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ADMISSIBLE NILPOTENT ORBITS OF REAL AND p-ADIC SPLIT EXCEPTIONAL LIE GROUPS

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Abstract. We determine the admissible nilpotent coadjoint orbits of real and p-adic split exceptional Lie groups of types G_2, F_4, E_6 and E_7. We find that all Lusztig-Spaltenstein special orbits are admissible. Moreover, there exist non-special admissible orbits, corresponding to "completely odd" orbits in Lusztig's special pieces.

1. Introduction

The orbit method, introduced by Kirillov [Ki], Moore [M1] and Duflo [Du], among many others, conjectures a deep relationship between irreducible unitary representations of a Lie group G (defined over a real or p-adic field) and the coadjoint orbits of G acting on the dual of its Lie algebra ($\mathfrak{g}^*$). As a result of much work by Lion-Perrin [LP], Auslander-Kostant [AK] and Vogan [V1, V2], the orbit method has been realized for all but: the orbits of reductive Lie groups over p-adic fields; and nilpotent orbits of reductive Lie groups over $\mathbb{R}$. (For reductive groups, we can and do identify the adjoint and coadjoint orbits of G in a natural way, which allows us in particular to define nilpotent orbits.)

In an effort to understand these remaining cases, Schwarz [Sch], Ohta [O] and Nevins [N1] determined the admissible nilpotent orbits of most groups of classical type over the real and p-adic fields. It was found that for all split groups (and some others), the set of admissible orbits coincides exactly with the set of special orbits [L1, Sp]. In this context, we define an orbit of the real or p-adic group to be special if the corresponding algebraic orbit is special.

Let k denote either $\mathbb{R}$ or a p-adic field. In this paper, we consider the nilpotent orbits of k-points of split simply connected Lie groups of exceptional types G_2, F_4, E_6 and E_7. For each algebraic non-even nilpotent orbit of these groups, we choose a k-rational representative such that the corresponding rational orbit is "split" over k. We then determine the admissibility of that rational orbit. For the groups of types G_2, F_4 and E_6, we go on to compute the number of other k-rational orbits of the given algebraic orbit and determine their admissibility as well.

We prove the following theorem (summarized here from Sections 6 to 9).

Main Theorem. *Let G denote the k-points of split simply connected Lie groups of exceptional types G_2, F_4, E_6 or E_7. Then*

(i) *every special orbit gives rise to a split admissible orbit;*
(ii) *there are non-special admissible orbits, occuring as* completely odd *members of special pieces (see Table 1)*

Now assume that $k = \mathbb{R}$ or has odd residual characteristic. Then for the groups of type G_2, F_4 and E_6:

(iii) *admissibility is independent of the choice of rational orbit within a given algebraic orbit, except for the B_2-orbit of F_4.*

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For an algebraic group $\mathbb{G}$, one defines a *special piece* P_s as a union of a unique special nilpotent orbit $\mathcal{O}_s$ and those nonspecial nilpotent orbits contained in its closure $\overline{\mathcal{O}_s}$, but not in the closure of any smaller special orbit. This defines a partition of the set of nilpotent orbits. In [L2], Lusztig gives a parametrization of the nilpotent orbits in each special piece P_s by the conjugacy classes of a finite group he denotes $A(\mathcal{O}_s)$. This group is (with few exceptions) equal to the equivariant fundamental group $\pi_1(\mathcal{O}_s)$ of the orbit.

The admissible orbits are exactly those orbits $\mathcal{O}_x$ in a special piece P_s for which (a) $A(\mathcal{O}_s)$ is a symmetric group on n letters ($n \geq 3$) and (b) the conjugacy class identifying $\mathcal{O}_x$ is identified by a completely odd partition of n. This is automatic for the special orbits (whose corresponding conjugacy class (the identity element) is represented by n ones. See Table 1 for a list of the nonspecial admissible orbits and their parametrization within special pieces.

Group	Orbit	$\mathcal{O}_s$	$A(\mathcal{O}_s)$	Lusztig parameter
G_2	$\widetilde{A_1}$	$G_2(a_1)$	S_3	(3)
F_4	$\widetilde{A_2}+A_1$	$F_4(a_3)$	S_4	$(3,1)$
E_6	$2A_2+A_1$	$D_4(a_1)$	S_3	(3)
E_7	$2A_2+A_1$	$D_4(a_1)$	S_3	(3)
E_7	A_5+A_1	$E_7(a_5)$	S_3	(3)

TABLE 1. The non-special admissible orbits of G_2, F_4, E_6 and E_7.

We remark that this identification of admissible orbits as completely odd members of special pieces is consistent with known results for classical groups (where in particular the $A(\mathcal{O}_s)$ are never symmetric groups).

This result clarifies the heretofore mysterious link between the algebraically defined special orbits and the geometrically defined admissible ones. It remains to show this link in a natural way; proving perhaps a conjecture of Vogan that admissibility is some mod $\mathbb{Z}/2\mathbb{Z}$ reduction of an intrinsic object related to the geometry of the special pieces.

Conjecture. *We expect that the admissible non-special nilpotent orbits of E_8 will be:*

Orbit	$\mathcal{O}_s$	$A(\mathcal{O}_s)$	*Lusztig parameter*
$2A_2+A_1$	$D_4(a_1)$	S_3	(3)
$2A_2+2A_1$	$D_4(a_1)+A_1$	S_3	(3)
$E_6(a_3)+A_1$	$E_8(a_7)$	S_5	$(3,1,1)$
A_4+A_3	$E_8(a_7)$	S_5	(5)
E_6+A_1	$E_8(b_5)$	S_3	(3)

Note that A. Noël [No1, No2] has independently computed the admissible nilpotent coadjoint orbits for *all* **real** exceptional groups, using techniques of Ohta [O]. His results agree with ours where they overlap, and moreover, they support the above conjecture for E_8 over $\mathbb{R}$. In addition, he has found a handful of orbits, like B_2 of F_4, for which admissibility is a non-stable criterion. For some non-split exceptional real Lie groups, he has shown that there are non-admissible special orbits (just as for some non-split classical groups over $\mathbb{R}$; see [Sch, O]).

In this paper we have excluded the study of the nonsplit rational orbits of E_7 due to their sheer number and diversity. Fields of residual characteristic equal to 2 are excluded in the discussion of nonsplit rational orbits because many of the results used in Appendix B either fail directly in that case or at least would require separate arguments.

The structure of the paper is as follows. In Section 2, we set our notation for the remainder of the paper and recall the definition of an admissible nilpotent orbit. In Section 3, we describe the procedure for determining the admissibility of "split" nilpotent orbits of the exceptional groups considered here.

Our discussion of the occurence of other rational orbits within (the k-points of) each algebraic orbit begins in Section 4. There, we compute a bound on their number using Galois cohomology, and go on to describe a method for obtaining representatives of these additional orbits. In Section 5 we give constraints on the structure of the groups G^ϕ that can arise (for ϕ classifying a nonsplit orbit in a given algebraic one).

Sections 6 to 9 are devoted to studying the orbits individually and recording their admissibility. We relegate our explicit computations with respect to additional rational orbits to the appendices. In Appendix A we give explicit representatives of each rational orbit and show that in each case our bound, computed using Galois cohomology, was minimal. In Appendix B we summarize results needed to determine the admissibility of the non-split rational orbits occuring here.

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2. Admissibility

In this section, let us set our notation for the remainder of the paper, and recall the definition of admissibility (originally defined by [Du] over $\mathbb{R}$).

Let k be a real or p-adic field. Let $\mathbb{G}$ be a linear algebraic group of exceptional type, defined and split over k. Write $G = \mathbb{G}(k)$. Let Φ denote the set of roots of G, and Δ a set of simple roots.

Identify the adjoint orbits of G with its coadjoint orbits via a nondegenerate invariant form $\langle , \rangle$ on $\mathfrak{g}$. For each nilpotent orbit $G \cdot E$ in $\mathfrak{g}$, choose $H, F \in \mathfrak{g}$ so that $\phi = \operatorname{span}\{E, H, F\}$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,k)$. Define $\mathfrak{g}^\phi$ to be the centralizer in $\mathfrak{g}$ of ϕ (*i.e.* the span of the trivial subrepresentations of ϕ acting on $\mathfrak{g}$), and G^ϕ to be the corresponding subgroup of G. Let $\mathfrak{g}[-1]$ denote the subspace of -1 weight vectors of $\mathfrak{g}$ with respect to H. It is a symplectic vector space, endowed with the canonical Kirillov-Kostant symplectic form $\omega_E(X,Y) = \langle E, [X,Y]\rangle$, $X, Y \in \mathfrak{g}[-1]$. Then G^ϕ acts, via the adjoint action, on $\mathfrak{g}[-1]$, and preserves ω_E.

The orbit $G \cdot E$ is *admissible* if the cover of G^ϕ defined by the diagram

$$\begin{array}{ccc} (G^\phi)^{mp} & \longrightarrow & Mp(\mathfrak{g}[-1]) \\ \big\downarrow & & \big\downarrow \\ G^\phi & \longrightarrow & Sp(\mathfrak{g}[-1]) \end{array} \tag{2.1}$$

splits (*i.e.* admits a smooth section) over G^ϕ_e, where G^ϕ_e is the topological identity component when $k = \mathbb{R}$, and an open normal subgroup containing I otherwise (see [N2]).

Remarks. (i) Each even orbit is automatically admissible, since in that case $\mathfrak{g}[-1] = \{0\}$; these orbits are also all special. Therefore we need only consider noneven orbits in this paper.

(ii) We will need to assume that the residual characteristic of k is odd to determine the admissibility of non-split rational orbits; see Appendix B.

3. Admissibility of split orbits

For each orbit $\mathcal{O}$ of the algebraic group $\mathbb{G}$, its set of k-rational points $\mathcal{O}(k)$ is a union of one or more orbits of G. We call a rational orbit in $\mathcal{O}(k)$ *split* if the corresponding reductive group G^ϕ is split over k. Each orbit $\mathcal{O}$ of a split group $\mathbb{G}$ has one or more split rational orbits, and it is these orbits that we wish to consider in this section.

Our method for choosing a k-rational representative E of $\mathcal{O}$ such that $G \cdot E$ is split is as follows; see [CMcG] for an overview of the subject. Note that all these computations were made feasible by programming them as functions for use with MatLab [M].

First set up a Chevalley basis $\{h_\alpha, x_\beta \mid \alpha \in \Delta, \beta \in \Phi\}$ for $\mathfrak{g}$, where $[x_\beta, x_{-\beta}] = h_\beta$ and the other structure constants are obtained from [GS].

Given the weighted Dynkin Diagram of $\mathcal{O}$, reconstruct the neutral element $H \in \mathfrak{g}$ of a Lie triple classifying $\mathcal{O}$, uniquely chosen to lie in the dominant chamber of the maximal split torus. In fact, H will be a non-negative integral linear combination of the root h_α's. Then identify $\mathfrak{g}[2]$, the 2-weight space of H acting on $\mathfrak{g}$; this contains a dense subset of representatives for $\mathcal{O}$. Now the Bala-Carter label for $\mathcal{O}$ identifies the semisimple part of a Levi subgroup $\mathfrak{l}$ of $\mathfrak{g}$, and $\mathfrak{g}[2]$ contains the span of its simple roots. The orbit $\mathcal{O}$ is the saturation of a distinguished orbit of $\mathfrak{l}$ (also identified by the Bala-Carter label). We can choose a "standard" representative of this orbit using techniques of [CMcG, Ch.5]; this is E, the desired representative of a split orbit in $\mathcal{O}(k)$. Finally, one can deduce the value of $F \in \mathfrak{g}[-2]$ such that $\phi = span\{E, H, F\}$ forms an $\mathfrak{sl}(2,k)$-subalgebra of $\mathfrak{g}$, which in turn classifies the rational orbit through E.

Now compute the subalgebra $\mathfrak{g}^\phi = \{Z \in \mathfrak{g}[0] \mid [E,Z] = 0\}$. (This subalgebra is called $\underline{\mathrm{C}}$ in [E]. Elkington's tables contain some errors, however; among them: F_4: orbit $\widetilde{A_1}$; E_6: orbit $A_2 + 2A_1$; E_7: orbits $(3A_1)'$, $2A_2 + A_1$, and $(A_5)'$.). It is split over k, and we can easily decompose the subspace $\mathfrak{g}[-1]$ into irreducibles under $\mathfrak{g}^\phi$. One can often use this to deduce the structure of (the algebraic identity component of) the corresponding group G^ϕ. For example, if all positive weights occur in representations of the Lie algebra, then $\mathbb{G}^\phi$ must be simply connected. Where this reasoning does not apply, we can find other simple arguments. (We elaborate on this step in Sections 6 through 9.)

At this point in the computation, we have all the data necessary to determine the admissibility of the orbit as per the definition in Section 2. For the remainder of this Section, we describe some criteria for admissibility in our particular settings.

The Steinberg Cocycle. To simplify notation, let $\mathbb{G}$ temporarily denote any simply connected simple Chevalley group (split over k), and G its set of k-points. (In practice, $\mathbb{G}$ will be $\mathbb{G}^\phi$ or some subgroup thereof.) Choose a split Cartan subgroup $\mathcal{H}$.

Steinberg cocycles were defined by Moore in [M2, Ch.III] as explicit representatives of the cohomology classes in $H^2(k, \mathbb{G})$. In the case of the metaplectic covering group of a symplectic group $Sp(V)$, the Steinberg cocycle is defined as follows ([R], [LV, appendix]).

The usual cocycle c_l of the metaplectic cover — obtained by restriction of the cocycle of the $\mathbb{C}^1$-cover of the symplectic group — is defined relative to a choice of lagrangian (*i.e.* maximally isotropic) subspace l of V (see [LV, A.9]). For our purposes, it suffices to note that for $g, g' \in Sp(V)$, $c_l(g, g') = 1$ if either g or g' preserves l. In general, $c_l(g, g')$ takes values in the set of eighth roots of unity $\mu_8 \subset \mathbb{C}$.

