# Structure theory of p-adic groups via the Bruhat-Tits building 

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#### Abstract

In this paper, we will introduce the $p$-adic numbers and their associated prime ideal and integer ring. The parahoric subgroups for points in the standard apartment of $S O(4)$ are computed, as well as the tori in $S O(4)$ presented in [3]. The tori are compared with the parahoric subgroups. Then, the stabilizer subgroups and the points in the standard apartment fixed by the tori are determined.


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## 1 The $p$-adic numbers

The extension of the rationals through the usual absolute value norm is not its unique such extension. The completion of $\mathbb{Q}$ with respect to the $p$-adic metric yields the $p$-adic numbers.

### 1.1 The $p$-adic norm

We can define the notion of a norm abstractly.
Definition 1.1. A norm on a field $\mathbb{F}$ is a function $|\cdot|: \mathbb{F} \rightarrow \mathbb{R}^{+}$satisfying,
(1) $|x|=0 \Longrightarrow x=0$;
(2) $|x y|=|x||y|$ for all $x, y \in \mathbb{F}$;
(3) $|x+y| \leq|x|+|y|$ for all $x, y \in \mathbb{F}$;

Definition 1.2. A norm is called non-archimedean if it satisfies the Ultrametric Inequality, which is a stronger version of the Triangle Inequality,

$$
|x+y| \leq \max \{|x|,|y|\} \quad \forall x, y \in \mathbb{F},
$$

and archimedean otherwise.
The usual absolute value on $\mathbb{Q}$ is archimedean. A field is non-archimedean if it is equipped with a non-archimedean norm. Now, there exists a characterization of non-archimedean norms on $\mathbb{Q}$.

Lemma 1.3. A norm $|\cdot|$ on $\mathbb{Q}$ is non-archimedean if and only if $|x| \leq 1 \forall x \in \mathbb{N}$.
The proof of the above lemma can be found in [4].
Remark 1.4. (The Archimedean Property). A norm $|\cdot|$ on $\mathbb{Q}$ is archimedean if and only if for all $x, y \in \mathbb{Q}$, there exists $n \in \mathbb{N}$ such that $|n x|>|y|$.
Now, we define the norm which allows for the construction of the $p$-adic numbers.

Definition 1.5. The valuation of $n$ given a prime $p, v_{p}(n)$, is the unique natural number such that $n=p^{v_{p}(n)} n^{\prime}$ where $p \nmid n^{\prime}$. Extend this definition to $\mathbb{Q}$ by letting $v_{p}\left(\frac{a}{b}\right)=v_{p}(a)-v_{p}(b)$.

We will use the valuation to obtain the defining properties of a norm. Define, for any $x \in \mathbb{Q}$

$$
|x|_{p}=p^{-v_{p}(x)}
$$

Then, $|\cdot|_{p}$ is a non-archimedean norm on $\mathbb{Q}$ called the $p$-adic norm. Notice that the $p$-adic norm of a rational number is entirely determined by the prime factorization of the numerator and denominator. The induced metric $d(x, y)=|x-y|_{p}$ satisfies the ultrametric inequality.

To illustrate the notion of distance, consider the following example. The distance between $p$ and 0 is,

$$
|p-0|_{p}=|p|_{p}=\frac{1}{p}
$$

The distance between $p$ and 1 is, $|p-1|_{p}=|1-p|_{p}=1$. Also notice, $\left\{x:|x|_{p}<1\right\}=\left\{x:|x|_{p} \leq 1 / p\right\}$.
We can see that the $p$-adic norm of an integer $x$ expressed in base $p$ depends on the least significant nonzero digit of $x$. This is opposite from the standard absolute value, where an integer's size depends on its most significant digit.

### 1.2 Properties of non-archimedean fields

We will show some topological properties of non-archimedean field. As stated above, the field $\mathbb{Q}$, equipped with the $p$-adic norm is non-archimedean.

Proposition 1.6. Let $\mathbb{F}$ be a non-archimedean field. Then every point in an open ball is a center, so $b \in B_{r}(a) \Longrightarrow B_{r}(a)=B_{r}(b)$.

Proof. For any $b \in B_{r}(a)$, we know $|b-a|<r$. For any $x \in B_{r}(a)$,

$$
|x-b|=|(x-a)+(a-b)| \leq \max \{|x-a|,|a-b|\}<r .
$$

Hence, $x \in B_{r}(b)$ as $|x-b|<r \Longrightarrow B_{r}(a) \subseteq B_{r}(b)$. Similarly, we obtain $B_{r}(a) \subseteq B_{r}(b)$.
From this result, we have the following consequence.
Corollary 1.7. Let $\mathbb{F}$ be a non-archimedean field. Then any 2 open balls are either disjoint or one is contained in the other, so

$$
x \in B_{r}(a) \wedge x \in B_{s}(b) \Longrightarrow B_{r}(a) \subseteq B_{s}(b) \text { or } B_{s}(b) \subseteq B_{r}(a)
$$

Proof. Suppose, without loss of generality, $r<s$. If $x \in B_{r}(a)$, then $B_{r}(a)=B_{r}(x)$ and if $x \in B_{s}(b)$, then $B_{s}(b)=B_{s}(x)$, by Proposition 1.6. Then,

$$
B_{r}(a)=B_{r}(x) \subseteq B_{s}(x)=B_{s}(b)
$$

as required.
We have shown that the complement of an open ball is the union of all the open balls in its complement. However a union of open balls is open. Hence, every open ball in a non-archimedean field is both open and closed.

### 1.3 The construction of the $p$-adic numbers

Any positive rational number $r$ can be represented with a decimal expansion, $r=\sum_{i=k}^{\infty} a_{i} 10^{-i}$, where $0 \leq a_{i}<10$. Now, we can also represent any positive rational number $r$ by a $p$-adic integer, which is given by, $r=\sum_{i=0}^{\infty} a_{i} p^{i}$, where $p$ is prime and $0 \leq a_{i}<p$. This representation of the rational numbers allows us to consider congruences modulo $p^{n}$.

Definition 1.8. The $p$-adic integers are defined to be the set of the series,

$$
\mathbb{Z}_{p}=\left\{\sum_{i=0}^{\infty} a_{i} p^{i}: 0 \leq a_{i}<p\right\}
$$

Addition of $p$-adic integers is defined component-wise modulo $p$, carrying remainders. Notice that $\mathbb{Z} \subset \mathbb{Z}_{p}$.
The $p$-adic integers form an additive group, as additive inverses can be defined for every $p$-adic integer. First, we will show $\sum_{i=0}^{\infty}(p-1) p^{i}=-1$. This series converges in the $p$-adic norm as for all $n \geq 0$, $\left|\frac{(p-1) p^{n+1}}{(p-1) p^{n}}\right|=|p|=\frac{1}{p}$. Now, as $1=p^{0}$,

$$
p^{0}+\sum_{i=0}^{\infty}(p-1) p^{i}=0
$$

For any $p$-adic integer $\alpha=\sum_{i=1}^{\infty} a_{i} p^{i}$, define,

$$
\sigma(\alpha)=\sum_{i=0}^{\infty}\left(p-1-a_{i}\right) p^{i}
$$

Then,

$$
\begin{aligned}
\alpha+\sigma(\alpha) & =\sum_{i=0}^{\infty} a_{i} p^{i}+\sum_{i=0}^{\infty}\left(p-1-a_{i}\right) p^{i} \\
& =\sum_{i=0}^{\infty}\left(p-1-a_{i}+a_{i}\right) p^{i} \\
& =-1
\end{aligned}
$$

Hence, $-\alpha=\sigma(\alpha)+1$. Also, $\sum_{i=1}^{\infty} 2^{i}=-1$ for $p=2$. Notice, for any $p, \sum_{i=0}^{\infty} p^{i}=\frac{1}{1-p}$.
The sequence $\sum_{i=1}^{\infty} 2^{i}=-1$, where $p=2$ is Cauchy with respect to $|\cdot|_{p}$, but diverges in $\mathbb{R}$. The geometric series identity holds for any $p>1$.
Notice the result of the division of the $p$-adic expansions of two integers is, in general, an infinite series with finitely many negative $p$-power terms. Now, suppose we wanted to define a $p$-adic expansion for a rational number $\frac{a}{b}$. If $p$ does not divide the denominator $b$, then simply represent numerator and denominator by the $p$-adic integer expansion and divide the polynomials. If $p$ divides the denominator $b$, then we need to use negative powers of $p$ to represent the rational number.

### 1.4 The completion of $\mathbb{Q}$ with respect to the $p$-adic norm.

Definition 1.9. The completion of a field $\mathbb{F}$ with respect to an absolute value $|\cdot|$ is the set,

$$
\left\{\left[\left(a_{n}\right)\right]:\left(a_{n}\right) \text { is a Cauchy sequence in } \mathbb{F}\right\}
$$

which is the set of all equivalence classes of Cauchy sequences, where the equivalence relation is,

$$
\left(a_{n}\right) \equiv\left(b_{n}\right) \text { if } \forall \varepsilon \in \mathbb{R}, \varepsilon>0, \exists N \in \mathbb{N} \text { such that } \forall n \geq N,\left|a_{n}-b_{n}\right|<\varepsilon
$$

A field is embedded in its completion (consider the sequence $(x, x, \ldots)$ for all $x \in \mathbb{F})$.
We can form the completion of $\mathbb{Q}$ with respect to $|\cdot|_{p}$.
Definition 1.10. Let $\mathbb{Q}_{p}$ be the completion of $\mathbb{Q}$ with respect to $|\cdot|_{p}$. We call $\mathbb{Q}_{p}$ the $p$-adic numbers and denote the extended absolute value on $\mathbb{Q}_{p}$ still by $|\cdot|_{p}$. For $x=\left[\left(x_{n}\right)\right] \in \mathbb{Q}_{p},|x|_{p}=\lim \left|x_{n}\right|_{p}$.
The completion of $\mathbb{Q}$ with respect to the $p$-adic norm is the set of all series of the form $\sum_{i=k}^{\infty} a_{i} p^{i}$, where $0 \leq a_{i}<p$ and $k \in \mathbb{Z}$.
Theorem 1.11. $\mathbb{Q}_{p}$ is complete with respect to $|\cdot|_{p}$.
The proof of the above theorem can be found in [4].
Remark 1.12. We can think of $\mathbb{Q}_{p}$ as the set of all Laurent series in $p$, with the addition and multiplication inherited from $\mathbb{Q}$ as above. Then, $\mathbb{Z}_{p}$ is the set of power series, and is the ball of radius 1 centered at 0 .

### 1.5 The integer ring $\mathcal{O}$ and the subgroups $\mathcal{P}^{n}$

To the $p$-adic numbers, we can associate an integer ring. Consider the ring given by,

$$
\mathcal{O}=\mathbb{Z}_{p}=\bar{B}_{1}(0)=\left\{x \in \mathbb{Q}_{p}:|x| \leq 1\right\}
$$

Notice $|x| \leq 1 \Longleftrightarrow \operatorname{val}(x) \geq 0$. Equivalently,

$$
\mathcal{O}=\left\{\sum_{i=0}^{\infty} a_{i} p^{i}: a_{i} \in\{0, \ldots, p-1\}\right\}
$$

Proposition 1.13. For $\mathcal{O}=\{x \in F:|x| \leq 1\}, \mathcal{P}=\{x \in \mathcal{O}:|x|<1\}$ is a maximal ideal, called the prime ideal, and every $y \in \mathcal{O} \backslash \mathcal{P}$ is invertible in $\mathcal{O}$.

Proof. To show $\mathcal{P}$ is an ideal, let $x, y \in \mathcal{P}$. Then, $|x+y| \leq \max \{|x|,|y|\} \leq 1$, so $x+y \in \mathcal{P}$. Now, let $x \in \mathcal{O}, y \in \mathcal{P}$, and $|x y|=|x||y|<1$ so $x y \in \mathcal{P}$. Also, $|x|=|-x|$ and $|0|=0$, so for every $x \in \mathcal{P}$, $-x \in \mathcal{P}$ and $0 \in \mathcal{P}$.

Now, if $x_{1}, x_{2} \in \mathcal{O}$ and $x_{1} x_{2} \in \mathcal{P}$, then one of $x_{1}$ or $x_{2}$ are in $\mathcal{P}$, as otherwise $\left|x_{1} x_{2}\right|=1 \notin \mathcal{P}$, hence $\mathcal{P}$ is a prime ideal of $\mathcal{O}$.

To show $\mathcal{P}$ is maximal, consider $\mathcal{O} \backslash \mathcal{P}=\left\{x \in \mathbb{Q}_{p}:|x|=1\right\}$. We have $x^{-1} \in \mathcal{O} \backslash \mathcal{P}$ as $\left|x x^{-1}\right|=1$, which implies $\left|x^{-1}\right| \cdot 1=1$. Thus, $\mathcal{O} \backslash \mathcal{P}=\mathcal{O}^{\times}$. Since $\mathcal{P}$ is prime and everything outside $\mathcal{P}$ is invertible, every ideal of $\mathcal{O}$ is contained in $\mathcal{P}$. Thus, $\mathcal{P}$ is the unique maximal ideal of $\mathcal{O}$.

