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Branching Rules for Principal Series Representations of $SL(2)$ over a p-adic Field

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Abstract. We explicitly describe the decomposition into irreducibles of the restriction of the principal series representations of $SL(2,k)$, for k a p-adic field, to each of its two maximal compact subgroups (up to conjugacy). We identify these irreducible subrepresentations in the Kirillov-type classification of Shalika. We go on to explicitly describe the decomposition of the reducible principal series of $SL(2,k)$ in terms of the restrictions of its irreducible constituents to a maximal compact subgroup.

1 Introduction

Let G be (the k-points of) a reductive group defined over a p-adic field k of characteristic zero, and odd residual characteristic, and K a maximal compact subgroup. We are interested in the decomposition of admissible representations of G upon restriction to K, in analogy with the case of $k = \mathbb{R}$. In this paper, we consider the group $G = SL(2,k)$ and its two nonconjugate maximal compact subgroups K and $\widetilde{K}$. Here K denotes the subgroup $SL(2,\mathcal{O})$, where $\mathcal{O}$ is the integer ring of k, and $\widetilde{K}$ denotes the subgroup $\omega K \omega^{-1}$, where $\omega = \left[\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right] \in GL(2,k)$, for ϖ a uniformizing element of k.

We have three main results. The first is contained in Propositions 4.4 and 4.5, and is valid also with K replaced by $\widetilde{K}$ (Corollary 4.6).

Theorem 1 *Let χ be a primitive character mod $\mathfrak{p}^m$. Then $V_\chi = \operatorname{Ind}_B^G \chi$ decomposes into a direct sum of K-representations as*

$$\operatorname{Res}_K V_\chi = V_\chi^{K_m} \oplus \bigoplus_{n>m} \left(W_{\chi,n}^+ \oplus W_{\chi,n}^- \right)$$

where $V_\chi^{K_m}$ is irreducible of degree $q^m + q^{m-1}$ if $\chi^2 \neq 1$, and for all $n > m$, the $W_{\chi,n}^\pm$ are irreducible, pairwise inequivalent, representations of degree $\frac{1}{2}q^{n-2}(q^2-1)$. Moreover, for $n \geq 2m$, the representations $W_{\chi,n}^\pm$ depend only on the central character of V_χ.

Our next result (Theorems 7.2 and 7.4) is the explicit matching of the irreducible K-representations arising in Theorem 1 with representations constructed by Shalika using induction from compact open subgroups to K. This construction is reviewed in Section 5; see particularly (5.3) and (5.9). It is an orbit method construction, in the sense of [H, LP], although effectively it is a construction for finite groups.

Our matching theorem may be summarized for K as follows. Here, $\mathfrak{g}$ denotes the maximal parahoric subalgebra $\mathfrak{sl}(2,\mathcal{O})$ of the Lie algebra of G.

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Theorem 2 *Suppose χ is primitive mod $\mathfrak{p}^m$. Then if $m > 1$, $V_\chi^{K_m} \simeq \mathcal{D}_m(\chi, X)$, where $X \in \mathfrak{g}\backslash\mathfrak{p}\mathfrak{g}$ represents a split K-orbit. For all $n > m$, $W_{\chi,n}^+ \oplus W_{\chi,n}^- \simeq \mathcal{D}_n(\rho_0, x_0) \oplus \mathcal{D}_n(\rho_1, x_1)$, where the characters ρ_i and the elements $x_i \in \mathfrak{g}\backslash\mathfrak{p}\mathfrak{g}$ (representing ramified K-orbits) are determined explicitly in terms of the character χ.*

Theorems 7.2 and 7.4 also give an explicit description of the $\widetilde{K}$-representations occuring in $\mathrm{Ind}_B^G \chi$, using a generalization of Shalika's construction presented in Section 6. For $n \geq 2m$, the orbits arising in Theorem 2 are nilpotent ones, as explained in Remark 7.5, and are no longer dependent on even the choice of additive character η used in Shalika's construction.

Theorem 2 fits in well in the theory of types. This theory asserts that the irreducible admissible representations are classified by the types they contain. A type is a pair (J, ρ) of a compact open subgroup J and a representation ρ of J; π contains (J, ρ) if $\mathrm{Res}_J(\pi)$ contains ρ as a summand. Alan Roche has computed types of the principal series representations and in [R, Example 3.5] explicitly describes a type (J_χ, ρ_χ) for $\mathrm{Ind}_B^G \chi$. This type is exactly the inducing datum for Shalika's constructions of the "primary" irreducible V^{K_m}; that is, $\mathcal{D}_m(\chi, X) = \mathrm{Ind}_{J_\chi}^K \rho_\chi$.

Most principal series representations are irreducible. With our normalizations (Section 3), the reducible ones are $\mathrm{Ind}_B^G \mathrm{sgn}_\tau$ for $\tau \in \{\varepsilon, \varpi, \varepsilon\varpi\}$. Each of these decomposes into two irreducible constituents, which we denote $H^\pm$ (depending on a choice of additive character). Our third result, an application of Theorem 2, is the identification of the irreducible K-representations occuring in each constituent.

The following is a summary of the results obtained for $\mathrm{Ind}_B^G \mathrm{sgn}_\varepsilon$, from Theorems 9.1 and 9.2 and Corollary 9.3 (with some extra notation suppressed for readability).

Theorem 3 *Consider $V = \mathrm{Ind}_B^G \mathrm{sgn}_\varepsilon$. Set $x_t = \left[\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right]$, and let $\mathbf{1}_K$ and St_K denote the trivial and Steinberg representations of K, respectively. Then*

$$\mathrm{Res}_K(V) = \mathbf{1}_K \oplus \mathrm{St}_K \oplus \bigoplus_{n>1} \{\mathcal{D}_n(\mathbf{1}, x_1) \oplus \mathcal{D}_n(\mathbf{1}, x_\varepsilon)\},$$

with $\mathrm{St}_K, \mathcal{D}_{2k+1}(\mathbf{1}, x_t) \subseteq H^+$ and $\mathbf{1}_K, \mathcal{D}_{2k}(\mathbf{1}, x_t) \subseteq H^-$, for $t \in \{1, \varepsilon\}$. Similarly,

$$\mathrm{Res}_{\widetilde{K}}(V) = \mathbf{1}_{\widetilde{K}} \oplus \mathrm{St}_{\widetilde{K}} \oplus \bigoplus_{n>1} \left\{ \widetilde{\mathcal{D}}_n(\mathbf{1}, x_{\varpi^{-1}}) \oplus \widetilde{\mathcal{D}}_n(\mathbf{1}, x_{\varepsilon\varpi^{-1}}) \right\},$$

with $\mathbf{1}_{\widetilde{K}}, \widetilde{\mathcal{D}}_{2k}(\mathbf{1}, x_{t\varpi^{-1}}) \subseteq H^+$ and $\mathrm{St}_{\widetilde{K}}, \widetilde{\mathcal{D}}_{2k+1}(\mathbf{1}, x_{t\varpi^{-1}}) \subseteq H^-$, for $t \in \{1, \varepsilon\}$.

The results for $\mathrm{Ind}_B^G \mathrm{sgn}_\tau$, $\tau = \varpi, \varepsilon\varpi$ are similar in flavour.

Theorems 2 and 3 tie in well with the "orbit method" interpretation of the classification of irreducible constituents of principal series, as follows. In [K], Wentang Kuo establishes an explicit connection between irreducible constituents of principal series representations of split reductive p-adic groups G and (classes of) principal nilpotent orbits. Specifically, he constructs, up to certain choices, a map ρ from the set of principal nilpotent orbits to the set of (generic) irreducible constituents of unitary principal series $\pi_\chi = \mathrm{Ind}_B^G \chi$ [K, 4.1.7]. This map ρ has the property that for $\mathcal{O}$,

a principal nilpotent orbit, the set of principal nilpotent orbits arising with nonzero coefficient in the Harish–Chandra–Howe local character expansion of $\rho(\mathcal{O})$ is exactly $Q_\chi \cdot \mathcal{O}$ [K, Thm 4.2.5]. Here Q_χ can be viewed as a subgroup of the maximal split torus of the (algebraic) adjoint group of G [K, 5.2.5].

