Unitary Space-Time Group Codes: Diversity Sums From Character Tables

Terasan Niyomsataya[1], Ali Miri[1] and Monica Nevins[2]

School of Information Technology and Engineering (SITE)[1]

Department of Mathematics and Statistics[2]

University of Ottawa, Ottawa, Canada K1N 6N5

email: {tniyomsa,samiri}@site.uottawa.ca, mnevins@uottawa.ca

Abstract

Diversity sum, which is calculated from the Frobenius norm of the difference of two distinct elements in a signal constellation, is the significant parameter to predict a unitary group constellation having a good performance in low SNR.

In this paper, we propose a method to compute the diversity sum of a unitary group constellation using a character table. Some examples of using our method to different unitary groups: cyclic groups, dicyclic groups and the special linear groups are given. We also define the term faithful group constellation, that is, its diversity sum is greater than 0. One example, in this paper, is the projective special linear group PSL_2.

1 Introduction

1.1 Differential Unitary Space-Time Modulation

Consider a multiple-antenna system in a Rayleigh falt fading channel with M transmitter antennas and N receiver antennas. The $M \times N$ received signal matrix X_τ is

$$X_\tau = \sqrt{\rho} S_\tau H_\tau + W_\tau, \quad \tau = 0, 1, \ldots \tag{1}$$

where S_τ is the $M \times M$ transmitted signal matrix , W_τ is the $M \times N$ additive noise matrix and H_τ is the $M \times N$ channel matrix. The fading coefficients h_{mn} of a channel matrix and the additive noise

w_{mn} on receiver antenna are independent complex Gaussian variables with mean zero and variance one. Here ρ is a signal-to-noise ratio (SNR) at each receiver antenna.

We consider in a case of unknown channel where we use *differential unitary space-time modulation* [2, 3] to transmit signals, assumed that $H_\tau \approx H_{\tau-1}$. Let $\mathcal{V}$ be a set of $M \times M$ unitary matrices, called a *signal constellation*, and $V_{z_\tau} \in \mathcal{V}$ be a message matrix at block τ which is chosen from an alphabet $z_\tau \in \{0, 1, \dots, L-1\}$. The transmitted signal matrix is

$$S_\tau = V_{z_\tau} S_{\tau-1} \tag{2}$$

with $S_0 = I_M$. The rate is $R = \log_2 L/M$. The maximum-likelihood (ML) decoder is used for the receiver antenna to decode a message $\hat{z_\tau}$ to be

$$\hat{z_\tau} = \arg \min_{V \in \mathcal{V}} ||X_\tau - V X_{\tau-1}||. \tag{3}$$

The pairwise probability of error that the receiver antenna decodes an error from V to V' is approximated as [2, 3, 7]

$$P_e \leq \frac{1}{2} \prod_{m=1}^{M} \left[1 + \frac{\rho^2}{4(1+2\rho)} \sigma_m^2 (V - V')\right]^{-N} \tag{4}$$

where σ_m denotes the m^{th} singular value of $(V - V')$. At a high SNR, the upper bound of P_e in (4) can be estimated by

$$P_e \leq \frac{1}{2} \left(\frac{8}{\rho}\right)^{MN} |\det(V - V')|^{-2N}. \tag{5}$$

It is well known that to minimize P_e in (5) at a high SNR, we need to maximize a *diversity product* defined as [7]

$$\zeta_\mathcal{V} = \frac{1}{2} \min_{V, V' \in \mathcal{V}} |\det(V - V')|^{\frac{1}{M}}. \tag{6}$$

We say a constellation has *full diversity* if it has $|\det(V - V')| > 0$ for all distinct $V, V' \in \mathcal{V}$.

At a low SNR, using a first order approximation, P_e in (4) can be approximated by [6]

$$P_e \leq \frac{1}{2} \left[1 + \frac{\rho^2}{4(1+2\rho)} \sum_{m=1}^{M} \sigma_m^2 (V - V')\right]^{-N}. \tag{7}$$

Thus in this case we want to minimize P_e by maximizing the *diversity sum* which is defined as [6]

$$\xi_\mathcal{V} = \frac{1}{2\sqrt{M}} \min_{V, V' \in \mathcal{V}} ||V - V'||_F. \tag{8}$$

Here, $||V - V'||_F$ denotes the *Frobenius norm* of the matrix $(V - V')$, which is computed as

$$||V - V'||_F = \left(\sum_{m=1}^{M} \sigma_m^2 (V - V')\right)^{\frac{1}{2}} = \sqrt{\operatorname{trace}(V - V')(V - V')^*}. \tag{9}$$

The finite simple unitary groups having full diversity (nonzero diversity product) are called *fixed-point free groups*, and have been classified in [7]. Fixed-point free group constellations and their best diversity products have been found by a computer search [7]. Recently, Shokrollahi [8] has proposed a method to compute the diversity product of a unitary group constellation, using a characteristic polynomial which is derived from the character table of the group.