Let γ denote the *Weil index* — a unitary character of the group k^*/k^{*2}. It satifies the integral equation ([W, §14])

$$\frac{\gamma(1)\gamma(ab)}{\gamma(a)\gamma(b)} = (a/b)_k$$

where $(\cdot/\cdot)_k$ denotes the (2-)Hilbert symbol of k (see, for example, [Neu, III §5]). Define $det(\alpha_{l,g\cdot l})$ to be the determinant of the linear transformation of l to $g \cdot l$ as in [LV, A.13].

Then one can normalize the usual cocycle c_l via the formula

$$S(g,h) = \frac{c_l(g,h)t(gh)}{t(g)t(h)} = \pm 1, \tag{3.1}$$

where

$$t(g) = \gamma(1)^{1-dim(l)+dim(g\cdot l\cap l)}\gamma(det(\alpha_{g\cdot l,l}))^{-1}.$$

This is the Steinberg cocycle.

Moore proved [M2, III,Lemma 8.4]

Lemma 3.1 (Moore). *If α is a long root, and $\mathcal{H}_\alpha$ is its corresponding one-parameter subgroup in $\mathcal{H}$, then any Steinberg cocycle is determined by its restriction to $\mathcal{H}_\alpha$.*

In particular, this implies that the covering of G induced by such a cocycle splits if and only if its restriction to $\mathcal{H}_\alpha$ is trivial.

Note that $G = SL(2,k)$ is a group to which this theory applies, and that Moore's lemma reduces the question of splitting for general G in this class to the splitting over a root $SL(2,k)$ subgroup. Bearing in mind the admissibility calculation we wish to carry out (*cf* (2.1)), let us compute the Steinberg cocycle arising through the special case of k-representations V of $SL(2,k)$ such that $SL(2,k) \to Sp(V,k)$.

Theorem 3.2. *Suppose V is an even-dimensional representation of $SL(2,k)$ affording an invariant symplectic form. Then the metaplectic cover of $SL(2,k)$ induced by the map $\varphi\colon SL(2,k) \to Sp(V,k)$ is trivial exactly when the total number of subrepresentations of V having dimension of the form $4n+2$ (for some n) is even.*

Proof. Since the symplectic form on V is $SL(2,k)$-invariant, the decomposition of V into irreducible representations under $SL(2,k)$ preserves orthogonality. Thus, non-isomorphic irreducible representations are orthogonal and the isotypic subspaces of V are (nondegenerate) symplectic subspaces. Denote the unique n-dimensional irreducible representation of $SL(2,k)$ by V^n. Note that it admits a unique nondegenerate invariant symplectic form when n is even, and is isotropic with respect to any invariant symplectic form otherwise.

Let us first consider the case of $V = V^{2n}$. Denote by $h(t)$ the image of the matrix

$$\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$$

in $Sp(V,k)$; then

$$h(t) = diag(t^{2n-1}, t^{2n-3}, \ldots, t^1, t^{-2n+1}, t^{-2n+3}, \ldots, t^{-1}).$$

This is the semisimple element to which we should restrict the Steinberg cocycle to determine whether or not the cover of $SL(2,k)$ induced by the metaplectic cover of $Sp(V,k)$ splits.

Note that $h(t)$ preserves the Lagrangian l spanned by the weight vectors of positive weight, and so $c_l(h(t), h(s)) = 1$ for all $s, t \in k$. We compute $det(\alpha_{h(t)\cdot l, l}) = det(h(t)|_l) = t^{n^2}$; and so (3.1) simplifies to

$$S(h(t), h(s)) = \frac{\gamma(s^{n^2})\gamma(t^{n^2})}{\gamma(1)\gamma((st)^{n^2})} = (s^{n^2}/t^{n^2})_k = (s/t)_k^{n^4}. \tag{3.2}$$

Hence S is trivial (for all s,t) if and only if n is even.

Now suppose V contains an odd-dimensional irreducible subrepresentation V^{2n+1}. Since each isotypic space under $SL(2)$ must be a symplectic space, it follows that V^{2n+1} occurs with even multiplicity. We may choose the Lagrangian l to be a direct sum of half of these irreducibles. Since $h(t)|_{V^{2n+1}} = diag(t^{2n}, t^{2n-2}, \ldots, t^{-2n})$, it follows that $det(h(t)|_{V^{2n+1}}) = 1$ for all $t \in k$. Consequently, the Steinberg cocycle takes value identically 1 on the isotypic space of any odd-dimensional irreducible.

Finally, consider the general case, where we have a decomposition of V into irreducibles under $SL(2,k)$ of the form $V = \bigoplus_{n'=1}^{N} m_{n'} V^{n'}$. We deduce from the above that the Steinberg cocycle will be $S(h(t), h(s)) = (s/t)_k^{M^2}$, where $M = \sum_n m_{2n} n^2$. It is thus trivial exactly when M is even, as we were required to show. □

Of course, in what follows, G^ϕ need not be simply connected or split over k. Let us recall a splitting theorem from [N2] which is sometimes applicable in such cases.

Theorem 3.3. *(a) If G^ϕ preserves a lagrangian subspace of $\mathfrak{g}[-1]$, then the corresponding metaplectic cover splits over G_e^ϕ. (b) If G^ϕ preserves complementary lagrangians and there exists a G^ϕ-invariant intertwining operator between them, then the cover splits over all of G^ϕ.*

4. Occurence of other Rational Orbits

Using the data generated in the course of determining the admissibility of the split orbits (see Section 3), let us bound the number of rational orbits for each algebraic orbit using Galois cohomology.

Proposition 4.1. *Let $\mathbb{G}$ be a simply connected exceptional algebraic group, E a nilpotent k-rational element of its Lie algebra, and ϕ an $\mathfrak{sl}(2,k)$-subalgebra of $\mathfrak{g}$. Then the number of k-rational orbits of $G = \mathbb{G}(k)$ in the k-points of the algebraic orbit $\mathbb{G} \cdot E$ is equal to the order of $H^1(k, \mathbb{G}^E)$. It is bounded above by the order of $H^1(k, \mathbb{G}^\phi)$.*

Proof. Note first that by [S, III.4.4], the number of rational p-adic orbits in the k-points of a given algebraic orbit $\mathcal{O} = \mathbb{G} \cdot E$ is finite. To compute this number, we begin with the short exact sequence of sets $1 \to \mathbb{G}^E \to \mathbb{G} \to \mathbb{G}/\mathbb{G}^E \to 1$. Take Galois cohomology [S, I-64,I-65], to obtain the long exact sequence

$$1 \to \mathbb{G}^E(k) \to \mathbb{G}(k) \to (\mathbb{G}/\mathbb{G}^E)(k) \to H^1(k, \mathbb{G}^E) \to H^1(k, \mathbb{G}).$$

The k-rational orbits are the $G = \mathbb{G}(k)$ orbits on $(\mathbb{G}/\mathbb{G}^E)(k)$. Hence their number is measured by the quotient $(\mathbb{G}/\mathbb{G}^E)(k)/G$, which is in bijection with

$$\ker(\alpha\colon H^1(k, \mathbb{G}^E) \to H^1(k, \mathbb{G})).$$

In our setting, $\mathbb{G}$ is simply connected linear algebraic group, and hence by [KnI, KnII], we have that $H^s(k, G) = 0$ for all $s \geq 1$. In particular, the number of rational orbits is given by the order of $H^1(k, \mathbb{G}^E)$.

We can make a further reduction. Since E is nilpotent, $\mathbb{G}^E$ is the semidirect product of its reductive part $\mathbb{G}^\phi$ and a unipotent part U^E. The first cohomology group of the unipotent group U^E is trivial by [S, III.2.1], so the short exact sequence $1 \to U^E \to \mathbb{G}^E \to \mathbb{G}^\phi \to 1$ yields the following exact sequence in cohomology

$$0 \to H^1(k, \mathbb{G}^E) \to H^1(k, \mathbb{G}^\phi).$$

Thus, the number of rational orbits in $\mathbb{G}^E(k)$ is bounded above by the number of elements in $H^1(k, \mathbb{G}^\phi)$. □

In practice, it is far easier to determine $H^1(k, \mathbb{G}^\phi)$ than $H^1(k, \mathbb{G}^E)$, as $\mathbb{G}^\phi$ is often a semisimple group. It may not be connected; however, its algebraic component group is well-known (see, for example, [CMcG, Ch.8.4]). We have

$$\mathbb{G}^\phi / \mathbb{G}^\phi_0 \simeq \pi_1(\mathbb{G} \cdot E),$$

where $\mathbb{G}^\phi_0$ is the algebraic connected component of the identity (and known to us from the calculations of the preceding section), and $\pi_1(\mathbb{G} \cdot E)$ is the $\mathbb{G}$-equivariant fundamental group of the orbit.

Corollary 4.2. *In the setting of Proposition 4.1, suppose that $\mathbb{G}^\phi_0$ is a simply connected algebraic group. Then $H^1(k, \mathbb{G}^\phi)$ injects into $H^1(k, \pi_1(\mathbb{G} \cdot E))$.*

Proof. The short exact sequence $1 \to \mathbb{G}^\phi_0 \to \mathbb{G}^\phi \to \pi_1(\mathbb{G} \cdot E) \to 1$ gives rise to the long exact sequence

$$\cdots \to H^1(k, \mathbb{G}^\phi_0) \to H^1(k, \mathbb{G}^\phi) \to H^1(k, \pi_1(\mathbb{G} \cdot E)) \tag{4.1}$$

in cohomology. By [KnI], $H^1(k, \mathbb{G}^\phi_0) = 0$ when $\mathbb{G}^\phi_0$ is simply connected. □

For those cases for which $\mathbb{G}^\phi_0$ is not simply connected, we note the following immediate lemma from [Kn, Ch. IV].

Lemma 4.3 (Kneser)**.** *If $\mathbb{G}$ is a semisimple connected algebraic group defined over k, and $\tilde{\mathbb{G}}$ is its simply connected covering group, with kernel F, then $H^1(k, \mathbb{G}) \simeq H^2(k, F)$.*

We have from [S]:

- If k contains all nth roots of unity, then $H^1(k, \mu_n) = k^*/k^{*n}$, and $H^2(k, \mu_n) = \mathbb{Z}/n\mathbb{Z}$.
- If $\mathbb{G}_m$ denotes the multiplicative group of the field, then $H^1(k, \mathbb{G}_m) = 0$.

Thus, Proposition 4.1 gives an effective means of computing an upper bound on the number of rational orbits in the k-points of a given algebraic orbit. We can then use the following more direct method to determine if this bound on the number of rational orbits is optimal.

By a Theorem of Mal'cev [CMcG, Thm 3.4.12], we know that the stabilizer of H in $\mathbb{G}$ acts transitively on the dense subset $\mathcal{P}$ of orbit representatives of $\mathcal{O}$ in $\mathfrak{g}[2]$. Now $\mathcal{P}(k)$ will decompose into one or more orbits under $\mathbb{G}^H(k) = G^H$.

Proposition 4.4. *The rational orbits of G^H acting on $\mathcal{P}$ are in one-to-one correspondence with the rational orbits of G acting on $\mathcal{O}(k) \subset \mathfrak{g}$.*

Proof. Given a rational Lie triple $\phi = \{E, H, F\}$ representing the orbit $\mathcal{O}$ (as in Section 3), suppose $\{E', H', F'\}$ is another rational Lie triple ϕ' representing an orbit in $\mathcal{O}$. Then ϕ and ϕ' are conjugate under an element $g \in \mathbb{G}$. In particular, their neutral elements H and H' are both diagonalizable over k, and hence conjugate under $G = \mathbb{G}(k)$. WLOG assume $H = H'$, so that E and E' both lie in $\mathcal{P}(k)$. Whence ϕ and ϕ' are conjugate under G if and only if E and E' are conjugate under G^H. □

The Lie algebra of G^H is simply $\mathfrak{g}[0]$, the zero-weight space of $\mathfrak{g}$ under H. It contains G^ϕ as a subgroup which fixes E. This lemma thus reduces the problem of determining the rational orbits of an exceptional Lie group acting on its Lie algebra down to determining the rational orbits of a much smaller group acting on a vector space. This is often feasible (see Appendix A), but nonetheless somewhat unsatisfactory — we have "reduced" from a simple group and a specific irreducible representation to a reductive group and a (generally) non-irreducible representation.

Remark. In Sections 6 to 8, we record the fundamental group of each orbit and the number of real rational classes (obtained from [CMcG]). We computed a bound on the number of p-adic rational classes using Proposition 4.1, and then verified it on a case-by-case basis in Appendix A using Proposition 4.4.

5. Possible forms of non-split rational orbits

For those algebraic orbits admitting more than one k-rational orbit, we proceed to decide admissibility as follows. If E' is a representative of such an orbit, then the $\mathfrak{sl}(2,k)$-subalgebra ϕ' it defines must be conjugate under $\mathbb{G}$ to ϕ. Hence in particular, $G^{\phi'}$ must be conjugate to G^ϕ under an element of $\mathbb{G}$, and in fact must be a (possibly different) k-form of $\mathbb{G}^\phi$. We can further constrain the possibilities by noting that, upon tensoring over k with an algebraic closure of k, the action of $G^{\phi'}$ on the corresponding $\mathfrak{g}\,[-1]$ will be equivalent to the split orbit case. Let us explore this constraint now.

In the following let $\mathbb{G}$ be a reductive algebraic group acting on a finite-dimensional vector space V, such that the group of k-points of $\mathbb{G}$ act rationally on the vector space of k-points of V. (In application, this $\mathbb{G}$ will be $\mathbb{G}^{\phi'}$.)