For $G = SL(2, k)$, we can identify Q_χ with a subgroup of $k^\times$ which acts by scaling on the nilpotent orbits. When π_χ is irreducible, $Q_\chi = k^\times$, so all orbits occur; whereas when $\chi = \mathrm{sgn}_\tau$, $Q_{\mathrm{sgn}_\tau} = \{u \in k^\times \mid \mathrm{sgn}_\tau(u) = 1\}$.

In our case, we have produced a list of representations of K and $\widetilde{K}$ occuring in π_χ, such that each representation is constructed via an orbit method (in the sense of Kirillov) and is associated to an adjoint orbit of K or $\widetilde{K}$. The nilpotent adjoint orbits of K which occur are represented by X_t, $t \in \mathcal{O}^\times$, whereas those of $\widetilde{K}$ which occur are represented by X_t, $t \in \varpi^{-1}\mathcal{O}^\times$. Taking their G-saturations gives the four principal nilpotent orbits of G, and thus one would wish to say that π_χ is associated to the nilpotent orbits occuring in its K and $\widetilde{K}$ decompositions. A glance at Theorem 3, however, reveals this is far too naive. Nonetheless, one can see from Theorem 9.2 and Corollary 9.3 that the collection of nilpotent orbits which occur among the primitive modulo $\mathfrak{p}^n$ representations of K and $\widetilde{K}$ for fixed $n > 1$ in a given irreducible constituent of π_χ is invariant under Q_χ. It seems quite reasonable to expect that this relationship can be made more precise, a problem the author hopes to address in a subsequent paper.

The organization of this paper is as follows. We establish our notation in Section 2. In Section 3, we define the class of principal series representations V_χ and recall some useful results about induced representations. The decomposition of $\mathrm{Res}_K V_\chi$ into irreducible subrepresentations is given in Section 4. In Section 5 we recall pertinent results of [ShII], in which Shalika constructs irreducible representations of K starting from adjoint orbits of K on its $\mathcal{O}$-Lie algebra. Section 6 is devoted to describing how conjugation by ω gives equivalent constructions of representations of $\widetilde{K}$, with some small modifications. In Section 7 we match the irreducible representations of K (and $\widetilde{K}$) occuring in V_χ with those presented in Sections 5 and 6. In Section 8, we recall a description of the irreducible constituents of reducible principal series of $SL(2, k)$ given in [GGPS]. We conclude in Section 9 with the identification of the irreducible representations of K and $\widetilde{K}$ ocurring in each of these irreducible constituents.

Principal series representations of $PGL(2, k)$ were decomposed by Silberger relative to $K = PGL(2, \mathcal{O})$ in [Si] and relative to the other nonconjugate maximal compact $\widetilde{K}$ in [Si2]. Casselman considered related questions for all irreducible admissible representations of the group $GL(2, k)$ in [C]. In Section 4, we are inspired by Silberger's approach in [Si].

2 Notation

Let k be a p-adic field (of characteristic 0) of residual characteristic $p \neq 2$. Denote its integer ring by $\mathcal{O}$ and its prime ideal $\mathfrak{p}$. The order of the residue field $\kappa = \mathcal{O}/\mathfrak{p}$ is denoted $q = p^f$. Let ϖ be a uniformizing element of $\mathfrak{p}$, and normalize the p-adic valuation so that $\mathrm{val}(\varpi) = 1$. Fix a nonsquare element $\varepsilon \in \mathcal{O}^\times \backslash \mathcal{O}^{\times 2}$. Let sgn denote the sign character on $\kappa^\times$: $\mathrm{sgn}(a^2) = 1$ and $\mathrm{sgn}(\varepsilon a^2) = -1$ for all $a \in \kappa^\times$. Extend

this to a character on $\mathcal{O}^\times$ by $\mathrm{sgn}(1+\mathfrak{p}) = 1$.

On $k^\times$, the sign characters are defined by $\mathrm{sgn}_\tau(a) = (a/\tau)$, for $\tau \in k^\times/k^{\times 2}$, $\tau \neq 1$, where $(\cdot/\cdot)$ denotes the (2-)Hilbert symbol. Thus since $p \neq 2$, $\mathrm{sgn}_\varepsilon(m\varpi^k) = (-1)^k$ for all $m \in \mathcal{O}^\times$ and $k \in \mathbb{Z}$, and for $\tau \in \{\varpi, \varepsilon\varpi\}$, sgn_τ coincides with sgn on $\mathcal{O}^\times$.

Let $G = SL(2,k)$, the group of 2×2 unimodular matrices over the field k. Let $K = SL(2,\mathcal{O})$, the subgroup of matrices of G, all of whose entries lie in $\mathcal{O}$, and

$$\widetilde{K} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, d \in \mathcal{O},\ c \in \mathfrak{p},\ b \in \mathfrak{p}^{-1},\ ad - bc = 1 \right\}.$$

These are representatives of the two conjugacy classes of maximal compact subgroups of G. If we set

$$\omega = \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix},$$

representing an element of the affine Weyl group, then $\widetilde{K} = \omega K \omega^{-1} = K^\omega$.

We make use of the standard filtration of K by normal subgroups:

$$K = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$$

where K_i, $i \geq 1$, consists of those matrices in K which are congruent to the identity matrix I mod $\mathfrak{p}^i$. Let $\widetilde{K}_i = K_i^\omega$; this defines a filtration of $\widetilde{K}$.

Let K be either K or $\widetilde{K}$. Let (π, V) be an admissible smooth representation of G, and denote by V^{K_i} the set of K_i-fixed vectors in V. Then we have

$$V = \bigcup_{i \geq 0} V^{K_i} \tag{2.1}$$

and each subspace V^{K_i} is finite dimensional. Moreover, by normality of the K_i in K, each V^{K_i} is K-invariant. Hence, to decompose the restriction to K of (π, V) into irreducibles, it suffices to do so for each V^{K_i}.

Let χ' be a continuous multiplicative character of $k^\times$ and consider its restriction χ to a character of $\mathcal{O}^\times$. If $|\cdot|$ denotes the norm on k, we have $\chi'(r\varpi^k) = |\varpi^k|^s \chi(r)$ (for some $s \in \mathbb{C}$) whenever $r \in \mathcal{O}^\times$. The character χ' is called *unramified* if $\chi = \mathbf{1}$, the trivial character, and *ramified* otherwise. Let m be the least positive integer such that $1 + \mathfrak{p}^m \subseteq \ker(\chi)$; then we say χ is *primitive mod* $\mathfrak{p}^m$.

3 Principal Series Representations of $SL(2,k)$

Let B be the Borel subgroup of upper triangular matrices in G. Write $B = TU$, where T is the maximal split torus, consisting of diagonal matrices, and U is the unipotent radical, consisting of those matrices u in G such that $u - I$ is strictly upper triangular. Since the commutator of B lies in U, a character of B is defined by its restriction to T (and any character of T defines one of B). These characters may be identified with characters of $k^\times$ via

$$\chi' \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \chi'(a).$$

Then a *principal series representation* of G is a (normalized) induced representation denoted $\pi_{\chi'} = \operatorname{Ind}_B^G \chi'$, on the vector space

$$V_{\chi'} = \{f\colon G \to \mathbb{C} \mid f(bg) = \chi'(b)|b|f(g)\ \forall g \in G, b \in B, \text{ and } f \in C^\infty\}. \tag{3.1}$$

The action of G on $V_{\chi'}$ is given by right translations: $\pi_{\chi'}(g)f(h) = f(hg)$ for all $g, h \in G$. The normalization by $|b|$ in (3.1) ensures that unitarity is preserved and that $\operatorname{Ind}_B^G \chi' = \operatorname{Ind}_B^G \chi'^{-1}$ (that is, invariance under the action of the Weyl group). The principal series representation $(\pi_{\chi'}, V_{\chi'})$ is called unramified (respectively, ramified) if χ' is an unramified (respectively, ramified) character of T.