The ideal $\mathcal{P}$ is also principal, as for $a, b \in \mathcal{P}$ with maximal norm, $a b^{-1} \in \mathcal{O} \backslash \mathcal{P}=\mathcal{O}^{\times}$. As $\left|a b^{-1}\right|=1$, both $a$ and $b$ generate $P$. In particular, for the $p$-adic numbers $\mathbb{Q}_{p}, \mathcal{O}=\mathbb{Z}_{p}$ and $\mathcal{P}=p \mathbb{Z}_{p}$.

Definition 1.14. A generator $\varpi$ of $\mathcal{P}$ is called a uniformizer for $\mathbb{Q}_{p}$.
Fix a generator $\varpi$ of $\mathcal{P}$, which has maximal norm. Now, we will prove every element $a \in \mathbb{Q}_{p}$ can be written as $u \varpi^{n}$ for some $u \in \mathcal{O}^{\times}, n \in \mathbb{Z}$. Choosing the largest $n \in \mathbb{Z}$ such that $\left|a \varpi^{n}\right| \leq 1$, we have, $|\varpi|<\left|a \varpi^{-n}\right| \leq 1$. However, $\varpi$ is is the element of $\mathcal{P}$ with maximal norm, so $\left|a \varpi^{-n}\right|=1$, and $a \varpi^{-n}=u$ for some $u \in \mathcal{O}^{\times}$.

Now, we can define a chain of subgroups based on the norm of the $p$-adic field.
Definition 1.15. Let $\mathcal{P}^{0}=\mathcal{O}$. For any integer $n$, let $\mathcal{P}^{n}$ denote the set,

$$
\mathcal{P}^{n}=\left\{x \in \mathbb{Q}_{p}: \operatorname{val}(x) \geq n\right\} .
$$

Every $\mathcal{P}^{n}$ is a group, but $\mathcal{P}^{n}$ is only an ideal of $\mathcal{O}$ when $n \geq 0$. If $n \leq-1$, then $\mathcal{P}^{n}$ is a group, but not a subset of $\mathcal{O}$.

We obtain the chain,

$$
\cdots \subseteq \mathcal{P}^{2} \subseteq \mathcal{P} \subseteq \mathcal{O} \subseteq \mathcal{P}^{-1} \subseteq \mathcal{P}^{-2} \subseteq \cdots
$$

Definition 1.16. For the integer ring $\mathcal{O}$ and the prime ideal $\mathcal{P}, \mathfrak{F}=\mathcal{O} / \mathcal{P}$ is the residue field of $\mathbb{F}$.
If $\mathbb{F}=\mathbb{Q}_{p}$, the residue field is $\mathbb{Z} / p \mathbb{Z}$.

## 2 Field extensions of $\mathbb{Q}_{p}$

In this section, we describe unramified and ramified field extensions. Also, in this section, we can set $\mathbb{F}=\mathbb{Q}_{p}$ and $\mathcal{O}=\mathbb{Z}_{p}$.

Proposition 2.1. For every prime $p$ and $n \in \mathbb{N}$, there exists a unique field of size $p^{n}$ constructed by $\mathbb{F}_{p}[x] /\langle m\rangle$, where $m$ is an irreducible polynomial of degree $n$.

Refer to Theorem 9.14 in [6] for the proof of the above proposition.
We can consider two kinds of extensions on the $p$-adic numbers.

### 2.1 Unramified extensions

First, we discuss field extensions of a field $\mathbb{F}$ which arise from extensions of the residue field.
Definition 2.2. Suppose $p \neq 2$. Let $m(x) \in \mathcal{O}[x]$ be an irreducible monic polynomial. Let $\bar{m}(x)=$ $m(x) \bmod p$, so the polynomial $\bar{m}(x) \in \mathfrak{F}[x]$. If $\bar{m}(x)$ is irreducible, then, $E=\mathbb{F}[x] /\langle m\rangle$ is an unramified extension of $\mathbb{F}$.

A consequence of Proposition 2.1 is that any two irreducible polynomials with equal degree in $\mathcal{O}[x]$ whose reduction modulo $P$ is also irreducible will give the same extension $E$, which is not the case for all fields.

Example 2.3. We give an example of an unramified quadratic extension. Let $m(x)=x^{2}-\varepsilon$ be an irreducible polynomial in $\mathcal{O}[x]$, where $\varepsilon \in \mathcal{O}$ is a non-square and suppose $\bar{m}(x)=x^{2}-\bar{\varepsilon}$, where $\bar{\varepsilon} \in \mathfrak{F}$ is a non-square. As $\bar{\varepsilon}$ is a non-square, $\bar{m}(x)$ is irreducible. Then,

$$
E=\mathbb{F}[\sqrt{\varepsilon}]=\{a+b \sqrt{\varepsilon}: a, b \in \mathbb{F}\}
$$

is an unramified extension field of $\mathbb{F}$. As $\bar{\varepsilon} \neq 0$ and $\bar{\varepsilon} \in \mathcal{O} / \mathcal{P}, \operatorname{val}(\varepsilon)=0$ and $\varepsilon \in \mathcal{O}^{\times}$. Hence, $|\sqrt{\varepsilon}|=1$. If $E=\mathbb{F}[\sqrt{\varepsilon}]$, we can define the integer ring and maximal prime ideal of $E$ by,

$$
\mathcal{O}_{E}=\{a+b \sqrt{\varepsilon}:|a+b \sqrt{\varepsilon}| \leq 1\}=\{a+b \sqrt{\varepsilon}: a, b \in \mathcal{O}\}
$$

and,

$$
\mathcal{P}_{E}=\{a+b \sqrt{\varepsilon}:|a+b \sqrt{\varepsilon}|<1\}=\{a+b \sqrt{\varepsilon}: a, b \in \mathcal{P}\}
$$

The residue field $\mathcal{O}_{E} / \mathcal{P}_{E} \cong \mathfrak{F}[\sqrt{\bar{\varepsilon}}]$.

### 2.2 Ramified extensions

Now, we consider a second type of field extension.
Definition 2.4. Any extension field $E$ that is not unramified is called ramified.
Example 2.5. We give an example of a ramified quadratic extension. Let $\varpi$ be a uniformizer of $\mathbb{F}$, so $\varpi$ has the maximal norm which is strictly less than 1 and $|\varpi|<|\sqrt{\varpi}|<1$. If $\mathbb{F}=\mathbb{Q}_{p}$, then $|\varpi|=\frac{1}{p}$ and $|\sqrt{\varpi}|=\frac{1}{\sqrt{p}}$. Then,

$$
\mathcal{O}_{E}=\{a+b \sqrt{\varpi}:|a+b \sqrt{\varpi}| \leq 1\}=\{a+b \sqrt{\varpi}: a, b \in \mathcal{O}\},
$$

where $\operatorname{val}(b \sqrt{\varpi})=\frac{1}{2}$ and,

$$
\mathcal{P}_{E}=\{a+b \sqrt{\varpi}:|a+b \sqrt{\varpi}|<1\}=\{a+b \sqrt{\varpi}: a \in \mathcal{P}, b \in \mathcal{O}\},
$$

as we must have $\operatorname{val}(a) \geq 1$. One can show that $\mathcal{O}_{E} / \mathcal{P}_{E} \cong \mathcal{O} / \mathcal{P}$ in this case; when this happens, the extension is called purely ramified.

Every extension of $\mathbb{F}$ is either unramified, purely ramified or partially ramified. We give an example of a partially ramified quartic extension.

Example 2.6. The field extension $\mathbb{F}[\sqrt{\varepsilon}, \sqrt{\varpi}]$ is a partially ramified quartic extension. Its residue field is isomorphic to $\mathbb{F}[\sqrt{\bar{\varepsilon}}]$, and in fact it has exactly three intermediate field extensions: $\mathbb{F}[\sqrt{\varepsilon}]$ which is unramified, and $\mathbb{F}[\sqrt{\varpi}]$ and $\mathbb{F}[\sqrt{\varepsilon \varpi}]$ which are both (purely) ramified.

## 3 The root system of $\mathfrak{s o}$ (4)

In this section, we will introduce the root space decomposition of the Lie algebra $\mathfrak{s o}(4)$, as presented in [5].

### 3.1 Root systems

Consider a finite-dimensional vector space $V$ over $\mathbb{R}$ endowed with a positive-definite symmetric billinear form $\langle\cdot, \cdot\rangle$. Given $a \neq 0 \in V$, the set

$$
H_{a}=\{\lambda \in V:\langle\lambda, a\rangle=0\}
$$

is a hyperplane of $V$. A reflection on $V$ is a linear operator $s$ on $V$ such that there exists an element $\beta \in V$ such that $s(\beta)=-\beta$ and $s$ fixes the hyperplane perpendicular to $\mathbb{R} \beta$. Evidently, a reflection is orthogonal, meaning it preserves the inner product on $V$.

For any $\beta \in V, s_{\alpha}(\beta)=\beta-\frac{2\langle\beta, \alpha\rangle}{\langle\alpha, \alpha\rangle}$. For $c \in \mathbb{R}$ and $\nu \in H_{\alpha}$,

$$
s_{\alpha}(\alpha)=-\alpha, s_{\alpha}(c \alpha)=-c \alpha, s_{\alpha}(\nu)=\nu
$$

Finally, for any $\beta \in V, V=\mathbb{R} \alpha \oplus H_{\alpha}$.
Definition 3.1. A subset $\Phi$ of the vector space $V$ is a root system in $V$ if the following axioms are all satisfied:
(i) $\Phi$ is finite, $\Phi$ does not contain 0 and spans $V$.
(ii) If $\alpha \in \Phi,-\alpha \in \Phi$ and $\mathbb{R} \alpha \cap \Phi=\{\alpha,-\alpha\}$.
(iii) For any $\alpha \in \Phi, \Phi$ is invariant under reflection $s_{\alpha}$.
(iv) $\frac{2\langle\beta, \alpha\rangle}{\langle\alpha, \alpha\rangle} \in \mathbb{Z}$, for all $\alpha, \beta \in \Phi$.

Example 3.2. Consider $\mathbb{R}^{3}$ with the usual dot product, with the usual basis vectors denoted by $\left\{e_{1}, e_{2}, e_{3}\right\}$ and let $V$ be the plane orthogonal to $e_{1}+e_{2}+e_{3}$. Consider $\Phi=\left\{ \pm\left(e_{1}-e_{2}\right), \pm\left(e_{1}-\right.\right.$ $\left.\left.e_{3}\right), \pm\left(e_{2}-e_{3}\right)\right\}$. Notice,

$$
\begin{aligned}
& \left(e_{1}+e_{2}+e_{3}\right) \cdot\left(e_{1}-e_{2}\right)=0 \\
& \left(e_{1}+e_{2}+e_{3}\right) \cdot\left(e_{1}-e_{3}\right)=0 \\
& \left(e_{1}+e_{2}+e_{3}\right) \cdot\left(e_{2}-e_{3}\right)=0
\end{aligned}
$$

Now, we verify the invariance of $\Phi$ under reflections.

$$
\begin{aligned}
& s_{e_{1}-e_{2}}\left(e_{1}-e_{3}\right)=-\left(e_{1}-e_{3}\right) ; \\
& s_{e_{1}-e_{3}}\left(e_{2}-e_{3}\right)=-\left(e_{2}-e_{3}\right) ; \\
& s_{e_{1}-e_{3}}\left(e_{1}-e_{2}\right)=-\left(e_{1}-e_{2}\right) .
\end{aligned}
$$

Finally, we give the following example.

$$
2 \frac{\left(e_{1}-e_{2}\right) \cdot\left(e_{2}-e_{3}\right)}{\left(e_{2}-e_{3}\right) \cdot\left(e_{2}-e_{3}\right)}=\frac{2(-1)}{1+2}=-1 \in \mathbb{Z}
$$

Hence $\Phi$ is a root system and $V=\operatorname{span}(\Phi)$.
Definition 3.3. Let $\Phi$ be a root system in $V$. Denote by $W$ the subgroup of $G L(V)$ generated by the reflections $s_{\alpha}, \alpha \in \Phi$. $W$ permutes the elements of $\Phi$, so $W$ is a finite subgroup of the symmetric group of $\Phi . W$ is the Weyl group of $\Phi$.

The definition of the Weyl group can be extended to the affine Weyl group. For each $\alpha \in \Phi$, define the coroot of $\alpha$, denoted by $\check{\alpha}$, to be,

$$
\check{\alpha}=\frac{2 \alpha}{(\alpha, \alpha)} .
$$

Definition 3.4. Let $W$ be the Weyl group of the root system $\Phi$. Consider the lattice $\Lambda$ generated by the coroots $\check{\alpha}$, for $\alpha \in \Phi$. The semi-direct product $W_{a f}=W \ltimes \Lambda$ is the affine Weyl group.

