A New Unitary Space-Time Code with High Diversity Product [1]

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Abstract

In this paper, we propose new full diversity unitary space-time codes based on Hamiltonian constellation designs. Our proposed constellations can be used for any number of antennas and for any data rate. For two transmitter antennas, the constellations are constructed from cyclic group codes. For a larger number of transmitter antennas, the design employs the direct sum of 2×2 Hamiltonian matrices and roots of unity. We give some examples of proposed constellations, and also show that they outperform known design techniques in the literature.

Index terms: multiple-antenna wireless communications, differential unitary space-time modulations, cyclic group codes, Hamiltonian constellations

1 Introduction

Space-time coding was developed for use in multiple-antenna wireless communications to achieve high data rate and reliability, using a combination of techniques in error control coding and transmission diversity. The design of a good space-time code with high coding gain and simple encoding-decoding algorithm is still an open problem. Consider a *unitary space-time modulation (USTM)* [5] for a multiple-antenna system with M transmitter and N receiver antennas in a Rayleigh flat fading channel. Let $\mathcal{V}$ be a signal constellation consisting of L $M \times M$ unitary matrices. At a high SNR ρ, a

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pairwise error probability, P_e, that the receiver antenna decodes an error from V_l to $V_{l'} \in \mathcal{V}$, can be approximated by [5, 6]

$$P_e \leq \frac{1}{2}\left(\frac{2\alpha}{\rho}\right)^{MN} \frac{1}{|\det(V_l - V_{l'})|^{2N}} \tag{1}$$

where $\alpha = 1$ for known channel and $\alpha = 2$ for unknown channel using a *differential unitary space-time modulation (DUSTM)* [6, 7]. We define a *diversity product* $\zeta_{\mathcal{V}}$ which is computed from a constellation $\mathcal{V}$ as [10]

$$\zeta_{\mathcal{V}} = \frac{1}{2} \min_{0 \leq l < l' \leq L-1} |\det(V_l - V_{l'})|^{\frac{1}{M}}, \quad 0 \leq \zeta_{\mathcal{V}} \leq 1. \tag{2}$$

To minimize P_e in (1), consequently, the design criteria of DUSTM is to find a unitary constellation $\mathcal{V}$ which has $\zeta_{\mathcal{V}}$ as large as possible. The unitary constellation $\mathcal{V}$ which has $\zeta_{\mathcal{V}} > 0$ is said to have *full diversity.* Our goal in this paper is to find sets of unitary matrices which have diversity products as large as possible. The problem of constructing constellations with high diversity product has been studied in many prior works. For example, some of the group structures proposed to represent constellations are cyclic and dicyclic groups [6, 7, 8] and fixed-point free groups [10]. Some examples of nongroup constellations include products of fixed-point free groups [10], Cayley codes [4], parametric codes [9] and numerical methods [3]. These designs variously have limitation in performance, the number of transmitters used, and the data rate achieved.

In this paper, we propose new unitary space-time codes, called *Hamiltonian constellations*, for any number of antennas and for any data rate. Our proposed constellations are based on *group codes.*[2] They have full diversity and can be used for both known and also for unknown channels, using DUSTM [6, 7]. Although the proposed constellations do not form a group, the optimization of a diversity product will not be computationally intensive for large L. It only requires checking $L-1$ distinct matrices, making it comparable to those that use group constellations. This paper is organized as follows: Section 2 gives an overview of cyclic group codes and the motivation of using a 2×2 Hamiltonian matrix to construct a full diversity unitary constellation. Section 3 presents the designs of Hamiltonian constellations. Section 4 gives some examples of proposed constellations and

[2]We use the term *group codes* refered to the original definition of Slepian's group codes [11].

compares their performance to different designs. The conclusion is given in Section 5.

2 Cyclic Group Code and a 2×2 Hamiltonian Matrix

2.1 Cyclic Group codes

An (L, n) *cyclic group code* [2] is a set of L codewords in the Euclidean space of dimension n. Basically we can think that all L codewords are on the surface of a unit sphere in n dimensional space. Let $\{O_l\}_{l=0}^{L-1}$ be a cyclic group of $n \times n$ orthogonal matrices which has the block diagonal form [2] as $O_l = \text{diag}((-1)^l, A(lk_1), \ldots, A(lk_\nu))$ for L even and n odd $n = 2\nu + 1$; and $O_l = \text{diag}(A(lk_1), \ldots, A(lk_\nu))$ for any L and n even $n = 2\nu$ where $A(k_i)$ is defined by

$$A(k_i) = \begin{bmatrix} \cos\frac{2\pi}{L}k_i & \sin\frac{2\pi}{L}k_i \\ -\sin\frac{2\pi}{L}k_i & \cos\frac{2\pi}{L}k_i \end{bmatrix}, \quad k_i \in \{1, 2, \ldots, L-1\}. \tag{3}$$