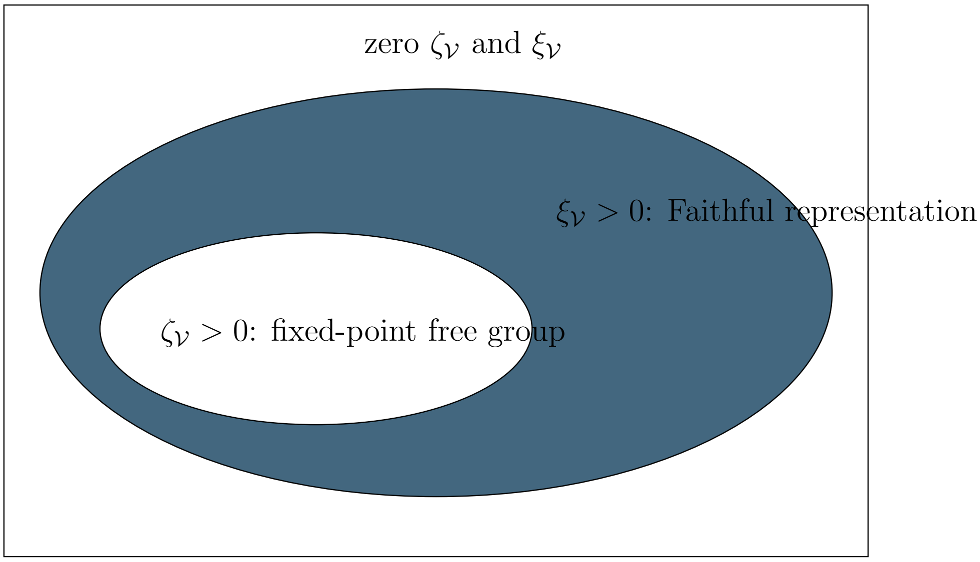

Figure 1: Classification of finite unitary groups

1.2 Motivation

Consider finite unitary groups. We can classify all their constellations based on the zero and nonzero terms of diversity product and diversity sum as shown in Figure 1. We can see that fixed-point free groups are a subset of the nonzero diversity sum groups, and some nonzero diversity sum groups do not have full diversity (the shaded area). When a unitary group constellation $\mathcal{V}$ has zero diversity product, that is, when there exist $V \neq V' \in \mathcal{V}$ such that $|\det(V - V')| = 0$, then the upper bound on P_e at high SNR as estimated by the equation (5) tends to infinity which is meaningless. For the case of low SNR, the upper bound on P_e given by the equation (7) is always finite, because $\sum_{m=1}^{M} \sigma_m^2(V - V')$ is always greater than or equal to zero. The main contribution of this paper is to consider those nonzero diversity sum groups which have good performance at low SNR, as predicted by the upper bound on P_e in the equation (7). The advantage of using a character table to choose a good performance group constellation is the character tables of finite groups are simple and can be found in most textbooks, while their representations are complicated to compute. Hence, using a character table, we can predict a good performance group constellation, and then compute its representation. Another advantage is fast computing the diversity product (6) and diversity product (8) which requires checking $h-1$ pairs of elements where h is a number of conjugacy classes of a finite group $\mathcal{G}$, normally $h \leq |\mathcal{G}| = |\mathcal{V}|$. For example, the group $SL_2(\mathbb{F}_5)$ of order 120, we need to check only 8 pairs of elements when we use its character table to compute the diversity sum.

In this paper, we present a method of computing the diversity sum of a unitary group constellation from its character table. In contrast to the case of computing the diversity product in [8], our method is very simple because it does not require the computation of characteristic polynomials, but only the values of the characters from the character table. We will call the unitary spce-time group constellations having $\xi_{\mathcal{V}} > 0$, *faithful constellations*, following the notion of faithful representation (as discussed in Section 3). Furthermore, we demonstrate with an example (the *special linear group* SL_2) how a constellation which is not faithful for a given group can be made into a faithful constellation for a quotient group (in this case, the *projective special linear group* PSL_2). We then show that non full diversity group constellations with high diversity sum outperform full diversity group constellations with low diversity sum in a specific range of low SNR.

2 Mathematical Preliminaries

Let $\mathcal{G}$ be a group. Two elements $g, g' \in \mathcal{G}$ are *conjugate* if there is an element $h \in \mathcal{G}$ such that $g = hg'h^{-1}$. The *conjugacy class* of g is the set of all elements in $\mathcal{G}$ which are conjugate to g. A *representation* ρ of a group $\mathcal{G}$ is a group homomorphism $\rho : \mathcal{G} \longmapsto GL_n(\mathbb{C})$ where $GL_n(\mathbb{C})$ is the general linear group consisting of $n \times n$ invertible matrices with complex entries. We call n the *degree* of the representation. A representation is irreducible if the only subspace of $\mathbb{C}^n$ preserved by all $\rho(g),\ g \in \mathcal{G}$, are the zero subspace and $\mathbb{C}^n$ itself; it is reducible otherwise. The number of irreducible representations of a group $\mathcal{G}$ equals the number of conjugacy classes of $\mathcal{G}$. Define the direct sum of representations ρ and ρ' via

$$(\rho \oplus \rho')(g) = \begin{bmatrix} \rho(g) & 0 \\ 0 & \rho'(g) \end{bmatrix}. \tag{10}$$

It has been prove that any reducible representation of a finite group may be written as a direct sum of two or more irreducible representations (with respect to a suitable choice of basis of $\mathbb{C}^n$). In this paper, we let $n = M$ a number of transmitter antennas, and we consider only *unitary representations*, that is, representation ρ such that $\rho(g)\rho(g)^* = I_M$ for all $g \in \mathcal{G}$. We say that a group constellation $\mathcal{V}$, coming from a unitary representation ρ, is a set of unitary representation matrices $\rho(g)$ for all $g \in \mathcal{G}$:

$$\mathcal{V} = \{\rho(g) \text{ for all } g \in \mathcal{G}\}. \tag{11}$$

We say ρ is a *faithful representation* if ρ is a one-to-one mapping between g and $\rho(g)$ for all $g \in \mathcal{G}$. Thus if there exist $g \neq h \in \mathcal{G}$ such that $\rho(g) = \rho(h)$ then ρ is not faithful. We use this to define a nonzero diversity sum unitary constellation.

The *character* χ of representation ρ of a group $\mathcal{G}$ is a mapping $\chi: G \to \mathbb{C}$ defined as

$$\chi(g) = \text{trace } \rho(g) = \sum_{i=1}^{n} \lambda_i \tag{12}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of $\rho(g)$. We call χ an *irreducible character* or a *reducible character* if ρ is an irreducible or a reducible representation respectively. We also call $n = M$ the *degree* of χ. Let ρ and ρ' be two representations of a group $\mathcal{G}$ with characters χ and χ' respectively. Then the character of $\rho(g) \oplus \rho'(g)$ is $\chi(g) + \chi'(g)$. This allows us to easily compute the character of a new reducible group constellation which is constructed by a direct sum of irreducible representation.

Hence the set of all irreducible characters can be described by an $h \times h$ matrix called a *character table* where h is a number of conjugacy classes of a group $\mathcal{G}$. The rows represent irreducible characters $\{\chi_i\}_{i=1}^h$, and the columns represent conjugacy classes $\{\mathcal{C}_i\}_{i=1}^h$. Let the first column correspond to the conjugacy class $\{e\}$ of the identity element of $\mathcal{G}$. The character is constant on conjugacy class of $\mathcal{G}$ (since the trace is) and it has been shown that the number of distinct irreducible representation of $\mathcal{G}$ is equal to the number of conjugacy classes. Note that for any irreducible character of $\mathcal{G}$, $\chi(e) =$ degree of $\rho = n = M$ and $|\chi(g)| \leq \chi(e)$ for all $g \in \mathcal{G}$. Moreover $|\mathcal{G}| = \sum_{i=1}^h \chi_i(e)^2$ and $\chi_i(e)$ divides $|\mathcal{G}|$ [5].

3 Computing a Diversity Sums From Character Tables

Let $\mathcal{V}$ be a unitary group constellation, that is, a set of unitary representation matrices of an abstract group $\mathcal{G}$ corresponding to a representation ρ with character χ (see the equation (11)). Let $V, V' \in \mathcal{V}$.

We simplify the term $||V - V'||_F$ of the diversity sum $\xi_{\mathcal{V}}$ in the equation (8) as follows:

$$\begin{aligned}
||V - V'||_F^2 &= \text{trace}((V - V')(V - V')^*) \\
&= \text{trace}(VV^* - VV'^* - V'V^* + V'V'^*) \\
&= \text{trace}(I - VV'^* - (VV'^*)^* + I) \quad \text{since } VV^* = I \\
&= \text{trace}(I - V'' - V''^* + I) \quad \text{from } VV'^* = V'' \in \mathcal{V} \\
&= \text{trace}[(I - V'') + (I - V'')^*] \\
&= \text{trace}(I - V'') + \text{trace}(I - V'')^* \\
&= 2\text{Re}\{\text{trace}(I - V'')\} \\
&= 2\text{Re}\{\text{trace}I - \text{trace}V''\} \\
&= 2(M - \text{Re}\{\chi(V'')\}).
\end{aligned}$$

$\text{Re}\{x\}$ denotes the real part of x.

