Admissible Nilpotent Orbits of p-adic Split Exceptional Groups

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The Orbit Method conjectures a deep relationship between

$$\left\{\begin{array}{c}\text{equivalence classes of}\\ \text{irreducible unitary}\\ \text{representations of G}\end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{c}\textit{admissible}\\ \text{coadjoint}\\ \text{orbits of G}\end{array}\right\}$$

Established for:

- nilpotent Lie groups over $\mathbb{R}$ (Kirillov, 1962)
- nilpotent algebraic groups over p-adic fields (C.C. Moore, 1965)
- real solvable type I Lie groups (Auslander-Kostant, 1971)
- compact Lie groups

Duflo, 1980: geometric definition of admissible orbits (for $\mathbb{R}$).

For **reductive** groups:

- hyperbolic orbits $\rightarrow$ parabolic induction
- elliptic orbits $\rightarrow$ cohomological induction (over $\mathbb{R}$),
  $\rightarrow$ Hecke algebra isomorphisms? (over p-adics)
- nilpotent orbits $\rightarrow$ area of active research

Admissible Nilpotent Orbits

- identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via κ
- $X \in \mathfrak{g}$, nilpotent
- $\phi = \{X, H, Y\}$, an $\mathfrak{sl}(2)$ triple
- G^ϕ = the centralizer of ϕ in G, a reductive group
- $\mathfrak{g}[-1]$ = the -1 weight space of H acting on $\mathfrak{g}$
- Then $\mathfrak{g}[-1]$ has a symplectic structure, preserved by G^ϕ, given by

$$\langle U, V \rangle = \kappa(X, [U, V]) \qquad \forall U, V \in \mathfrak{g}[-1]$$

- This gives a map $G^\phi \to Sp(\mathfrak{g}[-1])$.

Consider the pullback of the unique two-fold cover Mp to G^ϕ:

$$\begin{tikzcd} \widetilde{G^\phi} \arrow[r] \arrow[d] & Mp(\mathfrak{g}[-1]) \arrow[d] \\ G^\phi \arrow[r, "Ad"'] & Sp(\mathfrak{g}[-1]) \end{tikzcd}$$

Definition: The nilpotent orbit is **admissible** if the cover $\widetilde{G^\phi}$ *splits* over an open subgroup of G^ϕ.

Who's who in admissible nilpotent orbits

Classical groups:

- $\mathbb{R}$: Schwarz (1987), Ohta (1991)
- p-adic: Nevins (1998)

Exceptional groups:

- $\mathbb{R}$: Noël (2001) (all)
- p-adic: Nevins (2002) (split G_2, F_4, E_6, E_7)

Determining admissibility in exceptional groups:

Using MATLAB:

- Find X, ϕ using Collingwood and McGovern
- Compute $\mathfrak{g}^\phi$, and deduce G_0^ϕ
- Determine map G^ϕ to $Sp(\mathfrak{g}[-1])$
- Compute $\widetilde{G^\phi}$ using results of C.C.Moore (IHES) and others

But: An algebraic orbit may decompose into several rational orbits. Which one did we get?

Answer: a "split" orbit.

- X is defined over $\mathbb{Z}$
- G^ϕ is a split group

What about other rational orbits?

“Nonsplit” orbits (G_2, F_4, E_6)

Galois cohomology counts rational orbits in an algebraic orbit:

- $\#H^1(k, G^\phi)$ is the number of orbits (G simply connected)
- hard to compute directly
- can use exact sequences to find an upper bound

Theorem of Mal’cev helps us find representatives of other rational orbits:

- Each rational orbit intersects $\mathfrak{g}[2]$ in an open G^H orbit
- G^H is a reductive group, acting on the finite-dimensional vector space $\mathfrak{g}[2]$
- Number of rational orbits is field-dependent

Admissibility of "nonsplit" orbits

Given a rational orbit (X', ϕ') in the same algebraic orbit as the split orbit (X, ϕ):

- $G^{\phi'}$ will be another rational form (possibly nonsplit) of G^{ϕ}
- Can compute $G^{\phi'}$ directly, OR
- Can restrict possible rational forms from action of G^{ϕ} on $\mathfrak{g}[-1]$ using Tits (1971)
- Then determine $\widetilde{G^{\phi'}}$

Find: Admissibility is independent of choice of rational orbit in a given algebraic orbit, EXCEPT FOR the B_2 orbit of F_4.

B_2 orbit of F_4 $(2-0 \Rightarrow\!= 0-1)$ (Assume $p \neq 2$)

- split orbit is not admissible
- there are a total of $|k^*/k^{*2}|$ rational orbits
- if $-1 \in k^{*2}$, then all nonsplit orbits are admissible
- otherwise, exactly one nonsplit orbit is admissible

Now: What are the admissible split orbits?

Special Nilpotent Orbits (of an algebraic group):

Definition: A special nilpotent orbit is:

- one associated to a special representation of the Weyl group via the Springer correspondence (Lusztig, 1978)
- one in the image of a certain inclusion-reversing map d on the set of nilpotent orbits (Spaltenstein, 1982)

These are algebraic, combinatorial definitions.

Connections with representation theory:

- representations of Weyl groups: they parametrize the two-sided cells
- classification of irreducible complex representations of reductive groups over finite fields
- classification of primitive ideals in $U(\mathfrak{g})$

Closely related geometric notion: Special Pieces.

Definition (Spaltenstein, 1982): Let S be a nilpotent orbit. The *special piece* $\gamma(S)$ is the union of nilpotent orbits $C \subseteq \overline{S}$ contained in no smaller special orbit.

To each special orbit S, Lusztig (1997) associates a finite group $\mathcal{G}_S$.

$$\{\text{orbits in } \gamma(S)\} \hookrightarrow \{\text{conjugacy classes of } \mathcal{G}_S\}.$$

G exceptional group:

- $\mathcal{G}_S = S_r$, for some $r \leq 5$.
- Orbits in $\gamma(S)$ are parametrized by partitions of r.

Definition: A completely odd orbit is one whose Lusztig parametrization is by a completely odd partition.

Results: split p-adic groups G_2, F_4, E_6, E_7

Theorem:

- All special orbits are admissible.
- The split admissible orbits are the completely odd orbits.
- For $p \neq 2$, G_2, F_4 and E_6: Admissibility is a stable criterion except for the B_2 orbit of F_4.

These results are consistent with Noël's work. Moreover, he has proven the completely odd link for real split E_8.