Lemma 5.1. *Suppose $\mathbb{G}$ acts irreducibly on V. Then $\mathbb{G}(k)$ acts irreducibly on $V(k)$.*

Proof. Suppose W is an invariant proper k-subspace of $V(k)$. Then the subspace $W' = W \otimes_k \overline{k}$ is an invariant proper subspace of the action of $\mathbb{G}(k)$ on V. $\mathbb{G}(k)$ is Zariski-dense in $\mathbb{G}$. The set of all $g \in \mathbb{G}$ that preserve the subspace W' is a Zariski-closed set (since the action is algebraic) and contains $\mathbb{G}(k)$, hence is all of $\mathbb{G}$. Thus W' is an invariant subspace, and so much be $\{0\}$, by irreducibility. Thus there are no non-trivial invariant proper k-subspaces of $V(k)$, and the action is irreducible. □

The converse is not true, in general; an irreducible k-rational representation of a non-split k-form of $\mathbb{G}$ may decompose, upon passage to the algebraic (or even separable) closure, into a direct sum of irreducibles of $\mathbb{G}$. Let k_s denote the separable closure of k. More precisely, we have the following results of Tits [T2, Théorème 3.3, Théorème 7.2 et Lemme 7.4].

Theorem 5.2 (Tits). *Let $\mathbb{G}$ be a reductive group defined over k.*

(i) *Let λ be a dominant integral weight of $\mathbb{G}$ and let k_λ be the extension field of k corresponding to the stabilizer subgroup of λ in $Gal(k_s/k)$. Then λ gives rise to an absolutely irreducible representation ρ_λ of $\mathbb{G}$ over some central simple division algebra D_λ over k_λ.*
(ii) *Each k-rational irreducible k-representation of $\mathbb{G}(k)$ is isomorphic to some ρ^k_λ, where ρ^k_λ is obtained from ρ_λ by restriction of scalars (from D_λ and k_λ to k).*
(iii) *Let d_λ denote the degree of D_λ over k_λ (i.e. the square root of the index), and suppose the orbit of λ under the Galois group $Gal(k_s/k_\lambda)$ is $\{\lambda_1, \lambda_2, \dots \lambda_n\}$. Then, upon passage to k_s, ρ^k_λ decomposes into a direct sum of d_λ copies of ρ_{λ_1}, d_λ copies of ρ_{λ_2}, and so on.*

The orbit of λ under the Galois group can be read from the Tits diagram (see [T1]) of $\mathbb{G}$. Thus, whenever the decomposition of a (finite-dimensional) vector space V into irreducibles under $\mathbb{G}$ is known, we can apply (iii) of Theorem 5.2 to deduce whether or not this arises from a decomposition of $V(k)$ into irreducibles under $\mathbb{G}(k)$. In many cases, this allows us to exclude from possibility various groups $\mathbb{G}^{\phi'}(k)$ as occuring in other rational orbits. Where nonsplit orbits may occur, we compute their admissibility on a case-by-case basis (see Appendix B).

6. $G_{2,2}^0$

$\alpha \Rrightarrow \beta$

Bala-Carter Label: A_1; **Weighted Dynkin Diagram:** $1 \Rrightarrow 0$ **(not special)**
Lie Triple ϕ: $x_{2\alpha+3\beta}$, $2H_\alpha + H_\beta$, $x_{-2\alpha-3\beta}$
$\mathfrak{g}^\phi = \text{span}\ \{\ X_\beta,\ X_{-\beta},\ H_\beta\ \} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}$**[-1]** $=\ V^4$
We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: $\widetilde{A_1}$; **Weighted Dynkin Diagram:** $0 \Rrightarrow 1$ **(not special)**
Lie Triple ϕ: $x_{\alpha+2\beta}$, $3H_\alpha + 2H_\beta$, $x_{-\alpha-2\beta}$
$\mathfrak{g}^\phi = \text{span}\ \{\ X_\alpha,\ X_{-\alpha},\ H_\alpha\ \} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}$**[-1]** $=\ V^2$
We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

7. $F_{4,4}^0$

$\alpha — \beta \Rightarrow \gamma — \delta$

Bala-Carter Label: A_1; **Weighted Dynkin Diagram:** $1 — 0 \Rightarrow 0 — 0$ **(not special)**
Lie Triple ϕ: $x_{2\alpha+3\beta+4\gamma+2\delta}$, $2h_\alpha + 3h_\beta + 2h_\gamma + h_\delta$, $x_{-2\alpha-3\beta-4\gamma-2\delta}$
$\mathfrak{g}^\phi = \text{span}\ \{$ the subalgebra with simple roots $\beta,\ \gamma,\ \delta\ \} \simeq sp(6,k)$
$\mathfrak{g}$**[-1]** $=\ V^{14}$ (highest weight $L_1 + L_2 + L_3$ (notation of Fulton [FH, sec 17])
We have $G_0^\phi = Sp(6,k)$. The $SL(2,k)$-subgroup associated with the long root of $Sp(6,k)$ decomposes $\mathfrak{g}[-1]$ into five 2-dimensional irreducibles, and four copies of the trivial representation. Apply Lemma 3.1 and Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: $\widetilde{A_1}$; **Weighted Dynkin Diagram:** $0 — 0 \Rightarrow 0 — 1$ **(special)**
Lie Triple ϕ: $x_{\alpha+2\beta+3\gamma+2\delta}$, $2h_\alpha + 4h_\beta + 3h_\gamma + 2h_\delta$, $x_{-\alpha-2\beta-3\gamma-2\delta}$
$\mathfrak{g}^\phi = \text{span}\ \{\ x_\alpha,\ x_\beta,\ x_{\alpha+\beta},\ x_{\beta+2\gamma},\ x_{\alpha+\beta+2\gamma},\ x_{\alpha+2\beta+2\gamma},\ h_\alpha,\ h_\beta,\ h_\gamma,$ and the corresponding negative root spaces $\} \simeq \mathfrak{sl}(4,k)$
$\mathfrak{g}$**[-1]** $=\ V_{std} \oplus V_{std}^*$
We have $G_0^\phi = SL(4,k)$. Under any root $SL(2,k)$, $\mathfrak{g}[-1]$ decomposes as $2V^2 \oplus 4V^1$; apply Theorem 3.2. The orbit is **admissible**.
Fundamental Group: S_2; **#$\mathbb{R}$-orbits:** 2; **#p-adic orbits:** $|k^*/k^{*2}|$ (see Appendix A.1)
Note on rational classes: We see from Appendix B.3 that the only other form of G^ϕ could take is of a special unitary group and that the cover of G_0^ϕ would split in that case as well. All rational orbits are admissible.

Bala-Carter Label: $A_1 + \widetilde{A_1}$; **Weighted Dynkin Diagram:** $0 — 1 \Rightarrow 0 — 0$ **(special)**
Lie Triple ϕ: $x_{\alpha+2\beta+2\gamma+2\delta} + x_{\alpha+2\beta+3\gamma+\delta}$, $3h_\alpha + 6h_\beta + 4h_\gamma + 2h_\delta$, $x_{-\alpha-2\beta-2\gamma-2\delta} + x_{-\alpha-2\beta-3\gamma-\delta}$
$\mathfrak{g}^\phi = \text{span}\ \{\ x_\alpha,\ x_{-\alpha},\ h_\alpha;\ -2x_\gamma + 2x_\delta,\ -x_{-\gamma} + x_{-\delta},\ 2h_\gamma + 2h_\delta\ \} \simeq \mathfrak{sl}(2,k) \oplus \mathfrak{sl}(2,k)$
$\mathfrak{g}$**[-1]** $=\ 6V^2$ under first $\mathfrak{sl}(2,k)$; $\mathfrak{g}[-1] = 2V^1 \oplus 2V^5$ under second $\mathfrak{sl}(2,k)$

We have that the first $\mathfrak{sl}(2,k)$ corresponds necessarily to an $SL(2,k)$, since it has an even-dimensional irreducible representation. On the other hand, the second $\mathfrak{sl}(2,k)$ embeds into $\mathfrak{g}$ as a 3-dimensional representation (ie. the image lies irreducibly in the $\mathfrak{sl}(3,k)$ corresponding to the roots γ and δ), and hence necessarily lifts to $PGL(2,k)$ as a group. The two subalgebras commute; hence $G_0^\phi(k) = SL(2,k) \times PGL(2,k)$. Finally, note that the metaplectic cover splits over each group individually (using Theorem 3.2). The orbit is **admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 2; **#p-adic orbits:** 2 (see Appendix A.2)
Note on rational classes: By Appendix B.1 we see that the metaplectic cover will split over all other rational forms, and hence that all rational orbits are admissible.

Bala-Carter Label: $A_2 + \widetilde{A_1}$; **Weighted Dynkin Diagram:** $0 — 0 \Longrightarrow 1 — 0$ (**not special**)
Lie Triple ϕ: $x_{\alpha+\beta+2\gamma} + x_{\beta+2\gamma+2\delta} + x_{\alpha+2\beta+2\gamma+\delta}$, $4h_\alpha + 8h_\beta + 6h_\gamma + 3h_\delta$, $2x_{-\alpha-\beta-2\gamma} + 2x_{-\beta-2\gamma-2\delta} + x_{-\alpha-2\beta-2\gamma-\delta}$
$\mathfrak{g}^\phi = \text{span}\ \{\ -2x_\beta - x_{-\delta} + x_{-\alpha-\beta},\ x_\delta + 2x_{\alpha+\beta} - x_{-\beta},\ -2h_\alpha - h_\delta\ \} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}$[**-1**] $=\ V^2 \oplus V^4$
We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: B_2; **Weighted Dynkin Diagram:** $2 — 0 \Longrightarrow 0 — 1$ (**not special**)
Lie Triple ϕ: $x_{\alpha+\beta+\gamma} + x_{\beta+2\gamma+2\delta}$, $6h_\alpha + 10h_\beta + 7h_\gamma + 4h_\delta$, $3x_{-\alpha-\beta-\gamma} + 4x_{-\beta-2\gamma-2\delta}$
$\mathfrak{g}^\phi = \text{span}\ \{\ x_\beta,\ x_{-\beta},\ h_\beta,\ x_{\beta+2\gamma},\ x_{-\beta-2\gamma},\ h_\beta + h_\gamma\ \} \simeq \mathfrak{sl}(2,k) \oplus \mathfrak{sl}(2,k)$
$\mathfrak{g}$[**-1**] $=\ 2V^1 \oplus V^2$ under each $\mathfrak{sl}(2,k)$, with the two 2-dimensional subrepresentations complementary
We have that each $\mathfrak{sl}(2,k)$ corresponds to an $SL(2,k)$ at the group level, and their Lie algebras commute. Using Chevalley bases ([Ca, page97], we can argue directly that the two groups can admit no intersection. Hence $G_0^\phi(k) \simeq SL(2,k) \times SL(2,k)$. The metaplectic cover over each $SL(2,k)$ fails to split by Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: S_2; **#$\mathbb{R}$-orbits:** 2; **#p-adic orbits:** $|k^*/k^{*2}|$ (see Appendix A.3)
Note on rational classes: Each non-split rational orbit has corresponding group $G^\phi \simeq SL(2,E)$, for E varying over all the nontrivial quadratic extensions of k. As shown in Appendix B.2, the non-split rational orbits are thus not all admissible, and this variability depends on k.

Bala-Carter Label: $\widetilde{A_2} + A_1$; **Weighted Dynkin Diagram:** $0 — 1 \Longrightarrow 0 — 1$ (**not special**)
Lie Triple ϕ: $x_{\beta+2\gamma+\delta} + x_{\alpha+\beta+\gamma+\delta} + x_{\alpha+2\beta+2\gamma}$, $5h_\alpha + 10h_\beta + 7h_\gamma + 4h_\delta$, $2x_{-\beta-2\gamma-\delta} + 2x_{-\alpha-\beta-\gamma-\delta} + x_{-\alpha-2\beta-2\gamma}$
$\mathfrak{g}^\phi = \text{span}\ \{\ x_\alpha + x_\gamma,\ x_{-\alpha} + x_{-\gamma},\ h_\alpha + h_\gamma\ \} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}$[**-1**] $=\ 2V^2 \oplus V^4$
We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: $C_3(a_1)$; **Weighted Dynkin Diagram:** $1 — 0 \Longrightarrow 1 — 0$ (**not special**)
Lie Triple ϕ: $x_{\beta+2\gamma} + x_{\alpha+\beta+\gamma+\delta} + x_{\beta+2\gamma+2\delta}$, $6h_\alpha + 11h_\beta + 8h_\gamma + 4h_\delta$, $4x_{-\beta-2\gamma} + 3x_{-\alpha-\beta-\gamma-\delta} + x_{-\beta-2\gamma-2\delta}$
$\mathfrak{g}^\phi = \text{span}\ \{\ x_\beta,\ x_{-\beta},\ h_\beta\ \} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}$[**-1**] $=\ 3V^2$
We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: S_2; **#$\mathbb{R}$-orbits:** 2; **#p-adic orbits:** $|k^*/k^{*2}|$ (see Appendix A.4)
Note on rational classes: By Appendix B.1, we conclude that all rational orbits are split, and hence none are admissible.