The group codewords $\{X_l\}_{l=0}^{L-1}$ can be generated by $X_l = O_l X_0$ where $X_0 = (x_1, x_2, \ldots, x_n)$ is called *an initial vector*. The main problem of group codes is how to choose the best initial vector X_0 to minimize the error probability, or equivalent to maximize the minimum distance of nearest codewords. The square distance between X_0 and any codeword $X_l = O_l X_0$ is computed by

$$\begin{aligned} \|X_0 - O_l X_0\|^2 &= 2 - 2\sum_{i=1}^{\nu} \mu_i \cos\frac{2\pi}{L} l k_i && \text{for } n \text{ even, } n = 2\nu \\ &= 2 - 2(-1)^l \mu_0 - 2\sum_{i=1}^{\nu} \mu_i \cos\frac{2\pi}{L} l k_i && \text{for } n \text{ odd, } n = 2\nu + 1 \end{aligned} \tag{4}$$

where $\mu_0 = x_1^2$, and $\mu_i = x_{2i}^2 + x_{2i+1}^2$ or $\mu_i = x_{2i-1}^2 + x_{2i}^2$ for $n = 2\nu + 1$ and $n = 2\nu$ respectively.

2.2 A 2×2 Hamiltonian Matrix

A 2×2 Hamiltonian matrix can be used to design a full diversity unitary space-time constellation for $M = 2$ transmitter antennas. This matrix is defined by

$$H = \begin{bmatrix} x & -y^* \\ y & x^* \end{bmatrix} \tag{5}$$

where $x, y \in \mathbb{C}$ and $|x|^2 + |y|^2 = 1$. This differs from the 2×2 orthogonal design [1, 12] which requires $|x|^2 = |y|^2 = 1$. Here H is unitary. Let $\mathcal{H} = \{H_l\}_{l=0}^{L-1}$ be a *Hamiltonian space-time constellation.* From (2), the diversity product $\zeta_{\mathcal{H}}$ can be computed as

$$\zeta_{\mathcal{H}} = \frac{1}{2}|\det(H - H')|^{\frac{1}{2}} = \frac{1}{2}\det \begin{bmatrix} x - x' & -(y - y')^* \\ y - y' & (x - x')^* \end{bmatrix}^{\frac{1}{2}} = \frac{1}{2}\sqrt{|x - x'|^2 + |y - y'|^2}.$$

From (6), we can easily see that now $\zeta_{\mathcal{H}}$ equals one half of the Euclidean distance between two points (x, y) and (x', y') in $\mathbb{C}^2$. Consider a transformation from $\mathbb{R}^4$ to $\mathbb{C}^2$. If $A(a_1, a_2, a_3, a_4)$ is a point on the unit sphere in $\mathbb{R}^4$ where $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$, then we can convert this point onto the unit sphere in $\mathbb{C}^2$ space using the mapping:

$$A(a_1, a_2, a_3, a_4)_{\mathbb{R}^4} \longmapsto A(a_1 + ja_2, a_3 + ja_4)_{\mathbb{C}^2} = A(x, y)_{\mathbb{C}^2} \tag{6}$$

where $x = a_1 + ja_2, y = a_3 + ja_4$, $j^2 = -1$ and $a_1, a_2, a_3, a_4 \in \mathbb{R}$. Consequently the problem of constructing a 2×2 Hamiltonian constellation can be reduced to finding L points on a unit sphere in $\mathbb{R}^4$ space such that the minimum distance between two points as large as possible. A possible solution to this problem is to use *an* $(L, 4)$ *group code* to get these maximum equidistant L points on a unit sphere in $\mathbb{R}^4$. For simplicity, we choose a cyclic group of order L as explained in Section 2.1 to generate codewords, that is, *an* $(L, 4)$ *cyclic group code* which has

$$O_l = \mathrm{diag}(A(lk_1), A(lk_2)) \tag{7}$$

with the square distance between X_0 and $O_l X_0$ is $4\sum_{i=1}^{2} \mu_i \sin^2 \frac{\pi}{L} lk_i$ where $\mu_1 = x_1^2 + x_2^2$ and $\mu_2 = x_3^2 + x_4^2$.

3 Hamitonian Constellation Designs

3.1 The Case $M = 2$

From the distance of an $(L, 4)$ cyclic group code given as above, we observe that the distance between an initial vector X_0 and codeword X_l depends on the sum of squares of pair of entries (μ_1 and μ_2).

Thus, reducing from 4 to 2 unknown parameters , let's $X_0 = (\sqrt{x_1}, 0, \sqrt{x_2}, 0)$ be an initial vector with

$$x_1 + x_2 = 1 \quad \text{and} \quad x_1, x_2 \geq 0. \tag{8}$$

Generate the $(L, 4)$ cyclic group codewords by $\{X_l\}_{l=0}^{L-1} = O_l X_0$, where O_l is defined in (7). This gives

$$X_l = \left(\sqrt{x_1} \cos \frac{l2\pi k_1}{L}, -\sqrt{x_1} \sin \frac{l2\pi k_1}{L}, \sqrt{x_2} \cos \frac{l2\pi k_2}{L}, -\sqrt{x_2} \sin \frac{l2\pi k_2}{L} \right). \tag{9}$$