### 3.2 The root space decomposition of $\mathfrak{s o}(4)$

Let $\mathfrak{g}$ be a nontrivial semisimple finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero. Given a Cartan subalgebra $\mathfrak{h}$ and root system $\Phi$ of $\mathfrak{g}$, we can decompose $\mathfrak{g}=\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$, where $\mathfrak{g}_{\alpha}$ is the weight space associated with the root $\alpha$. Recall the normalizer of $\mathfrak{h}$, denoted by $N_{L}(\mathfrak{h})$ is given by, $N_{L}(\mathfrak{h})=\{x \in \mathfrak{g}:[x, \mathfrak{h}] \in \mathfrak{h}\}$. The classical Lie algebras satisfy the above definition, hence a root space decomposition exists for $\mathfrak{s o}(4)$.
We know $\mathfrak{s o}(4, \mathbb{F})=\left\{X \in \mathfrak{g l}(4, k): X^{\top} J=-J X\right\}$ where,

$$
J=\left[\begin{array}{cc}
0 & I_{2} \\
I_{2} & 0
\end{array}\right],
$$

where $I_{2}$ denotes the $2 \times 2$ identity matrix. The root system of $\mathfrak{s o}(4, \mathbb{F})$ is is $\Phi=\{ \pm \alpha, \pm \beta\}$, where $\alpha=e_{1}-e_{2}$ and $\beta=e_{1}+e_{2}$.

Consider the Cartan subalgebra,

$$
\mathfrak{h}=\left\{\left[\begin{array}{cccc}
a & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & -a & 0 \\
0 & 0 & 0 & -b
\end{array}\right]: a, b \in \mathbb{F}\right\} .
$$

For the root system $\Phi$ of $\mathfrak{s o}(4), \mathfrak{s o}(4, \mathbb{F})=\mathfrak{h} \oplus_{\alpha \in \Phi} L_{\alpha}$, where $L_{\alpha}$ is the root space defined by, $L_{\alpha}=$ $\{x \in \mathfrak{s o}(4, \mathbb{F}): \operatorname{ad}(H)=\alpha(H) X$ for all $H \in \mathfrak{h}\}$.

## 4 The Bruhat-Tits Building

We will introduce the Bruhat-Tits building as described in [7].

### 4.1 The apartment associated to the maximal torus

Let $\mathbf{G}$ be an algebraic group defined over a field $\mathbb{F}$ and let $G=\mathbf{G}(\mathbb{F})$. Every group $G$ has a maximal split torus over $\mathbb{F}$, denoted by $\mathbf{S}$.

Example 4.1. The maximal torus of $G L(2 n)$ is given by,

$$
\left\{\left[\begin{array}{cccc}
t_{1} & 0 & \cdots & 0 \\
0 & t_{2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & t_{n}
\end{array}\right]: t_{1}, t_{2}, \cdots, t_{n} \in \mathbb{F}\right\}
$$

as shown in [2]. From this, we know the maximal split torus of $S O(4, \mathbb{F})$ is given by,

$$
\mathbf{S}=\left\{\left[\begin{array}{cccc}
t_{1} & 0 & 0 & 0 \\
0 & t_{2} & 0 & 0 \\
0 & 0 & t_{1}^{-1} & 0 \\
0 & 0 & 0 & t_{2}^{-1}
\end{array}\right]: t_{1}, t_{2} \in \mathbb{F}\right\}
$$

The group of $\mathbb{F}$-points is $S=\mathbf{S}(\mathbb{F})$.
To every maximal split torus $\mathbf{S}$ over $\mathbf{G}$, we associate an apartment $\mathcal{A}=(\mathbf{G}, \mathbf{S}, \mathbb{F})$. The apartment $\mathcal{A}$ is a real vector space spanned by the coroots of the root system of $G$ with respect to $S, \Phi(G, S)$. In particular, from Section 3, the root system of $\mathfrak{s o}(4)$ is $\Phi=\{ \pm \alpha, \pm \beta\}$. Then, $\Phi \subseteq \mathcal{A}^{*}$ and for every $\alpha \in \Phi$,

$$
\alpha: \mathcal{A} \rightarrow \mathbb{R}, x \mapsto \alpha(x)
$$

To every point in the apartment, we can assign a parahoric subgroup $G_{x, 0}$, which depends on action of the root vectors on the point $x, \alpha(x), \alpha \in \Phi$. Apartments are "glued" together through the parahoric subgroups of $G$. Parahoric subgroups allow for the study of the affine structure theory of $G$. First, we define, for $\alpha \in \Phi$,

$$
x_{\alpha}(t)=\left[\begin{array}{cccc}
1 & t & 0 & 0  \tag{1}\\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -t & 1
\end{array}\right], \quad x_{-\alpha}(r)=\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
r & 1 & 0 & 0 \\
0 & 0 & 1 & -r \\
0 & 0 & 0 & 1
\end{array}\right]
$$

and similarly, for $\beta \in \Phi$,

$$
x_{\beta}(s)=\left[\begin{array}{cccc}
1 & 0 & 0 & s  \tag{2}\\
0 & 1 & -s & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right], \quad x_{-\beta}(v)=\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -v & 1 & 0 \\
v & 0 & 0 & 1
\end{array}\right]
$$

As shown in [7],

$$
G_{x, 0}=\left\langle\mathbf{S}(\mathbb{R}), x_{\gamma}(t): \gamma \in \Phi, \operatorname{val}(t)+\gamma(x) \geq 0\right\rangle
$$

Now, when $\mathbf{G}$ is not simply connected, compute the normalizer of $x$ in $\mathcal{A}$ under the action of $\mathbf{N}(F)$, where $\mathbf{N}=\operatorname{Norm}_{\mathbf{G}}(\mathbf{S})$. The normalizer acts by affine isometries on the apartment.
For $S O(4)$, the normalizer of a point $x$ is computed by the action on the apartment of the group generated by the affine Weyl group and translations by any integer linear combination of $e_{1}$ and $e_{2}$. The reader may refer to Appendix A of [7].
Notice that an element of the split torus $s \in \mathbf{S}, s=\operatorname{diag}\left(a, b, a^{-1}, b^{-1}\right)$ induces a translation in the building, $t_{-v(s)}$, where $v_{s}=(\operatorname{val}(a), \operatorname{val}(b))$, so $\mathbf{S}$ induces translations by $\mathbb{Z}^{2}$, which is greater than the translations induces by the Weyl group.

Notation 4.2. Let $r_{\alpha, n}=r_{\alpha} \circ t_{-n \check{\alpha}}$ be the reflections in the affine hyperplanes $H_{\alpha, n}$, where $H_{\alpha, n}=$ $\{x \in \mathcal{A}: \alpha(x)=n\}$

Remark 4.3. Reflections have the following properties:
(i) $r_{\alpha}(x, y)=(y, x)$,
(ii) $r_{\beta}(x, y)=(y,-x)$.

Now, we will illustrate a reflection of a point in an apartment. The point $\left(0, \frac{1}{2}\right)$ can be reflected by $r_{\alpha, 0}$, which gives $r_{\alpha}\left(0, \frac{1}{2}\right)=\left(\frac{1}{2}, 0\right)$ and translated by $t_{\frac{1}{2}, 0}$ to the point $(1,0)$. Notice $t_{\frac{1}{2}, 0} \notin W_{a f}$. Also, $(3,0)$ can be reflected to $(2,1)$ by $r_{\alpha, 2}$.


Figure 1: Reflection of points in an apartment

Every element in the affine Weyl group can be realized by the normalizer of the torus, however there exist elements of the normalier which are not contained in the affine Weyl group.

Definition 4.4. The Bruhat-Tits building is the set of equivalence classes $\mathrm{B}(\mathbf{G}, \mathbb{F})=(G \otimes \mathcal{A}) / \sim$ where $(g, x) \sim(h, y)$ if there exists an element $n \in N$ such that $n \cdot y=x$ and $g^{-1} h n \in G_{x, 0}$. Then $G$ acts on $\mathrm{B}(\mathbf{G}, \mathbb{F})$ by the rule $h \cdot(g, x)=(h g, x)$

Now, we may determine the stabilizer of a point, for example the point $(1, x)$. The stabilizer is the element $n \in N$ that sends $x$ to $x$ and $h n \in G_{x, 0}$.

### 4.2 The parahoric subgroups of $S O(4)$

In this section, will discuss the parahoric subgroups $G_{x, 0}$. First, we must note some consequences of the non-simply-connectedness of $S O(4)$.

It is known that $S O(4)$ is not a simply connected algebraic group. One consequence of this is that we cannot immediately deduce the valuations of all of the matrix entries of the parahoric subgroup directly from the valuations of the root subgroups.

### 4.2.1 A formula for the parahoric subgroups

We will introduce some notation. Given an $m \times m$ matrix, $M=\left\{\left[t_{i, j}\right]_{1 \leq i, j \leq m}: t_{i, j} \in \mathcal{P}^{n_{i, j}}\right\}$ for some $n_{i, j} \in \mathbb{Z}$, let $M$ be denoted by, $M:=\left[\mathcal{P}^{n_{i, j}}\right]_{1 \leq i, j \leq m}$.

Proposition 4.5. Let $x$ be a point in the standard apartment $\mathcal{A}$. The parahoric subgroup associated to $x$ is,

$$
G_{x, 0}=\left[\begin{array}{cccc}
\mathcal{O} & \mathcal{P}^{-\lfloor\alpha(x)\rfloor} & \mathcal{P}^{-\lfloor\alpha(x)\rfloor-\lfloor\beta(x)\rfloor} & \mathcal{P}^{-\lfloor\beta(x)\rfloor}  \tag{3}\\
\mathcal{P}^{\lceil\alpha(x)\rceil} & \mathcal{O} & \mathcal{P}^{-\lfloor\beta(x)\rfloor} & \mathcal{P}^{\lceil\alpha(x)\rceil-\lfloor\beta(x)\rfloor} \\
\mathcal{P}^{\lceil\alpha(x)\rceil+\lceil\beta(x)\rceil} & \mathcal{P}^{\lceil\beta(x)\rceil} & \mathcal{O} & \mathcal{P}^{\lceil\alpha(x)\rceil} \\
\mathcal{P}^{\lceil\beta(x)\rceil} & \mathcal{P}^{-\lfloor\alpha(x)\rfloor+\lceil\beta(x)\rceil} & \mathcal{P}^{-\lfloor\alpha(x)\rfloor} & \mathcal{O}
\end{array}\right] \cap G
$$

Proof. We have $G_{x, 0}=\left\langle\mathbf{S}(\mathcal{O}), x_{\gamma}(t) \mid \gamma \in \Phi, \operatorname{val}(t)+\gamma(x) \geq 0\right\rangle$. First, consider the following properties, which will give a formula for the parahoric subgroups of $S O(4)$,
(i) $x_{\alpha}(s) x_{\beta}(t)$ has entry $-s t$ in the $(1,3)$ matrix position;
(ii) $x_{-\alpha}(r) x_{\beta}(t)$ has entry $r t$ in the $(2,4)$ matrix position;
(iii) $x_{-\beta}(v) x_{-\alpha}(r)$ has entry $-v r$ in the $(3,1)$ matrix position;
(iv) $x_{-\beta}(v) x_{\alpha}(t)$ has entry $t v$ in the $(4,2)$ matrix position.

From the properties of the root subspaces above, (1) and (2), we can derive the restriction on the valuations of $x$.

As we allow only integer vaulations for $t$,

$$
\begin{aligned}
\operatorname{val}(t) \geq-\alpha(x) & \Longrightarrow \operatorname{val}(t) \geq-\lfloor\alpha(x)\rfloor \\
\operatorname{val}(s) \geq-\beta(x) & \Longrightarrow \operatorname{val}(s) \geq-\lfloor\beta(x)\rfloor \\
\operatorname{val}(r) \geq \alpha(x) & \Longrightarrow \operatorname{val}(r) \geq\lceil\alpha(x)\rceil \\
\operatorname{val}(v) \geq \beta(x) & \Longrightarrow \operatorname{val}(v) \geq\lceil\beta(x)\rceil
\end{aligned}
$$

Consider, $x_{\alpha}(t) x_{\beta}(s)$. By fact (i) above,

$$
\operatorname{val}(-s t)=\operatorname{val}(s)+\operatorname{val}(t) \geq-\lfloor\alpha(x)\rfloor-\lfloor\beta(x)\rfloor
$$

Consider, $x_{-\alpha}(r) x_{\beta}(s)$. By fact (ii) above,

$$
\operatorname{val}(r s)=\operatorname{val}(r)+\operatorname{val}(s) \geq\lceil\alpha(x)\rceil-\lfloor\beta(x)\rfloor
$$

Consider, $x_{-\beta}(v) x_{-\alpha}(r)$. By fact (iii) above,

$$
\operatorname{val}(v r)=\operatorname{val}(v)+\operatorname{val}(r) \geq\lceil\alpha(x)\rceil+\lceil\beta(x)\rceil
$$

Consider, $x_{-\beta}(v) x_{\alpha}(t)$. By fact (iv) above,

$$
\operatorname{val}(v t)=\operatorname{val}(v)+\operatorname{val}(t) \geq-\lfloor\alpha(x)\rfloor+\lceil\beta(x)\rceil
$$

We have solved for the subgroups $\mathcal{P}^{n}$ such that, $t \in \mathcal{P}^{n} \Longrightarrow \operatorname{val}(t) \geq-\gamma(x)$ for every $\gamma \in \Phi$. It is now routine to show (3) is closed under multiplication, so forms a group. Then, $G_{x, 0}$ contains $\left\langle\mathbf{S}(\mathcal{O}), x_{\gamma}(t) \mid \gamma \in \Phi, \operatorname{val}(t)+\gamma(x) \geq 0\right\rangle$. It can be shown from Bruhat-Tits theory, as developed in [1], that equality holds.