Since both K and $\widetilde{K}$ are good maximal compact subgroups, G admits the Iwasawa decompositions $G = BK = B\widetilde{K}$. If follows that for K denoting either K or $\widetilde{K}$,

$$\operatorname{Res}_K \operatorname{Ind}_B^G \chi' = \operatorname{Ind}_{B\cap K}^K \chi$$

where $\chi = \operatorname{Res}_{B\cap K} \chi'$. (In particular, the normalization factor in the induction from $B \cap K$ is identically 1 and may be omitted.) In the sequel, we assume without loss of generality that $\chi' = \chi$.

The following easy lemma implies that it suffices to consider the restriction to K.

Lemma 3.1 *Let K, $\widetilde{K}$ be as above. There exists a vector space automorphism Υ of $\operatorname{Ind}_B^G \chi$ such that $\pi_\chi(k)(f) = \Upsilon^{-1}\pi_\chi(\omega k \omega^{-1})(\Upsilon f)$ for all $k \in K$, $f \in \operatorname{Ind}_B^G \chi$. Thus*

$$\operatorname{Res}_K \operatorname{Ind}_B^G \chi \simeq \operatorname{Res}_{\widetilde{K}} \operatorname{Ind}_B^G \chi$$

(with the group isomorphism implicit).

Proof Let $\mathfrak{s} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. For each $f \in \operatorname{Ind}_B^G \chi$, define $\Upsilon f(g) = f(\mathfrak{s}\omega^{-1} g \omega)$. Since

$$\mathfrak{s}\omega^{-1} \begin{bmatrix} a & c \\ 0 & a^{-1} \end{bmatrix} \omega \mathfrak{s}^{-1} = \begin{bmatrix} a & -c\varpi \\ 0 & a^{-1} \end{bmatrix}$$

it follows that $\Upsilon f(bg) = f(\mathfrak{s}\omega^{-1} bg\omega) = f(\mathfrak{s}\omega^{-1} b\omega\mathfrak{s}^{-1}\mathfrak{s}\omega^{-1} g\omega) = \chi(b)\Upsilon f(g)$ for all $b \in B$, $g \in G$. Hence Υ is an automorphism. It is an intertwining operator between the usual action π_χ of G and the twisted action $\pi_\chi^\omega(g) = \pi_\chi(\omega g \omega^{-1})$, since for all $g, h \in G$ and $f \in \operatorname{Ind}_B^G \chi$,

$$\Upsilon(\pi_\chi(g)f)(h) = \pi_\chi(g)f(\mathfrak{s}\omega^{-1} h\omega) = f(\mathfrak{s}\omega^{-1} h\omega g)$$

and

$$\pi_\chi^\omega(g)(\Upsilon f)(h) = \Upsilon f(h\omega g\omega^{-1}) = f(\mathfrak{s}\omega^{-1}(h\omega g\omega^{-1})\omega) = f(\mathfrak{s}\omega^{-1} h\omega g)$$

are equal. The conclusions follow. ∎

Finally, let us recall the Intertwining Number Theorem for finite groups, which will be used extensively in the following sections.

Proposition 3.2 *Let G be a finite group and H, L two subgroups of G. Suppose χ and ρ are characters of H and L, respectively. Define $\mathcal{H} = \mathcal{H}(H\backslash G/L, \chi, \rho)$ to be the vector space*

$$\mathcal{H} = \{\mathcal{F}\colon G \to \mathbb{C} \mid \mathcal{F}(hgl) = \chi(h)\mathcal{F}(g)\rho(l) \quad \forall h \in H, g \in G, l \in L\}.$$

Then the intertwining operators from $\operatorname{Ind}_L^G \rho$ to $\operatorname{Ind}_H^G \chi$ are given by convolution with the functions in $\mathcal{H}$. This is a Hecke algebra *when $H = L$ and $\chi = \rho$.*

Proof Given $\mathcal{F} \in \mathcal{H}$, it is clear that the map T from $\operatorname{Ind}_L^G \rho$ to $\operatorname{Ind}_H^G \chi$ given by

$$Tf(g) = \sum_{k \in G} \mathcal{F}(k) f(k^{-1}g) \tag{3.2}$$

is an intertwining operator. Hence it suffices to prove that

$$\dim \mathcal{H} = \dim \operatorname{Hom}_G(\operatorname{Ind}_H^G \chi, \operatorname{Ind}_L^G \rho).$$

By Frobenius reciprocity, one has $\operatorname{Hom}_G(\operatorname{Ind}_H^G \chi, \operatorname{Ind}_L^G \rho) \simeq \operatorname{Hom}_H(\chi, \operatorname{Res}_H \operatorname{Ind}_L^G \rho)$. By [S, Prop. 22], one has

$$\operatorname{Res}_H \operatorname{Ind}_L^G \rho \simeq \bigoplus_{s \in S} \operatorname{Ind}_{L_s}^H \rho_s,$$

where S is a system of representatives for the double cosets $H\backslash G/L$, the group L_s is the intersection $sLs^{-1} \cap H$, and $\rho_s(x) = \rho(s^{-1}xs)$ for $x \in L_s$. Hence

$$\operatorname{Hom}_H(\chi, \operatorname{Res}_H \operatorname{Ind}_L^G \rho) \simeq \bigoplus_{s \in S} \operatorname{Hom}_H(\chi, \operatorname{Ind}_{L_s}^H \rho_s).$$

This latter term is isomorphic to $\bigoplus_{s \in S} \operatorname{Hom}_{L_s}(\chi, \rho_s)$, again by Frobenius reciprocity. Since these are characters, we conclude that for a given $s \in S$, the dimension of $\operatorname{Hom}_{L_s}(\chi, \rho_s)$ is at most 1. It is nonzero if and only if $\chi(h) = \rho_s(h)$ for all $h \in L_s$, that is, if and only if $\chi(h) = \rho(l)$ whenever $h = sls^{-1}$ (for $h \in H$ and $l \in L$). This is precisely the condition under which there exists a non-zero function $\mathcal{F} \in \mathcal{H}$ supported on the double coset HsL. Such functions form a basis for $\mathcal{H}$, which completes the proof. ∎

4 Branching Rules for (π_χ, V_χ) under K and $\widetilde{K}$

Let χ be as in Section 3 and write V_χ for $\operatorname{Res}_K V_\chi \simeq \operatorname{Ind}_{B\cap K}^K \chi$ (or simply V where there can be no confusion). Write $B^n = (B \cap K)/(B \cap K_n)$ and $K^n = K/K_n$. Suppose χ is primitive mod $\mathfrak{p}^m$. Then, for $\chi \neq \mathbf{1}$, χ factors through B^n if and only if $n \geq m$, in which case we denote the corresponding character of B^n by $\overline{\chi}$.

Definition 4.1 A representation π of K^n which does not factor through K^{n-1} is called *primitive*.

(One often adds the condition that a primitive representation must be irreducible (see, for example, [ShII]), but it is convenient for us not to.)