Bala-Carter Label: C_3; **Weighted Dynkin Diagram:** $1 — 0 \Longrightarrow 1 — 2$ **(special)**
Lie Triple ϕ**:** $x_\delta + x_{\alpha+\beta+\gamma} + x_{\beta+2\gamma}$, $10h_\alpha + 19h_\beta + 14h_\gamma + 8h_\delta$, $8x_{-\delta} + 5x_{-\alpha-\beta-\gamma} + 9x_{-\beta-2\gamma}$
$\mathfrak{g}^\phi$ = span { x_β, $x_{-\beta}$, h_β } $\simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}$[-1] = $2V^2$
We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

8. ${}^1E_{6,6}^0$

$$\begin{array}{ccccccccc} & & & & \alpha_2 & & & & \\ & & & & | & & & & \\ \alpha_1 & - & \alpha_3 & - & \alpha_4 & - & \alpha_5 & - & \alpha_6 \end{array}$$

Bala-Carter Label: A_1; **Weighted Dynkin Diagram:** $\begin{array}{c} 1 \\ 0-0-\overset{|}{0}-0-0 \end{array}$ **(special)**
Lie Triple ϕ**:** $x_{\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6}$, $h_{\alpha_1}+2h_{\alpha_2}+2h_{\alpha_3}+3h_{\alpha_4}+2h_{\alpha_5}+h_{\alpha_6}$, $x_{-\alpha_1-2\alpha_2-2\alpha_3-3\alpha_4-2\alpha_5-\alpha_6}$

$\mathfrak{g}^\phi$ = span { the subalgebra with simple roots α_1, α_3, α_4, α_5, α_6 } $\simeq \mathfrak{sl}(6,k)$
$\mathfrak{g}$[-1] = V^{20}
We have $G_0^\phi = SL(6,k)$ because E_6 admits a faithful 27-dimensional representation, which must decompose under G^ϕ with a 6-dimensional irreducible factor. To determine admissibility, apply Lemma 3.1. Each $SL(2,k)$ root subgroup decomposes $\mathfrak{g}[-1]$ into $6V^2 \oplus 8V^1$; apply Theorem 3.2. The orbit is **admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: $2A_1$; **Weighted Dynkin Diagram:** $\begin{array}{c} 0 \\ 1-0-\overset{|}{0}-0-1 \end{array}$ **(special)**
Lie Triple ϕ**:** $x_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6} + x_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6}$, $2h_{\alpha_1}+2h_{\alpha_2}+3h_{\alpha_3}+4h_{\alpha_4}+3h_{\alpha_5}+2h_{\alpha_6}$, $x_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5-\alpha_6} + x_{-\alpha_1-\alpha_2-\alpha_3-2\alpha_4-2\alpha_5-\alpha_6}$
$\mathfrak{g}^\phi$ = span { $h_{\alpha_1} + h_{\alpha_3} - h_{\alpha_6}$ and the subalgebra with simple root vectors x_{α_2}, x_{α_4}, $x_{\alpha_3} - x_{\alpha_5}$ } $\simeq$ V + so(7,k)
$\mathfrak{g}$[-1] = $2V_{spin}$ under the split $so(7,k)$
We have $G^\phi(k) = Spin(7,k) \ltimes k^*$ since the Lie algebra admits the 8-dimensional spin representation. We can apply Lemma 3.1; since their are two copies of V_{spin}, it follows immediately from Theorem 3.2 that the metaplectic cover of $Spin(7,k)$ will split. The metaplectic cover of k^* splits by Theorem 3.3. The orbit is **admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: $3A_1$; **Weighted Dynkin Diagram:** $\begin{array}{c} 0 \\ 0-0-\overset{|}{1}-0-0 \end{array}$ **(not special)**
Lie Triple ϕ**:** $x_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6} + x_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5} + x_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6}$, $2h_{\alpha_1} + 3h_{\alpha_2} + 4h_{\alpha_3} + 6h_{\alpha_4} + 4h_{\alpha_5} + 2h_{\alpha_6}$, $x_{-\alpha_1-\alpha_2-\alpha_3-2\alpha_4-\alpha_5-\alpha_6} + x_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5} + x_{-\alpha_2-\alpha_3-2\alpha_4-2\alpha_5-\alpha_6}$
$\mathfrak{g}^\phi$ = span { $\mathfrak{sl}(2,k)(\alpha_2)$, $\mathfrak{sl}(3,k)$ with simple root vectors $-x_{\alpha_1}+x_{\alpha_5}$ and $x_{\alpha_3}+x_{\alpha_6}$ } $\simeq \mathfrak{sl}(2,k)\oplus\mathfrak{sl}(3,k)$
$\mathfrak{g}$[-1] = $9V^2$ under the $\mathfrak{sl}(2,k)$; $\mathfrak{g}[-1] = 2V^1 \oplus 2V^8$ under the $\mathfrak{sl}(3,k)$

There is a decomposition of $\mathfrak{g}[-1]$ into 9 symplectic 2-dimensional irreducible representations under this $\mathfrak{sl}(2,k)$. Hence the cover fails to split over the $\mathfrak{sl}(2,k)$ piece, even though the decomposition into lagrangian subspaces under $\mathfrak{sl}(3,k)$ implies that the cover does split over the $\mathfrak{sl}(3,k)$ piece.
We have that the $\mathfrak{sl}(2,k)$ lifts to a copy of $SL(2,k)$ at the group level. The $\mathfrak{sl}(3,k)$ embeds into $\mathfrak{sl}(3,k)(\alpha_1,\alpha_3) \oplus \mathfrak{sl}(3,k)(\alpha_5,\alpha_6)$, and each of these $\mathfrak{sl}(3,k)$-subalgebras admit 3-dimensional irreducible representations. Thus our $\mathfrak{sl}(3,k)$ lifts to a copy of $SL(3,k)$ at the group level. Finally, we note that the roots of the $\mathfrak{sl}(2,k)$ are orthogonal to those of the $\mathfrak{sl}(3,k)$, and the groups can admit no intersection.

Thus $G^\phi = SL(2,k) \times SL(3,k)$. The metaplectic cover does not split over the $SL(2,k)$ factor by Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: 1; **$\#\mathbb{R}$-orbits:** 1; **$\#p$-adic orbits:** 1

Bala-Carter Label: $A_2 + A_1$; **Weighted Dynkin Diagram:** $\begin{array}{c} 1 \\ 1-0-\dot{0}-0-1 \end{array}$ **(special)**

Lie Triple ϕ: $x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} + x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6} + x_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6}$, $3h_{\alpha_1} + 4h_{\alpha_2} + 5h_{\alpha_3} + 7h_{\alpha_4} + 5h_{\alpha_5} + 3h_{\alpha_6}$, $2x_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5} + x_{-\alpha_1-\alpha_3-\alpha_4-\alpha_5-\alpha_6} + 2x_{-\alpha_2-\alpha_3-2\alpha_4-\alpha_5-\alpha_6}$
$\mathfrak{g}^\phi$ = span { $h_{\alpha_1} + 2h_{\alpha_4} - h_{\alpha_6}$, and the subalgebra with simple root vectors x_{α_3} and $x_{\alpha_4+\alpha_5}$ } $\simeq V \oplus \mathfrak{sl}(3,k)$
$\mathfrak{g}$[-1] = $2V^1 \oplus 4V^3$ under the $\mathfrak{sl}(3,k)$
We have $G^\phi = SL(3,k) \ltimes k^*$, since the $\mathfrak{sl}(3,k)$ admits a 3-dimensional irreducible representation, and the two subgroups do not intersect. Apply Lemma 3.1 and Theorem 3.2. The orbit is **admissible**.
Fundamental Group: 1; **$\#\mathbb{R}$-orbits:** 1; **$\#p$-adic orbits:** 1

Bala-Carter Label: $A_2 + 2A_1$; **Weighted Dynkin Diagram:** $\begin{array}{c} 0 \\ 0-1-\dot{0}-1-0 \end{array}$ **(special)**

Lie Triple ϕ: $x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5} + x_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} + x_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} + x_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6}$, $3h_{\alpha_1} + 4h_{\alpha_2} + 6h_{\alpha_3} + 8h_{\alpha_4} + 6h_{\alpha_5} + 3h_{\alpha_6}$, $2x_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5} + 2x_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6} + x_{-\alpha_2-\alpha_3-2\alpha_4-\alpha_5} + x_{-\alpha_1-\alpha_2-\alpha_3-2\alpha_4-\alpha_5-\alpha_6}$
$\mathfrak{g}^\phi$ = span { $-2x_{\alpha_4} - x_{\alpha_6} + x_{-\alpha_1} + x_{-\alpha_2-\alpha_4}$, $-x_{\alpha_1} - 2x_{\alpha_2+\alpha_4} + x_{-\alpha_4} + x_{-\alpha_6}$, $h_{\alpha_1} + 2h_{\alpha_2} + h_{\alpha_6}$, $h_{\alpha_2} - h_{\alpha_3} + h_{\alpha_5}$ } $\simeq \mathfrak{sl}(2,k) \oplus V$
$\mathfrak{g}$[-1] = $2V^2 \oplus 2V^4$
We have $G^\phi = SL(2,k) \ltimes k^*$; apply Theorem 3.2. The orbit is **admissible**.
Fundamental Group: 1; **$\#\mathbb{R}$-orbits:** 1; **$\#p$-adic orbits:** 1

Bala-Carter Label: A_3; **Weighted Dynkin Diagram:** $\begin{array}{c} 2 \\ 1-0-\dot{0}-0-1 \end{array}$ **(special)**

Lie Triple ϕ: $x_{\alpha_2+\alpha_4} + x_{\alpha_2+\alpha_3+\alpha_4+\alpha_5} + x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6}$, $4h_{\alpha_1} + 6h_{\alpha_2} + 7h_{\alpha_3} + 10h_{\alpha_4} + 7h_{\alpha_5} + 4h_{\alpha_6}$, $3x_{-\alpha_2-\alpha_4} + 3x_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5} + 4x_{-\alpha_1-\alpha_3-\alpha_4-\alpha_5-\alpha_6}$
$\mathfrak{g}^\phi$ = span { $x_{\alpha_3+\alpha_4}$, $x_{\alpha_5} + x_{-\alpha_3}$, $-x_{\alpha_4} + x_{\alpha_3+\alpha_4+\alpha_5}$, $x_{\alpha_4+\alpha_5}$, $x_{-\alpha_3-\alpha_4}$, $x_{\alpha_3} + x_{-\alpha_5}$, $-x_{-\alpha_4} + x_{-\alpha_3-\alpha_4-\alpha_5}$, $x_{-\alpha_2-\alpha_4-\alpha_5-\alpha_6}$, $h_{\alpha_3} + h_{\alpha_4}$, $-h_{\alpha_3} + h_{\alpha_5}$, $h_{\alpha_1} + h_{\alpha_6}$ } $\simeq so(5,k) \oplus V$
$\mathfrak{g}$[-1] = $2V^4$ under the split $so(5,k)$
We have $G^\phi = Sp(4,k) \ltimes k^*$, by the existence of a 4-dimensional irreducible representation. Apply Lemma 3.1 and Theorem 3.2. The orbit is **admissible**.
Fundamental Group: 1; **$\#\mathbb{R}$-orbits:** 1; **$\#p$-adic orbits:** 1

Bala-Carter Label: $2A_2 + A_1$; **Weighted Dynkin Diagram:** $\begin{array}{c} 0 \\ 1-0-\dot{1}-0-1 \end{array}$ **(not special)**

Lie Triple ϕ: $x_{\alpha_1+\alpha_3+\alpha_4} + x_{\alpha_4+\alpha_5+\alpha_6} + x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} + x_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} + x_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5}$, $4h_{\alpha_1} + 5h_{\alpha_2} + 7h_{\alpha_3} + 10h_{\alpha_4} + 7h_{\alpha_5} + 4h_{\alpha_6}$, $2x_{-\alpha_1-\alpha_3-\alpha_4} + 2x_{-\alpha_4-\alpha_5-\alpha_6} + 2x_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5} + 2x_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6} + x_{-\alpha_2-\alpha_3-2\alpha_4-\alpha_5}$
$\mathfrak{g}^\phi$ = span { $-x_{\alpha_3} + x_{\alpha_5} + x_{-\alpha_2}$, $x_{\alpha_2} - x_{-\alpha_3} + x_{-\alpha_5}$, $-h_{\alpha_2} + h_{\alpha_3} + h_{\alpha_5}$ } $\simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}$[-1] = $4V^2 \oplus V^4$
We have $G^\phi_0 = SL(2,k)$; apply Theorem 3.2. The orbit is **admissible**.
Fundamental Group: $\mathbb{Z}/3\mathbb{Z}$; **$\#\mathbb{R}$-orbits:** 1; **$\#p$-adic orbits:** $|k^*/{k^*}^3|$ (see Appendix A.5)
Note on rational classes: the existence of a unique 4-dimensional rational representation implies that all orbits must be split, and hence admissible.

Bala-Carter Label: A_3+A_1; **Weighted Dynkin Diagram:** $\begin{array}{c} 1 \\ 0-1-\overset{|}{0}-1-0 \end{array}$ **(not special)**

Lie Triple ϕ: $x_{\alpha_2+\alpha_3+\alpha_4}+x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_2+\alpha_4+\alpha_5+\alpha_6}+x_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}$, $4h_{\alpha_1}+6h_{\alpha_2}+8h_{\alpha_3}+11h_{\alpha_4}+8h_{\alpha_5}+4h_{\alpha_6}$, $3x_{-\alpha_2-\alpha_3-\alpha_4}+4x_{-\alpha_1-\alpha_3-\alpha_4-\alpha_5}+3x_{-\alpha_2-\alpha_4-\alpha_5-\alpha_6}+x_{-\alpha_3-\alpha_4-\alpha_5-\alpha_6}$
$\mathfrak{g}^\phi=$ span $\{\, x_{\alpha_4}, x_{-\alpha_4}, h_{\alpha_4}, 2h_{\alpha_1}+h_{\alpha_3}-h_{\alpha_5}+h_{\alpha_6}\,\}\simeq \mathfrak{sl}(2,k)\oplus V$
$\mathfrak{g}$[**-1**] $=$ $5V^2$ under the $\mathfrak{sl}(2,k)$
We have $G^\phi(k)=SL(2,k)\times k^*$, a direct product since the two subalgebras commute. Apply Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: A_4+A_1; **Weighted Dynkin Diagram:** $\begin{array}{c} 1 \\ 1-1-\overset{|}{0}-1-1 \end{array}$ **(special)**

Lie Triple ϕ: $x_{\alpha_1+\alpha_3}+x_{\alpha_2+\alpha_3+\alpha_4}+x_{\alpha_2+\alpha_4+\alpha_5}+x_{\alpha_3+\alpha_4+\alpha_5}+x_{\alpha_4+\alpha_5+\alpha_6}$, $6h_{\alpha_1}+8h_{\alpha_2}+11h_{\alpha_3}+15h_{\alpha_4}+11h_{\alpha_5}+6h_{\alpha_6}$, $6x_{-\alpha_1-\alpha_3}+4x_{-\alpha_2-\alpha_3-\alpha_4}+4x_{-\alpha_2-\alpha_4-\alpha_5}+x_{-\alpha_3-\alpha_4-\alpha_5}+6x_{-\alpha_4-\alpha_5-\alpha_6}$
$\mathfrak{g}^\phi=$ span $\{\, -2h_{\alpha_1}-h_{\alpha_3}-3h_{\alpha_4}+h_{\alpha_5}+2h_{\alpha_6}\,\}\simeq$ V
$\mathfrak{g}$[**-1**] $=$ trivially decomposed under $\mathfrak{g}^\phi$
We have $G^\phi=k^*$, which clearly preserves Lagrangian subspaces; apply Theorem 3.3. The orbit is **admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

Bala-Carter Label: A_5; **Weighted Dynkin Diagram:** $\begin{array}{c} 1 \\ 2-1-\overset{|}{0}-1-2 \end{array}$ **(not special)**

Lie Triple ϕ: $x_{\alpha_1}+x_{\alpha_6}+x_{\alpha_2+\alpha_3+\alpha_4}+x_{\alpha_2+\alpha_4+\alpha_5}+x_{\alpha_3+\alpha_4+\alpha_5}$, $8h_{\alpha_1}+10h_{\alpha_2}+14h_{\alpha_3}+19h_{\alpha_4}+14h_{\alpha_5}+8h_{\alpha_6}$, $8x_{-\alpha_1}+8x_{-\alpha_6}+5x_{-\alpha_2-\alpha_3-\alpha_4}+5x_{-\alpha_2-\alpha_4-\alpha_5}+9x_{-\alpha_3-\alpha_4-\alpha_5}$
$\mathfrak{g}^\phi=$ span $\{\, x_{\alpha_4}, x_{-\alpha_4}, h_{\alpha_4}\,\}\simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}$[**-1**] $=$ $3V^2$
We have $G_0^\phi=SL(2,k)$; apply Theorem 3.2. The orbit is **not admissible**.
Fundamental Group: $\mathbb{Z}/3\mathbb{Z}$; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** $|k^*/{k^*}^3|$ (see Appendix A.6)
Note on rational classes: By Appendix B.1, all orbits are split, hence none are admissible.