Let's say a group $\mathcal{G}$ has h conjugacy classes $\{\mathcal{C}_i\}_{i=1}^h$. Consequently we can define the diversity sum in (8) in the terms of a character as

$$\xi_{\mathcal{V}} = \frac{1}{\sqrt{2M}} \min_{\mathcal{C}_i \neq e} \sqrt{M - \text{Re}\{\chi_{\mathcal{C}_i}\}} \tag{13}$$

where $\chi_{\mathcal{C}_i} = \chi(g)$ for any $g \in \mathcal{C}_i$. Using (13), we now can choose a high performance group constellation at low SNR by simply looking at its character table.

Proposition 1 *The value of a diversity sum is* $0 \leq \xi_{\mathcal{V}} \leq 1$.

Proof: From Section 2, we know that $|\chi_{\mathcal{C}}| \leq \chi(e) = n$ where n is the degree of the representation, and $n = M$. Thus $-M \leq \text{Re}\{\chi_{\mathcal{C}}\} \leq M$. Substituting this value in (13) completes the proof. □

Theorem 1 *If* $\chi_e \neq \chi_{\mathcal{C}_i}$ *for any* $\mathcal{C}_i \neq \{e\}$ *then* $\xi_{\mathcal{V}} > 0$.

Proof: From the property of a character, we know that $\chi_e = \chi(e) = n$, and $n = M$. Oue hypothesis is that $\chi_{\mathcal{C}_i} \neq \chi_e$ for any $\mathcal{C}_i \neq \{e\}$. This implies $\chi_{\mathcal{C}_i}$ can not equal M. Hence, following the proof of Proposition 1, the right hand side of equation (13) can never be 0. The diversity sum $\xi_{\mathcal{V}}$ is always greater than zero. □

Remark: If $\chi_{\mathcal{C}} = \chi_e = M$ when $\mathcal{C} \neq e$ then $\xi_{\mathcal{V}} = 0$. Moreover if $\chi_{\mathcal{C}}$ then $\rho(g) = I_M = \rho(e)$ for all $g \in \mathcal{C}$, so we see that the representation associated to χ is not faithful. In particularly we have $|\mathcal{V}| < |\mathcal{G}|$.

Definition 1 *We say a finite unitary group constellation $\mathcal{V}$ is a* ***faithful constellation*** *if it has $\xi_{\mathcal{V}} > 0$.*

Example 1 *Consider the special linear group $SL_2(\mathbb{F}_5)$ over the finite field with 5 elements. The group $SL_2(\mathbb{F}_5)$ has 9 conjugacy classes which we denote by $e = I, B, C_1, D_1, D_2, E_1, E_{-1}, F_1$ and F_{-1}, and 9 irreducible characters $\{\chi_i\}_{i=1}^{9}$ (following [8]). Its character table is given in Table 1 below. The irreducible characters $\chi_2, \chi_3, \chi_5, \chi_6$ and χ_9 do not correspond to faithful representations because their $\chi_B = \chi_I$. The faithful constellations of $SL_2(\mathbb{F}_5)$ are χ_1, χ_4, χ_7 and χ_8. We note that, from computing the characteristic polynomials in [8], only the irreducible characters χ_7 and χ_8 correspond to fixed-point free groups ($\zeta_{\mathcal{V}} > 0$). Hence the irreducible characters χ_1 and χ_4 correspond to faithful constellations without full diversity which are in the shaded area in Figure 1.*

	I	B	C_1	D_1	D_2	E_1	E_{-1}	F_1	F_{-1}
χ_1	6	-6	0	0	0	1	1	-1	-1
χ_2	5	5	1	-1	-1	0	0	0	0
χ_3	4	4	0	1	1	-1	-1	-1	-1
χ_4	4	-4	0	-1	1	-1	-1	1	1
χ_5	3	3	-1	0	0	$-\frac{(-1+j\sqrt{5})}{2}$	$\frac{-1-j\sqrt{5}}{2}$	$-\frac{(-1+j\sqrt{5})}{2}$	$\frac{-1-j\sqrt{5}}{2}$
χ_6	3	3	-1	0	0	$\frac{-1-j\sqrt{5}}{2}$	$-\frac{(-1+j\sqrt{5})}{2}$	$\frac{-1-j\sqrt{5}}{2}$	$-\frac{(-1+j\sqrt{5})}{2}$
χ_7	2	-2	0	1	-1	$-\frac{(-1-j\sqrt{5})}{2}$	$\frac{-1+j\sqrt{5}}{2}$	$\frac{-1-j\sqrt{5}}{2}$	$-\frac{(-1+j\sqrt{5})}{2}$
χ_8	2	-2	0	1	-1	$\frac{-1+j\sqrt{5}}{2}$	$-\frac{(-1-j\sqrt{5})}{2}$	$-\frac{(-1+j\sqrt{5})}{2}$	$\frac{-1-j\sqrt{5}}{2}$
χ_9	1	1	1	1	1	1	1	1	1

Table 1: Character table of the group $SL_2(\mathbb{F}_5)$

From the Example 1, we see that the irreducible characters χ_2, χ_3, χ_5 and χ_6 do not have faithful constellations. We do not consider χ_9 because χ_9 is the trivial representation of degree 1, that is, $\chi_9(g) = 1$ for all $g \in \mathcal{G}$. We deduce that the kernels of these unfaithful representations consist of all elements in the conjugacy classes I and B. The conjugacy class of B consists of exact one element of $SL_2(\mathbb{F}_5)$, namely $-I$. Hence to obtain a new faithful constellation, by group theory, we consider the quotient group $PSL_2(\mathbb{F}_5) = SL_2(\mathbb{F}_5)/\{\pm I\}$, called the *projective special linear group*. These four representations will factor through PSL_2, giving faithful representations of PSL_2. Basically $PSL_2(\mathbb{F}_5)$

are the group of cosets of $\pm I \in SL_2(\mathbb{F}_5)$, with $|PSL_2(\mathbb{F}_5)| = \frac{1}{2}|SL_2(\mathbb{F}_5)|$. More properties of $PSL_2(\mathbb{F}_q)$ for an odd prime q are summarized in Appendix A.

Example 2 *We construct an irreducible faithful representation of $PSL_2(\mathbb{F}_5)$ from the irreducible character χ_3 of $SL_2(\mathbb{F}_5)$ (in Table 1 which has unfaithful constellation). The group $PSL_2(\mathbb{F}_5)$ has 5 conjugacy classes and $120/2 = 60$ elements. The value of a the faithful character $\bar{\chi}_3$ of $PSL_2(\mathbb{F}_5)$ can be read from Table 2. From (13), the diversity sum of this new 4-dimensional representation of $PSL_2(\mathbb{F}_5)$ is*

$$\xi_{\mathcal{V}} = \frac{1}{\sqrt{2 \cdot 4}}\sqrt{4-1} = 0.6124. \tag{14}$$

The faithful $PSL_2(\mathbb{F}_5)$ constellation obtained from χ_2, χ_5 and χ_6 can be done similarly.