Lemma 4.2 *Suppose $\chi \neq \mathbf{1}$ is primitive mod $\mathfrak{p}^m$. If $0 \leq n < m$, then $V_\chi^{K_n} = \{0\}$. For any $n \geq m$,*

$$V_\chi^{K_n} \simeq Ind_{B^n}^{K^n} \overline{\chi}$$

where both sides are primitive representations of K^n.

Proof Let $n \geq 0$. If $B \cap K_n \not\subseteq \ker(\chi)$, then $V_\chi^{K_n} = \{0\}$; this happens exactly for $n < m$ (unless $\chi = \mathbf{1}$). The indicated isomorphism of K^n-representations for $n \geq m$ is immediate. This implies in particular that $V_\chi^{K_{n-1}} \subsetneq V_\chi^{K_n}$ for all $n \geq m$, whence the primitivity of $V_\chi^{K_n}$. ■

Note that Lemma 4.2, together with (2.1), implies that in order to decompose V_χ into irreducibles under K, it suffices to decompose principal series representations of the finite groups K^n. We begin with the group K^1, which may be identified with $SL(2, \kappa)$, a finite group of Lie type. Its representation theory is well understood, and we deduce the following result from [DM, §15.9].

Lemma 4.3 *Suppose χ is primitive mod $\mathfrak{p}$. If $\chi = \mathbf{1}$ is the trivial character, then $V_{\mathbf{1}}^{K_1} = \mathbf{1} \oplus \mathrm{St}$, where St denotes the q-dimensional Steinberg representation of $SL(2, \kappa)$. If $\chi = \mathrm{sgn}$, then $V_{\mathrm{sgn}}^{K_1}$ decomposes into two inequivalent irreducible representations with characters $\Xi_{\mathrm{sgn}}^{\pm}$, each of degree $(q+1)/2$. For any other χ, $V_\chi^{K_1}$ is irreducible.*

The last statement of Lemma 4.3 is a special case of the following Proposition.

Proposition 4.4 *Let χ be a primitive character mod $\mathfrak{p}^m$, and $V = V_\chi$. If $\chi^2 \neq \mathbf{1}$, then V^{K_m} is an irreducible representation of degree $q^m + q^{m-1}$. For all $n > m$, $W_n = V^{K_n}/V^{K_{n-1}}$ decomposes into two inequivalent irreducible representations of degree $q^{n-2}(q^2-1)/2$.*

Proof The dimensions of V^{K_m} and W_n can be computed directly (for instance by counting left cosets of B^n in K^n). One can show the irreducibility of V^{K_m} using the Mackey irreducibility criterion, but let us instead deduce both assertions by applying Proposition 3.2, with $\mathcal{H} = \mathcal{H}(B^n \backslash K^n / B^n, \chi)$.

A basis for $\mathcal{H}$ consists of functions with support on a double coset of B^n in K^n. Let $\varepsilon \in \mathcal{O}^\times \backslash \mathcal{O}^{\times 2}$ be a fixed nonsquare (as in Section 2). Then a set of representatives for these double cosets is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathfrak{s} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ s_k^t = \begin{bmatrix} 1 & 0 \\ t\varpi^k & 1 \end{bmatrix} : t \in \{1, \varepsilon\}, \ 1 \leq k < n \right\} \tag{4.1}$$

(or, more accurately, their corresponding cosets in $K^n = K/K_n$).

For each $k \in K^n$, a nonzero function $\mathcal{F}_k \in \mathcal{H}$ with support on $B^n k B^n$ exists if and only if, whenever $bkb' = k$ (with $b, b' \in B^n$) we have $\chi(b)\chi(b') = 1$. Thus the identity double coset supports a nonzero function (namely, $\mathcal{F}_I|_{B^n} = \chi$), but a

nonzero $\mathcal{F}_{\mathfrak{s}} \in \mathcal{H}$ supported on the double coset $B^n \mathfrak{s} B^n$ exists if and only if $\chi^2 = \mathbf{1}$. It also follows from direct calculation that the double cosets $B^n s_k^t B^n$ support nonzero functions if and only if $1 + \mathfrak{p}^k$ is contained in the kernel of χ, that is, for $k \geq m$.

Hence $\dim \mathcal{H} = 1 + 2(n - m)$ if $\chi^2 \neq \mathbf{1}$; otherwise, $\dim \mathcal{H} = 2 + 2(n - m)$. The irreducibility of V^{K_m} (for $\chi^2 \neq \mathbf{1}$), and the decomposition of each W_n into two inequivalent irreducible subrepresentations, follows.

Let us write this decomposition for each $n > m$ as $W_n = W_n^+ \oplus W_n^-$. We would like to deduce that $W_n^{\pm}$ have the same degree. Note that W_n is the restriction to $SL(2, \mathcal{O})$ of a representation of $GL(2, \mathcal{O})$ — namely, consider the principal series representation corresponding to the character $\chi \otimes \mathbf{1}$ on diagonal subgroup of $GL(2, \mathcal{O})$ (identified with $k^\times \times k^\times$). By the same method as above, we deduce that this subspace is irreducible under $GL(2, \mathcal{O})$. Hence there must exist an element $g \in GL(2, \mathcal{O})$, $g \notin SL(2, \mathcal{O})$ mapping W_n^+ bijectively onto W_n^-. ■

We thus have

$$V_\chi = V_\chi^{K_m} \oplus (W_{\chi,m+1}^+ \oplus W_{\chi,m+1}^-) \oplus (W_{\chi,m+2}^+ \oplus W_{\chi,m+2}^-) \oplus \cdots ,$$

and this is an orthogonal decomposition of V_χ into pairwise inequivalent irreducible (when $\chi^2 \neq \mathbf{1}$) subrepresentations of K.

The following proposition shows that the "tail end" of V_χ depends only on the central character of the representation. More precisely, let ϑ be a character of $k^\times$ satisfying $\vartheta(-1) = -1$. (For example, take $\vartheta = \mathrm{sgn}$ if $-1 \notin \mathcal{O}^{\times 2}$.)

Proposition 4.5 *Let χ be primitive modulo $\mathfrak{p}^m$. The orthogonal complement of $V_\chi^{K_{2m-1}}$ in V_χ is equivalent to the orthogonal complement of $V_{\mathbf{1}}^{K_{2m-1}}$ in $V_{\mathbf{1}}$ if $\chi(-1) = 1$ and to the orthogonal complement of $V_\vartheta^{K_{2m-1}}$ in V_ϑ if $\chi(-1) = -1$.*

Proof Let $n \geq m$. We apply Proposition 3.2. Let $\mathcal{H} = \mathcal{H}(B^n \backslash K^n / B^n, \chi, \mathbf{1})$ if $\chi(-1) = 1$ and $\mathcal{H} = \mathcal{H}(B^n \backslash K^n / B^n, \chi, \vartheta)$ if $\chi(-1) = -1$. The condition on χ is clearly necessary for $\mathcal{H}$ to be nontrivial.

Assume without loss of generality that $\chi \notin \{\mathbf{1}, \vartheta, \vartheta^{-1}\}$ (since $V_\vartheta \simeq V_{\vartheta^{-1}}$). Then neither B^n nor $B^n \mathfrak{s} B^n$ (notation of (4.1)) support a nonzero function in $\mathcal{H}$. For the remaining double cosets, a nonzero function $\mathcal{F} \in \mathcal{H}$ supported on $B^n s_k^t B^n$ exists if and only if whenever $b s_k^t b' = s_k^t$, the diagonal elements of b and b' either all lie in $1 + \mathfrak{p}^m$ or in $-1 + \mathfrak{p}^m$. This occurs exactly for those s_k^t such that neither the parameter k nor the difference $n - k$ are less than m. It follows that $\dim \mathcal{H} = 2(n + 1 - 2m)$ for $n \geq 2m$, and is zero otherwise, as required. ■

Corollary 4.6 *The statements of Lemma 4.2, Lemma 4.3, Proposition 4.4 and Proposition 4.5 are also true with K replaced by $\widetilde{K}$.*

This follows from Lemma 3.1.