Transforming these codewords to the new codewords in $\mathbb{C}^2$, $X_{l\mathbb{C}^2}$, by (6) gives

$$X_{l\mathbb{C}^2} = \begin{bmatrix} \sqrt{x_1}(\cos \frac{l2\pi k_1}{L} - j \sin \frac{l2\pi k_1}{L}) \\ \sqrt{x_2}(\cos \frac{l2\pi k_2}{L} - j \sin \frac{l2\pi k_2}{L}) \end{bmatrix} = \begin{bmatrix} \sqrt{x_1} e^{-j\frac{l2\pi k_1}{L}} \\ \sqrt{x_2} e^{-j\frac{l2\pi k_2}{L}} \end{bmatrix}. \tag{10}$$

Substitute in the form of a Hamiltonian matrix in (5) to get a 2×2 Hamiltonian space-time constellation $\mathcal{H}_{2\times 2} = \{H_l\}_{l=0}^{L-1}$ with

$$H_l = \begin{bmatrix} \sqrt{x_1} e^{-j\frac{l2\pi k_1}{L}} & -\sqrt{x_2} e^{j\frac{l2\pi k_2}{L}} \\ \sqrt{x_2} e^{-j\frac{l2\pi k_2}{L}} & \sqrt{x_1} e^{j\frac{l2\pi k_1}{L}} \end{bmatrix}. \tag{11}$$

We can also write H_l of (11) in terms of $H_{l=0} = H_0$ as $H_l = e^{j\frac{2\pi l k_1}{L}} R_l H_0 T_l$ where

$$R_l = \begin{bmatrix} e^{-j\frac{2\pi l k_1}{L}} & 0 \\ 0 & e^{-j\frac{2\pi l k_2}{L}} \end{bmatrix}, \; H_0 = \begin{bmatrix} \sqrt{x_1} & -\sqrt{x_2} \\ \sqrt{x_2} & \sqrt{x_1} \end{bmatrix} \text{ and } T_l = \begin{bmatrix} e^{-j\frac{2\pi l k_1}{L}} & 0 \\ 0 & e^{j\frac{2\pi l k_2}{L}} \end{bmatrix}.$$

Both $\{R_l\}_{l=0}^{L-1}$ and $\{T_l\}_{l=0}^{L-1}$ form cyclic groups of order at most L. In Appendix A, we will prove that if $l < l'$, then $|\det(H_l - H_{l'})| = |\det(H_0 - H_{l'-l})|$. Thus the expression for diversity product of a Hamiltonian constellation $\zeta_{\mathcal{H}}$ simplifies to

$$\begin{aligned} \zeta_{\mathcal{H}} &= \frac{1}{2} \min_{0 \leq l < l' \leq L-1} |\det(H_l - H_{l'})|^{\frac{1}{2}} && (12) \\ &= \frac{1}{2} \min_{l=1,2,\ldots,L-1} |\det(H_0 - H_l)|^{\frac{1}{2}} && (13) \\ &= \frac{1}{2} \min_{l=1,2,\ldots,L-1} \left\{ 4 \left(x_1 \sin^2 \frac{\pi k_1 l}{L} + x_2 \sin^2 \frac{\pi k_2 l}{L} \right) \right\}^{\frac{1}{2}}. && (14) \end{aligned}$$

The values of $x = (x_1, x_2)$ satisfied (8) and $k = (k_1, k_2) \in \{1, 2, \ldots, L-1\}$ are chosen to maximize $\zeta_{\mathcal{H}}$ of (14).

3.2 The Case M Even, $M > 2$

An $M \times M$ constellation for a case where M is even can be constructed by a direct sum of $\frac{M}{2}$ 2×2 Hamiltonian matrices. The $\mathcal{H}_{M\times M} = \{J_l\}_{l=0}^{L-1}$, has the block diagonal form

$$J_l = \mathrm{diag}(H_l^{1,2}, H_l^{3,4}, \ldots, H_l^{M-1,M}) \tag{15}$$

where $H_l^{m,n}$ is defined as

$$H_l^{m,n} = \begin{bmatrix} \sqrt{x_1}e^{-j\frac{l2\pi k_m}{L}} & -\sqrt{x_2}e^{j\frac{l2\pi k_n}{L}} \\ \sqrt{x_2}e^{-j\frac{l2\pi k_n}{L}} & \sqrt{x_1}e^{j\frac{l2\pi k_m}{L}} \end{bmatrix}. \tag{16}$$

Using the similar derivation as above, the diversity product can be computed by

$$\zeta_{\mathcal{H}} = \frac{1}{2} \min_{l=1,2,\ldots,L-1} \left| 2^M \prod_{j=1}^{M/2} \sum_{i=1}^{2} x_i \sin^2 \frac{\pi k_{2j-2+i} l}{L} \right|^{\frac{1}{M}}. \tag{17}$$

Again, the values of $x = (x_1, x_2)$ satisfied (8) and $k = (k_1, k_2, \ldots, k_M) \in \{1, 2, \ldots, L-1\}$ are chosen to maximize $\zeta_{\mathcal{H}}$ of (17).