Bala-Carter Label: $D_5(a_1)$; **Weighted Dynkin Diagram:** $\begin{array}{c} 2 \\ 1-1-\overset{|}{0}-1-1 \end{array}$ **(special)**

Lie Triple ϕ: $x_{\alpha_2}+x_{\alpha_1+\alpha_3}+x_{\alpha_1+\alpha_3+\alpha_4}+x_{aac+\alpha_4+\alpha_5}+x_{\alpha_4+\alpha_5+\alpha_6}$, $7h_{\alpha_1}+10h_{\alpha_2}+13h_{\alpha_3}+18h_{\alpha_4}+13h_{\alpha_5}+7h_{\alpha_6}$, $10x_{-\alpha_2}+2x_{-\alpha_1-\alpha_3}-5x_{-\alpha_2-\alpha_4}+2x_{-\alpha_5-\alpha_6}+5x_{-\alpha_1-\alpha_3-\alpha_4}+6x_{-\alpha_3-\alpha_4-\alpha_5}+7x_{-\alpha_4-\alpha_5-\alpha_6}$
$\mathfrak{g}^\phi=$ span $\{\, -h_{\alpha_1}+h_{\alpha_3}-h_{\alpha_5}+h_{\alpha_6}\,\}\simeq V$
$\mathfrak{g}$[**-1**] $=$ trivially decomposed under $\mathfrak{g}^\phi$
We have $G^\phi=k^*$, which clearly preserves Lagrangian subspaces; apply Theorem 3.3. The orbit is **admissible**.
Fundamental Group: 1; **#$\mathbb{R}$-orbits:** 1; **#p-adic orbits:** 1

9. $E^0_{7,7}$

$$\begin{array}{c} \alpha_2 \\ \alpha_1-\alpha_3-\overset{|}{\alpha_4}-\alpha_5-\alpha_6-\alpha_7 \end{array}$$

In this section, let $x_{n_1n_2n_3n_4n_5n_6n_7}$ denote the root vector $x_{n_1\alpha_1+n_2\alpha_2+n_3\alpha_3+n_4\alpha_4+n_5\alpha_5+n_6\alpha_6+n_7\alpha_7}$ and $x_{-n_1n_2n_3n_4n_5n_6n_7}$ the corresponding negative root vector.

Bala-Carter Label: A_1; **Weighted Dynkin Diagram:** $\begin{array}{c} 0 \\ 1-0-\dot{0}-0-0-0 \end{array}$ **(special)**

Lie Triple ϕ**:** $x_{2234321}$, $2h_{\alpha_1}+2h_{\alpha_2}+3h_{\alpha_3}+4h_{\alpha_4}+3h_{\alpha_5}+2h_{\alpha_6}+h_{\alpha_7}$, $x_{-2234321}$
$\mathfrak{g}^{\phi}$ = span { subalgebra with simple roots α_2, α_3, α_4, α_5, α_6, α_7 } $\simeq so(12,k)$
$\mathfrak{g}$**[-1]** $= V^{32}$
We have $G^{\phi}=Spin(12,k)$, since it admits a spin representation. A root $SL(2,k)$ corresponding to a long root decomposes $\mathfrak{g}[-1]$ into 16 copies of the trivial representation and 8 copies of the 2-dimensional representation. Thus the cover splits over each long root by Theorem 3.2, and consequently over G^{ϕ} by Lemma 3.1. The orbit is **admissible**.

Bala-Carter Label: $2A_1$; **Weighted Dynkin Diagram:** $\begin{array}{c} 0 \\ 0-0-\dot{0}-0-1-0 \end{array}$ **(special)**

Lie Triple ϕ**:** $x_{1223221}+x_{1123321}$, $2h_{\alpha_1}+3h_{\alpha_2}+4h_{\alpha_3}+6h_{\alpha_4}+5h_{\alpha_5}+4h_{\alpha_6}+2h_{\alpha_7}$, $x_{-1223221}+x_{-1123321}$
$\mathfrak{g}^{\phi}$ = span { x_{α_7},$x_{-\alpha_7}$, h_{α_7}; B_4 with positive roots x_{α_1}, x_{α_3},x_{α_4}, and $x_{\alpha_5}-x_{\alpha_2}$ } $\simeq \mathfrak{sl}(2,k)\oplus so(9,k)$
$\mathfrak{g}$**[-1]** $= 16V^2$ under the $\mathfrak{sl}(2,k)$; $\mathfrak{g}[-1]=2V^{16}$ under the split $so(9,k)$

The 16-dimensional representations are complementary lagrangians. Consequently, the cover of G^{ϕ} splits over each piece individually.
We have $G^{\phi}=SL(2,k)\times Spin(9,k)$, since each group admits representations of the simply connected group, they admit no intersection, and their Lie algebras are orthogonal. Applying Lemma 3.1 and Theorem 3.2, we deduce that the cover of G^{ϕ} splits over each piece individually. The orbit is **admissible**.

Bala-Carter Label: $(3A_1)'$; **Weighted Dynkin Diagram:** $\begin{array}{c} 0 \\ 0-1-\dot{0}-0-0-0 \end{array}$ **(not special)**

Lie Triple ϕ**:** $x_{1122221}+x_{1123211}+x_{1223210}$, $3h_{\alpha_1}+4h_{\alpha_2}+6h_{\alpha_3}+8h_{\alpha_4}+6h_{\alpha_5}+4h_{\alpha_6}+2h_{\alpha_7}$, $x_{-1122221}+x_{-1123211}+x_{-1223210}$
$\mathfrak{g}^{\phi}$ = span { x_{α_1}, $x_{-\alpha_1}$, h_{α_1}; C3 with simple roots x_{α_5}, $-x_{\alpha_4}+x_{\alpha_6}$, $x_{\alpha_2}+x_{\alpha_7}$ } $\simeq \mathfrak{sl}(2,k)\oplus sp(6,k)$
$\mathfrak{g}$**[-1]** $= 15V^2$ under the $\mathfrak{sl}(2,k)$; $\mathfrak{g}[-1]=2V^1\oplus 2V^{14}$ under $sp(6,k)$
We have that the $\mathfrak{sl}(2,k)$ lifts to $SL(2,k)$. With the help of explicit calculations to determine which copy of C_3 in E_7 we have, we deduce from [LS, Table 8.6] that it admits a 6-dimensional irreducible representation, and hence lifts to $Sp(6,k)$ at the group level. The two groups commute and admit no intersection, so $G^{\phi}=SL(2,k)\times Sp(6,k)$. The cover does not split over the $SL(2,k)$ piece of G^{ϕ} (although it does split over $Sp(6,k)$, since the representations are complementary lagrangians). The orbit is **not admissible**.

Bala-Carter Label: $4A_1$; **Weighted Dynkin Diagram:** $\begin{array}{c} 1 \\ 0-0-\dot{0}-0-0-1 \end{array}$ **(not special)**

Lie Triple ϕ**:** $x_{1111111}+x_{0112221}+x_{1123211}+x_{1223210}$, $3h_{\alpha_1}+5h_{\alpha_2}+6h_{\alpha_3}+9h_{\alpha_4}+7h_{\alpha_5}+5h_{\alpha_6}+3h_{\alpha_7}$, $x_{-1111111}+x_{-0112221}+x_{-1123211}+x_{-1223210}$
$\mathfrak{g}^{\phi}$ = span { h_{α_3}, h_{α_5}, $h_{\alpha_1}+h_{\alpha_4}+h_{\alpha_6}$, $x_{\alpha_6}-x_{1011000}$, $x_{0001100}-x_{-0011000}$, x_{α_3}, $x_{\alpha_1}-x_{0001110}$, $x_{0011100}-x_{-\alpha_4}$, x_{α_5}, $x_{1010000}+x_{0011110}$, $x_{0000110}+x_{1011100}$, $x_{1011110}$, and the corresponding negative root vectors } $\simeq sp(6,k)$
$\mathfrak{g}$**[-1]** $= 2V^6\oplus V^{14}$
We have $G^{\phi}=Sp(6,k)$, by the existence of the 6-dimensional irreducible representation. The restriction of the representation $\mathfrak{g}[-1]$ to the $SL(2,k)$ arising from the long root decomposes as seven 2-dimensional representations and twelve trivial representations. Hence the cover does not split over this $SL(2,k)$, and therefore, by Lemma 3.1, it doesn't split over $G^{\phi}=Sp(6,k)$. The orbit is **not admissible**.

Bala-Carter Label: A_2+A_1; **Weighted Dynkin Diagram:** $\begin{array}{c} 0 \\ 1-0-\dot{0}-0-1-0 \end{array}$ **(special)**

Lie Triple ϕ**:** $x_{1011110}+x_{0112221}+x_{1223211}$, $4h_{\alpha_1}+5h_{\alpha_2}+7h_{\alpha_3}+10h_{\alpha_4}+8h_{\alpha_5}+6h_{\alpha_6}+3h_{\alpha_7}$,

$2x_{-1011110}+x_{-0112221}+2x_{-1223211}$
$\mathfrak{g}^{\phi}$ = span { The copy of A3 generated by α_3, α_4, α_5, and a Cartan part through $h_{\alpha_2}-h_{\alpha_7}$. } $\simeq \mathfrak{sl}(4,k)\oplus V$
$\mathfrak{g}$**[-1]** $=\ 6V^4$
We have $G_0^{\phi}=SL(4,k)\ltimes k^*$, since the H part lifts to a copy of k^* inside the Cartan subgroup of G and so normalizes $SL(4)$. Restricting to a root $SL(2,k)$ of $SL(4,k)$ gives a decomposition of $\mathfrak{g}[-1]$ into 6 two-dimensional irreducible representations. By Lemma 3.1 and Theorem 3.2, the cover splits. The orbit is **admissible**.

Bala-Carter Label: A_2+2A_1; **Weighted Dynkin Diagram:** $\begin{matrix} & & 0 & & & \\ 0- & 0- & 1- & 0- & 0- & 0\end{matrix}$ **(special)**

Lie Triple ϕ**:** $x_{1122110}+x_{0112211}+x_{1112111}+x_{1112210}$, $4h_{\alpha_1}+6h_{\alpha_2}+8h_{\alpha_3}+12h_{\alpha_4}+9h_{\alpha_5}+6h_{\alpha_6}+3h_{\alpha_7}$, $2x_{-1122110}+2x_{-0112211}+x_{-1112111}+x_{-1112210}$
$\mathfrak{g}^{\phi}$ = span { x_{α_2},$x_{-\alpha_2}$,h_{α_2}; $2x_{\alpha_1}+x_{\alpha_3}-x_{\alpha_5}+x_{\alpha_7}$, $x_{-\alpha_1}+2x_{-\alpha_3}-x_{-\alpha_5}+x_{-\alpha_7}$, $2h_{\alpha_1}+2h_{\alpha_3}+h_{\alpha_5}+h_{\alpha_7}$; $x_{0000110}-x_{0000011}$,$x_{-0000110}-x_{-0000011}$, $h_{\alpha_5}+2h_{\alpha_6}+h_{\alpha_7}$ } $\simeq \mathfrak{sl}(2,k)\oplus\mathfrak{sl}(2,k)\oplus\mathfrak{sl}(2,k)$
$\mathfrak{g}$**[-1]** $=\ 12V^2$ under the first and third; $\mathfrak{g}[-1]=4V^2\oplus 4V^4$ under the second
We have that the groups all are $SL(2,k)$s, and all commute with one another. However, the second and third $SL(2,k)$ share a common center. So $G_0^{\phi}=SL(2,k)\times(SL(2,k)\times SL(2,k))/\mathbb{Z}/2\mathbb{Z}$. By Theorem 3.2, the cover splits over each $SL(2,k)$ individually. Moreover, the common intersection of the second and third $SL(2,k)$ is the center, and the two splitting maps agree over this center. Hence the cover splits. The orbit is **admissible**.