5 Shalika's Classification of Irreducible Representations of K

Propositions 4.4 and 4.5 give a description of the decomposition of $\mathrm{Res}_K V_\chi$ into irreducible representations. It is of interest to further identify these subrepresentations in an explicit way, using the classification of irreducible representations of K given by Shalika in [ShI, ShII]. In this Section, we recall the pertinent results from [ShII].

Let $k \geq 1$ and fix η a primitive additive character of $\mathcal{O}/\mathfrak{p}^k$ (that is, a character of $\mathcal{O}$ satisfying $\mathfrak{p}^{k-1} \not\subseteq \ker(\eta)$). Note that this primitivity implies that the set of additive characters of $\mathcal{O}/\mathfrak{p}^k$ is exactly $\{\eta^a \mid a \in \mathcal{O}/\mathfrak{p}^k\}$ where $\eta^a(b) = \eta(ab)$ for all $b \in \mathcal{O}/\mathfrak{p}^k$.

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathcal{O})$ be the Lie algebra (over $\mathcal{O}$) of traceless 2×2 matrices with coefficients in $\mathcal{O}$. Denote by $\mathfrak{g}_k$ the quotient space $\mathfrak{g}/\mathfrak{p}^k\mathfrak{g}$; this is a Lie algebra over $\mathcal{O}/\mathfrak{p}^k$. The trace defines a nondegenerate K-invariant bilinear form on $\mathfrak{g}$ (or on $\mathfrak{g}_k$) via $B(X, Y) = \mathrm{tr}(XY)$.

An element $X \in \mathfrak{g}$ is called *primitive* if $X \notin \mathfrak{p}\mathfrak{g}$. Fix a nonsquare $\varepsilon \in \mathcal{O}^\times$. Representatives of the K-orbits of primitive elements of $\mathfrak{g}$ are:

split orbits: $\left[\begin{smallmatrix} \lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right]$, where $\lambda \in \mathcal{O}^\times$ (taken modulo ± 1);

ramified orbits: $\left[\begin{smallmatrix} 0 & t \\ a\varpi & 0 \end{smallmatrix}\right]$, where $t \in \{1, \varepsilon\}$ and $a \in \mathcal{O}$;

unramified orbits: $\left[\begin{smallmatrix} 0 & 1 \\ \varepsilon a^2 & 0 \end{smallmatrix}\right]$, where $a \in \mathcal{O}^\times$.

The centralizer T_X of a split element X is the diagonal split torus; whereas if $X = \left[\begin{smallmatrix} 0 & \sigma \\ \tau & 0 \end{smallmatrix}\right]$ then

$$T_X = \left\{ \begin{bmatrix} a & b \\ b\tau\sigma^{-1} & a \end{bmatrix} : a^2 - b^2\tau\sigma^{-1} = 1 \right\}.$$

Denote by $T_{X,n}$ the image of each of these in K^n; they coincide with the centralizers in K^n of the image of X in $\mathfrak{g}_n$.

Shalika constructs irreducible primitive representations of K^n for $n \geq 2$ (that is, representations which do not factor through K^{n-1}) using the following result from Clifford theory (see [ShII, Thm 1.3], for example): *Given an irreducible representation τ of a normal subgroup N of a finite group G, and an extension of τ to an irreducible representation θ of the stabilizer S of τ under G, then* $\mathrm{Ind}_S^G\, \theta$ *is irreducible and its restriction to N contains τ.*

Shalika's construction proceeds as follows. First suppose $n = 2k$ is even. Define the normal subgroup N_k of $K^{2k} = K/K_{2k}$ as

$$N_k = \left\{ \mathbf{n} = \begin{bmatrix} 1 + c\varpi^k & d\varpi^k \\ e\varpi^k & 1 - c\varpi^k \end{bmatrix} : c, d, e \in \mathcal{O}/\mathfrak{p}^k \right\}. \tag{5.1}$$

In particular, an element $\mathbf{n} \in N_k$ can be written as $\mathbf{n} = 1 + z\varpi^k$, with $z \in \mathfrak{g}_k$. Then for each $x \in \mathfrak{g}_k$, define a character η_x of N_k via

$$\eta_x(1 + z\varpi^k) = \eta(\mathrm{tr}(zx)). \tag{5.2}$$

As x runs over the K-orbits in $\mathfrak{g}_k$, η_x gives all (classes of) characters of the abelian normal subgroup N_k of K^{2k}. Choose any $X \in \mathfrak{g}_{2k}$ whose reduction modulo $\mathfrak{p}^k$ equals x;

then Shalika shows that the stabilizer S of η_x under K^{2k} is $T_{X,2k}N_k$. Hence if ρ is a character of $T_{X,2k}$ coinciding with η_x on $T_{X,2k} \cap N_k$, then

$$\mathcal{D}_{2k}(\rho, x) = \mathrm{Ind}_{T_{X,2k}N_k}^{K^{2k}} \rho \otimes \eta_x \tag{5.3}$$

defines a primitive irreducible representation of K^{2k} (where $\rho \otimes \eta_x$ denotes the character of $T_{X,2k}N_k$ given by $\rho \otimes \eta_x(\mathbf{tn}) = \rho(\mathbf{t})\eta_x(\mathbf{n})$).

The construction for odd n, $n = 2k + 1$, is more complex; we describe it only for split and ramified orbits. One defines for $x \in \mathfrak{g}_k$ characters η_x of the abelian normal subgroup N_{k+1} of K^{2k+1} via $\eta_x(1 + z\varpi^{k+1}) = \eta(\mathrm{tr}(zx))$ as before. Now, however, the stabilizer of η_x in K^{2k+1} is $T_{X,2k+1}I_k$, where $X \in \mathfrak{g}_{2k+1}$ is any lift of x, and the subgroup I_k is defined to be

$$I_k = \left\{ \mathbf{n} = \begin{bmatrix} 1 + c\varpi^k & d\varpi^k \\ e\varpi^{k+1} & 1 - c\varpi^k + c^2\varpi^{2k} \end{bmatrix} : c, d \in \mathcal{O}/\mathfrak{p}^{k+1}, \quad e \in \mathcal{O}/\mathfrak{p}^k \right\}. \tag{5.4}$$

Choose a character ξ of $\mathfrak{p}^k/\mathfrak{p}^{2k+1}$ satisfying

$$\xi(\lambda\varpi^{k+1}) = \eta(\lambda) \quad \forall \lambda \in \mathcal{O}/\mathfrak{p}^k \tag{5.5}$$

and define an extension $\eta_{x,\xi}$ of η_x to I_k via

$$\eta_{x,\xi}(1 + z\varpi^k) = \xi(\varpi^k \,\mathrm{tr}(zX)), \tag{5.6}$$

where we have written $\mathbf{n} = 1 + z\varpi^k$ as in (5.4). This expression can be simplified somewhat, as follows.