3.3 The Case M Odd, $M \geq 3$

For a case where M is odd, an $M \times M$ constellation is constructed by using a direct sum of $\frac{M-1}{2}$ 2×2 Hamiltonian matrices and the Lth roots of unity. Thus a block diagonal matrix of $\mathcal{H}_{M\times M} = \{J_l\}_{l=0}^{L-1}$ for odd M is

$$J_l = \mathrm{diag}(e^{j2\pi k_1 l/L}, H_l^{2,3}, H_l^{4,5}, \ldots, H_l^{M-1,M}) \tag{18}$$

where $H_l^{m,n}$ is also defined in (16). Using a derivation similar to the above, the diversity product can be given by

$$\zeta_{\mathcal{H}} = \frac{1}{2} \min_{l=1,2,\ldots,L-1} \left| 2^M \sin \frac{\pi k_1 l}{L} \prod_{j=2}^{(M+1)/2} \sum_{i=1}^{2} x_i \sin^2 \frac{\pi k_{2j-3+i} l}{L} \right|^{\frac{1}{M}}. \tag{19}$$

The values of $x = (x_1, x_2)$ satisfied (8) and $k = (k_1, k_2, \ldots, k_M) \in \{1, 2, \ldots, L-1\}$ are found such that they maximize $\zeta_{\mathcal{H}}$ of (19).

4 Examples and Performance

Table 1 on page 14 to 16 shows some of proposed Hamiltonian constellations with their best diversity products compared to different unitary space-time constellation designs: orthogonal designs [12], dicyclic and cyclic groups in [6, 7, 8], fixed-point free groups and nongroups in [10], parametric codes [9] and numerical methods [3]. Our constellations were found by computer-search. We can see that the proposed constellations for every M even case and $L = 2$ to 5 are optimal constellations whose diversity products achieve the upper bound [9] as given by $\zeta_{\text{upper}} = \sqrt{\frac{L}{2(L-1)}}$ for $L = 2, 3, 4$ and 5. For $M = 2$ transmitter antennas, Hamiltonian constellations have diversity products higher than dicyclic, cyclic group and orthogonal designs. Hamiltonian constellations for $M = 3, 4, 5$ and 6 also have diversity product much higher than cyclic group designs. For example, for $M = 3$, the diversity product of Hamiltonian with $L = 9$ is 0.6632, which is even higher than that obtained by the fixed-point free group $G_{9,1}$. The diversity product of a Hamiltonian constellation for $M = 6$ transmitter antennas is only 0.5185 while the diversity product of the cyclic group is 0.3792 at the same data rate $R = 1.00$. The performance is considered by plotting the block error rate (bler), against SNR in dB. All plots are considered in unknown Rayleigh flat fading channel using DUSTM with various combinations of the numbers of transmitter and receiver antennas. The fading coefficient and additive noise are independent $\mathcal{CN}(0, 1)$, and the channel matrix is assumed to be constant within two consecutive time periods. The maximum likelihood is used for decoding.

4.1 Hamiltonian vs. Group Designs

Figure 1 shows the block error rate performance of Hamiltonian $\mathcal{H}_{3\times 3} = \{J_l\}_{l=0}^{8}$ of (18) with $x_1 = 0.4659, k = (1, 4, 3)$ and the fixed-point free group $G_{9,1}$ [10] for $M = 3$ transmitter antennas and $N = 1$ and 2 receiver antennas at the same $L = 9$ and $R = \log_2 L/M = \log_2 9/3 = 1.06$. We can see that our proposed constellation outperforms the fixed-point free group design.

4.2 Hamiltonian vs. Orthogonal and Group Designs

Figure 2 compares the block error rate performance of Hamiltonian $\mathcal{H}_{2\times 2} = \{J_l\}_{l=0}^{63}$ of (11), $x_1 = 0.6281, k = (1, 27)$, orthogonal design with the 8th roots of unity [12], dicyclic group Q_5 [7] and cyclic group $u = (1, 19)$ [6] at the same $R = 3.00$ and $L = 64$ for $M = 2$ transmitter antennas and $N = 2$ receiver antennas. Figure 3 also compares the block error rate performance of different 2×2 constellation designs for $M = 2$ transmitter antennas but $N = 1$ receiver antenna at $R \approx 3.45$: Hamiltonian $\mathcal{H}_{2\times 2} = \{J_l\}_{l=0}^{120}$ of (11) with $x_1 = 0.5590, k = (1, 22)$ $L = 121, R = 3.46$, orthogonal design with the 11th-root of unity [12] $L = 121, R = 3.46$, dicyclic group Q_6 [7] with $L = 128, R = 3.50$ and cyclic group $u = (1, 43)$ [6] with $L = 120, R = 3.45$. From both Figure 2 and 3, it is clear that Hamiltonian constellations outperform the other three designs.