	$I = B$	C_1	$D_1 = D_2$	$E_1 = F_1$	$E_{-1} = F_{-1}$
$\bar{\chi}_2$	5	1	-1	0	0
$\bar{\chi}_3$	4	0	1	-1	-1
$\bar{\chi}_5$	3	-1	0	$-\frac{(-1+j\sqrt{5})}{2}$	$\frac{-1-j\sqrt{5}}{2}$
$\bar{\chi}_6$	3	-1	0	$\frac{-1-j\sqrt{5}}{2}$	$-\frac{(-1+j\sqrt{5})}{2}$

Table 2: The character table of $PSL_2(\mathbb{F}_5)$

4 Examples and Performance

In this section, we give some examples to compute the diversity sum using a character table from different unitary groups. The performance is considered by plotting the block error rate (bler) against SNR in db. We use the differential space-time modulation [2, 3] as explained in Section 1.1 for signal transmission. The fading coefficients and additive noise are indepedent $\mathcal{CN}(0, 1)$, and the channel matrix is assumed to be constant within two consecutive time periods. The maximum-likelihood (3) is used for decoding.

4.1 Example: Cyclic Group

An $M \times M$ cyclic group constellation of order L, $\mathcal{V} = \{V_l\}_{l=0}^{L-1}$ is defined as [2, 3]

$$V_l = \text{diag}(e^{j2\pi k_1 l/L}, e^{j2\pi k_2 l/L}, \ldots, e^{j2\pi k_M l/L}) \tag{15}$$

where $k_i = 1, 2, \ldots, L-1$. The character of a cyclic group is $\chi_{V_l} = \sum_{i=1}^{M} e^{j2\pi k_i l/L}$ so $\mathrm{Re}\{\chi_{V_l}\} = \sum_{i=1}^{M} \cos\frac{2\pi k_i l}{L}$. The L conjugacy classes of a cyclic group each consist of only one element. Thus a diversity sum of a cyclic group of order L is

$$\xi_{\mathcal{V}} = \frac{1}{\sqrt{2M}} \min_{l=1,2,\ldots,L-1} \sqrt{M - \sum_{i=1}^{M} \cos\frac{2\pi k_i l}{L}}. \tag{16}$$

4.2 Example: Dicyclic Group

The dicyclic group constellation [4] of order L and even M is defined as

$$\mathcal{V} = \{V_0^l V_1^m : \ 0 \le l < L/2; \ m = 0, 1\} \tag{17}$$

where

$$V_0 = \mathrm{diag}(e^{j4\pi k_1/L}, e^{j4\pi k_2/L}, \ldots, e^{j4\pi k_M/L}) \quad \text{and} \quad V_1 = \begin{bmatrix} 0 & -I_{M/2} \\ I_{M/2} & 0 \end{bmatrix}. \tag{18}$$

The character of $\{V_0^l V_1\}$ equals 0 for every $l = 0, 1, \ldots, M/2$. So to compute a diversity sum, it is enough to consider only $\{V_0^l\}_{l=0}^{M/2}$. Using the similar computation in Section 4.1, the diversity sum of a dicyclic group of order L is

$$\xi_{\mathcal{V}} = \frac{1}{\sqrt{2M}} \min_{l=1,2,\ldots,L/2} \sqrt{M - \sum_{i=1}^{M} \cos\frac{4\pi k_i l}{L}}. \tag{19}$$

4.3 Performance: The Group $SL_2(\mathbb{F}_q)$ For Odd Prime q

The character table of $SL_2(\mathbb{F}_q)$ for an odd prime q is given in Table II [8], so we can use the equation (13) to compute the diversity sum very easily. We give some examples for the group $SL_2(\mathbb{F}_5)$, and also show that its non full diversity constellations outperform full diversity constellations in a specific range of low SNR.

4.3.1 $SL_2(\mathbb{F}_5)$: $\mathcal{V}_{\chi_4}$ vs. $\mathcal{V}_{\chi_7 \oplus \chi_7}$, $M = 4$

The formulas to compute the 4-dimensional representation of irreducible character χ_4 (setting $\alpha(2^n) = e^{\pi j n/2}$) and the 2-dimensional representation of irreducible character χ_7 are given in Appendix B. To compare a diversity sum in the equation (13), it is enough to consider only the term $M - \mathrm{Re}\{\chi_{\mathcal{C}}\} = 4 - \mathrm{Re}\{\chi_{\mathcal{C}}\}$. Table 3 shows the values of $4 - \mathrm{Re}\{\chi_{\mathcal{C}}\}$ of χ_4 and $\chi_7 \oplus \chi_7$. We note that the character of $\chi_7 \oplus \chi_7 = \chi_7 + \chi_7$ from Section 2. We can see that the minimum values of $4 - \mathrm{Re}\{\chi\}$ maximizing over their conjugacy classes $\mathcal{C}_i \neq I$ of χ_4 and $\chi_7 \oplus \chi_7$ are 3 and 2 respectively. Thus one can predict that the constellation $\mathcal{V}_{\chi_4}$ should outperform the constellation $\mathcal{V}_{\chi_7 \oplus \chi_7}$ at a low SNR. Simulation shows that this is true, as shown in Figure 2 with the number of receiver antenna $N = 1$, and that the specific range of low SNR $\approx$ 0-17 dB.

	I	B	C_1	D_1	D_2	E_1	E_{-1}	F_1	F_{-1}
χ_4	4	-4	0	-1	1	-1	-1	1	1
$4 - \mathrm{Re}\{\chi_4\}$	0	8	4	5	3	5	5	3	3
$\chi_7 \oplus \chi_7$	4	-4	0	2	-2	$1 + j\sqrt{5}$	$-1 + j\sqrt{5}$	$-1 - j\sqrt{5}$	$1 - j\sqrt{5}$
$4 - \mathrm{Re}\{\chi_7 \oplus \chi_7\}$	0	8	4	2	6	3	5	5	3