Each subgroup $G_{x, 0}$ is compact and open with respect to the induced topology on $G$ as a group of $p$-adic matrices.

### 4.2.2 The group $G_{x}$

Let $r_{\alpha}$ denote the reflection which fixes the hyperplane $H_{\alpha}$. Every affine reflection is the product of a linear reflection and a translation. For example,

$$
r_{e_{1}-e_{2}, 1}=\left[\begin{array}{cccc}
0 & \varpi^{-1} & 0 & 0 \\
\varpi & 0 & 0 & 0 \\
0 & 0 & 0 & \varpi \\
0 & 0 & \varpi^{-1} & 0
\end{array}\right]=\left[\begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array}\right]\left[\begin{array}{cccc}
\varpi & 0 & 0 & 0 \\
0 & \varpi^{-1} & 0 & 0 \\
0 & 0 & \varpi & 0 \\
0 & 0 & 0 & \varpi^{-1}
\end{array}\right]=r_{e_{1}-e_{2}} t_{(-1,1)}
$$

We wish to identify the points $x \in \mathcal{A}$ for which there exists $n \in N_{x}, n \notin G_{x, 0}$ which fixes $x$. This corresponds to the points $x=(a, b)$ for which there exists a vector $v$ such that $v(s)=(c, d)$ for some $s \in \mathbf{S}(\mathbb{F})$, where $(c, d) \in N_{x}$ and $t_{(c, d)} \notin W$ such that $r_{\alpha} r_{\beta} t_{(c, d)}(a, b)=(a, b)$.
The translations from the torus are by elements of $\mathbb{Z}^{2}$, but the translations by elements of the affine Weyl group are just those generated by the coroots $\check{\alpha}=(1,-1)$ and $\check{\beta}=(1,1)$ so they span a sublattice consisting of all $(a, b)$ such that $a+b$ is even.

Proposition 4.6. If there exists $v \in \mathbb{Z}^{2} \backslash \operatorname{span}_{\mathbb{Z}} \Phi^{\vee}$ such that $r_{\alpha} r_{\beta} t_{v}(a, b)=(a, b)$, then exactly one of $a$ or $b$ are contained in $\frac{1}{2} \mathbb{Z}$.

Proof. We identify all the points $(a, b)$ such that there exists $v \in \mathbb{Z}^{2}$ such $r_{\alpha} r_{\beta} t_{v}(a, b)=(a, b)$. Suppose $r_{\alpha} r_{\beta} t_{v}(a, b)=(a, b)$ for some $v=(c, d)$, where $c+d=2 k+1, k \in \mathbb{Z}$.

$$
\begin{aligned}
r_{\alpha} r_{\beta} t_{v}(a, b)=(a, b) \Longrightarrow & r_{\alpha} r_{\beta}(-d+2 k+1+a, d+b)=(a, b) \\
\Longrightarrow & (-(a-d+2 k+1),-(b+d))=(a, b) \\
\Longrightarrow & -a+d-2 k+1=a \\
& -b-d=b
\end{aligned}
$$

The points $(a, b)$ which satisfy $r_{\alpha} r_{\beta} t_{v}(a, b)=(a, b)$ must satisfy,

$$
-2 k+1=2 a+2 b,
$$

which implies $a \in \mathbb{Z}$ and $b \in \frac{1}{2} \mathbb{Z}$ or $a \in \frac{1}{2} \mathbb{Z}$ and $b \in \mathbb{Z}$.
Remark 4.7. Notice that $t_{(c, d)} r_{\alpha} r_{\beta}=r_{\alpha} r_{\beta} t_{(-c,-d)}$.
Now, consider the following lemma, which will allow us to identify points fixed by tori of $S O(4)$.
Lemma 4.8. If $r_{\alpha} r_{\beta} t_{(c, d)}$ fixes a point $x$, then there exists an element $n \in N_{x}$ that acts by $r_{\alpha} r_{\beta} t_{(c, d)}$. In fact, $G_{x}=\left\langle n, G_{x, 0}\right\rangle$.

Proof. If $r_{\alpha} r_{\beta} t_{(c, d)}$ fixes a point $x$, then it is realized by,

$$
n_{c, d}=\left[\begin{array}{cccc}
0 & 0 & \varpi^{-c} & 0 \\
0 & 0 & 0 & \varpi^{-d} \\
\varpi^{c} & 0 & 0 & 0 \\
0 & \varpi^{d} & 0 & 0
\end{array}\right]
$$

Suppose the matrix associated to the parahoric subgroup $G_{x, 0}$ is, $\left[G_{x, 0}\right]_{i j}=\left[\mathcal{P}^{k_{i j}}\right]$, where $k_{i j} \in \mathbb{Z}$, $1 \leq i, j \leq 4$. Thus, as $G_{x}=\left\langle n_{c, d} \mathbf{S}, G_{x, 0}\right\rangle$, we have,

$$
n=\left[\begin{array}{cccc}
0 & 0 & \varpi^{-c} & 0 \\
0 & 0 & 0 & \varpi^{-d} \\
\varpi^{c} & 0 & 0 & 0 \\
0 & \varpi^{d} & 0 & 0
\end{array}\right]
$$

As shown in Appendix A of [7], $G_{x}$ is generated by $n$ and $G_{x, 0}$.
We will illustrate this fact explicitly with a class of tori.

## 5 Elliptic tori in $S O(4)$

In this section, we will consider three classes of tori of $S O(4)$, compute their matrices, and determine the point associated to a subgroup of $S O(4)$ associated to each torus.

### 5.1 Matrix representations of elliptic tori

As presented in [3], we consider three classes of tori in $S O(4)$, all arising from the field $E=F[\sqrt{\varepsilon}, \sqrt{\varpi}]$, but arising from different subfields.

| Extension fields | Witt basis $\left\{e_{1}, e_{2}, f_{1}, f_{2}\right\}$ |
| :---: | :---: |
| $F(\sqrt{\varepsilon}) \subset F(\sqrt{\varepsilon, \varpi})$ | $\left\{1, \sqrt{\varpi}, \sqrt{\varepsilon}^{-1},-\sqrt{\varepsilon \varpi}\right.$ |
| $F(\sqrt{\varpi}) \subset F(\sqrt{\varepsilon, \varpi})$ | $\left\{1, \sqrt{\varepsilon}, \sqrt{\varpi}^{-1},-\sqrt{\varepsilon \varpi^{-1}}\right\}$ |
| $F(\sqrt{\varepsilon \varpi}) \subset F(\sqrt{\varepsilon, \varpi})$ | $\left\{1, \sqrt{\varepsilon}^{-1},{\sqrt{\varepsilon \varpi^{\varpi}}}^{-1},-\sqrt{\varpi}^{-1}\right\}$ |

Table 1: Three classes of tori in $S O(4)$
We begin with the second and third bases. Consider the basis $\mathcal{B}_{2}=\left\{e_{1}, e_{2}, f_{1}, f_{2}\right\}=\left\{1, \sqrt{\varepsilon}, \sqrt{\bar{\varpi}}^{-1},-{\sqrt{\varepsilon \varpi^{\omega}}}^{-1}\right\}$ of $E$ over $F$. We can write any element of $\mathcal{O}_{E}$ as,

$$
u=u_{1}+u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1}
$$

where $u_{1}, u_{2} \in \mathcal{O}$, and $u_{3}, u_{4} \in \mathcal{P}$. We compute the matrix,

$$
M_{[u]_{\mathcal{B}_{2}}}=\left[\begin{array}{cccc}
u_{1} & u_{2} & u_{3} \varpi^{-1} & u_{4}(\varepsilon \varpi)^{-1} \\
u_{2} & u_{1} & -u_{4} \varpi^{-1} & -u_{3} \varpi^{-1} \\
u_{3} & -u_{4} & u_{1} & -u_{2} \\
u_{4} & -u_{3} \varepsilon & -u_{2} \varepsilon & u_{1}
\end{array}\right]
$$

Then, according to the shorthand presented in Section 4.2,

$$
M_{[u]_{\mathcal{B}_{2}}} \subset\left[\begin{array}{cccc}
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\
\mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O} \\
\mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O}
\end{array}\right]
$$

this matrix $M_{[u]_{\mathcal{B}_{2}}}$ is contained in the parahoric subgroup $G_{(1 / 4,1 / 4), 0}$.
Consider the basis $\mathcal{B}_{3}=\left\{e_{1}, e_{2}, f_{1}, f_{2}\right\}=\left\{1, \sqrt{\varepsilon}^{-1},{\sqrt{\varepsilon \varpi^{-1}}}^{-1},-\sqrt{\varpi}^{-1}\right\}$ of $E$ over $F$. We can write any element of $\mathcal{O}_{E}$ as,

$$
u=u_{1}+u_{2} \sqrt{\varepsilon}^{-1}+u_{3} \sqrt{\varepsilon \varpi}^{-1}-u_{4} \sqrt{\varpi}^{-1}
$$

where $u_{1}, u_{2} \in \mathcal{O}$, and $u_{3}, u_{4} \in \mathcal{P}$.

$$
M_{[u]_{\mathcal{B}_{3}}} i=\left[\begin{array}{cccc}
u_{1} & u_{2} \varepsilon^{-1} & u_{3}(\varepsilon \varpi)^{-1} & u_{4} \varpi^{-1} \\
u_{2} & u_{1} & -u_{4} \varpi^{-1} & -u_{3} \varpi^{-1} \\
u_{3} & -u_{4} & u_{1} & -u_{2} \\
u_{4} & -u_{3} \varepsilon^{-1} & -u_{2} \varepsilon^{-1} & u_{1}
\end{array}\right]
$$

Then, according to the shorthand presented in Section 4.2,

$$
M_{[u]_{\mathcal{B}_{3}}} \subset\left[\begin{array}{llll}
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\
\mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O} \\
\mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O}
\end{array}\right]
$$

The matrix $M_{[u]_{\mathcal{B}_{3}}}$ is contained in the parahoric subgroup $G_{(1 / 4,1 / 4), 0}$.
Now, consider the first basis, $\mathcal{B}_{1}=\left\{e_{1}, e_{2}, f_{1}, f_{2}\right\}=\left\{1, \sqrt{\varpi}, \sqrt{\varepsilon}^{-1},{\sqrt{\varepsilon \varpi^{-}}}^{-1}\right\}$ of $E$ over $F$. We can write any element of $\mathcal{O}_{E}$ as,

$$
u=u_{1}+u_{2} \sqrt{\varpi}+u_{3} \sqrt{\varepsilon}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1}
$$

where $u_{1}, u_{2}, u_{3} \in \mathcal{O}$, and $u_{4} \in \mathcal{P}$. The intermediate field is $F(\sqrt{\varepsilon})$. The reader may refer to Appendix A for the detailed computation of the tori. We compute the matrix of the multiplication of an element of $E$ by the element $u$ relative to this basis, which is given by,

$$
M_{[u]_{\mathcal{B}_{1}}}=\left[\begin{array}{cccc}
u_{1} & u_{2} \varpi & u_{3} \varepsilon^{-1} & u_{4}(\varepsilon \varpi)^{-1} \\
u_{2} & u_{1} & -u_{4}(\varepsilon \varpi)^{-1} & -u_{3}(\varepsilon \varpi)^{-1} \\
u_{3} & -u_{4} & u_{1} & -u_{2} \\
u_{4} & -u_{3} \varpi & -u_{2} \varpi & u_{1}
\end{array}\right] .
$$

Then, according to the shorthand presented in Section 4.2,

$$
M_{[u]_{\mathcal{B}_{1}}} \subset\left[\begin{array}{cccc}
\mathcal{O} & \mathcal{P} & \mathcal{O} & \mathcal{O} \\
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{P}^{-1} \\
\mathcal{O} & \mathcal{P} & \mathcal{O} & \mathcal{O} \\
\mathcal{P} & \mathcal{P} & \mathcal{P} & \mathcal{O}
\end{array}\right]
$$

Now, $M_{[u]_{\mathcal{B}_{1}}} \nsubseteq G_{x, 0}$ for any points in the standard apartment, but in the next section we find $x$ such that $M_{[u]_{\mathcal{B}_{1}}} \subset G_{x}$.