4.3 Hamiltonian vs. Cayley, Orthogonal and Group Designs

Figure 4 shows the block error rate performance for $M = 4$ transmitter antennas, $N = 1$ receiver antenna of Hamiltonian $\mathcal{H}_{4\times 4} = \{J_l\}_{l=0}^{255}$ of (15) with $x_1 = 0.4834, k = (1, 121, 79, 87)$ at $R = 2.00$, cyclic group with $u = (1, 25, 97, 107)$ [6] at $R = 2.00$, cayley code with $Q = 7, r = 2$ [4] at $R = 1.75$ and 4×4 orthogonal design with z_1, z_2, z_3 are chosen from 6-PSK [14] at $R = 1.94$. We can see that Hamiltonian constellation also outperforms these other three designs.

4.4 Hamiltonian Constellations at $R = 1.00$ for $M = 2, 3, 4$

Figure 5 displays the block error rate performance of Hamiltonian constellations at the same $R = 1.00$ for $M = 2, 3, 4$ transmitter antennas and $N = 1$ receiver antenna of $\mathcal{H}_{2\times 2}$ with $x_1 = 0.6667, k = (1, 2), L = 4$, $\mathcal{H}_{3\times 3}$ with $x_1 = 0.8089, k = (1, 3, 4), L = 8$ and $\mathcal{H}_{4\times 4}$ with $x_1 = 0.0.3680, k = (1, 3, 7, 5), L = 16$. One can see that $\mathcal{H}_{4\times 4}$ performs very well for high SNR.

5 Conclusion

We have constructed new full diversity, unitary space-time constellations which can be used for any number of transmitter antennas, receiver antennas and for any data rate. Our constellations can also be used for both unknown and known channels, and have shown that they have low computational complexity. Furthermore the Hamiltonian constellations achieve the optimal theoretical bound for diversity product in certain cases. Our simulations show good performance as compared with prior designs. In this paper, we set the initial vectors $X_0^{1,2} = X_0^{3,4} = \cdots = X_0^{m,n} = X_0 = (\sqrt{x_1}, 0, \sqrt{x_2}, 0)$ satisfied (8) for all constituents $H_l^{m,n}$ of $\mathcal{H}_{M\times M}$, $M > 2$. The different values of initial vectors for different $H_l^{m,n}$ which may give higher diversity product can be considered as a possible extension work (for example, when $M = 4$, we need to search $X_0^{1,2} = (\sqrt{x_1}, 0, \sqrt{x_2}, 0)$ and $X_0^{3,4} = (\sqrt{x_3}, 0, \sqrt{x_4}, 0)$ for $H_l^{1,2}$ and $H_l^{3,4}$ respectively).

Appendix A

We prove that to compute the diversity product of (13) it suffices to choose $l = 0$ for H_l and arbitrary $l' = 1, \ldots, L-1$ for $H_{l'}$. This proof can also be worked for the general $M \times M$ case. We show that $|\det(H_l - H_{l'})| = |\det(H_0 - H_{l'-l})|$ for $0 \le l < l' \le L-1$.

$$\begin{aligned}
|\det(H_l - H_{l'})| &= |\det(e^{j\frac{2\pi l k_1}{L}} R_l H_0 T_l - e^{j\frac{2\pi l' k_1}{L}} R_{l'} H_0 T_{l'})| \qquad \text{from } H_l = e^{j2\pi l k_1/L} R_l H_0 T_l. \\
&= |e^{j\frac{2\pi l k_1}{L}}| |\det R_l| |\det(H_0 - e^{j\frac{2\pi (l'-l) k_1}{L}} R_{l'-l} H_0 T_{l'-l})| |\det T_l| \\
&= |\det(H_0 - H_{l'-l})| \qquad \text{from } |e^{j\frac{2\pi l k_1}{L}}| = |\det R_l| = |\det T_l| = 1.
\end{aligned}$$

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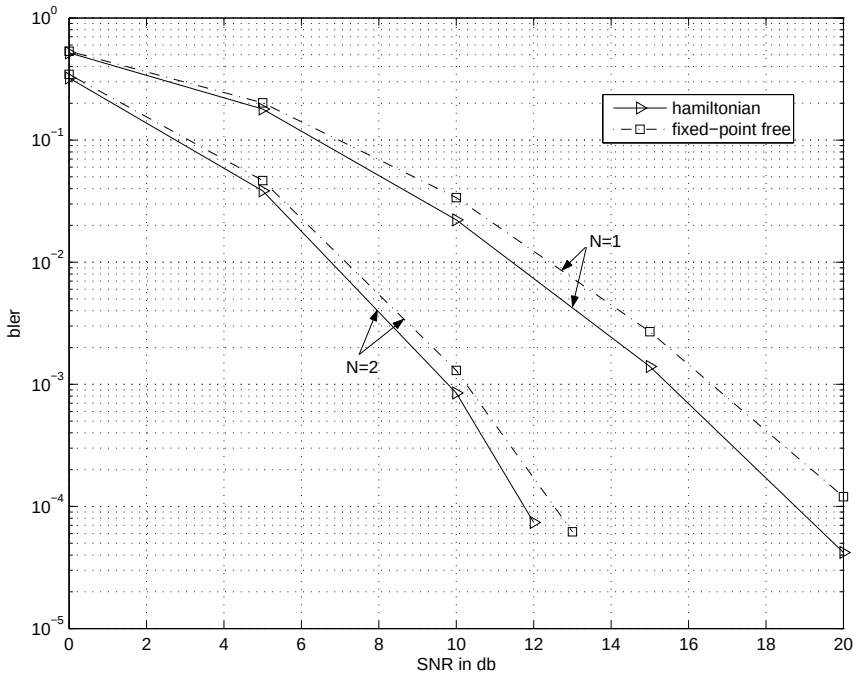