When $x \in \mathfrak{g}_k$ is a split element of the form $x = \left[\begin{smallmatrix} \lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right]$ with $X \in \mathfrak{g}_{2k+1}$ covering x, (5.6) becomes

$$\eta_{x,\xi}(1 + z\varpi^k) = \xi(2\lambda c\varpi^k - \lambda c^2\varpi^{2k}). \tag{5.7}$$

When $x \in \mathfrak{g}_k$ is a ramified element of the form $x = \left[\begin{smallmatrix} 0 & t \\ a\varpi & 0 \end{smallmatrix}\right]$ (with $t \in \{1, \varepsilon\}$, $a \in \mathcal{O}$), then (5.6) becomes

$$\eta_{x,\xi}(1 + z\varpi^k) = \eta(ad + te) = \eta(\varpi^{-1} \,\mathrm{tr}(zX)). \tag{5.8}$$

In these two cases, Shalika deduces that the representations

$$\mathcal{D}_{2k+1}(\rho, x, \xi) = \mathrm{Ind}_{T_{X,2k+1}I_k}^{K^{2k+1}} \rho \otimes \eta_{x,\xi} \tag{5.9}$$

for ρ any character of $T_{X,2k+1}$ agreeing with $\eta_{x,\xi}$ on $T_{X,2k+1} \cap I_k$, are irreducible and primitive. Note that since (5.8) is independent of the choice of ξ, for x ramified we may abbreviate $\mathcal{D}_{2k+1}(\rho, x, \xi)$ to $\mathcal{D}_{2k+1}(\rho, x)$ where convenient.

Remark 5.1 This method for the case of n odd and x unramified is also discussed in [ShII]. In contrast with the above, it does not produce a complete list of primitive irreducible representations of K associated to unramified elements; to do this, Shalika uses a more geometric construction in [ShI]. We omit this case and will have no need for those representations here.

The representations (5.3) and (5.9) are primitive, irreducible and equivalent only if their parameters x are in the same K-orbit [ShII, Theorems 2.1 and 2.5]. Their dimensions are as follows:

split case:	$q^{n-1}(q+1)$,
ramified case:	$\frac{1}{2}q^{n-1}(q^2-1)$,
unramified case:	$q^{n-1}(q-1)$,

and they, together with the unramified representations constructed in [ShI], exhaust all irreducible primitive (mod K_n) representations of K, $n \geq 2$.

6 Representations of $\widetilde{K}$

We would now like to carry out the same Kirillov-type construction to explicitly produce representations of $\widetilde{K}$. These representations would then be associated to primitive orbits, not in $\mathfrak{g}$, but in the $\mathcal{O}$-Lie algebra

$$\widetilde{\mathfrak{g}} = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a \in \mathcal{O}, b \in \mathfrak{p}^{-1}, c \in \mathfrak{p} \right\} = \mathfrak{g}^{\omega}.$$

There is essentially no work to be done; one simply conjugates all the components in Shalika's construction by ω and this produces the required Kirillov-type construction of representations of $\widetilde{K}$.

More specifically, the primitive orbits in $\widetilde{\mathfrak{g}}$ are:

split orbits:	$\left[\begin{smallmatrix} \lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right]^{\omega} = \left[\begin{smallmatrix} -\lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right]$, $\lambda \in \mathcal{O}^{\times}$ (taken modulo ± 1);
ramified orbits:	$\left[\begin{smallmatrix} 0 & t \\ a\varpi & 0 \end{smallmatrix}\right]^{\omega} = \left[\begin{smallmatrix} 0 & a \\ t\varpi & 0 \end{smallmatrix}\right]$, with $t \in \{1, \varepsilon\}$, $a \in \mathcal{O}$; and
unramified orbits:	$\left[\begin{smallmatrix} 0 & 1 \\ \varepsilon a^2 & 0 \end{smallmatrix}\right]^{\omega} = \left[\begin{smallmatrix} 0 & \varepsilon a^2 \varpi^{-1} \\ \varpi & 0 \end{smallmatrix}\right]$ with $a \in \mathcal{O}^{\times}$.

Let $\widetilde{T}_{X,n}$ denote the stabilizer of $X \in \widetilde{\mathfrak{g}}$ in $\widetilde{K}^n$. Define the normal subgroup $\widetilde{N_k}$ of $\widetilde{K}^{2k}$ via $\widetilde{N_k} = \{1 + z\varpi^k \mid z \in \widetilde{\mathfrak{g}}_k\}$ and the subgroup $\widetilde{I_k}$ of $\widetilde{K}^{2k+1}$ via (5.4), with the necessary modification that $d \in \mathcal{O}/\mathfrak{p}^k$ and $e \in \mathcal{O}/\mathfrak{p}^{k+1}$.

Choose $x \in \widetilde{\mathfrak{g}}_k$ as above, and $X \in \widetilde{\mathfrak{g}}_n$ lifting x. Then the corresponding character $\widetilde{\eta}_x$ of $\widetilde{N_k}$ is given by $\widetilde{\eta}_x(1 + z\varpi^k) = \eta(\mathrm{tr}(zx))$. For $n = 2k+1$, choose ξ as in (5.5) and define $\widetilde{\eta}_{x,\xi}(1 + z\varpi^k) = \xi(\varpi^k \,\mathrm{tr}(zX))$.

Then, for $\widetilde{\rho}$ a character of $\widetilde{T}_{X,n}$ agreeing with $\widetilde{\eta}_x$, $\widetilde{\eta}_{x,\xi}$ as appropriate, we define:

$$\widetilde{\mathcal{D}}_{2k}(\widetilde{\rho}, x) = \mathrm{Ind}_{\widetilde{T}_{X,2k}\widetilde{N}_k}^{\widetilde{K^{2k}}} \widetilde{\rho} \otimes \widetilde{\eta}_x;$$

$$\widetilde{\mathcal{D}}_{2k+1}(\widetilde{\rho}, x, \xi) = \mathrm{Ind}_{\widetilde{T}_{X,2k+1}\widetilde{I}_k}^{\widetilde{K^{2k+1}}} \widetilde{\rho} \otimes \widetilde{\eta}_{x,\xi}.$$

Proposition 6.1 *The representations above are irreducible and primitive, and under the isomorphism $K \simeq \widetilde{K} = K^{\omega} = \omega K \omega^{-1}$, we have*

$$\widetilde{\mathcal{D}}_{2k}(\rho^{\omega}, x^{\omega}) \simeq \mathcal{D}_{2k}(\rho, x)$$

and

$$\widetilde{\mathcal{D}}_{2k+1}(\rho^{\omega}, x^{\omega}, \xi) \simeq \mathcal{D}_{2k+1}(\rho, x, \xi)$$

where $\rho^{\omega}(\mathrm{t}) = \rho(\omega^{-1}\mathrm{t}\omega)$ and $x^{\omega} = \omega x \omega^{-1}$.

Proof First note that for any subgroup $H \subseteq K$, and character ϕ of H, $\mathrm{Ind}_H^K \phi \simeq \mathrm{Ind}_{H^\omega}^{K^\omega} \phi^\omega$ via the map $f \mapsto f^\omega$, where $f^\omega(g) = f(\omega^{-1} g \omega)$. This map satisfies $(k \cdot f)^\omega = k^\omega \cdot f^\omega$, which is the statement that the representations of the two isomorphic groups are equivalent.

Since $\widetilde{T}_{X^\omega,n} = T_{X,n}^\omega$, $\widetilde{N_k} = N_k^\omega$, $\widetilde{I}_k = I_k^\omega$, $\widetilde{\eta}_{x^\omega} = (\eta_x)^\omega$ and $\widetilde{\eta}_{x^\omega,\xi} = (\eta_{x,\xi})^\omega$, the indicated isomorphisms, and thus the first statement of the proposition, follow from (5.3), (5.9) and the definitions preceding the proposition. ■

The representatives for the primitive orbits of $\widetilde{K}$ given above are in some sense not the most natural. For instance, when x represents a split orbit, both $x^\omega = -x$ and x itself represent the same $\widetilde{K}$ orbit.

Corollary 6.2 *Let* $x = \left[\begin{smallmatrix} \lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right]$, *with* $\lambda \in \mathcal{O}^\times$. *Then*

$$\widetilde{\mathcal{D}}_{2k}(\rho, x) \simeq \mathcal{D}_{2k}(\rho, x)$$

and

$$\widetilde{\mathcal{D}}_{2k+1}(\rho, x, \xi) \simeq \mathcal{D}_{2k+1}(\rho, x, \xi)$$

(with the group isomorphism implicit).

