullback of the unique two-fold cover Mp to G^f:

$$\begin{array}{ccc} \widetilde{G^f} & \longrightarrow & Mp(\mathfrak{g}/\mathfrak{g}^f) \\ \Big\downarrow p & & \Big\downarrow \\ G^f & \xrightarrow[Ad]{} & Sp(\mathfrak{g}/\mathfrak{g}^f) \end{array}$$

Definition: A nilpotent coadjoint orbit of a reductive group G is **admissible** if the cover $\widetilde{G^f}$ *splits* over an open subgroup of G^f.

Some results for the Orbit Method on nilpotent orbits:

- construction of representations associated with minimal orbits of simple Lie groups over $\mathbb{R}$ and p-adic fields (Torasso, 1997)
- over $\mathbb{R}$: K-types of associated representation known in many cases (Vogan, see PCMI notes 1998)
- classification of admissible nilpotent orbits
    - Schwarz 1987, Ohta 1991: classical groups over $\mathbb{R}$
    - Nevins 1998: classical groups over p-adic fields

Two remarkable things about the list of admissible nilpotent orbits:

1. For all classical groups, an orbit is admissible **if and only if** every other orbit in its algebraic class is admissible.
2. For all split groups (and most non-split ones as well), the admissible orbits coincide with the set of **special orbits**.

Special Nilpotent Orbits (of an algebraic group):

Definition: A special nilpotent orbit is:

- one associated to a special representation of the Weyl group via the Springer correspondence (Lusztig, 1978)
- one in the image of a certain inclusion-reversing map d on the set of nilpotent orbits (Spaltenstein, 1982)

Connections with representation theory:

- representations of Weyl groups: they parametrize the two-sided cells
- classification of irreducible complex representations of reductive groups over finite fields
- classification of primitive ideals in $U(\mathfrak{g})$

Characterized by:

	Special orbits
A_n	all
B_n	all partitions of $2n+1$ such that even parts occur with even multiplicity and there are an even number of odd parts less than any even part.
C_n	all partitions of $2n$ such that odd parts occur with even multiplicity and there are an even number of even parts greater than any odd part.
D_n	all partitions of $2n$ such that even parts occur with even multiplicity and an even number of odd parts less than any even part.
E, F, G	listed in tables (*cf.* Carter, Collingwood-McGovern)

Closely related geometric notion: Special Pieces.

Definition (Spaltenstein, 1982): A special piece, corresponding to a special nilpotent orbit S, is the union of nilpotent orbits C such that

- $C \subset \overline{S}$ and
- if $C \subset \overline{T}$ for some other special orbit T, then $\overline{S} \subset \overline{T}$.

Theorem (Lusztig, 1997): Two nilpotent orbits lie in the same special piece if and only if their corresponding Springer representations lie in the same two-sided cell.

Parametrization of orbits in special pieces:

special orbit S $\rightsquigarrow$ a finite group $\mathcal{G}_S$ (usually the component group of the orbit)

Set $\gamma(S)$ = set of orbits in the special piece associated to S. Then

$$\gamma(S) \hookrightarrow \{\text{conjugacy classes of } \mathcal{G}_S\}.$$

G exceptional group:

- $\mathcal{G}_S = S_r$, for some $r \leq 5$.
- Elements of $\gamma(S)$ are parametrized by partitions of r.

Newest results: split simply connected real and p-adic exceptional Lie groups G_2, F_4, E_6, E_7.

For each orbit of the algebraic group, choose one rational orbit $G \cdot f$. Determine the reductive part of the group G^f and its action on $\mathfrak{g}/\mathfrak{g}^f$.

Using the conjecture that admissibility is independent of the choice of orbit in each algebraic class, we find:

Theorem: All special orbits are admissible. Moreover, there are some nonspecial admissible orbits. They correspond to orbits in special pieces parametrized by completely odd partitions. Since special orbits are also parametrized by an odd partition, this description is uniform.

Group	Orbit	Special piece	partition parametrization
G_2	A_1	$G_2(a_1)$	(3)
F_4	$\widetilde{A_2} + A_1$	$F_4(a_3)$	$(3,1)$
E_6	$2A_2 + A_1$	$D_4(a_1)$	(3)
E_7	$2A_2 + A_1$	$D_4(a_1)$	(3)
E_7	$A_5 + A_1$	$E_7(a_5)$	(3)

Table 1: Nonspecial admissible nilpotent orbits

The geometric notion of admissibility should be related to the algebraic notion of specialness for all simply connected, split Lie groups (real and p-adic).

Conjecture (Vogan):

- There is a canonical extension of the isotropy group

$$1 \to C \to (G^f)' \to G^f \to 1$$

  for some finite group C (related to how the orbit of f sits in a special piece), that measures failure to be special: the extension is trivial for f special.
- The metaplectic extension $\widetilde{G^f}$ is a reduction of $(G^f)'$ mod 2 (replace C by C/2C).

Alfred Noël (University of Massachusetts) has recently computed the admissible nilpotent orbits of **all** adjoint exceptional groups over $\mathbb{R}$, using techniques of Ohta.

In E_8, there are 5 nonspecial orbits having completely odd parametrization:

Orbit	Special piece	Lusztig parameter
$2A_2 + A_1$	$D_4(a_1)$	(3)
$2A_2 + 2A_1$	$D_4(a_1) + A_1$	(3)
$E_6(a_3) + A_1$	$E_8(a_7)$	$(3, 1, 1)$
$A_4 + A_3$	$E_8(a_7)$	(5)
$E_6 + A_1$	$E_8(b_5)$	(3)

These are all admissible over $\mathbb{R}$ [Noël]; it is unknown if they are admissible over p-adic fields.