### 5.2 Points of the standard apartment fixed by the tori

From $M_{[u]_{\mathcal{B}_{1}}}$, applying Proposition 4.5, we obtain the following restrictions on $\alpha(x), \beta(x)$,

$$
\begin{aligned}
\lceil\alpha(x)\rceil & =0 \\
-\lfloor\alpha(x)\rfloor & =1 ; \\
\lceil\beta(x)\rceil & =1 ; \\
-\lfloor\beta(x)\rfloor & =0 ;
\end{aligned}
$$

which implies $-1 \leq \alpha(x) \leq 0$ and $0 \leq \beta(x) \leq 1$. We will show the point $x=\left(0, \frac{1}{2}\right)$ will generate $G_{x}$ such that $M_{[u]_{\mathcal{B}_{1}}} \subseteq G_{x}$. We know $t_{(0,1)} r_{\alpha} r_{\beta}\left(0, \frac{1}{2}\right)=\left(0, \frac{1}{2}\right)$ and, let

$$
n=t_{(0,1)} r_{\alpha} r_{\beta}\left(0, \frac{1}{2}\right)=\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \varpi^{-1} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \varpi
\end{array}\right]=\left[\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \varpi^{-1} \\
1 & 0 & 0 & 0 \\
0 & \varpi & 0 & 0
\end{array}\right]
$$

The parahoric subgroup $G_{x, 0}$ associates with $x=\left(0, \frac{1}{2}\right)$, according to Proposition 4.5 , is,

$$
G_{x, 0}=\left[\begin{array}{cccc}
\mathcal{O} & \mathcal{P} & \mathcal{P} & \mathcal{O} \\
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\
\mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O} \\
\mathcal{P} & \mathcal{P}^{2} & \mathcal{P} & \mathcal{O}
\end{array}\right]
$$

Making use of Lemma 4.8 and $G_{x}=\left\langle n, G_{x, 0}\right\rangle$, with the shorthand presented above, we have,

$$
G_{x}=\left[\begin{array}{cccc}
\mathcal{O} & \mathcal{P} & \mathcal{O} & \mathcal{O} \\
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{P}^{-1} \\
\mathcal{O} & \mathcal{P} & \mathcal{O} & \mathcal{O} \\
\mathcal{P} & \mathcal{P} & \mathcal{P} & \mathcal{O}
\end{array}\right]
$$

Hence, for $x=\left(0, \frac{1}{2}\right), M_{[u]_{\mathcal{B}_{1}}} \subseteq G_{x}$. The point in the standard apartment fixed by the torus $M_{[u]_{\mathcal{B}_{1}}}$ is $x=\left(0, \frac{1}{2}\right)$.

## References

[1] K. S. Abramenko, P. Brown. Buildings. Springer, New York, 2008.
[2] A. Borel. Linear Algebraic Groups. Springer-Verlag, New York, 1991.
[3] T. Chinner. Elliptic Tori in p-adic Orthogonal Groups. Master's thesis, University of Ottawa, Ottawa, Canada, 2021.
[4] F. Q. Gouvea. p-adic Numbers, volume 2. Springer, Berlin, Heidelberg, 1997.
[5] J. E. Humphreys. Introduction to Lie Algebras and Representation Theory. Springer-Verlag, New York, 1972.
[6] A. W. Knapp. Basic Algebra. Birkhauser, Boston, 2006.
[7] J. Rabinoff. The Bruhat-Tits building of a p-adic Chevalley group and an application to representation theory. 2005.

## A Computation of the tori

In this appendix, we will give the explicit computation of the matrices of the tori from Section 5 , which are introduced in [3], along with their eigenvalues. The eigenvalues of the tori can identify the points of the building attached to each torus.

## A. 1 The matrix of the torus with respect to Witt basis $\mathcal{B}_{1}=\left\{1, \sqrt{\varpi}, \sqrt{\varepsilon}^{-1}, \sqrt{\varepsilon \varpi}^{-1}\right\}$

Consider the basis $\mathcal{B}_{1}=\left\{e_{1}, e_{2}, f_{1}, f_{2}\right\}=\left\{1, \sqrt{\varpi}, \sqrt{\varepsilon}^{-1}, \sqrt{\varepsilon \varpi}^{-1}\right\}$. We can write,

$$
u=u_{1}+u_{2} \sqrt{\varpi}+u_{3} \sqrt{\varepsilon}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1}
$$

where $u_{1}, u_{2}, u_{3} \in \mathcal{O}$, and $u_{4} \in \mathcal{P}$. The intermediate field is $F(\sqrt{\varepsilon})$. The second column of the matrix is given by multiplying $u$ by the second element of the basis $B_{1}$, and writing out the result with respect to $B_{1}$. We will show this computation as follows,

$$
\begin{aligned}
u \sqrt{\varpi} & =u_{1} \sqrt{\varpi}+u_{2} \varpi+u_{3} \sqrt{\varpi} \sqrt{\varepsilon}^{-1}-u_{4} \sqrt{\varepsilon}^{-1} \\
& =u_{1}(\sqrt{\varpi})+u_{2} \varpi(1)+u_{4}(-1)\left(\sqrt{\varepsilon}^{-1}\right)+u_{3}(-\varpi)\left(-\sqrt{\varepsilon \varpi}^{-1}\right)
\end{aligned}
$$

The third column of the matrix is given by multiplying $u$ by the third element of the basis $B_{1}$, and writing out the result with respect to $B_{1}$. We will show this computation as follows,

$$
\begin{aligned}
u \sqrt{\varepsilon} & =u_{1} \sqrt{\varepsilon}^{-1}+u_{2} \sqrt{\varpi} \sqrt{\varepsilon}^{-1}+u_{3} \varepsilon^{-1}-u_{4} \varepsilon^{-1} \sqrt{\varpi}^{-1} \\
& =u_{1}\left(\sqrt{\varepsilon}^{-1}\right)+u_{2}(-\varpi)\left(-\sqrt{\varepsilon \varpi}^{-1}\right)+u_{3} \varepsilon^{-1}(1)+u_{4}(-\varepsilon \varpi)^{-1}(\sqrt{\varpi})
\end{aligned}
$$

The fourth column of the matrix of the matrix is given by multiplying $u$ by the fourth element of the basis $B_{1}$, and writing out the result with respect to $B_{1}$. We will show this computation as follows,

$$
\begin{aligned}
-u \sqrt{\varepsilon \varpi}^{-1} & =-u_{1} \sqrt{\varepsilon \varpi^{-1}}-u_{2} \sqrt{\varepsilon}^{-1}-u_{3} \varepsilon^{-1} \sqrt{\varpi}^{-1}+u_{4} \varepsilon^{-1} \varpi^{-1} \\
& =u_{1}\left(-{\sqrt{\varepsilon \varpi^{-1}}}^{-1}\right)-u_{2}\left(\sqrt{\varepsilon}^{-1}\right)-u_{3}\left(\varepsilon^{-1} \varpi^{-1}\right)(\sqrt{\varpi})+u_{4}(\varepsilon \varpi)^{-1}
\end{aligned}
$$

The matrix of of the torus given by the Witt basis $\mathcal{B}_{1}$ is,

$$
M_{[u]_{\mathcal{B}_{1}}}=\left[\begin{array}{cccc}
u_{1} & u_{2} \varpi & u_{3} \varepsilon^{-1} & u_{4}(\varepsilon \varpi)^{-1} \\
u_{2} & u_{1} & -u_{4}(\varepsilon \varpi)^{-1} & -u_{3}(\varepsilon \varpi)^{-1} \\
u_{3} & -u_{4} & u_{1} & -u_{2} \\
u_{4} & -u_{3} \varpi & -u_{2} \varpi & u_{1}
\end{array}\right] .
$$

Then,

Remark A.1. Since the matrices in Section 5.1 give a torus, they are simultaneously diagonalizable over the field $E$. The eigenvectors can be deduced from the intermediate extension field. If $t \in T$ is the element of a torus corresponding to the field element,

$$
u_{t}=a e_{1}+b e_{2}+c f_{1}+d f_{2}
$$

with the basis $\mathcal{B}=\left\{e_{1}, e_{2}, f_{1}, f_{2}\right\}$, then the eigenvalues of $t$ will be exactly $\left\{\sigma\left(z_{t}\right): z_{t} \in \operatorname{Gal}\left(E^{\prime} / F\right)\right\}$.
Now, we find an eigenvector for $\lambda_{1}=u_{1}+u_{2} \sqrt{\varpi}+u_{3} \sqrt{\varepsilon}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1}$.

$$
\left.\begin{array}{rl} 
& {\left[\begin{array}{llll}
u_{1} & u_{2} \varpi & u_{3} \varepsilon^{-1} & u_{4}(\varepsilon \varpi)^{-1}
\end{array}\right]\left[\begin{array}{l}
1 \\
a \\
b \\
c
\end{array}\right]}
\end{array}\right]=u_{1}+u_{2} \sqrt{\varpi}+u_{3} \sqrt{\varepsilon} \bar{\varepsilon}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1},
$$

and since this holds for all $u_{i}$, this implies $a=\sqrt{\varpi}^{-1}, b=\sqrt{\varepsilon}, c=-\sqrt{\varepsilon \varpi}$. Now, we verify the product of the eigenvector with the second row of $M_{[u]_{\mathcal{B}_{1}}}$.

$$
\left.\begin{array}{rl} 
& {\left[\begin{array}{llll}
u_{2} & u_{1} & -u_{4}(\varepsilon \varpi)^{-1} & -u_{3}(\varepsilon \varpi)^{-1}
\end{array}\right]\left[\begin{array}{c}
1 \\
\sqrt{\varpi}^{-1} \\
\sqrt{\varepsilon} \\
-\sqrt{\varepsilon \varpi}
\end{array}\right]}
\end{array}\right]=u_{1} \sqrt{\varpi}^{-1}+u_{2}+u_{3} \sqrt{\varepsilon \varpi}^{-1}-u_{4} \varpi^{-1} \sqrt{\varepsilon}^{-1},
$$

as required. Now, we check the product of the eigenvector with the third row of $M_{[u]_{\mathcal{B}_{1}}}$.

$$
\begin{aligned}
& {\left[\begin{array}{llll}
u_{3} & -u_{4} & u_{1} & -u_{2}
\end{array}\right]\left[\begin{array}{c}
1 \\
\sqrt{\varpi}^{-1} \\
\sqrt{\varepsilon} \\
-\sqrt{\varepsilon \varpi}
\end{array}\right]=u_{1} \sqrt{\varepsilon}+u_{2} \sqrt{\varepsilon \varpi}+u_{3}-u_{4} \sqrt{\varpi}^{-1} } \\
\Longrightarrow \quad u_{3}-u_{4} \sqrt{\varpi}^{-1}+u_{1} \sqrt{\varepsilon}+u_{2} \sqrt{\varepsilon \varpi} & =u_{1} \sqrt{\varepsilon}+u_{2} \sqrt{\varepsilon \varpi}+u_{3}-u_{4} \sqrt{\varpi}^{-1},
\end{aligned}
$$

as required. Now, we check the product of the eigenvector with the fourth row of $M_{[u]_{\mathcal{B}_{1}}}$.

$$
\begin{aligned}
& {\left[\begin{array}{llll}
u_{4} & -u_{3} \varpi & -u_{2} \varpi & u_{1}
\end{array}\right]\left[\begin{array}{c}
1 \\
\sqrt{\varpi}^{-1} \\
\sqrt{\varepsilon} \\
-\sqrt{\varepsilon \varpi}
\end{array}\right] }
\end{aligned}=-u_{1} \sqrt{\varepsilon \varpi}-u_{2} \sqrt{\varepsilon} \varpi-u_{3} \sqrt{\varpi}+u_{4},
$$

as required. The eigenvector associated to $\lambda_{1}=u_{1}+u_{2} \sqrt{\varpi}+u_{3} \sqrt{\varepsilon}^{-1}-u_{4}{\sqrt{\varepsilon \varpi^{\omega}}}^{-1}$ is $\left[\begin{array}{c}1 \\ \sqrt{\varpi}^{-1} \\ \sqrt{\varepsilon} \\ -\sqrt{\varepsilon \varpi}\end{array}\right]$. Now, we consider the next 3 eigenvectors of $M_{[u]_{\mathcal{B}_{1}}}$.