Figure 1: Block error rate performance for $M = 3, N = 1$ and 2, $R = 1.06$ of Hamiltonian and the fixed-point free group

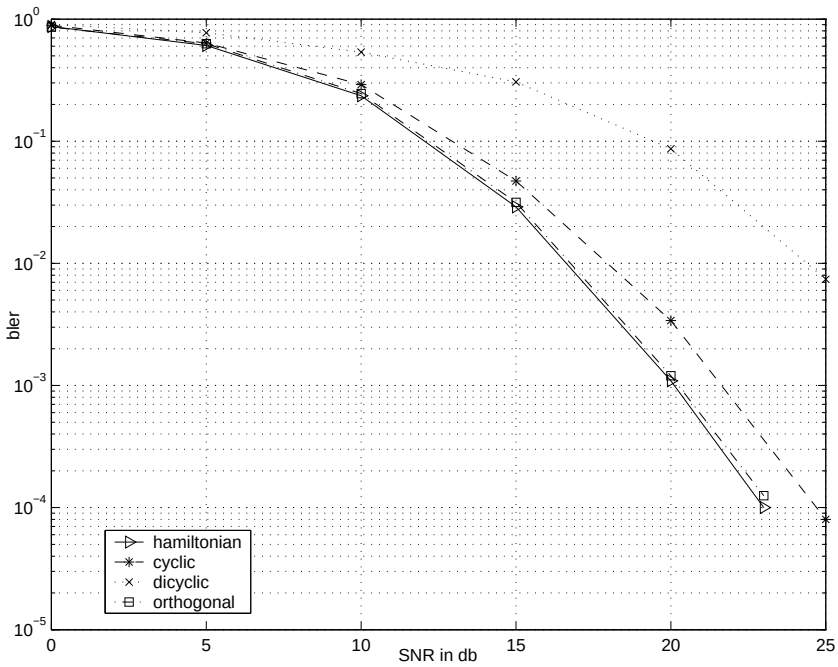

Figure 2: Block error rate performance for $M = 2, N = 2, R = 3.00$ of Hamiltonian, orthogonal design, dicyclic group and cyclic group

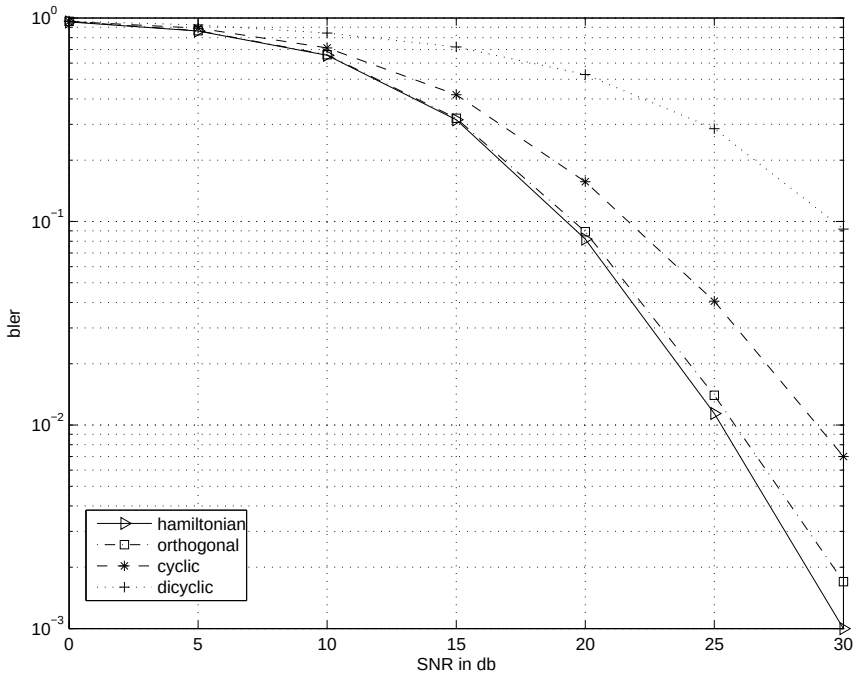

Figure 3: Block error rate performance for $M = 2, N = 1$ of Hamiltonian $R = 3.46$, orthogonal design $R = 3.46$, cyclic group $R = 3.45$ and dicylic group $R = 3.50$