Bala-Carter Label: A_3; **Weighted Dynkin Diagram:** $\begin{matrix} & & 0 & & & \\ 2- & 0- & 0- & 0- & 1- & 0\end{matrix}$ **(special)**

Lie Triple ϕ**:** $x_{1111000}+x_{1011100}+x_{0112221}$, $6h_{\alpha_1}+7h_{\alpha_2}+10h_{\alpha_3}+14h_{\alpha_4}+11h_{\alpha_5}+8h_{\alpha_6}+4h_{\alpha_7}$, $3x_{-1111000}+3x_{-1011100}+4x_{-0112221}$
$\mathfrak{g}^{\phi}$ = span { x_{α_7}, $x_{-\alpha_7}$, h_{α_7}; B_3 with simple root vectors: x_3, x_4, x_2+x_5 } $\simeq \mathfrak{sl}(2,k)\oplus so(7,k)$
$\mathfrak{g}$**[-1]** $=\ 8V^2$ under the $\mathfrak{sl}(2,k)$; $2V^8$ under the split $so(7,k)$
We have $G_0^{\phi}=SL(2,k)\times Spin(7)$, because their Lie algebras commute and $so(7,k)$ admits an 8-dimensional irreducible (spin) representation. The cover splits over the $SL(2,k)$ by Theorem 3.2. The two 8-dimensional representations are seen to be complementary invariant Lagrangians, and so the cover splits over $Spin(7,k)$ by Theorem 3.3. The orbit is **admissible**.

Bala-Carter Label: $2A_2+A_1$; **Weighted Dynkin Diagram:** $\begin{matrix} & & 0 & & & \\ 0- & 1- & 0- & 0- & 1- & 0\end{matrix}$ **(not special)**

Lie Triple ϕ**:** $x_{0011111}+x_{1111111}+x_{1112110}+x_{1122100}+x_{0112210}$, $5h_{\alpha_1}+7h_{\alpha_2}+10h_{\alpha_3}+14h_{\alpha_4}+11h_{\alpha_5}+8h_{\alpha_6}+4h_{\alpha_7}$, $2x_{-0011111}+2x_{-1111111}+2x_{-1112110}+x_{-1122100}+2x_{-0112210}$
$\mathfrak{g}^{\phi}$ = span { $-x_{\alpha_1}+x_{\alpha_5}+x_{-\alpha_2}$, $h_{\alpha_1}-h_{\alpha_2}+h_{\alpha_5}$, $-x_{-\alpha_1}+x_{-\alpha_5}+x_{\alpha_2}$; $x_{\alpha_4}+x_{\alpha_7}+x_{0101100}$, $h_{\alpha_2}+2h_{\alpha_4}+h_{\alpha_5}+h_{\alpha_7}$, $x_{-\alpha_4}+x_{-\alpha_7}+x_{-0101100}$ } $\simeq \mathfrak{sl}(2,k)\oplus\mathfrak{sl}(2,k)$
$\mathfrak{g}$**[-1]** $=\ 8V^2\oplus V^4$ under the first; $\mathfrak{g}[-1]=8V^1\oplus 4V^3$ under the second
We have that the first A_1 is an $SL(2,k)$; the second embeds inside of a copy of A_5 in E_7, and decomposes the standard representation of $\mathfrak{sl}(6,k)$ into even-dimensional irreducibles. Thus it is an $SL(2,k)$ as well. There is no intersection between them. Moreover, they commute. Thus $G_0^{\phi}=SL(2,k)\times SL(2,k)$ By Theorem 3.2, the cover splits over each piece individually. The orbit is **admissible**.

Bala-Carter Label: $(A_3+A_1)'$; **Weighted Dynkin Diagram:** $\begin{matrix} & & 0 & & & \\ 1- & 0- & 1- & 0- & 0- & 0\end{matrix}$ **(not special)**

Lie Triple ϕ**:** $x_{1011000}+x_{1111111}+x_{0112111}+x_{0112210}$, $6h_{\alpha_1}+8h_{\alpha_2}+11h_{\alpha_3}+16h_{\alpha_4}+12h_{\alpha_5}+8h_{\alpha_6}+4h_{\alpha_7}$, $3x_{-1011000}+3x_{-1111111}+x_{-0112111}+4x_{-0112210}$

$\mathfrak{g}^\phi = \text{span}\ \{\ x_{\alpha_3},\ x_{-\alpha_3},\ h_{\alpha_3},\ x_{\alpha_6},\ x_{-\alpha_6},\ h_{\alpha_6},\ x_{0000111} + x_{-\alpha_2},\ x_{\alpha_2} + x_{-0000111},\ -h_{\alpha_2} + h_{\alpha_5} + h_{\alpha_6} + h_{\alpha_7}\ \} \simeq \mathfrak{sl}(2,k) \oplus \mathfrak{sl}(2,k) \oplus \mathfrak{sl}(2,k)$
$\mathfrak{g}$**[-1]** $=\ 9V^2$ under the first; $\mathfrak{g}\,[-1] = 10V^1 \oplus 4V^2$ under the second; and $\mathfrak{g}\,[-1] = 4V^1 \oplus 4V^2 \oplus 2V^3$ under the third
We have $G_0^\phi = SL(2) \times SL(2) \times SL(2)$ since the three $\mathfrak{sl}(2,k)$ subalgebras commute and admit no intersection. By Theorem 3.2, the cover splits on the second and third pieces, but not on first copy of $SL(2)$. The orbit is **not admissible**.

Bala-Carter Label: $A_3 + 2A_1$; **Weighted Dynkin Diagram:** $\begin{array}{ccccccccccc} &&&&0&&&&&& \\ 1&-&0&-&0&-&1&-&0&-&1 \end{array}$ **(not special)**

Lie Triple ϕ: $x_{1011100} + x_{0101111} + x_{0011111} + x_{0112210} + x_{1122110}\ ,\ 6h_{\alpha_1} + 8h_{\alpha_2} + 11h_{\alpha_3} + 16h_{\alpha_4} + 13h_{\alpha_5} + 9h_{\alpha_6} + 5h_{\alpha_7}\ ,\ 3x_{-1011100} + 4x_{-0101111} + x_{-0011111} + x_{-0112210} + 3x_{-1122110}$
$\mathfrak{g}^\phi = \text{span}\ \{\ x_{\alpha_4},\ x_{-\alpha_4},\ h_{\alpha_4},\ x_{0111000} - x_{-\alpha_6},\ -x_{\alpha_6} + x_{-0111000},\ h_{\alpha_2} + h_{\alpha_3} + h_{\alpha_4} - h_{\alpha_6}\ \} \simeq \mathfrak{sl}(2,k) \oplus \mathfrak{sl}(2,k)$
$\mathfrak{g}$**[-1]** $=\ 8V^1 \oplus 5V^2$ under the first $\mathfrak{sl}(2,k)$; $\mathfrak{g}\,[-1] = 4V^1 \oplus 4V^2 \oplus 2V^3$ under the second $\mathfrak{sl}(2,k)$
We have $G_0^\phi = SL(2,k) \times SL(2,k)$, since their Lie algebras commute, and they admit no intersection. By Theorem 3.2, the first copy of $SL(2)$ doesn't split (although the second one does). The orbit is **not admissible**.

Bala-Carter Label: $D_4(a_1) + A_1$; **Weighted Dynkin Diagram:** $\begin{array}{ccccccccccc} &&&&1&&&&&& \\ 0&-&1&-&0&-&0&-&0&-&1 \end{array}$ **(special)**

Lie Triple ϕ: $x_{1111000} + x_{0101111} + x_{0011111} + x_{1011111} + x_{0112210}\ ,\ 6h_{\alpha_1} + 9h_{\alpha_2} + 12h_{\alpha_3} + 17h_{\alpha_4} + 13h_{\alpha_5} + 9h_{\alpha_6} + 5h_{\alpha_7}\ ,\ -2x_{-0111000} + 4x_{-1111000} + x_{-0101111} + 2x_{-0011111} + 2x_{-1011111} + 4x_{-0112210} - 2x_{-1112210}$
$\mathfrak{g}^\phi = \text{span}\ \{\ x_{\alpha_4},\ x_{-\alpha_4},\ h_{\alpha_4},\ x_{\alpha_6},\ x_{-\alpha_6},\ h_{\alpha_6}\ \ \} \simeq \mathfrak{sl}(2,k) \oplus \mathfrak{sl}(2,k)$
$\mathfrak{g}$**[-1]** $=\ 8V^1 \oplus 4V^2$ under each $\mathfrak{sl}(2,k)$ (with the two dimensional representations all complementary)
We have $G_0^\phi = SL(2,k) \times SL(2,k)$, since they admit no intersection and they commute. By Theorem 3.2, the cover splits over each piece of G^ϕ. The orbit is **admissible**.

Bala-Carter Label: $A_3 + A_2$; **Weighted Dynkin Diagram:** $\begin{array}{ccccccccccc} &&&&0&&&&&& \\ 0&-&0&-&1&-&0&-&1&-&0 \end{array}$ **(special)**

Lie Triple ϕ: $x_{0001110} + x_{0111110} + x_{0101111} + x_{1011111} + x_{1122100}\ ,\ 6h_{\alpha_1} + 9h_{\alpha_2} + 12h_{\alpha_3} + 18h_{\alpha_4} + 14h_{\alpha_5} + 10h_{\alpha_6} + 5h_{\alpha_7}\ ,\ 3x_{-0001110} + 2x_{-0111110} + 3x_{-0101111} + 2x_{-1011111} + 4x_{-1122100}$
$\mathfrak{g}^\phi = \text{span}\ \{\ x_{\alpha_5},\ x_{-\alpha_5},\ h_{\alpha_5},\ 2h_{\alpha_1} + h_{\alpha_2} - h_{\alpha_7}\ \ \} \simeq \mathfrak{sl}(2,k) \oplus V$
$\mathfrak{g}$**[-1]** $=\ 8V^2$ under the $\mathfrak{sl}(2,k)$
We have $G_0^\phi = SL(2,k) \times k^*$, since the Lie algebras of the two pieces commute, and admit no intersection. By Theorem 3.2 and Theorem 3.3, the cover splits over both pieces of G_e^ϕ. The orbit is **admissible**.

Bala-Carter Label: $D_4 + A_1$; **Weighted Dynkin Diagram:** $\begin{array}{ccccccccccc} &&&&1&&&&&& \\ 2&-&1&-&0&-&0&-&0&-&1 \end{array}$ **(not special)**

Lie Triple ϕ: $x_{\alpha_1} + x_{0111000} + x_{0101111} + x_{0011111} + x_{0112210}\ ,\ 10h_{\alpha_1} + 13h_{\alpha_2} + 18h_{\alpha_3} + 25h_{\alpha_4} + 19h_{\alpha_5} + 13h_{\alpha_6} + 7h_{\alpha_7}\ ,\ 10x_{-\alpha_1} + 6x_{-0111000} + x_{-0101111} + 6x_{-0011111} + 6x_{-0112210}$
$\mathfrak{g}^\phi = \text{span}\ \{\ h_{\alpha_4}, h_{\alpha_6},\ x_{-\alpha_6},\ x_{0001110} + x_{-\alpha_5},\ x_{0001100} - x_{-0000110},\ x_{\alpha_4},\ x_{\alpha_6},\ x_{-0001110} + x_{\alpha_5},\ x_{-0001100} - x_{0000110},\ x_{-\alpha_4}\ \ \} \simeq so(5,k)$
$\mathfrak{g}$**[-1]** $=\ 3V^4$
We have $G^\phi = Sp(4,k)$ by the existence of a 4-dimensional irreducible representation. So we have $Sp(4,k)$ mapping into 3 copies of itself, which does not split. The orbit is **not admissible**.

Bala-Carter Label: $A_4 + A_1$; **Weighted Dynkin Diagram:** $\begin{array}{ccccccccccc} &&&&0&&&&&& \\ 1&-&0&-&1&-&0&-&1&-&0 \end{array}$ **(special)**

Lie Triple ϕ: $x_{1011000} + x_{0101110} + x_{1111100} + x_{0011111} + x_{0112100}\ ,\ 8h_{\alpha_1} + 11h_{\alpha_2} + 15h_{\alpha_3} + 22h_{\alpha_4} +$

$17h_{\alpha_5}+12h_{\alpha_6}+6h_{\alpha_7}$, $4x_{-1011000}+6x_{-0101110}+4x_{-1111100}+6x_{-0011111}+x_{-0112100}$
$\mathfrak{g}^\phi=$ span $\{\ h_{\alpha_2}+h_{\alpha_3}-h_{\alpha_5},\ h_{\alpha_3}-h_{\alpha_7}\ \}\simeq V^2$
$\mathfrak{g}$[-1] $=$ decomposes trivially under $\mathfrak{g}^\phi$
We have $G_0^\phi=k^*\times k^*$, and G^ϕ acts diagonalizably. By Theorem 3.3, the cover will split over G_0^ϕ, since we can intertwine the H-actions across 2 Lagrangians. The orbit is **admissible**.

Bala-Carter Label: $D_5(a_1)$; **Weighted Dynkin Diagram:** $\begin{array}{c}0\\ 2-0-1-0-1-0\end{array}$ **(special)**

Lie Triple ϕ**:** $x_{\alpha_1}+x_{1010000}+3x_{0001110}+2x_{0011110}+x_{0101111}+x_{0112100}$, $10h_{\alpha_1}+13h_{\alpha_2}+18h_{\alpha_3}+26h_{\alpha_4}+20h_{\alpha_5}+14h_{\alpha_6}+7h_{\alpha_7}$, $2x_{-\alpha_1}+8x_{-1010000}+x_{-0001110}+2x_{-0011110}+7x_{-0101111}+6x_{-0112100}+2x_{-0111111}$
$\mathfrak{g}^\phi=$ span $\{\ x_{\alpha_5},\ x_{-\alpha_5},\ h_{\alpha_5},\ h_{\alpha_2}-h_{\alpha_7}\ \}\simeq\mathfrak{sl}(2,k)\oplus V$
$\mathfrak{g}$[-1] $=$ $6V^2$ under the $\mathfrak{sl}(2,k)$
We have $G_0^\phi=SL(2,k)\times k^*$, since their Lie subalgebras commute and they admit no intersection. By Theorem 3.2 and Theorem 3.3, the cover splits over each piece. The orbit is **admissible**.