The intermediate field for $\mathcal{B}_{1}$ is $F(\sqrt{\varepsilon})$, hence,

$$
\begin{aligned}
\sigma_{1}: 1 & \mapsto 1 \\
\sqrt{\varpi} & \mapsto-\sqrt{\varpi} \\
\sqrt{\varepsilon} & \mapsto \sqrt{\varepsilon} \\
\sqrt{\varepsilon \varpi} & \mapsto-\sqrt{\varepsilon \varpi}
\end{aligned}
$$

The second eigenvector is $\lambda_{2}=\sigma_{1}\left(u_{1}+u_{2} \sqrt{\varpi}+u_{3} \sqrt{\varepsilon}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1}\right)=u_{1}-u_{2} \sqrt{\varpi}+u_{3} \sqrt{\varepsilon}^{-1}+u_{4} \sqrt{\varepsilon \varpi}^{-1}$.

$$
\begin{aligned}
& {\left[\begin{array}{llll}
u_{1} & u_{2} \varpi & u_{3} \varepsilon^{-1} & u_{4}(\varepsilon \varpi)^{-1}
\end{array}\right]\left[\begin{array}{l}
1 \\
a \\
b \\
c
\end{array}\right]=u_{1}-u_{2} \sqrt{\varpi}+u_{3} \sqrt{\varepsilon^{-1}}+u_{4} \sqrt{\varepsilon \varpi}^{-1} } \\
\Longrightarrow \quad & u_{1}+a u_{2} \varpi+b u_{3} \varepsilon^{-1}+c u_{4}(\varepsilon \varpi)^{-1}=u_{1}-u_{2} \sqrt{\varpi}+u_{3} \sqrt{\varepsilon}^{-1}+u_{4} \sqrt{\varepsilon \varpi}^{-1},
\end{aligned}
$$

and since this holds for all $u_{i}$, this implies $a=-\sqrt{\varpi}^{-1}, b=\sqrt{\varepsilon}, c=\sqrt{\varepsilon \varpi}$. Now, we verify the product of the eigenvector with the second row of $M_{[u]_{\mathcal{B}_{1}}}$.

$$
\begin{aligned}
& {\left[\begin{array}{llll}
u_{2} & u_{1} & -u_{4}(\varepsilon \varpi)^{-1} & -u_{3}(\varepsilon \varpi)^{-1}
\end{array}\right]\left[\begin{array}{c}
1 \\
-\sqrt{\varpi}^{-1} \\
\sqrt{\varepsilon} \\
\sqrt{\varepsilon \varpi}
\end{array}\right]=-\sqrt{\varpi}^{-1}\left(u_{1}-u_{2} \sqrt{\varpi}+u_{3} \sqrt{\varepsilon}^{-1}+u_{4} \sqrt{\varepsilon \varpi}^{-1}\right)} \\
& \Longrightarrow \quad u_{2}-u_{1} \sqrt{\varpi}^{-1}-u_{4} \sqrt{\varepsilon}^{-1} \varpi^{-1}-u_{3} \sqrt{\varepsilon \varpi}^{-1}=-u_{1} \sqrt{\varpi}^{-1}+u_{2}-u_{3}{\sqrt{\varepsilon \varpi^{\varpi}}}^{-1}-u_{4} \sqrt{\varepsilon}^{-1} \varpi^{-1}
\end{aligned}
$$

as required. Now, we check the product of the eigenvector with the third row of $M_{[u]_{\mathcal{B}_{1}}}$.

$$
\left.\begin{array}{rl} 
& {\left[\begin{array}{llll}
u_{3} & -u_{4} & u_{1} & -u_{2}
\end{array}\right]\left[\begin{array}{c}
1 \\
-\sqrt{\varpi}^{-1} \\
\sqrt{\varepsilon} \\
\sqrt{\varepsilon \varpi}
\end{array}\right]}
\end{array}\right]=\sqrt{\varepsilon}\left(u_{1}-u_{2} \sqrt{\varpi}+u_{3} \sqrt{\varepsilon}^{-1}+u_{4}{\sqrt{\varepsilon \varpi^{\varpi}}}^{-1}\right) .
$$

as required. We can do this similarly for the fourth row of $M_{[u]_{\mathcal{B}_{1}}}$.
Now, consider $\sigma_{2}$.

$$
\begin{aligned}
\sigma_{2}: 1 & \mapsto 1 \\
\sqrt{\varpi} & \mapsto \sqrt{\varpi} \\
\sqrt{\varepsilon} & \mapsto-\sqrt{\varepsilon} \\
\sqrt{\varepsilon \varpi} & \mapsto-\sqrt{\varepsilon \varpi}
\end{aligned}
$$

The third eigenvector is $\lambda_{3}=\sigma_{2}\left(u_{1}+u_{2} \sqrt{\varpi}+u_{3} \sqrt{\varepsilon}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1}\right)=u_{1}+u_{2} \sqrt{\varpi}-u_{3} \sqrt{\varepsilon}^{-1}+u_{4} \sqrt{\varepsilon \varpi}^{-1}$.

$$
\begin{aligned}
& {\left[\begin{array}{llll}
u_{1} & u_{2} \varpi & u_{3} \varepsilon^{-1} & u_{4}(\varepsilon \varpi)^{-1}
\end{array}\right]\left[\begin{array}{l}
1 \\
a \\
b \\
c
\end{array}\right]=u_{1}+u_{2} \sqrt{\varpi}-u_{3} \sqrt{\varepsilon}^{-1}+u_{4} \sqrt{\varepsilon \varpi}^{-1} } \\
\Longrightarrow \quad & u_{1}+a u_{2} \varpi+b u_{3} \varepsilon^{-1}+c u_{4}(\varepsilon \varpi)^{-1}=u_{1}+u_{2} \sqrt{\varpi}-u_{3} \sqrt{\varepsilon} \bar{\varepsilon}^{-1}+u_{4} \sqrt{\varepsilon \varpi}^{-1},
\end{aligned}
$$

and since this holds for all $u_{i}$, this implies $a=\sqrt{\varpi}^{-1}, b=-\sqrt{\varepsilon}, c=\sqrt{\varepsilon \varpi}$. Now, consider $\sigma_{3}$.

$$
\begin{aligned}
\sigma_{3}: \sqrt{\varepsilon} & \mapsto-\sqrt{\varepsilon} \\
\sqrt{\varpi} & \mapsto-\sqrt{\varpi}
\end{aligned}
$$

The fourth eigenvector is $\lambda_{4}=\sigma_{3}\left(u_{1}+u_{2} \sqrt{\varpi}+u_{3} \sqrt{\varepsilon}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1}\right)=u_{1}-u_{2} \sqrt{\varpi}-u_{3} \sqrt{\varepsilon}^{-1}-u_{4}{\sqrt{\varepsilon \varpi^{-1}}}^{-1}$.

$$
\begin{aligned}
{\left[\begin{array}{llll}
u_{1} & u_{2} \varpi & u_{3} \varepsilon^{-1} & u_{4}(\varepsilon \varpi)^{-1}
\end{array}\right]\left[\begin{array}{l}
1 \\
a \\
b \\
c
\end{array}\right] } & =u_{1}-u_{2} \sqrt{\varpi}-u_{3} \sqrt{\varepsilon}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1} \\
\Longrightarrow \quad u_{1}+a u_{2} \varpi+b u_{3} \varepsilon^{-1}+c u_{4}(\varepsilon \varpi)^{-1} & =u_{1}-u_{2} \sqrt{\varpi}-u_{3} \sqrt{\varepsilon}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1},
\end{aligned}
$$

and since this holds for all $u_{i}$, this implies $a=-\sqrt{\varpi}^{-1}, b=-\sqrt{\varepsilon}, c=-\sqrt{\varepsilon \varpi}$.
The eigenvectors and eigenvalues are,

$$
\begin{array}{ll}
\lambda_{1}=u_{1}+u_{2} \sqrt{\varpi}+u_{3} \sqrt{\varepsilon}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1} & v_{1}=\left[\begin{array}{c}
1 \\
\sqrt{\varpi}^{-1} \\
\sqrt{\varepsilon} \\
-\sqrt{\varepsilon \varpi}
\end{array}\right] ; \\
\lambda_{2}=u_{1}-u_{2} \sqrt{\varpi}+u_{3} \sqrt{\varepsilon}^{-1}+u_{4} \sqrt{\varepsilon \varpi}^{-1} & v_{2}=\left[\begin{array}{c}
1 \\
-\sqrt{\varpi}^{-1} \\
\sqrt{\varepsilon} \\
\sqrt{\varepsilon \varpi}
\end{array}\right] ; \\
\lambda_{3}=u_{1}+u_{2} \sqrt{\varpi}-u_{3} \sqrt{\varepsilon}^{-1}+u_{4} \sqrt{\varepsilon \varpi}^{-1} & v_{3}=\left[\begin{array}{c}
1 \\
\sqrt{\varpi}^{-1} \\
-\sqrt{\varepsilon}^{\sqrt{\varepsilon \varpi}}
\end{array}\right] ; \\
\lambda_{4}=u_{1}-u_{2} \sqrt{\varpi}-u_{3} \sqrt{\varepsilon}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1} & v_{4}=\left[\begin{array}{c}
1 \\
-\sqrt{\varpi}^{-1} \\
-\sqrt{\varepsilon}^{-\sqrt{\varepsilon \varpi}}
\end{array}\right] .
\end{array}
$$

These vectors are linearly independent.

## A. 2 The matrix of the torus with respect to Witt basis Basis $\mathcal{B}_{2}=\left\{1, \sqrt{\varepsilon}, \sqrt{\varpi}^{-1},-\sqrt{\varepsilon \varpi}^{-1}\right\}$

Consider the basis $\mathcal{B}_{2}=\left\{e_{1}, e_{2}, f_{1}, f_{2}\right\}=\left\{1, \sqrt{\varepsilon}, \sqrt{\varpi}^{-1},-\sqrt{\varepsilon \varpi}^{-1}\right\}$. We can write,

$$
u=u_{1}+u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1}
$$

where $u_{1}, u_{2} \in \mathcal{O}$, and $u_{3}, u_{4} \in \mathcal{P}$. The intermediate field is $E=F(\sqrt{\varpi})$. The second column of the matrix is given by multiplying $u$ by the second element of the basis $B_{2}$, and writing out the result with respect to $B_{2}$. We will show this computation as follows,

$$
\begin{aligned}
\sqrt{\varepsilon}\left(u_{1}+u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1}\right) & =u_{1} \sqrt{\varepsilon}+u_{2} \varepsilon+u_{3} \sqrt{\varepsilon} \cdot \sqrt{\varpi}^{-1}-u_{4} \sqrt{\varpi}^{-1} \\
& =u_{2} \varepsilon(1)+u_{1} \sqrt{\varepsilon}+u_{4}(-1)\left(\sqrt{\varpi}^{-1}\right)+u_{3}(-\varepsilon)\left(-{\sqrt{\varepsilon \varpi^{-1}}}^{-1}\right)
\end{aligned}
$$

The third column of the matrix is given by multiplying $u$ by the third element of the basis $B_{2}$, and writing out the result with respect to $B_{2}$. We will show this computation as follows,

$$
\begin{aligned}
\sqrt[\varpi]{\varpi}^{-1}\left(u_{1}+u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}-u_{4}{\sqrt{\varepsilon \varpi^{-1}}}^{-1}\right) & =u_{1} \sqrt{\varpi}^{-1}+u_{2} \sqrt{\varepsilon} \cdot \sqrt{\varpi}^{-1}+u_{3} \varpi^{-1}-u_{4} \sqrt{\varepsilon} \varpi^{-1} \\
& =u_{3} \varpi^{-1}+u_{4}\left(-\varpi^{-1}\right)(\sqrt{\varepsilon})+u_{1} \sqrt{\varpi}^{-1}+u_{2}(-\varepsilon)\left(-{\sqrt{\varepsilon} \varpi^{-1}}^{-1}\right.
\end{aligned}
$$

The fourth column of the matrix is given by multiplying $u$ by the fourth element of the basis $B_{2}$, and writing out the result with respect to $B_{2}$. We will show this computation as follows,

$$
\begin{aligned}
&-\sqrt{\varepsilon \varpi}^{-1}\left(u_{1}+u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}-u_{4}{\sqrt{\varepsilon \varpi^{-1}}}^{-1}\right)=-u_{1} \sqrt{\varepsilon \varpi}^{-1}-u_{2} \sqrt{\varpi}^{-1}-u_{3}{\sqrt{\varepsilon} \varpi^{-1}+u_{4} \varepsilon^{-1} \varpi^{-1}} \\
&=u_{4}(\varepsilon \varpi)^{-1}+u_{3}\left(-\varpi^{-1}\right)(\sqrt{\varepsilon})+u_{2}(-1)\left(\sqrt{\varpi}^{-1}\right)+u_{1}\left({\sqrt{\varepsilon} \varpi^{-1}}^{-1}\right.
\end{aligned}
$$