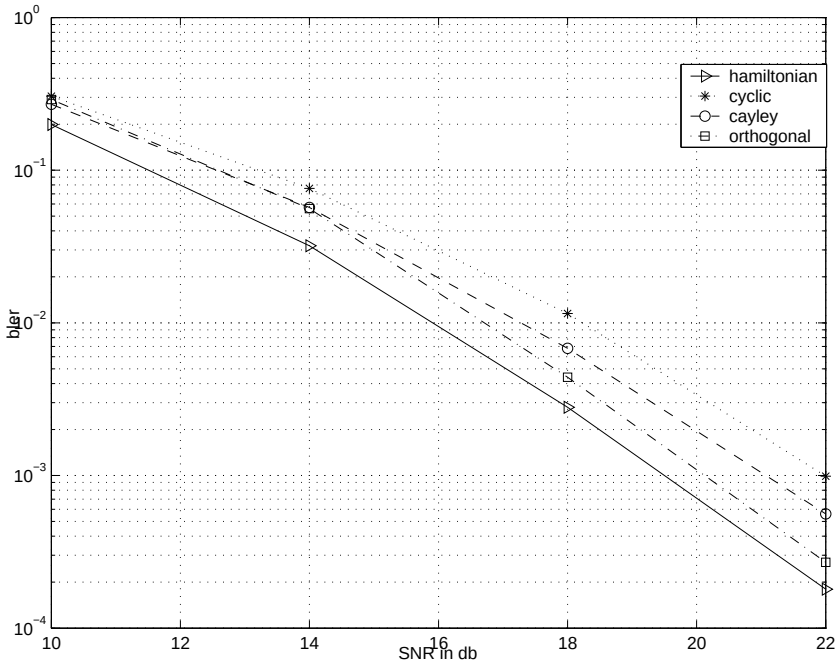

Figure 4: Block error rate performance for $M = 4, N = 1$ of Hamiltonian $R = 2.00$, cayley code $R = 1.75$, orthogonal design $R = 1.94$ and cyclic group $R = 2.00$

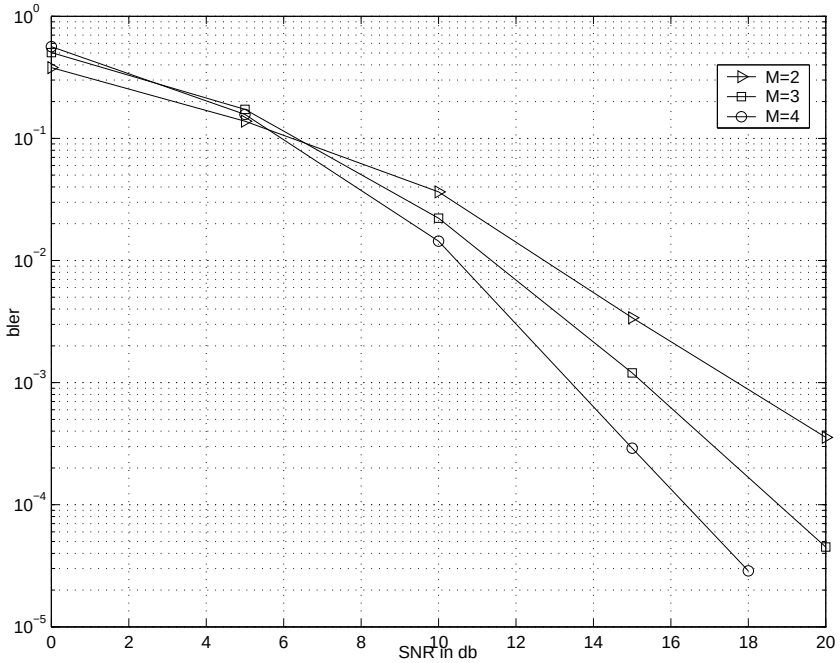

Figure 5: Block error rate performance of Hamiltonian constellations at $R = 1.00$ for $M = 2, 3$ and 4 transmitter antennas and $N = 1$ receiver antenna

M	L	R	ζ	Constellation designs
2	4	1.00	0.7071	dicyclic group Q_1
2	4	1.00	0.7071	cyclic group $u = (1, 1)$
2	4	1.00	0.7071	orthogonal with 2th-roots of unity
2	4	1.00	0.8165	$\mathcal{H}$ $x_1 = 0.6667,\ k = (1, 2)$ Figure 5
2	8	1.50	0.7071	dicyclic group Q_2
2	8	1.50	0.5946	cyclic group $u = (1, 3)$
2	8	1.50	0.7071	$\mathcal{H}$ $x_1 = 0.5000,\ k = (1, 3)$
2	16	2.00	0.3827	dicyclic group Q_3
2	16	2.00	0.3827	cyclic group $u = (1, 7)$
2	16	2.00	0.5000	orthogonal with 4th-roots of unity
2	16	2.00	0.5098	$\mathcal{H}$ $x_1 = 0.5198,\ k = (1, 4)$
2	32	2.50	0.1951	dicyclic group Q_4
2	32	2.50	0.2494	cyclic group $u = (1, 7)$
2	32	2.50	0.3827	parametric code, $k = (7, 8, 2)$
2	32	2.50	0.3827	$\mathcal{H}$ $x_1 = 0.4953,\ k = (1, 7)$
2	64	3.00	0.0980	dicyclic group Q_5 Figure 2
2	64	3.00	0.1985	cyclic group $u = (1, 19)$ Figure 2
2	64	3.00	0.2706	orthogonal with 8th-roots of unity Figure 2
2	64	3.00	0.2816	$\mathcal{H}$ $x_1 = 0.6281,\ k = (1, 27)$ Figure 2
2	120	3.45	0.1353	cyclic group $u = (1, 43)$ Figure 3
2	121	3.46	0.1922	orthogonal with 11th-roots of unity Figure 3
2	121	3.46	0.2106	$\mathcal{H}$ $x_1 = 0.5590,\ k = (1, 22)$ Figure 3