Table 3: Comparison of χ_4 and $\chi_7 \oplus \chi_7$

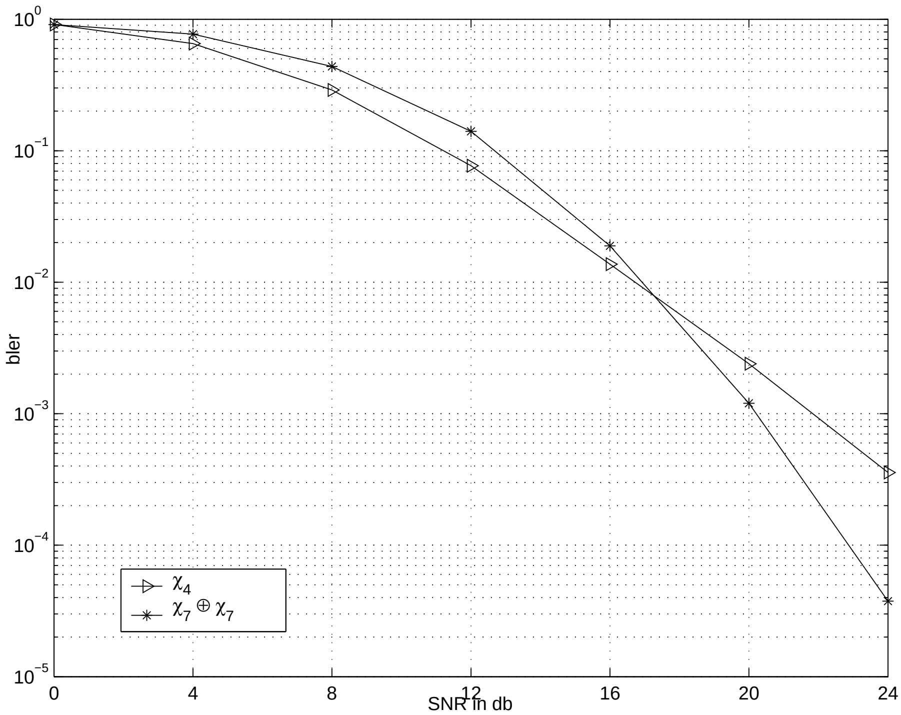

Figure 2: χ_4 vs. $\chi_7 \oplus \chi_7$. $M = 4, N = 1$ and $R = 1.73$

4.3.2 $SL_2(\mathbb{F}_5)$: $\mathcal{V}_{\chi_4 \oplus \chi_7}$ vs.$\mathcal{V}_{\chi_7 \oplus \chi_7 \oplus \chi_7}$, $M = 6$

The values of their $M - \mathrm{Re}\{\chi_{\mathcal{C}}\} = 6 - \mathrm{Re}\{\chi_{\mathcal{C}}\}$ are given in Table 4. Their minimum values of $\chi_4 \oplus \chi_7$ and $\chi_7 \oplus \chi_7 \oplus \chi_7$ are 5.5 and 3 respectively. From a simulation shown in Figure 3, we can see that the constellation $\mathcal{V}_{\chi_4 \oplus \chi_7}$ performs very well at low SNR 0-20 db. It outperforms the constellation $\mathcal{V}_{\chi_7 \oplus \chi_7 \oplus \chi_7} \approx 3$ db.

	I	B	C_1	D_1	D_2	E_1	E_{-1}	F_1	F_{-1}
$\chi_4 \oplus \chi_7$	6	-6	0	0	0	$\frac{-1+j\sqrt{5}}{2}$	$\frac{-3+j\sqrt{5}}{2}$	$\frac{1-j\sqrt{5}}{2}$	$\frac{3-j\sqrt{5}}{2}$
$6 - \mathrm{Re}\{\chi_4 \oplus \chi_7\}$	0	12	6	6	6	6.5	7.5	5.5	4.5
$\chi_7 \oplus \chi_7 \oplus \chi_7$	6	-6	0	3	-3	$\frac{3+j\sqrt{5}}{2}$	$\frac{-3+j\sqrt{5}}{2}$	$\frac{-3-j\sqrt{5}}{2}$	$\frac{3-j\sqrt{5}}{2}$
$6 - \mathrm{Re}\{\chi_7 \oplus \chi_7 \oplus \chi_7\}$	0	12	6	3	9	4.5	7.5	7.5	4.5

Table 4: Comparison of $\chi_4 \oplus \chi_7$ and $\chi_7 \oplus \chi_7 \oplus \chi_7$

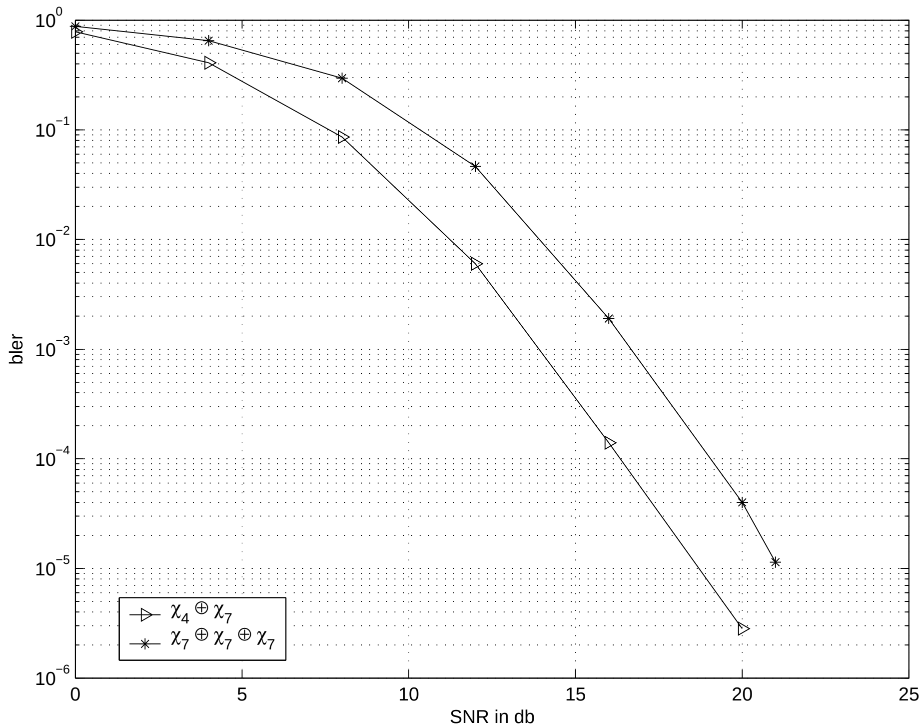

Figure 3: $\chi_4 \oplus \chi_7$ vs. $\chi_7 \oplus \chi_7 \oplus \chi_7$. $M = 6, N = 1$ and $R = 1.15$

4.4 Performance: The Group $PSL_2(\mathbb{F}_q)$ For Odd Prime q

4.4.1 $PSL_2(\mathbb{F}_5)$: $\bar{\chi_4}$, $M = 4$

From the Example 2, the constellation obtained from irreducible character χ_4 of $PSL_2(\mathbb{F}_5)$ has $\xi_{\mathcal{V}} = 0.6124$. The rate is $R = \log_2(60)/4 = 1.48$, using for 4 transmitter antennas. The 60 faithful representation matrices of $PSL_2(\mathbb{F}_5)$ can be generated from $(PQ)^j X$ where $j = 0, 1, 2, 3, 4$ (see

Appendix B, by setting $\alpha(2^n) = e^{\pi jn}$). Figure 4 shows the block error rate perfromance of $\mathcal{V}_{\bar{\chi}_4}$ for $N = 1, 2, 3$ receiver antennas.