Proof For any subgroup $H \subseteq K$, $\mathfrak{s} \in K$ and character ϕ of H, $\mathrm{Ind}_H^K \phi \simeq \mathrm{Ind}_{H^\mathfrak{s}}^K \phi^\mathfrak{s}$ via the map $f \mapsto f_\mathfrak{s}$ where $f_\mathfrak{s}(k) = f(\mathfrak{s}^{-1}k)$. Let $\mathfrak{s} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Then since $(T_{X,n}N_k)^\mathfrak{s} = T_{-X,n}N_k$, and $\rho^\mathfrak{s} \otimes \eta_x^\mathfrak{s} = \rho^{-1} \otimes \eta_{-x}$, we have $\mathcal{D}_{2k}(\rho, x) \simeq \mathcal{D}_{2k}(\rho^{-1}, -x)$. Since $\rho^\omega = \rho^{-1}$ and $x^\omega = -x$, we are done by Proposition 6.1. (This part also follows from [ShII, Theorem 2.1].)

When $n = 2k+1$, the subgroup I_k is not normal, and in fact $I_k^\mathfrak{s} = I_k^T$, the transpose. We wish to prove that the two representations,

$$\mathcal{D}_n(\rho^{\omega^{-1}}, x^{\omega^{-1}}, \xi) = \mathcal{D}_n(\rho^{-1}, -x, \xi) = \mathrm{Ind}_{T_{-X,n}I_k}^{K^n} \rho^{-1} \otimes \eta_{-x,\xi}$$

(from Proposition 6.1) and $\mathcal{D}_n(\rho, x, \xi) \simeq \mathrm{Ind}_{T_{-X,n}I_k^\mathfrak{s}}^{K^n} \rho^{-1} \otimes \eta_{x,\xi}^\mathfrak{s}$ (from Corollary 6.2), are isomorphic. We use Proposition 3.2. Since $T_{-X,n}I_k \cap T_{-X,n}I_k^\mathfrak{s} = I_k \cap I_k^\mathfrak{s}$, one sees that the two characters $\eta_{-x,\xi}$ and $\eta_{x,\xi}^\mathfrak{s}$ agree on the intersection. Consequently there exists a nonzero intertwining operator (supported on the identity double coset) between these two irreducible representations of K^n. ■

For nilpotent x, x^ω represents also a nilpotent orbit of $\widetilde{K}$. An upper triangular representative of this orbit, which compares more directly with x, is preferable. Let $X = X_t = \left[\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right] \in \mathfrak{g}_{2k+1}$ and $x = x_t$, its image in $\mathfrak{g}_k$. Set

$$\mathfrak{r} = \begin{bmatrix} 0 & \alpha^{-1}r \\ r & 0 \end{bmatrix}, \text{ with } \begin{cases} r = \sqrt{-1},\ \alpha = 1 & \text{if } -1 \in k^2, \\ r = \sqrt{-\varepsilon},\ \alpha = \varepsilon & \text{if } -1 \notin k^2. \end{cases}$$

(An obvious choice for the nonsquare element ε in the latter case would be $\varepsilon = -1$, giving $\mathfrak{r} = \mathfrak{s}$.) Then for $\eta_{x^{\mathfrak{r}},\xi}$ defined by (5.6) and any character ρ agreeing with $\eta_{x^{\mathfrak{r}},\xi}$ on the intersection, define

$$\mathcal{D}_{2k+1}(\rho, x^{\mathfrak{r}}, \xi) = \mathrm{Ind}_{T_{X^{\mathfrak{r}},2k+1}I_k}^{K^{2k+1}} \rho \otimes \eta_{x^{\mathfrak{r}},\xi}.$$

Note that in this case, the expression for $\eta_{x^{\mathfrak{r}},\xi}$ does not simplify as it did in (5.8).

Lemma 6.3 *For X, ρ, ξ as above, we have $\mathcal{D}_{2k+1}(\rho, x^{\mathfrak{r}}, \xi) \simeq \mathcal{D}_{2k+1}(\rho, x, \xi)$.*

Proof We have $\mathcal{D}_{2k+1}(\rho, x, \xi) = \mathrm{Ind}_{T_{X,2k+1}I_k}^{K^{2k+1}} \rho \otimes \eta_{x,\xi} \simeq \mathrm{Ind}_{T_{X^{\mathfrak{r}},2k+1}I_k^{\mathfrak{r}}}^{K^{2k+1}} \rho^{\mathfrak{r}} \otimes \eta_{x,\xi}^{\mathfrak{r}}$. The structure of $T_{X,2k+1}$ for this X implies that $\rho^{\mathfrak{r}} = \rho$. As in the proof of Corollary 6.2, we apply Proposition 3.2 to see that there is an intertwining operator between this last representation and $\mathcal{D}_{2k+1}(\rho, x^{\mathfrak{r}}, \xi)$, supported on the identity double coset. Hence $\mathcal{D}_{2k+1}(\rho, x, \xi)$ occurs in $\mathcal{D}_{2k+1}(\rho, x^{\mathfrak{r}}, \xi)$; by dimension count, they are equivalent. ■

One then defines, for $t \in \{\varpi^{-1}, \varepsilon\varpi^{-1}\}$, the analogous representations of $\widetilde{K}$ via $\widetilde{\mathcal{D}}_{2k+1}(\rho, x_t, \xi) = \mathrm{Ind}_{\widetilde{T}_{X_t,2k+1}\widetilde{I}_k}^{\widetilde{K}^{2k+1}} \rho \otimes \eta_{x_t,\xi}$.

Corollary 6.4 *Let $t \in \{1, \varepsilon\}$, $X_t \in \mathfrak{g}_{2k+1}$, $x_t \in \mathfrak{g}_k$, ρ, ξ and $\alpha \in \{1, \varepsilon\}$ be as above. Then*

$$\widetilde{\mathcal{D}}_{2k}(\rho, x_{\alpha t\varpi^{-1}}) \simeq \mathcal{D}_{2k}(\rho, x_t) \quad \textit{and} \quad \widetilde{\mathcal{D}}_{2k+1}(\rho, x_{\alpha t\varpi^{-1}}, \xi) \simeq \mathcal{D}_{2k+1}(\rho, x_t, \xi).$$

Proof Let $\widetilde{\mathfrak{r}} = \mathfrak{r}\left[\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi^{-1} \end{smallmatrix}\right]$; then $\widetilde{\mathfrak{r}} \in \widetilde{K}$ conjugates X_t^{ω} to $X_{\alpha t\varpi^{-1}}$. We have $\widetilde{T}_{X,n}^{\widetilde{\mathfrak{r}}} = \widetilde{T}_{X^{\widetilde{\mathfrak{r}}},n}$ and $\rho^{\widetilde{\mathfrak{r}}} = \rho = \rho^{\omega}$. Since $\widetilde{\mathfrak{r}} \in \widetilde{K}^n$, $\widetilde{N_k}^{\widetilde{\mathfrak{r}}} = \widetilde{N_k}$, but $\widetilde{I}_k^{\widetilde{\mathfrak{r}}} \neq \widetilde{I}_k$. As in the proof of Corollary 6.2, the isomorphism $\widetilde{\mathcal{D}}_{2k}(\rho^{\omega}, x_t{}^{\omega}) \simeq \widetilde{\mathcal{D}}_{2k}(\rho, x_{\alpha t\varpi^{-1}})$ follows directly, whereas the isomorphism $\widetilde{\mathcal{D}}_{2k+1}(\rho^{\omega}, x_t{}^{\omega}, \xi) \simeq \widetilde{\mathcal{D}}_{2k+1}(\rho, x_{\alpha t\varpi^{-1}}, \xi)$ is equivalent to Lemma 6.3. ■

7 Identification of Irreducible Components and Adjoint Orbits

In this section, we identify explicitly, in the context of Shalika's classification, the irreducible representations occuring in the decomposition of $V = V_{\chi}$ with respect to both K and $\widetilde{K}$.