Table 1: Comparison of different constellation designs: our constellations (highlighted in grey), orthogonal design [12], dicyclic and cyclic groups [6, 7], fixed-point free groups and nongroups [10], parametric codes [9] and numerical methods [3].

M	L	R	ζ	Constellation designs
2	128	3.50	0.0491	dicyclic group Q_6 Figure 3
2	128	3.50	0.1498	cyclic group $u = (1, 47)$
2	128	3.50	0.2031	$\mathcal{H}$ $x_1 = 0.5142,\ k = (1, 12)$
2	240	3.95	0.1045	cyclic group $u = (1, 151)$
2	240	3.95	0.1511	$\mathcal{H}$ $x_1 = 0.4173,\ k = (1, 85)$
2	256	4.00	0.0245	dicyclic group Q_7
2	256	4.00	0.0988	cyclic group $u = (1, 75)$
2	256	4.00	0.1379	orthogonal with 16th-roots of unity
2	256	4.00	0.1477	$\mathcal{H}$ $x_1 = 0.5526,\ k = (1, 119)$
3	3	0.53	0.8660	$\mathcal{H}$ $x_1 = 0.5000,\ k = (1, 1, 1)$
3	5	0.77	0.7183	numerical method
3	5	0.77	0.7673	$\mathcal{H}$ $x_1 = 0.2316,\ k = (1, 1, 2)$
3	8	1.00	0.5134	cyclic group $u = (1, 1, 3)$
3	8	1.00	0.6588	$\mathcal{H}$ $x_1 = 0.8089,\ k = (1, 3, 4)$ Figure 5
3	9	1.06	0.6004	fixed-point free group $G_{9,1}$ with $u = (1, 2, 5)$ Figure 1
3	9	1.06	0.6632	$\mathcal{H}$ $x_1 = 0.4679,\ k = (1, 4, 3)$ Figure 1
3	63	1.99	0.3301	cyclic group $u = (1, 17, 26)$
3	63	1.99	0.3498	$\mathcal{H}$ $x_1 = 0.3758,\ k = (1, 20, 27)$
3	64	2.00	0.2765	cyclic group $u = (1, 11, 27)$
3	64	2.00	0.3478	$\mathcal{H}$ $x_1 = 0.6994,\ k = (1, 23, 30)$

M	L	R	ζ	Constellation designs
4	3	0.40	0.8660	$\mathcal{H}$ $x_1 = 0.5000,\ k = (1,1,1,1)$
4	4	0.50	0.8165	$\mathcal{H}$ $x_1 = 0.6667,\ k = (1,2,1,2)$
4	5	0.58	0.7906	$\mathcal{H}$ $x_1 = 0.5000,\ k = (1,2,1,2)$
4	9	0.79	0.5904	numerical method
4	9	0.79	0.7119	$\mathcal{H}$ $x_1 = 0.4094,\ k = (1,2,6,5)$
4	16	1.00	0.5453	cyclic group $u = (1,3,5,7)$
4	16	1.00	0.6377	$\mathcal{H}$ $x_1 = 0.3680,\ k = (1,3,7,5)$ Figure 5
4	256	2.00	0.2208	cyclic group $u = (1,25,97,107)$ Figure 4
4	256	2.00	0.3320	$\mathcal{H}$ $x_1 = 0.4834,\ k = (1,121,79,87)$ Figure 4
4	289	2.04	0.3105	nongroup, $L_A = 17, u = (1,3,4,11)$
4	289	2.04	0.3287	$\mathcal{H}$ $x_1 = 0.4646,\ k = (1,126,12,67)$
5	32	1.00	0.4095	cyclic group $u = (1,5,7,9,11)$
5	32	1.00	0.5444	$\mathcal{H}$ $x_1 = 0.4500,\ k = (1,11,13,15,7)$
6	3	0.26	0.8660	$\mathcal{H}$ $x_1 = 0.5000,\ k = (1,1,1,1,1,1)$
6	4	0.33	0.8165	$\mathcal{H}$ $x_1 = 0.6667,\ k = (1,2,1,2,1,2)$
6	5	0.39	0.7906	$\mathcal{H}$ $x_1 = 0.5000,\ k = (1,2,1,2,1,2)$
6	64	1.00	0.3792	cyclic group $u = (1,7,15,23,25,31)$
6	64	1.00	0.5185	$\mathcal{H}$ $x_1 = 0.4549$ $k = (1,19,3,57,23,31)$