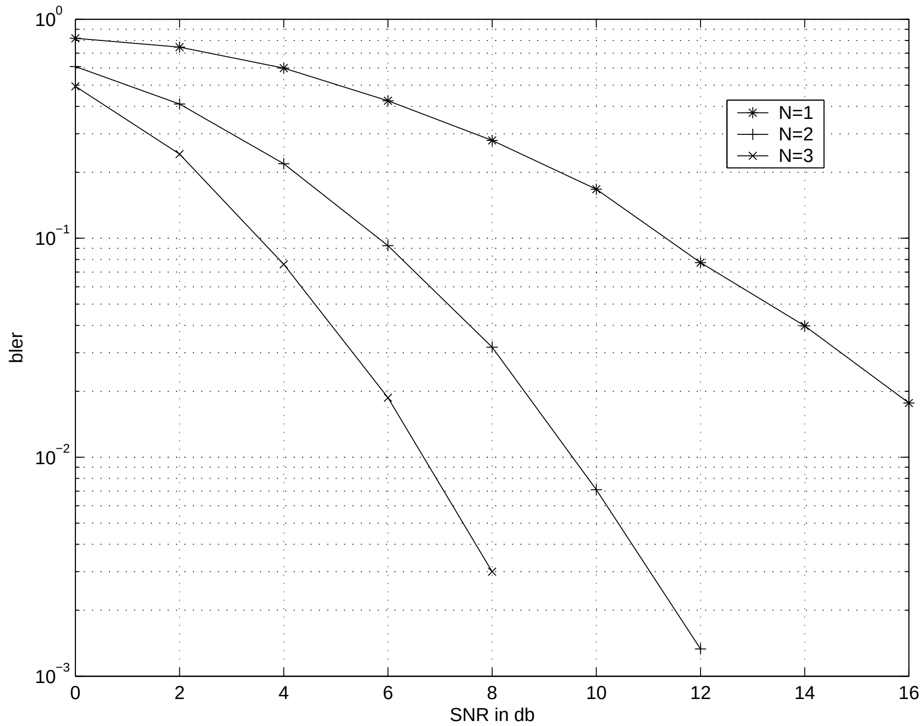

Figure 4: $PSL_2(\mathbb{F}_5)$

4.4.2 $PSL_2(\mathbb{F}_7)$

From the equation (24) in Appendix A, the group $PSL_2(\mathbb{F}_7)$ has

$$3 + \left\lceil \frac{7-3}{4} \right\rceil + \left\lceil \frac{7-1}{4} \right\rceil = 3 + 1 + 2 = 6 \tag{20}$$

conjugacy classes with $7(7^2 - 1)/2 = 168$ elements. The character table of $PSL_2(\mathbb{F}_7)$ computing from Table 6 (see also pp.314 in [5]) is shown in Table 5. Consider the irreducible character χ_4, using for 3 transmitter antennas with the rate $R = \log_2(168)/3 = 2.46$. The value of diversity sum is

$$\xi_{\mathcal{V}} = \frac{1}{\sqrt{2 \cdot 3}} \sqrt{3 - 1} = 0.557. \tag{21}$$

	$A=B$	$C_1=C_2$	$D_1=D_2$	D_3	$E_1=F_{-1}$	$E_{-1}=F_1$
$\bar{\chi_1}$	8	-1	0	0	1	1
$\bar{\chi_2}$	7	1	-1	-1	0	0
$\bar{\chi_3}$	6	0	0	2	-1	-1
$\bar{\chi_4}$	3	0	1	-1	$\frac{-1-j\sqrt{7}}{2}$	$\frac{-1+j\sqrt{7}}{2}$
$\bar{\chi_5}$	3	0	1	-1	$\frac{-1+j\sqrt{7}}{2}$	$\frac{-1-j\sqrt{7}}{2}$
$\bar{\chi_6}$	1	1	1	1	1	1

Table 5: Character table of the group $PSL_2(\mathbb{F}_7)$

5 Conclusion

We have presented how to compute the diversity sum of a unitary group constellation using its character table. Our method is shown to be very simple and save time to compute the diversity sum. Simulations show that our method can predict a good performance group constellation in low SNR. We have also proposed how to obatin a faithful group constellation (its diversity sum is greater than 0), that is the group PSL_2, from an unfaithful group constellation SL_2.

Appendix A: The Group $PSL_2(\mathbb{F}_q)$ For Odd Prime q

Using the description of the group $SL_2(\mathbb{F}_q)$, for odd prime q, given in [8]. Let us summarize the number and the sizes of the conjugacy classes and the character table of $PSL_2(\mathbb{F}_q)$ for an odd prime q below.

Let ρ denote a representation of $SL_2(\mathbb{F}_q)$. If $\rho(I) = \rho(-I)$ then by the group homomorphism property, $\rho(g) = \rho(-g)$ for all $g \in SL_2(\mathbb{F}_q)$. Hence rho may be thought of as a representation of $PSL_2(\mathbb{F}_q) = SL_2(\mathbb{F}_q)/\{\pm I\}$, denoted $\bar{\rho}$ for clarity. The order of $PSL_2(\mathbb{F}_q)$ is one half of the order of $SL_2(\mathbb{F}_q)$, that is

$$|PSL_2(\mathbb{F}_q)| = \frac{1}{2}q(q^2-1). \tag{22}$$

The group $SL_2(\mathbb{F}_q)$ has $q+4$ conjugacy classes which are denoted $A, B, C_1, \ldots, C_{(q-3)/2}, D_1, \ldots, D_{(q-1)/2}, E_1, E_{-}$ and F_{-1}. Since we are quotienting out by the center of $SL_2(\mathbb{F}_q)$, the conjugacy classes of $PSL_2(\mathbb{F}_q)$ are formed union of conjugacy classes of $SL_2(\mathbb{F}_q)$. Namely, if $g \in \mathcal{C}_i$ and $-g \in \mathcal{C}_j$ then $\mathcal{C}_i \cup \mathcal{C}_j$ lifts to a single conjugacy class of $PSL_2(\mathbb{F}_q)$. It can be shown that

$$\begin{aligned}
&A = B \\
&\begin{cases} C_1 = C_2, C_3 = C_4, \ldots, C_{(q-3)/2-1} = C_{(q-3)/2} \ \text{ if } (q-3)/2 \text{ is an even number } \ \text{or} \\ C_1 = C_2, C_3 = C_4, \ldots, C_{(q-3)/2} \ \text{ if } (q-3)/2 \text{ is an odd number} \end{cases} \\
&\begin{cases} D_1 = D_2, D_3 = D_4, \ldots, D_{(q-1)/2-1} = D_{(q-1)/2} \ \text{ if } (q-1)/2 \text{ is an even number } \quad \text{or} \\ D_1 = D_2, D_3 = D_4, \ldots, D_{(q-1)/2} \ \text{ if } (q-1) \text{ is an odd number} \end{cases} \\
&E_1 = F_{-1} \\
&E_{-1} = F_1.
\end{aligned} \tag{23}$$

Conseqently the number of conjucay classes of the gropp $PSL_2(\mathbb{F}_q)$ is

$$3 + \left\lceil \frac{q-3}{4} \right\rceil + \left\lceil \frac{q-1}{4} \right\rceil \tag{24}$$

where $\lceil \ \rceil$ denotes the upper floor integer.