Bala-Carter Label: $(A_5)'$; **Weighted Dynkin Diagram:** $\begin{array}{c}0\\ 1-0-1-0-2-0\end{array}$ **(not special)**

Lie Triple ϕ**:** $x_{0000110}+x_{0000011}+x_{1011000}+x_{1111100}+x_{0112100}$, $10h_{\alpha_1}+14h_{\alpha_2}+19h_{\alpha_3}+28h_{\alpha_4}+22h_{\alpha_5}+16h_{\alpha_6}+8h_{\alpha_7}$, $8x_{-0000110}+8x_{-0000011}+5x_{-1011000}+5x_{-1111100}+9x_{-0112100}$
$\mathfrak{g}^\phi=$ span $\{\ x_{\alpha_3},\ x_{-\alpha_3},\ h_{\alpha_3},\ x_{\alpha_5}+x_{\alpha_7}+x_{-\alpha_2},\ x_{\alpha_2}+x_{-\alpha_5}+x_{-\alpha_7},\ -h_{\alpha_2}+h_{\alpha_5}+h_{\alpha_7}\ \}\simeq\mathfrak{sl}(2,k)\oplus\mathfrak{sl}(2,k)$
$\mathfrak{g}$[-1] $=$ $5V^2$ under the first $\mathfrak{sl}(2,k)$; $\mathfrak{g}[-1]=4V^1\oplus 2V^3$ under the second $\mathfrak{sl}(2,k)$
We have that both subalgebras lift to $SL(2,k)$ at the group level, since the second embeds into a subalgebra of type A_5 in E_7 and decomposes its standard representation into even dimensional irreducibles. They commute and admit no intersection, so $G_0^\phi=SL(2,k)\times SL(2,k)$. By Theorem 3.2, the cover splits over the second $SL(2,k)$ but not the first. The orbit is **not admissible**.

Bala-Carter Label: A_5+A_1; **Weighted Dynkin Diagram:** $\begin{array}{c}0\\ 1-0-1-0-1-2\end{array}$ **(not special)**

Lie Triple ϕ**:** $x_{\alpha_7}+x_{1011000}+x_{0101110}+x_{0011110}+x_{1111100}+x_{0112100}$, $10h_{\alpha_1}+14h_{\alpha_2}+19h_{\alpha_3}+28h_{\alpha_4}+22h_{\alpha_5}+16h_{\alpha_6}+9h_{\alpha_7}$, $9x_{-\alpha_7}+5x_{-1011000}+8x_{-0101110}+8x_{-0011110}+5x_{-1111100}+x_{-0112100}$
$\mathfrak{g}^\phi=$ span $\{\ x_{\alpha_5}+x_{-\alpha_2}-x_{-\alpha_3},\ x_{\alpha_2}-x_{\alpha_3}+x_{-\alpha_5}\ -h_{\alpha_2}-h_{\alpha_3}+h_{\alpha_5}\ \}\simeq\mathfrak{sl}(2,k)$
$\mathfrak{g}$[-1] $=$ $4V^2\oplus V^4$
We have $G_0^\phi=SL(2,k)$, and the cover splits by Theorem 3.2. The orbit is **admissible**.

Bala-Carter Label: $D_6(a_2)$; **Weighted Dynkin Diagram:** $\begin{array}{c}1\\ 0-1-0-1-0-2\end{array}$ **(not special)**

Lie Triple ϕ**:** $x_{\alpha_7}+2x_{0000011}-x_{0111000}+x_{0101100}+5x_{0011100}+2x_{1111000}+4x_{0101110}+3x_{0011110}+4x_{1011110}$, $10h_{\alpha_1}+15h_{\alpha_2}+20h_{\alpha_3}+29h_{\alpha_4}+23h_{\alpha_5}+16h_{\alpha_6}+9h_{\alpha_7}$, $5x_{-\alpha_7}+2x_{-0000011}+x_{-0101100}+2x_{-0011100}+3x_{-1111000}+2x_{-0101110}+x_{-1011110}$
$\mathfrak{g}^\phi=$ span $\{\ x_{\alpha_4},\ x_{-\alpha_4},\ h_{\alpha_4}\ \}\simeq\mathfrak{sl}(2,k)$
$\mathfrak{g}$[-1] $=$ $5V^2$
We have $G_0^\phi=SL(2,k)$; apply Theorem 3.2. The orbit is **not admissible**.

Bala-Carter Label: D_5+A_1; **Weighted Dynkin Diagram:** $\begin{array}{c}1\\ 2-1-0-1-1-0\end{array}$ **(special)**

Lie Triple ϕ**:** $x_{\alpha_1}+x_{0000110}+x_{0111000}+x_{0101100}+x_{0011100}+x_{0001111}$, $14h_{\alpha_1}+19h_{\alpha_2}+26h_{\alpha_3}+37h_{\alpha_4}+29h_{\alpha_5}+20h_{\alpha_6}+10h_{\alpha_7}$, $14x_{-\alpha_1}+10x_{-0000110}+18x_{-0111000}+x_{-0101100}+8x_{-0011100}+10x_{-0001111}$
$\mathfrak{g}^\phi=$ span $\{\ x_{\alpha_7}+x_{-\alpha_4},\ x_{\alpha_4}+x_{-\alpha_7},\ -h_{\alpha_4}+h_{\alpha_7}\ \}\simeq\mathfrak{sl}(2,k)$
$\mathfrak{g}$[-1] $=$ $4V^2$

We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **admissible**.

Bala-Carter Label: $D_6(a_1)$; **Weighted Dynkin Diagram:** $\begin{array}{c} 1 \\ 2-1-\overset{|}{0}-1-0-2 \end{array}$ **(special)**

Lie Triple ϕ: $x_{\alpha_1}+x_{\alpha_7}+x_{0000011}+x_{0111000}+x_{0101100}+2x_{0011100}-x_{0011110}$, $14h_{\alpha_1}+19h_{\alpha_2}+26h_{\alpha_3}+37h_{\alpha_4}+29h_{\alpha_5}+20h_{\alpha_6}+11h_{\alpha_7}$, $14x_{-\alpha_1}-9x_{-\alpha_7}+20x_{-0000011}+8x_{-0111000}+11x_{-0101100}+9x_{-0011100}+20x_{-0101110}$
$\mathfrak{g}^\phi =$ span $\{ x_{\alpha_4}, x_{-\alpha_4}, h_{\alpha_4} \} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}$[**-1**] $= 4V^2$
We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **admissible**.

Bala-Carter Label: D_6; **Weighted Dynkin Diagram:** $\begin{array}{c} 1 \\ 2-1-\overset{|}{0}-1-2-2 \end{array}$ **(not special)**

Lie Triple ϕ: $x_{\alpha_1}+x_{\alpha_6}+x_{\alpha_7}+x_{0111000}+x_{0101100}+x_{0011100}$, $18h_{\alpha_1}+25h_{\alpha_2}+34h_{\alpha_3}+49h_{\alpha_4}+39h_{\alpha_5}+28h_{\alpha_6}+15h_{\alpha_7}$, $18x_{-\alpha_1}+28x_{-\alpha_6}+15x_{-\alpha_7}+10x_{-0111000}+15x_{-0101100}+24x_{-0011100}$
$\mathfrak{g}^\phi =$ span $\{ x_{\alpha_4}, x_{-\alpha_4}, h_{\alpha_4} \} \simeq \mathfrak{sl}(2,k)$
$\mathfrak{g}$[**-1**] $= 3V^2$
We have $G_0^\phi = SL(2,k)$; apply Theorem 3.2. The orbit is **admissible**.

Appendix A. Finding the given number of rational orbits

A.1. **The $\widetilde{A_1}$ orbit of $F^0_{4,4}$.** Using the weighted Dynkin Diagram of this orbit, we deduce that $\mathfrak{g}[0]$ is the direct sum of a split Lie algebra of type B_3 (with simple roots α, β and γ) and kh_δ (notation of Section 7). The subspace $\mathfrak{g}[2]$ has basis

$$B = \{x_{\beta+2\gamma+2\delta}, x_{\alpha+\beta+2\gamma+2\delta}, x_{\alpha+2\beta+2\gamma+2\delta}, x_{\alpha+2\beta+3\gamma+2\delta}, x_{\alpha+2\beta+4\gamma+2\delta}, x_{\alpha+3\beta+4\gamma+2\delta}, x_{2\alpha+3\beta+4\gamma+2\delta}\}.$$

The split $so(7)$ acts on $\mathfrak{g}[2]$ as the standard representation, preserving an isotropic quadratic form. With respect to the basis B above, this form is $Q(\vec{x}) = x_1x_7 + x_2x_6 + x_3x_5 + x_4^2$. The subspace spanned by h_δ acts diagonally with respect to B.

The representative E chosen for the split orbit satisfies $Q(E) \neq 0$, whence $\mathcal{P}$ may be identified as the subset of non-isotropic elements of $\mathfrak{g}[2]$. Given a representative E' of a rational orbit, conjugation by $SO(7)$ will preserve $Q(E')$, and (we check) the action by the one-parameter subgroup corresponding to h_δ will modify $Q(E')$ by a square. Thus under G^H, $\mathcal{P}$ decomposes into $|k^*/k^{*2}|$ orbits. Explicitly, representatives for these orbits are given in coordinates with respect to B as $(m,0,0,0,0,0,1)$, where m runs over the square classes in k^*.

A.2. $A_1 + \widetilde{A_1}$ **orbit of F_4.** In this case, $\mathfrak{g}[0] = \mathfrak{sl}(3,k)_{\gamma,\delta} \oplus \mathfrak{sl}(2,k)_\alpha \oplus kh_\beta$. A basis for $\mathfrak{g}[2]$ is

$$B = \{x_{\alpha+2\beta+2\gamma}, x_{\alpha+2\beta+2\gamma+\delta}, x_{\alpha+2\beta+2\gamma+2\delta}, x_{\alpha+2\beta+3\gamma+\delta}, x_{\alpha+2\beta+3\gamma+2\delta}, x_{\alpha+2\beta+4\gamma+2\delta}\}.$$

The $\mathfrak{sl}(2,k)$ acts trivially on $\mathfrak{g}[2]$, the kh_β acts diagonaly (with respect to B), and the $\mathfrak{sl}(3,k)$ acts on $\mathfrak{g}[2]$ as the symmetric square of the standard representation of $\mathfrak{sl}(3,k)$. Applying Proposition 4.1, we deduce that there are at most 2 rational orbits, for any choice of k (real or p-adic).

We have two immediate choices of orbit representatives $E = x_{\alpha+2\beta+2\gamma+2\delta} + x_{\alpha+2\beta+3\gamma+\delta}$ and $E' = x_{\alpha+2\beta+2\gamma} + x_{\alpha+2\beta+2\gamma+2\delta} + x_{\alpha+2\beta+4\gamma+2\delta}$ (obtained by considering the G^H-action on $\mathfrak{g}[2]$). Let ϕ' be the standard Lie triple corresponding to E'; then $\mathfrak{g}^{\phi'} = \mathfrak{sl}(2,k) \oplus so(Q)$, where the quadratic form Q is represented by the 3×3 identity matrix. When $k = \mathbb{R}$ or k has residual characteristic even and $(-1/-1)_k = -1$, $so(Q)$ is not equivalent to $\mathfrak{sl}(2,k)$, which shows that the two orbits are distinct.

Otherwise, the groups G^ϕ and $G^{\phi'}$ are isomorphic; however, one can (laboriously) prove directly that the two orbits are not rationally conjugate under G^H (for any choice of k).

A.3. B_2 **orbit of** F_4**.** Here, $\mathfrak{g}[0] = so(5) \oplus kh_\alpha \oplus kh_\delta$, where $so(5)$ is the split subalgebra with simple roots β and γ. The subspace $\mathfrak{g}[2]$ is 6-dimensional, with basis

$$B = \{x_\alpha, x_{\alpha+\beta}, x_{\alpha+\beta+\gamma}, x_{\alpha+\beta+2\gamma}, x_{\alpha+2\beta+2\gamma}, x_{\beta+2\gamma+2\delta}\}.$$

With respect to this basis B, h_α and h_δ act diagonally and $so(5)$ acts as $V_{std} \oplus V^1$. As before, the subgroup of G^H corresponding to $so(5)$ preserves a quadratic form, which is given by $Q(\vec{x}) = x_1x_5+x_2x_4+x_3^2$ in coordinates with respect to B. Moreover, the one-parameter subgroups corresponding to h_α and h_δ can only change the value of $Q(\vec{x})$ by a square, whereas they can scale the last coordinate by any value in k^*. Whence there must be $|k^*/k^{*2}|$ orbits. Representatives of this other orbits are given in coordinates with respect to B as $(m, 0, 0, 0, 1, 1)$, where m runs over the square classes in k^*.

A more careful look at the groups G^ϕ arising from each of these rational orbits shows that in fact each different value $m \notin k^{*2}$ chosen leads to $G^\phi \simeq SL(2, F(\sqrt{m}))$.

A.4. $C_3(a_1)$ **orbit of** F_4**.** In this case, $\mathfrak{g}[0] = \mathfrak{sl}(2,k)_\beta \oplus \mathfrak{sl}(2,k)_\delta \oplus kh_\alpha \oplus kh_\gamma$, and $\mathfrak{g}[2]$ is spanned by

$$B = \{x_{\alpha+\beta+\gamma}, x_{\beta+2\gamma}, x_{\beta+2\gamma+\delta}, x_{\alpha+\beta+\gamma+\delta}, x_{\beta+2\gamma+2\delta}\}.$$

Hence $\mathfrak{sl}(2,k)_\beta$ acts trivially, whereas $\mathfrak{sl}(2,k)_\delta$ decomposes $\mathfrak{g}[2]$ into irreducibles $V^2 \oplus V^3$. It acts transitively on the former; its action on the latter is equivalent to the adjoint representation of $SL(2,k)$ on its Lie algebra. The one-parameter subgroup of G^H corresponding to h_α will act by scalars (in k^*) on each of these subrepresentations; the subgroup corresponding to h_γ acts diagonalizably.

Under the identification of V^3 with the adjoint representation, the representative $X = x_{\beta+2\gamma} + x_{\beta+2\gamma+2\delta}$ corresponds to the matrix $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$. Its orbit under G^H consists of all nonzero hyperbolic (diagonalizable over k) matrices. Choosing representatives $X(\xi) = x_{\beta+2\gamma} + \xi x_{\beta+2\gamma+2\delta}$, as ξ ranges over the square classes of k^*, yields representatives for the remaining rational orbits of maximal dimension.