The matrix of of the torus given by the Witt basis $\mathcal{B}_{2}$ is,

$$
M_{[u]_{\mathcal{B}_{2}}}=\left[\begin{array}{cccc}
u_{1} & u_{2} & u_{3} \varpi^{-1} & u_{4}(\varepsilon \varpi)^{-1} \\
u_{2} & u_{1} & -u_{4} \varpi^{-1} & -u_{3} \varpi^{-1} \\
u_{3} & -u_{4} & u_{1} & -u_{2} \\
u_{4} & -u_{3} \varepsilon & -u_{2} \varepsilon & u_{1}
\end{array}\right]
$$

Then,

$$
M_{[u]_{\mathcal{B}_{2}}} \subseteq\left[\begin{array}{cccc}
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\
\mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O} \\
\mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O}
\end{array}\right]
$$

To compute the eigenvalues and eigenvectors of $M_{[u]_{\mathcal{B}_{2}}}$, we proceed according to Remark A.1. We find an eigenvector for $\lambda_{1}=u_{1}+u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}-u_{4}{\sqrt{\varepsilon \varpi^{*}}}^{-1}$.

$$
\begin{aligned}
& {\left[\begin{array}{llll}
u_{1} & u_{2} & u_{3} \varpi^{-1} & u_{4}(\varepsilon \varpi)^{-1}
\end{array}\right]\left[\begin{array}{l}
1 \\
a \\
b \\
c
\end{array}\right] }=u_{1}+u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1} \\
& \Longrightarrow \quad u_{1}+a u_{2}+b u_{3} \varpi^{-1}+c u_{4}(\varepsilon \varpi)^{-1}=u_{1}+u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1},
\end{aligned}
$$

which implies $a=\sqrt{\varepsilon}, b=\sqrt{\varpi}, c=-\sqrt{\varepsilon \varpi}$. Hence, the first eigenvector and eigenvalue of $M_{[u]_{\mathcal{B}_{2}}}$ are,

$$
\lambda_{1}=u_{1}+u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1}, \quad v_{1}=\left[\begin{array}{c}
1 \\
\sqrt{\varepsilon} \\
\sqrt{\varpi} \\
-\sqrt{\varepsilon \varpi}
\end{array}\right]
$$

Now, we find an eigenvector for $\lambda_{2}=\sigma_{1}\left(u_{1}+u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1}\right)=u_{1}+u_{2} \sqrt{\varepsilon}-u_{3} \sqrt{\varpi}^{-1}+$ $u_{4} \sqrt{\varepsilon \varpi}^{-1}$.

$$
\begin{aligned}
& {\left[\begin{array}{llll}
u_{1} & u_{2} & u_{3} \varpi^{-1} & u_{4}(\varepsilon \varpi)^{-1}
\end{array}\right]\left[\begin{array}{l}
1 \\
a \\
b \\
c
\end{array}\right]=u_{1}+u_{2} \sqrt{\varepsilon}-u_{3} \sqrt{\varpi}^{-1}+u_{4} \sqrt{\varepsilon \varpi}^{-1}} \\
& \Longrightarrow \quad u_{1}+a u_{2}+b u_{3} \varpi^{-1}+c u_{4}(\varepsilon \varpi)^{-1}=u_{1}+u_{2} \sqrt{\varepsilon}-u_{3} \sqrt{\varpi}^{-1}+u_{4} \sqrt{\varepsilon \varpi}^{-1},
\end{aligned}
$$

which implies $a=\sqrt{\varepsilon}, b=-\sqrt{\varpi}, c=\sqrt{\varepsilon \varpi}$. Hence, the second eigenvector and eigenvalue of $M_{[u]_{\mathcal{B}_{2}}}$ are,

$$
\lambda_{2}=u_{1}+u_{2} \sqrt{\varepsilon}-u_{3} \sqrt{\varpi}^{-1}+u_{4} \sqrt{\varepsilon \varpi}^{-1}, \quad v_{2}=\left[\begin{array}{c}
1 \\
\sqrt{\varepsilon} \\
-\sqrt{\varpi} \\
\sqrt{\varepsilon \varpi}
\end{array}\right]
$$

Now, we find an eigenvector for $\lambda_{3}=\sigma_{2}\left(u_{1}+u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1}\right)=u_{1}-u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}+$ $u_{4} \sqrt{\varepsilon \varpi}^{-1}$.

$$
\begin{aligned}
& {\left[\begin{array}{llll}
u_{1} & u_{2} & u_{3} \varpi^{-1} & u_{4}(\varepsilon \varpi)^{-1}
\end{array}\right]\left[\begin{array}{l}
1 \\
a \\
b \\
c
\end{array}\right]=u_{1}-u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}+u_{4} \sqrt{\varepsilon \varpi}^{-1}} \\
& \Longrightarrow \quad u_{1}+a u_{2}+b u_{3} \varpi^{-1}+c u_{4}(\varepsilon \varpi)^{-1}=u_{1}-u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}+u_{4} \sqrt{\varepsilon \varpi}^{-1},
\end{aligned}
$$

which implies $a=-\sqrt{\varepsilon}, b=\sqrt{\varpi}, c=\sqrt{\varepsilon \varpi}$. Hence,

$$
\lambda_{3}=u_{1}-u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}+u_{4} \sqrt{\varepsilon \varpi}^{-1}, \quad v_{3}=\left[\begin{array}{c}
1 \\
-\sqrt{\varepsilon} \\
\sqrt{\varpi} \\
\sqrt{\varepsilon \varpi}
\end{array}\right]
$$

Now, we find an eigenvector for $\lambda_{4}=\sigma_{3}\left(u_{1}+u_{2} \sqrt{\varepsilon}+u_{3} \sqrt{\varpi}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1}\right)=u_{1}-u_{2} \sqrt{\varepsilon}-u_{3} \sqrt{\varpi}^{-1}-$ $u_{4} \sqrt{\varepsilon \varpi}^{-1}$.

$$
\begin{aligned}
{\left[\begin{array}{llll}
u_{1} & u_{2} & u_{3} \varpi^{-1} & u_{4}(\varepsilon \varpi)^{-1}
\end{array}\right]\left[\begin{array}{l}
1 \\
a \\
b \\
c
\end{array}\right] } & =u_{1}-u_{2} \sqrt{\varepsilon}-u_{3} \sqrt{\varpi}^{-1}-u_{4}{\sqrt{\varepsilon \varpi^{\varpi}}}^{-1} \\
\Longrightarrow \quad u_{1}+a u_{2}+b u_{3} \varpi^{-1}+c u_{4}(\varepsilon \varpi)^{-1} & =u_{1}-u_{2} \sqrt{\varepsilon}-u_{3} \sqrt{\varpi}^{-1}-u_{4}{\sqrt{\varepsilon \varpi^{\varpi}}}^{-1}
\end{aligned}
$$

which implies $a=-\sqrt{\varepsilon}, b=-\sqrt{\varpi}, c=-\sqrt{\varepsilon \varpi}$. Hence,

$$
\lambda_{4}=u_{1}-u_{2} \sqrt{\varepsilon}-u_{3} \sqrt{\varpi}^{-1}-u_{4} \sqrt{\varepsilon \varpi}^{-1}, \quad v_{4}=\left[\begin{array}{c}
1 \\
-\sqrt{\varepsilon} \\
-\sqrt{\varpi} \\
-\sqrt{\varepsilon \varpi}
\end{array}\right]
$$

The matrix $M_{[u]_{\mathcal{B}_{2}}}$ is contained in the parahoric subgroup of $G_{0,(1 / 4,1 / 4)}$.

## A. 3 The matrix of the torus with respect to Witt basis Basis $\mathcal{B}_{3}=\left\{1, \sqrt{\varepsilon}^{-1},{\sqrt{\varepsilon \varpi^{-1}}}^{-1},-\sqrt{\varpi}^{-1}\right\}$

We pick the basis $\mathcal{B}_{3}=\left\{e_{1}, e_{2}, f_{1}, f_{2}\right\}=\left\{1, \sqrt{\varepsilon}^{-1}, \sqrt{\varepsilon \varpi}^{-1},-\sqrt{\varpi}^{-1}\right\}$. We can write

$$
u=u_{1}+u_{2} \sqrt{\varepsilon}^{-1}+u_{3} \sqrt{\varepsilon \varpi}^{-1}-u_{4} \sqrt{\varpi}^{-1}
$$

where $u_{1}, u_{2} \in \mathcal{O}$, and $u_{3}, u_{4} \in \mathcal{P}$. The intermediate field is $F(\sqrt{\varepsilon \varpi})$. The second column of the matrix is given by multiplying $u$ by the second element of the basis $B_{3}$, and writing out the result with respect to $B_{3}$. We will show this computation as follows,

$$
\begin{aligned}
\sqrt{\varepsilon}^{-1}\left(u_{1}+u_{2} \sqrt{\varepsilon}^{-1}+u_{3}{\sqrt{\varepsilon \varpi^{-1}}}^{-1}-u_{4} \sqrt{\varpi}^{-1}\right) & =u_{1} \sqrt{\varepsilon}^{-1}+u_{2} \varepsilon^{-1}+u_{3} \varepsilon^{-1} \sqrt{\varpi}^{-1}-u_{4}{\sqrt{\varepsilon \varpi^{-1}}}^{-1} \\
& =u_{2}\left(\varepsilon^{-1}\right)+u_{1}\left(\sqrt{\varepsilon}^{-1}\right)+u_{4}(-1)\left(\sqrt{\varepsilon \varpi}^{-1}\right)+u_{3}\left(-\varepsilon^{-1}\right) \sqrt{\varpi}^{-1}
\end{aligned}
$$

The third column of the matrix is given by multiplying $u$ by the third element of the basis $B_{3}$, and writing out the result with respect to $B_{3}$. We will show this computation as follows,

$$
\begin{aligned}
\sqrt{\varepsilon \varpi}^{-1}\left(u_{1}+u_{2} \sqrt{\varepsilon}^{-1}+u_{3} \sqrt{\varepsilon \varpi}^{-1}-u_{4} \sqrt{\varpi}^{-1}\right) & =u_{1} \sqrt{\varepsilon \varpi}^{-1}+u_{2} \varepsilon^{-1} \sqrt{\varpi}^{-1}+u_{3} \varepsilon^{-1} \varpi^{-1}-u_{4} \sqrt{\varepsilon}^{-1} \varpi^{-1} \\
& =u_{3}(\varepsilon \varpi)^{-1}-u_{4}\left(\varpi^{-1}\right)\left(\sqrt{\varepsilon}^{-1}\right)+u_{1} \sqrt{\varepsilon \varpi}^{-1}+u_{2}\left(-\varepsilon^{-1}\right)\left(-\sqrt{\varpi}^{-1}\right)
\end{aligned}
$$

The fourth column of the matrix is given by multiplying $u$ by the fourth element of the basis $B_{3}$, and writing out the result with respect to $B_{3}$. We will show this computation as follows,

$$
\begin{gathered}
-\sqrt{\varpi}^{-1}\left(u_{1}+u_{2} \sqrt{\varepsilon}^{-1}+u_{3} \sqrt{\left.\varepsilon \varpi^{-1}-u_{4} \sqrt{\varpi}^{-1}\right)} \begin{array}{rl} 
& =-u_{1} \sqrt{\varpi}^{-1}-u_{2}{\sqrt{\varepsilon \varpi^{-1}}-u_{3} \sqrt{\varepsilon}^{-1} \varpi^{-1}+u_{4} \varpi^{-1}}=u_{4} \varpi^{-1}+u_{3}\left(-\varpi^{-1}\right)\left(\sqrt{\varepsilon}^{-1}\right)+u_{2}(-1) \sqrt{\varepsilon \varpi^{-1}}-u_{1} \sqrt{\varpi}^{-1} . \\
M_{[u]_{\mathcal{B}_{3}}}=\left[\begin{array}{cccc}
u_{1} & u_{2} \varepsilon^{-1} & u_{3}\left(\varepsilon \varpi^{-1}\right. & u_{4} \varpi^{-1} \\
u_{2} & u_{1} & -u_{4} \varpi^{-1} & -u_{3} \varpi^{-1} \\
u_{3} & -u_{4} & u_{1} & -u_{2} \\
u_{4} & -u_{3} \varepsilon^{-1} & -u_{2} \varepsilon^{-1} & u_{1}
\end{array}\right] .
\end{array} . .\right.
\end{gathered}
$$

Then,

To compute the eigenvalues and eigenvectors of $M_{[u]_{\mathcal{B}_{3}}}$, we proceed according to Remark A.1. We find an eigenvector for $\lambda_{1}=u_{1}+u_{2} \sqrt{\varepsilon}^{-1}+u_{3} \sqrt{\varepsilon \varpi}^{-1}-u_{4} \sqrt{\varpi}^{-1}$.