The sizes of those conjugacy classes in $PSL_2(\mathbb{F}_q)$ are

$$\begin{aligned}
&|A| = |B| = 1 \\
&|C_i| = q(q+1) \text{ , and } |C_{(q-3)/2}| = q(q+1)/2 \text{ if } (q-3)/2 \text{ is an odd number} \\
&|D_i| = q(q-1) \text{ , and } |D_{(q-1)/2}| = q(q-1)/2 \text{ if } (q-1)/2 \text{ is an odd number} \\
&|E_1| = |F_{-1}| = (q^2-1)/2 \\
&|E_{-1}| = |F_1| = (q^2-1)/2.
\end{aligned} \tag{25}$$

The character table of $PSL_2(\mathbb{F}_q)$ can then be computed from the character table of $SL_2(\mathbb{F}_q)$ (Table II in [8]). For example, the irreducible character $R_T^G(\alpha)$ of $SL_2(\mathbb{F}_q)$ has $\chi_A = q+1$ and $\chi_B = \alpha(-1)(q+1)$. Hence to have $\chi_A = \chi_B$ and hence a representation of $PSL_2(\mathbb{F})q)$, we need to restrict to the case $\alpha(-1) = 1$. Other irreducible characters can be done in the same way. The character table of $PSL_2(\mathbb{F}_q)$ is given in Table 6. ω denotes a generator of $\mathbb{F}_q^\times$.

Appendix B: 4-Dimensional Representations of $SL_2(\mathbb{F}_5)$

We compute some 4-dimensional representations for of $SL_2(\mathbb{F}_5)$ following the construction in [9]. We first define an additive character Φ of $\mathbb{F}_5$ via $\Phi(u) = e^{2\pi j \frac{u}{5}}$ for $u \in \{0,1,2,3,4\}$. Secondly we define a multiplicative character α of $\mathbb{F}_5^*$ which can have four choices: i) $\alpha(z) = 1$ for all $z \in \{1,2,3,4\}$, the

	$A=B$	C_i	D_i	$E_1=F_{-1}$	$E_{-1}=F_1$
Id	1	1	1	1	1
$R_T^G(\alpha)$	$q+1$	$\alpha(\omega^i)+\alpha(\omega^{-i})$	0	1	1
St	q	1	-1	0	0
$R_T^G(\beta)$	$q-1$	0	$-\beta(\theta^i)-\beta(\theta^{-i})$	-1	-1
$\chi_{\alpha_0}^\epsilon$	$(q+1)/2$	$\alpha_0(\omega^i)$	0	$\frac{1+j\sqrt{q}}{2}$	$\frac{1-j\sqrt{q}}{2}$
$\chi_{\beta_0}^\epsilon$	$(q-1)/2$	0	$-\beta_0(\theta^i)$	$\frac{-1+j\sqrt{q}}{2}$	$\frac{-1-j\sqrt{q}}{2}$

Table 6: Character table of the group $PSL_2(\mathbb{F}_q)$

trivial character; ii) $\alpha(1)=\alpha(2^2)=1$, $\alpha(2)=\alpha(2^3)=-1$; iii) $\alpha(1)=1,\alpha(2)=j,\alpha(4)=-1,\alpha(3)=-j$; and iv) $\alpha(1)=1,\alpha(2)=-j,\alpha(4)=-1,\alpha(3)=j$. We note that (iii) and (iv) are inverses of one another.

Thirdly, consider the field $\mathbb{F}_{25}=\mathbb{F}_5(\sqrt{2})=\{c+d\sqrt{2}|c,d\in F_5\}$. $\mathbb{F}_{25}^*$ is cyclic, with generator $\epsilon=2+4\sqrt{2}$. We define a character Π of $\mathbb{F}_{25}^*$ which has $24=|\mathbb{F}_{25}^*|$ choices, but to get discrete series representations, there is a condition. Let N^1 denote those elements in $\mathbb{F}_{25}^*$ whose *norm* is equal to 1. The norm of an element of $\mathbb{F}_{25}^*$ is defined by $N(c+d\sqrt{2})=(c+d\sqrt{2})(c-d\sqrt{2})=c^2-2d^2$. From direct calculations, we deduce that $N^1=\{1,\epsilon^4,\epsilon^8,\ldots,\epsilon^{20}\}$. The condition on Π is that its restriction to N^1 is not allowed to be trivial. Moreover, if two choices of Π give the same restriction to N^1, or restrictions to N^1 which are exactly inverses of each other, then the resulting discrete series representations will be the same. Finally, there is a condition relating Π and α. Namely, $\Pi(2)=\alpha(2)$ (so that Π and α agree on $\mathbb{F}_5^*$). Since $2=\epsilon^6$, this implies we have the following choices: i) $\Pi(\epsilon^n)=e^{\pi jn/12}$, $\alpha(2^n)=e^{\pi jn/2}$; ii) $\Pi(\epsilon^n)=e^{\pi jn/6}$, $\alpha(2^n)=e^{\pi jn}$; and iii) $\Pi(\epsilon^n)=e^{\pi jn/4}$, $\alpha(2^n)=e^{\pi j3n/2}$. The first two will give irreducible discrete series (characters χ_3 and χ_4, in some order) and the last will be reducible, with two inequivalent 2-dimensional components. We need one technical definition, which will simplify the notation in what follows. For each $n\in\{0,1,\ldots,23\}$, write $\epsilon^n=c_n+d_n\sqrt{2}$ with $c_n,d_n\in\{0,1,2,3,4\}$. Thus, for example, $c_0=1$, $c_1=2$, $c_2=1$, $c_3=0$, $c_4=3$, $c_5=2$. Since multiplication by ϵ^6 is multiplication by a number in $\mathbb{F}_5^*$, we deduce that $c_{6+n}=2c_n$, $c_{12+n}=4c_n$ and $c_{18+n}=3c_n$, and so the rest are easily computed. For each $N\in\{0,1,2,3\}$, define a complex number $F(N)$ as follows:

$$F(N)=\sum_{0\leq n\leq 23,n\equiv N \mod 4}\Pi(\epsilon^n)\Phi(2c_n).$$

So for example, if $N=0$ and we take $\Pi(\epsilon^n)=e^{\pi j n/12}$, then $F(0)=e^{2\pi j\frac{2}{5}}+e^{2\pi j\frac{4}{24}}e^{2\pi j\frac{1}{5}}+e^{2\pi j\frac{8}{24}}e^{2\pi j\frac{4}{5}}+e^{2\pi j\frac{12}{24}}e^{2\pi j\frac{8}{5}}+e^{2\pi j\frac{16}{24}}e^{2\pi j\frac{4}{5}}+e^{2\pi j\frac{20}{24}}e^{2\pi j\frac{6}{5}}$.