Finally, using Section 4, we conclude that the elements $E(\xi) = x_{\alpha+\beta+\gamma+\delta} + X(\xi)$ must exhaust a set of representatives for all rational orbits in $\mathbb{G} \cdot E$.

A.5. $2A_2 + A_1$ **orbit of** E_6**.** In this case, $\mathfrak{g}[2]$ is a 9-dimensional space, with basis

$$\begin{aligned} B \;=\; \{ & x_{\alpha_1+\alpha_3+\alpha_4}, x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}, x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}, \\ & x_{\alpha_4+\alpha_5+\alpha_6}, x_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}, x_{\alpha_2+\alpha_4+\alpha_5+\alpha_6}, x_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}, x_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6}\} \end{aligned}$$

and

$$\mathfrak{g}[0] = \mathfrak{sl}(2,k)_2 \oplus \mathfrak{sl}(2,k)_3 \oplus \mathfrak{sl}(2,k)_5 \oplus kh_1 \oplus kh_4 \oplus kh_6.$$

Note that the three $\mathfrak{sl}(2,k)$ subalgebras commute, and that the Cartan pieces will act diagonalizably on $\mathfrak{g}[2]$. Let us compute the rational orbit of the chosen representative E, which equals $(1,0,0,1,1,0,0,1,1)$ in coordinates with respect to the basis for $\mathfrak{g}[2]$ above.

First note that the $SL(2,k)_5$-action on $\mathfrak{g}[2]$ decomposes as $2V^2 + 5V^1$, with the two two copies of the standard representation lying on the first four vectors of the basis above. Hence, for $g_5 = \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \in SL(2,k)_5$, we have $g_5 \cdot E = (a,c,b,d,1,0,0,1,1)$, with $ad - bc = 1$. Similarly, an element $g_3 = \begin{smallmatrix} e & f \\ g & h \end{smallmatrix} \in SL(2,k)_3$ acts nontrivially only on the next four coordinates, so we have $g_3g_5 \cdot E = (a,c,b,d,e,g,f,h,1)$, with $ad - bc = 1$ and $eh - fg = 1$. The action of $SL(2,k)_2$ does not further enlarge the orbit.

Now action by the one-parameter subgroups corresponding to h_1, h_4, and h_6 (denoted by $h_1(r)$, $h_4(s)$ and $h_6(t)$, respectively) scales these values as follows

$$h_1(r)h_4(s)h_6(t)g_3g_5 \cdot E = (rsa, rt^{-1}c, rb, rs^{-1}t^{-1}d, ste, r^{-1}tg, tf, r^{-1}s^{-1}th, r^{-1}st^{-1}).$$

This gives a degree of freedom of r^2t^{-1} in the first four coordinates, of $r^{-1}t^2$ in the second four coordinates, and $r^{-1}st^{-1}$ in the last coordinate.

To solve for an arbitary element of $\mathfrak{g}[2]$ (except possibly some in a subvariety of lower dimension), one can choose g_3 and g_5 to give the correct values up to scaling in the first four, second four, and last coordinate, respectively. However, to solve $r^2t^{-1} = m$ and $r^{-1}t^2 = n$ implies solving $t^3 = n^2m$, which is possible only if $k^{*3} = k^*$.

Whence there are $|k^*/k^{*3}|$ rational orbits in $\mathbb{G}{\cdot}E$. Representatives for these orbits are $(m, 0, 0, 1, 1, 0, 0, 1, 1)$, with m running over the cube classes in k^*.

A.6. A_5 **orbit of** E_6**.** We have that $\mathfrak{g}[2]$ is spanned by

$$B = \{x_{\alpha_1}, x_{\alpha_6}, x_{\alpha_2+\alpha_3+\alpha_4}, x_{\alpha_2+\alpha_4+\alpha_5}, x_{\alpha_3+\alpha_4+\alpha_5}\}$$

and that $\mathfrak{g}[0]$ is the direct sum of $\mathfrak{sl}(2,k)_4$ (which acts trivially on $\mathfrak{g}[2]$) and the rest of the Cartan subalgebra. Our orbit representative $E = (1,1,1,1,1)$ with respect to the above basis B for $\mathfrak{g}[2]$; since the action of G^H is diagonal with respect to B, any orbit representative must have all non-zero coordinates. Computing the Cartan action directly and then solving for any possible orbit representative leads to a cubic equation in k^*. Once again, we deduce there are $|k^*/k^{*3}|$ rational orbits. One possible set of representatives would be $\{(1,1,1,d,1)\}$, where d runs over the cube classes in k^*.

Appendix B. Admissibility of non-split rational orbits

Those rational orbits for which we cannot by way of the previous section exclude the possibility that non-split forms of G^ϕ arise must be treated on a case-by-case basis. The admissibility of these orbits is known over $\mathbb{R}$ by [No1, No2]. In this section, we discuss the (very few) cases arising in G_2, F_4 and E_6.

B.1. **k-forms of** $SL(2)$**.** Suppose $\mathbb{G}^\phi$ is $SL(2)$. It admits a unique non-split k-form, which we identify with $SL(1,D) = D^\times$, for D the quaternions over k [T1] when k is a p-adic field, and with $SU(2)$ when $k = \mathbb{R}$. Note that both these groups are compact, each being isomorphic to a special orthogonal group preserving an anisotropic quadratic form.

Theorem 5.2 implies that these groups do not admit 2-dimensional k-rational representations. Rather, their standard 4-dimensional k-rational representations decompose into two 2-dimensional irreducibles upon passage to the algebraic closure.

Lemma B.1. *The metaplectic cover of a compact k-from of $SL(2)$ arising from its 4-dimensional irreducible k-rational representation splits.*

Proof. First let k denote a p-adic field (of residual characteristic different from 2), and D the skew field of quaternions over k. Realize the given representation of $D^\times$ as left multiplication on D. This action preserves a symplectic structure on $D = F^4$, hence gives a map $D^\times \to Sp(4,k)$. The image of $D^\times$ is a compact subgroup of $Sp(4,k)$, which preserves a self-dual lattice in F^4. Hence by [MVW, Ch2.II.8], the cover splits.

Now let $k = \mathbb{R}$. The standard representation of $U(2)$ preserves a hermitian form on $\mathbb{C}^2$. The imaginary part of this form defines a symplectic form on $\mathbb{R}^4$ preserved by $U(2)$, and hence gives a map $U(2) \to Sp(4,\mathbb{R})$. This realizes $U(2)$ as part of a dual reductive pair in $Sp(4,\mathbb{R})$, whose full $\mathbb{C}^1$ metaplectic cover therefore splits by [MVW, 3.I.1]. Restriction to $SU(2)$ gives a splitting over that subgroup; since $SU(2)$ is equal to its group of commutators, the image of this splitting map must lie in $Mp(4,\mathbb{R})$ [MVW, 2.II], as required. □

B.2. **k-forms of** $SL(2) \times SL(2)$**.** Suppose $\mathbb{G}^\phi$ is $SL(2) \times SL(2)$. Its non-split k-forms include not only direct product groups, with one or both factors non-split forms of $SL(2)$, but also $SL(2,E)$, for E a quadratic extension field of k, viewed as a k-group. The former cases may be understood with the help of Section B.1.

For $k = \mathbb{R}$, the group $SL(2,\mathbb{C})$ is simply connected, and so any cover of it will split, implying the orbit is admissible. For k a p-adic field, the answer is less straightforward. We compute the metaplectic cover in the fundamental case of the standard representation of $SL(2,E)$ (a k-rational representation by restriction of scalars).

Lemma B.2. *Let k-be a p-adic field of residual characteristic different from 2. Let $E = k(\sqrt{\alpha})$ be a quadratic extension field of k (where $\alpha \in k^*$ is not a square). View $SL(2,E)$ as a k-group; then it* *admits a 4-dimensional irreducible k-rational representation, coming from the standard representation of $SL(2,E)$ on $E^2 \simeq k^4$. This gives a homomorphism*

$$\varphi\colon SL(2,E) \to Sp(4,k).$$

The metaplectic cover of $SL(2,E)$ determined by the lift of φ to $Mp(4,k)$ splits when either $-1 \in k^{*2}$, or $\alpha = -1 \notin k^{*2}$. It does not split otherwise.

Proof. Note that Lemma 3.1 applies to $SL(2,E)$. The calculation of the restriction of the metaplectic cocycle of $Sp(4,k)$ to the diagonal subgroup of $SL(2,E)$ is as follows. Let $h = diag(z, z^{-1}) \in \mathcal{H} \subset SL(2,E)$, with $z \in E^*$. Then $\varphi(h) = blockdiag(A, {}^tA^{-1})$, where $A = \left[\begin{smallmatrix} a & \alpha b \\ b & a \end{smallmatrix}\right]$ and $z = a + \sqrt{\alpha}b$.

The Steinberg cocycle (3.1) of $Sp(4,k)$, restricted to $\varphi(\mathcal{H})$, is $S(h,h') = t(hh')t(h)^{-1}t(h')^{-1}$ since $\varphi(\mathcal{H})$ preserves lagrangian subspaces. Moreover, $t(h) = \gamma(1)\gamma(\det(A))^{-1}$, so as before (*cf* (3.2)), we have $S(h,h') = (\det(A)/\det(A'))_k$. Now $\det(A) = a^2 - \alpha b^2 = N_{E/k}(z)$, the norm of z. The image of these norms in k^*/k^{*2} has index 2. Since k has residual characteristic different from 2, this implies that $\det(A)$ can take on only 2 different values modulo k^{*2}: denote these values $\{1, a\}$.

If $(a/a)_k = (a/-1)_k = 1$, then the Steinberg cocycle is identically 1 on $\varphi(\mathcal{H})$, implying that the cover splits. This occurs whenever $-1 \in k^{*2}$, or when $\alpha = -1 \notin k^{*2}$.

The remaining cases have $k^*/k^{*2} = \{1, -1, \varpi, -\varpi\}$, for $\varpi \in k$ an element of minimal positive valuation, and $\alpha \in \{\varpi, -\varpi\}$. It follows that $a = -\alpha$ and $(a/a)_k = -1$ and the Steinberg cocycle is nontrivial. (In fact, one can verify directly that the cocycle $(N_{E/k}(z), N_{E/k}(z'))_k$ is equivalent to the non-trivial Steinberg cocycle $(z/z')_E$.) Thus the cover does not split. □

Remark. The (2-)Hilbert symbol is not as simply understood for fields of the residual characteristic equal to 2. Part of the difficulty lies in the 2^{N+2} square classes in k^* (where N is the valuation of 2 in k); the triviality of the Steinberg cocycle is far less likely. For $\mathbb{Q}_2$, for example, one can show that for any nontrivial quadratic extension E, $SL(2,E)$ inherits a nontrivial metaplectic cover from φ.

B.3. **k-forms of $SL(4)$.** Finally, suppose $\mathbb{G}^\phi$ is $SL(4)$. Its nonsplit rational p-adic forms are (see [T1]): $SL(1,D)$, for D a central simple division algebra over k such that $[D:k] = 16$; $SL(2,D)$, for D the quaternions over k, *i.e.* $[D:k] = 4$; and $SU(E,h)$ for (two kinds of) degree 4 hermitian forms h over a quadratic extension field E of k. Over $k = \mathbb{R}$, its nonsplit rational forms are $SU(4)$, $SU(1,3)$, $SU(2,2)$ and $SL(2,\mathbb{H})$, where $\mathbb{H}$ denotes the quaternions over $\mathbb{R}$.

Denote the three fundamental weights in $SL(4)$ by λ_1, λ_2, and λ_3.

Lemma B.3. *The special unitary group forms of $SL(4)$ admit k-rational representations which decompose into $V_{\lambda_1} \oplus V_{\lambda_3}$ at the level of the algebraic closure, whereas the special linear group forms of $SL(4)$ do not.*

Proof. First let k denote a p-adic field. Let us apply Theorem 5.2. For the special unitary groups, we have [T2, Section 6] that d_{λ_2} is either 1 or 2, and that $d_{\lambda_1} = d_{\lambda_3} = 1$. Furthermore, we deduce from the Tits index of these groups that $\{\lambda_1, \lambda_3\}$ form the only nontrivial orbit of the Galois group. In particular, there is an 8-dimensional irreducible k-rational representation of G which decomposes into $V_{\lambda_1} \oplus V_{\lambda_3}$ at the level of the algebraic closure — namely, the standard representation.

For the special linear groups defined over central simple division algebras over k, we note that the orbits of the Galois group on the set of fundamental weights is trivial, and so in particular every irreducible representation $\rho^k_{\lambda_i}$ of is isotypic at the level of the algebraic closure. In particular, the standard representation ρ_{std} of $\mathbb{G}(k)$ is irreducible over k and decomposes over the algebraic closure as a direct sum of 4 (for $SL(1,D)$, $[D:k] = 16$) and 2 (for $SL(2,D)$, $[D:k] = 4$) copies of ρ_{λ_1}. Hence $\rho^k_{\lambda_1} = \rho_{std}$ in both cases, and $d_{\lambda_1} = 4$ (respectively, 2). Thus ρ_{λ_1} is not k-rational, nor does it occur with multiplicity 1 as an absolutely irreducible component of a k-rational representation of $\mathbb{G}(k)$. □

Hence, whenever $\mathfrak{g}[-1]$ decomposes as $V_{std} \oplus V^*_{std}$ under $G^\phi = SL(4,k)$, it follows that the only other k-rational forms of $\mathbb{G}^\phi$ that can occur correspond to special unitary groups.

Lemma B.4. *Let E be a quadratic extension field of k, and h a Hermitian form on E^4. The standard representation of $SU(h,E)$ gives a map $\varphi\colon SU(h,E) \to Sp(8,k)$. Then the metaplectic cover of $SU(h,E)$ determined by φ splits.*

Proof. As in the second part of the proof of Lemma B.1, the unitary group arises as half of a dual reductive pair, and hence admits a splitting over the special unitary group into the (two-fold) metaplectic group. □

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