$$
\left[\begin{array}{llll}
u_{1} & u_{2} \varepsilon^{-1} & u_{3}(\varepsilon \varpi)^{-1} & u_{4} \varpi^{-1}
\end{array}\right]\left[\begin{array}{l}
1 \\
a \\
b \\
c
\end{array}\right]=u_{1}+u_{2} \sqrt{\varepsilon}^{-1}+u_{3} \sqrt{\varepsilon \varpi}^{-1}-u_{4} \sqrt{\varpi}^{-1}
$$

which implies $a=\sqrt{\varepsilon}, b=\sqrt{\varepsilon \varpi}, c=-\sqrt{\varpi}$. Hence, the first eigenvector and eigenvalue of $M_{[u]_{\mathcal{B}_{3}}}$ are,

$$
\lambda_{1}=u_{1}+u_{2} \sqrt{\varepsilon}^{-1}+u_{3} \sqrt{\varepsilon \varpi}^{-1}-u_{4} \sqrt{\varpi}^{-1}, \quad v_{1}=\left[\begin{array}{c}
1 \\
\sqrt{\varepsilon} \\
\sqrt{\varepsilon \varpi} \\
-\sqrt{\varpi}
\end{array}\right]
$$

Now, we find an eigenvector for $\lambda_{2}=\sigma_{1}\left(u_{1}+u_{2} \sqrt{\varepsilon}^{-1}+u_{3} \sqrt{\varepsilon \varpi}^{-1}-u_{4} \sqrt{\varpi}^{-1}\right)=u_{1}+u_{2} \sqrt{\varepsilon}^{-1}-$ $u_{3} \sqrt{\varepsilon \varpi}^{-1}+u_{4} \sqrt{\varpi}^{-1}$.

$$
\left[\begin{array}{llll}
u_{1} & u_{2} \varepsilon^{-1} & u_{3}(\varepsilon \varpi)^{-1} & u_{4} \varpi^{-1}
\end{array}\right]\left[\begin{array}{l}
1 \\
a \\
b \\
c
\end{array}\right]=u_{1}+u_{2} \sqrt{\varepsilon}^{-1}-u_{3} \sqrt{\varepsilon \varpi}^{-1}+u_{4} \sqrt{\varpi}^{-1}
$$

which implies $a=\sqrt{\varepsilon}, b=-\sqrt{\varepsilon \varpi}, c=\sqrt{\varpi}$. Hence,

$$
\lambda_{2}=u_{1}+u_{2} \sqrt{\varepsilon}^{-1}-u_{3} \sqrt{\varepsilon \varpi}^{-1}+u_{4} \sqrt{\varpi}^{-1}, \quad v_{2}=\left[\begin{array}{c}
1 \\
\sqrt{\varepsilon} \\
-\sqrt{\varepsilon \varpi} \\
\sqrt{\varpi}
\end{array}\right]
$$

Now, we find an eigenvector for $\lambda_{3}=\sigma_{2}\left(u_{1}+u_{2} \sqrt{\varepsilon}^{-1}+u_{3}{\sqrt{\varepsilon \varpi^{-1}}}^{-1}-u_{4} \sqrt{\varpi}^{-1}\right)=u_{1}-u_{2} \sqrt{\varepsilon}^{-1}-$ $u_{3} \sqrt{\varepsilon \varpi}^{-1}-u_{4} \sqrt{\varpi}^{-1}$.

$$
\left[\begin{array}{llll}
u_{1} & u_{2} \varepsilon^{-1} & u_{3}(\varepsilon \varpi)^{-1} & u_{4} \varpi^{-1}
\end{array}\right]\left[\begin{array}{l}
1 \\
a \\
b \\
c
\end{array}\right]=u_{1}-u_{2} \sqrt{\varepsilon}^{-1}-u_{3} \sqrt{\varepsilon \varpi}^{-1}-u_{4} \sqrt{\varpi}^{-1}
$$

which implies $a=-\sqrt{\varepsilon}, b=-\sqrt{\varepsilon \varpi}, c=\sqrt{\varpi}$. Hence,

$$
\lambda_{3}=u_{1}-u_{2} \sqrt{\varepsilon}^{-1}-u_{3} \sqrt{\varepsilon \varpi}^{-1}-u_{4} \sqrt{\varpi}^{-1}, \quad v_{3}=\left[\begin{array}{c}
1 \\
-\sqrt{\varepsilon} \\
-\sqrt{\varepsilon \varpi} \\
\sqrt{\varpi}
\end{array}\right]
$$

Finally, we find an eigenvector for $\lambda_{4}=\sigma_{3}\left(u_{1}+u_{2} \sqrt{\varepsilon}^{-1}+u_{3} \sqrt{\varepsilon \varpi}^{-1}-u_{4} \sqrt{\varpi}^{-1}\right)=u_{1}-u_{2} \sqrt{\varepsilon}^{-1}+$ $u_{3} \sqrt{\varepsilon \varpi}^{-1}+u_{4} \sqrt{\varpi}^{-1}$.

$$
\left[\begin{array}{llll}
u_{1} & u_{2} \varepsilon^{-1} & u_{3}(\varepsilon \varpi)^{-1} & u_{4} \varpi^{-1}
\end{array}\right]\left[\begin{array}{l}
1 \\
a \\
b \\
c
\end{array}\right]=u_{1}-u_{2} \sqrt{\varepsilon}^{-1}+u_{3} \sqrt{\varepsilon \varpi}^{-1}+u_{4} \sqrt{\varpi}^{-1}
$$

which implies $a=-\sqrt{\varepsilon}, b=\sqrt{\varepsilon \varpi}, c=\sqrt{\varpi}$. Hence,

$$
\lambda_{4}=u_{1}-u_{2} \sqrt{\varepsilon}^{-1}+u_{3} \sqrt{\varepsilon \varpi}^{-1}+u_{4} \sqrt{\varpi}^{-1}, \quad v_{4}=\left[\begin{array}{c}
1 \\
-\sqrt{\varepsilon} \\
\sqrt{\varepsilon \varpi} \\
\sqrt{\varpi}
\end{array}\right] .
$$

The matrix $M_{[u]_{\mathcal{B}_{3}}}$ is contained in the parahoric subgroup of $G_{0,(1 / 4,1 / 4)}$.

## B Computation the parahoric subgroups

In this appendix, we give a detailed calculation showing the reader how to compute the parahoric subgroups in Section 5 . We find the parahoric subgroups for 9 points in the standard apartment.

Point in standard apartment: $x=(0,0)$. The matrix is, $G_{x, 0}=[\mathcal{O}]_{4 \times 4}$.
Point in standard apartment: $x=(1 / 2,0)$. We compute,

$$
\begin{aligned}
\alpha(x) & =\left(e_{1}-e_{2}\right)(x)=\frac{1}{2} \\
\beta(x) & =\left(e_{1}+e_{2}\right)(x)=\frac{1}{2} \\
-\alpha(x) & =-\frac{1}{2} \\
-\beta(x) & =-\frac{1}{2}
\end{aligned}
$$

The parahoric subgroup associated to $x$ is,

$$
G_{x, 0}=\left[\begin{array}{cccc}
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\
\mathcal{P} & \mathcal{O} & \mathcal{O} & \mathcal{P} \\
\mathcal{P}^{2} & \mathcal{P} & \mathcal{O} & \mathcal{P} \\
\mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O}
\end{array}\right]
$$

Point in standard apartment: $x=(1 / 4,1 / 4)$. We compute,

$$
\begin{aligned}
\alpha(x) & =\left(e_{1}-e_{2}\right)(x)=0 ; \\
\beta(x) & =\left(e_{1}+e_{2}\right)(x)=1 / 2 ; \\
-\alpha(x) & =0 ; \\
-\beta(x) & =-1 / 2 .
\end{aligned}
$$

The parahoric subgroup associated to $x$ is,

$$
G_{x, 0}=\left[\begin{array}{llll}
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\
\mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O} \\
\mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O}
\end{array}\right]
$$

Point in standard apartment: $x=(1 / 2,1 / 2)$. We compute,

$$
\begin{aligned}
\alpha(x) & =\left(e_{1}-e_{2}\right)(x)=0 \\
\beta(x) & =\left(e_{1}+e_{2}\right)(x)=1 \\
-\alpha(x) & =0 \\
-\beta(x) & =-1
\end{aligned}
$$

The parahoric subgroup associated to $x$ is,

$$
G_{x, 0}=\left[\begin{array}{cccc}
\mathcal{O} & \mathcal{O} & \mathcal{P}^{-1} & \mathcal{P}^{-1} \\
\mathcal{O} & \mathcal{O} & \mathcal{P}^{-1} & \mathcal{P}^{-1} \\
\mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O} \\
\mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O}
\end{array}\right]
$$

We have $G_{(1 / 4,1 / 4), 0} \subseteq G_{(1 / 2,1 / 2), 0}$.

Point in standard apartment: $x=(3 / 4,1 / 4)$. We compute,

$$
\begin{aligned}
\alpha(x) & =\left(e_{1}-e_{2}\right)(x)=1 / 2 \\
\beta(x) & =\left(e_{1}+e_{2}\right)(x)=1 \\
-\alpha(x) & =-1 / 2 \\
-\beta(x) & =-1
\end{aligned}
$$

The parahoric subgroup associated to $x$ is,

$$
G_{x, 0}=\left[\begin{array}{cccc}
\mathcal{O} & \mathcal{O} & \mathcal{P}^{-1} & \mathcal{P}^{-1} \\
\mathcal{P} & \mathcal{O} & \mathcal{P}^{-1} & \mathcal{O} \\
\mathcal{P}^{2} & \mathcal{P} & \mathcal{O} & \mathcal{P} \\
\mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O}
\end{array}\right]
$$

However, $G_{(3,4,1 / 4)} \nsubseteq G_{(1,0)}$
Point in standard apartment: $x=(1,0)$. We compute,

$$
\begin{aligned}
\alpha(x) & =\left(e_{1}-e_{2}\right)(x)=1 \\
\beta(x) & =\left(e_{1}+e_{2}\right)(x)=1 \\
-\alpha(x) & =-1 ; \\
-\beta(x) & =-1
\end{aligned}
$$

The parahoric subgroup associated to $x$ is,

$$
G_{x, 0}=\left[\begin{array}{cccc}
\mathcal{O} & \mathcal{P}^{-1} & \mathcal{P}^{-2} & \mathcal{P}^{-1} \\
\mathcal{P} & \mathcal{O} & \mathcal{P}^{-1} & \mathcal{O} \\
\mathcal{P}^{2} & \mathcal{P} & \mathcal{O} & \mathcal{P} \\
\mathcal{P} & \mathcal{O} & \mathcal{P}^{-1} & \mathcal{O}
\end{array}\right]
$$

Point in standard apartment: $x=(3 / 4,-1 / 4)$. We compute,

$$
\begin{aligned}
\alpha(x) & =\left(e_{1}-e_{2}\right)(x)=1 / 2 \\
\beta(x) & =\left(e_{1}+e_{2}\right)(x)=1 \\
-\alpha(x) & =-1 / 2 \\
-\beta(x) & =-1
\end{aligned}
$$

The parahoric subgroup associated to $x$ is,

$$
G_{x, 0}=\left[\begin{array}{cccc}
\mathcal{O} & \mathcal{O} & \mathcal{P}^{-1} & \mathcal{P}^{-1} \\
\mathcal{P} & \mathcal{O} & \mathcal{P}^{-1} & \mathcal{O} \\
\mathcal{P}^{2} & \mathcal{P} & \mathcal{O} & \mathcal{P} \\
\mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O}
\end{array}\right]
$$

Point in standard apartment: $x=(1 / 2,-1 / 2)$. We compute,

$$
\begin{aligned}
\alpha(x) & =\left(e_{1}-e_{2}\right)(x)=1 \\
\beta(x) & =\left(e_{1}+e_{2}\right)(x)=0 \\
-\alpha(x) & =-1 \\
-\beta(x) & =0
\end{aligned}
$$

The parahoric subgroup associated to $x$ is,

$$
G_{x, 0}=\left[\begin{array}{cccc}
\mathcal{O} & \mathcal{P}^{-1} & \mathcal{P}^{-1} & \mathcal{O} \\
\mathcal{P} & \mathcal{O} & \mathcal{O} & \mathcal{P} \\
\mathcal{P} & \mathcal{O} & \mathcal{O} & \mathcal{P} \\
\mathcal{O} & \mathcal{P}^{-1} & \mathcal{P}^{-1} & \mathcal{O}
\end{array}\right]
$$

Point in standard apartment: $x=(1 / 4,-1 / 4)$. We compute,

$$
\begin{aligned}
\alpha(x) & =\left(e_{1}-e_{2}\right)(x)=1 / 2 \\
\beta(x) & =\left(e_{1}+e_{2}\right)(x)=0 \\
-\alpha(x) & =-1 / 2 \\
-\beta(x) & =0
\end{aligned}
$$

The parahoric subgroup associated to $x$ is,

$$
G_{x, 0}=\left[\begin{array}{llll}
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\
\mathcal{P} & \mathcal{O} & \mathcal{O} & \mathcal{P} \\
\mathcal{P} & \mathcal{O} & \mathcal{O} & \mathcal{P} \\
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O}
\end{array}\right]
$$