Each element of $SL_2(\mathbb{F}_5)$ is either of the form

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} a & au \\ 0 & a^{-1} \end{bmatrix} \tag{26}$$

with $a\in\mathbb{F}_5^*=\{1,2,3,4\}$ and $u\in\mathbb{F}_5=\{0,1,2,3,4\}$, or of the form

$$\begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} -av & -auv+a^{-1} \\ -a & -au \end{bmatrix} \tag{27}$$

with $a\in\mathbb{F}_5^*$, and $u,v\in\mathbb{F}_5$. Thus there are 20 elements of the first kind and 100 of the second kind.

For elements in g of the first kind (26), the (i,k)th element of the matrix $\rho(g)$ is given by

$$\rho(g)_{i,k}=\alpha(a)\Phi(ku)\quad \text{if } ki^{-1}=a^2 \tag{28}$$

and is zero otherwise. The formula for the (i,k)th matrix coefficient of $\rho(g)$, when g is of the form (27), is

$$\rho(g)_{i,k}=-\frac{1}{5}\Phi(iv+ku)\alpha(ai^{-1})F(\log_2(ika^{-2})) \tag{29}$$

where $N=\log_2(ika^{-2})\in\{0,1,2,3\}$ is the number so that $2^N\equiv ika^{-2}\mod 5$.

The group $SL_2(\mathbb{F}_5)$ has two generators P and Q [7]. The 120 elements are given by $(PQ)^jX$, where $j=0,1,\dots,9$ and X runs over $\{I_2,P,Q,QP,QPQ,QPQP,QPQ^2,QPQPQ,QPQPQ^2,QPQPQ^2P,QPQPQ^2PQ,QPQPQ^2PQP\}$. We deduce from calculations that the two generators P and Q belong to elements of a second kind (27), namely

$$P=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\quad\text{when } a=1,u=0,v=0,$$

and

$$Q=\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\quad\text{when } a=1,u=-1,v=0.$$

Using the formula (29), the 4-dimensional representation of P is

$$\rho(P)_{i,k}=-\frac{1}{5}\alpha(i^{-1})F(\log_2(ik))\qquad\text{for } i,k=1,2,3,4. \tag{30}$$

This gives

$$\rho(P) = -\frac{1}{5}\begin{bmatrix} F(0) & F(1) & F(3) & F(2) \\ -jF(1) & -jF(2) & -jF(0) & -jF(3) \\ jF(3) & jF(0) & jF(2) & jF(1) \\ -F(2) & -F(3) & -F(1) & -F(0) \end{bmatrix}, \tag{31}$$

where

$$\begin{aligned}
F(0) &= e^{2\pi j\frac{2}{5}} + e^{2\pi j\frac{4}{24}}e^{2\pi j\frac{1}{5}} + e^{2\pi j\frac{8}{24}}e^{2\pi j\frac{4}{5}} + e^{2\pi j\frac{12}{24}}e^{2\pi j\frac{8}{5}} + e^{2\pi j\frac{16}{24}}e^{2\pi j\frac{4}{5}} + e^{2\pi j\frac{20}{24}}e^{2\pi j\frac{6}{5}} &(32)\\
F(1) &= e^{2\pi j\frac{1}{24}}e^{2\pi j\frac{4}{5}} + e^{2\pi j\frac{5}{24}}e^{2\pi j\frac{4}{5}} + e^{2\pi j\frac{9}{24}} + e^{2\pi j\frac{13}{24}}e^{2\pi j\frac{6}{5}} + e^{2\pi j\frac{17}{24}}e^{2\pi j\frac{6}{5}} + e^{2\pi j\frac{21}{24}} &(33)\\
F(2) &= e^{2\pi j\frac{2}{24}}e^{2\pi j\frac{2}{5}} + e^{2\pi j\frac{6}{24}}e^{2\pi j\frac{4}{5}} + e^{2\pi j\frac{10}{24}}e^{2\pi j\frac{2}{5}} + e^{2\pi j\frac{14}{24}}e^{2\pi j\frac{8}{5}} + e^{2\pi j\frac{18}{24}}e^{2\pi j\frac{6}{5}} + e^{2\pi j\frac{22}{24}}e^{2\pi j\frac{8}{5}} &(34)\\
F(3) &= e^{2\pi j\frac{3}{24}} + e^{2\pi j\frac{7}{24}}e^{2\pi j\frac{8}{5}} + e^{2\pi j\frac{11}{24}}e^{2\pi j\frac{8}{5}} + e^{2\pi j\frac{15}{24}} + e^{2\pi j\frac{19}{24}}e^{2\pi j\frac{2}{5}} + e^{2\pi j\frac{23}{24}}e^{2\pi j\frac{2}{5}}. &(35)
\end{aligned}$$

The 4-dimensional representation of Q (also using a formula (29)) is

$$\rho(Q)_{i,k} = -\frac{1}{5}e^{-j2\pi k/5}\alpha(i^{-1})F(\log_2(ik)) = e^{-2\pi j\frac{k}{5}}\rho(P) \qquad \text{for } i,k = 1,2,3,4. \tag{36}$$

This gives

$$\rho(Q) = -\frac{1}{5}\begin{bmatrix} e^{-2\pi j\frac{1}{5}}F(0) & e^{-2\pi j\frac{2}{5}}F(1) & e^{-2\pi j\frac{3}{5}}F(3) & e^{-2\pi j\frac{4}{5}}F(2) \\ -je^{-2\pi j\frac{1}{5}}F(1) & -je^{-2\pi j\frac{2}{5}}F(2) & -je^{-2\pi j\frac{3}{5}}F(0) & -je^{-2\pi j\frac{4}{5}}F(3) \\ je^{-2\pi j\frac{1}{5}}F(3) & je^{-2\pi j\frac{2}{5}}F(0) & je^{-2\pi j\frac{3}{5}}F(2) & je^{-2\pi j\frac{4}{5}}F(1) \\ -e^{-2\pi j\frac{1}{5}}F(2) & -e^{-2\pi j\frac{2}{5}}F(3) & -e^{-2\pi j\frac{3}{5}}F(1) & -e^{-2\pi j\frac{4}{5}}F(0) \end{bmatrix}. \tag{37}$$

For the 2-dimensional representations (χ_7), we have more directly from [7] that

$$\rho(P) = \frac{1}{\sqrt{5}}\begin{bmatrix} \eta^2-\eta^3 & \eta-\eta^4 \\ \eta-\eta^4 & \eta^3-\eta^2 \end{bmatrix}, \qquad \rho(Q) = \frac{1}{\sqrt{5}}\begin{bmatrix} \eta-\eta^2 & \eta^2-1 \\ 1-\eta^3 & \eta^4-\eta^3 \end{bmatrix} \tag{38}$$

where $\eta = e^{j2\pi/5}$.

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