Fix a nontrivial additive character ψ of k with kernel $\mathcal{O}$. For any $k > 0$, the additive characters of $\mathcal{O}/\mathfrak{p}^k$ are given by $a \mapsto \psi(ta)$, for some $t \in \mathfrak{p}^{-k}$. Without loss of generality, we may thus choose the family of primitive additive characters of $\mathcal{O}/\mathfrak{p}^k$, for each $k > 0$, via

$$\eta_k(x) = \psi(\varpi^{-k}x) \quad \text{for all } x \in \mathcal{O}/\mathfrak{p}^k \tag{7.1}$$

for use in Shalika's construction. In particular, the characters ξ_k of $\mathfrak{p}^k/\mathfrak{p}^{2k+1}$ (see (5.5)) may then simply be defined by

$$\xi_k(x) = \psi(\varpi^{-2k-1}x) \quad \text{for all } x \in \mathfrak{p}^k/\mathfrak{p}^{2k+1}. \tag{7.2}$$

We have the decomposition $\mathrm{Res}_K V = V^{K_m} \oplus \bigoplus_{n>m}(W_n^+ \oplus W_n^-)$; let us write $\mathrm{Res}_{\widetilde{K}} V = V^{\widetilde{K}_m} \oplus \bigoplus_{n>m}(\widetilde{W}_n^+ \oplus \widetilde{W}_n^-)$. By the dimension count in Proposition 4.4, it follows that the representation V^{K_m} (respectively $V^{\widetilde{K}_m}$) corresponds to a split orbit, and the $W_n^\pm$ (respectively, $\widetilde{W}_n^\pm$) correspond to ramified orbits. The following Lemma is the key to defining these orbit representatives (as will become clear in the proof of Theorem 7.2 below).

Lemma 7.1 *Assume χ is primitive mod $\mathfrak{p}^m$. If $m = 1$, set $\lambda_\chi = 0$. Otherwise, if $m = 2k$ is even, choose $\lambda_\chi \in \mathcal{O}^\times$ so that*

$$\chi(1 + c\varpi^k) = \psi(2\varpi^{-k} c\lambda_\chi) \quad \textit{for all } c \in \mathcal{O}/\mathfrak{p}^k; \tag{7.3}$$

if $m = 2k + 1$ is odd, choose $\lambda_\chi \in \mathcal{O}^\times$ so that

$$\chi(1 + c\varpi^k) = \psi(\varpi^{-k-1}(2c - c^2\varpi^k)\lambda_\chi) \quad \textit{for all } c \in \mathcal{O}/\mathfrak{p}^{k+1}\,. \tag{7.4}$$

In each of these cases, λ_χ is uniquely defined modulo $\mathfrak{p}^{m-k}$.

Proof One verifies directly that in each case, the map $c \mapsto \chi(1 + c\varpi^k)$ gives a well-defined primitive additive character of $\mathcal{O}/p^k$ (respectively, of $\mathcal{O}/\mathfrak{p}^{k+1}$). Consequently λ_χ is uniquely defined up to $\mathfrak{p}^k$ (respectively, $\mathfrak{p}^{k+1}$), and has valuation zero by primitivity. ■

Theorem 7.2 *Suppose $m \geq 2$. Then for $m = 2k$,*

$$V^{K_m} \simeq \mathcal{D}_m\left(\chi, \begin{bmatrix} \lambda_\chi & 0 \\ 0 & -\lambda_\chi \end{bmatrix}\right) \quad \textit{and} \quad V^{\widetilde{K}_m} \simeq \widetilde{\mathcal{D}}_m\left(\chi, \begin{bmatrix} \lambda_\chi & 0 \\ 0 & -\lambda_\chi \end{bmatrix}\right).$$

For $m = 2k + 1$,

$$V^{K_m} \simeq \mathcal{D}_m\left(\chi \begin{bmatrix} \lambda_\chi & 0 \\ 0 & -\lambda_\chi \end{bmatrix}, \xi_\chi\right) \quad \textit{and} \quad V^{\widetilde{K}_m} \simeq \widetilde{\mathcal{D}}_m\left(\chi, \begin{bmatrix} \lambda_\chi & 0 \\ 0 & -\lambda_\chi \end{bmatrix}, \xi_\chi\right).$$

Proof Write $\lambda = \lambda_\chi$, and let $\eta = \eta_k$. Let $x = \left[\begin{smallmatrix} \lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right] \in \mathfrak{g}_k$. For any lift X of x to $\mathfrak{g}_m$, $T_{X,m}$ is the subgroup of diagonal matrices in K; thus χ defines a character of $T_{X,m}$ by restriction.

We claim that for $m = 2k$ (respectively, $m = 2k+1$), $\chi \otimes \eta_x$ (respectively, $\chi \otimes \eta_{x,\xi}$) coincides with χ on the intersection of $T_{X,n}N_k$ (respectively, $T_{X,n}I_k$) with B^m. Since $T_{X,n} \subset B^m$, this would prove the "coherence condition" of Shalika's inducing character, and thus that $\mathcal{D}_m(\chi, x)$ (respectively, $\mathcal{D}_m(\chi, x, \xi)$) is well-defined. It would also prove the existence of a nonzero element in the corresponding intertwining algebra $\mathcal{H}$ (namely, one supported on the identity double coset). The irreducibility of the two representations would then imply their equivalence.

Suppose first that $m = 2k$. Then

$$N_k \cap B^m = \left\{ \mathbf{n} = \begin{bmatrix} 1 + c\varpi^k & d\varpi^k \\ 0 & 1 - c\varpi^k \end{bmatrix} : c, d \in \mathcal{O}/\mathfrak{p}^k \right\};$$

thus for all $\mathbf{n} \in N_k \cap B^m$, $\eta_x(\mathbf{n}) = \eta(2\lambda c) = \psi(\varpi^{-k}2\lambda c) = \chi(\mathbf{n})$ by (7.1) and (7.3).

Next, suppose $m = 2k+1$. Then we have

$$I_k \cap B^m = \left\{ \mathbf{n} = \begin{bmatrix} 1 + c\varpi^k & d\varpi^k \\ 0 & 1 - c\varpi^k + c^2\varpi^{2k} \end{bmatrix} : c, d \in \mathcal{O}/\mathfrak{p}^k \right\}.$$

Using now (5.7), (7.2) and (7.4), we deduce

$$\eta_{x,\xi}(\mathbf{n}) = \xi(2\lambda c\varpi^k - \lambda c^2\varpi^{2k}) = \psi(\varpi^{-2k-1}(2c\varpi^k - c^2\varpi^{2k})\lambda) = \chi(1 + c\pi^k) = \chi(\mathbf{n}),$$

as required.

The analogous statements for $\widetilde{K}$ follow directly by Corollary 6.2. ■

Let us now turn to the remaining irreducible representations in $\mathrm{Res}_K V$ (respectively, $\mathrm{Res}_{\widetilde{K}} V$). We begin with a definition to help simplify our notation.

Definition 7.3 Let m, $k = [m/2]$ and λ_χ be as in Lemma 7.1, and suppose $n > m$. Set $\gamma_0 = \lambda_\chi \varpi^{n-m}$ and $\gamma_1 = \varepsilon^{-1}\lambda_\chi\varpi^{n-m}$. When $m = 1$, $\gamma_i = 0$ for each i; otherwise, these elements of $\mathfrak{p}^{n-m}$ are uniquely defined modulo $\mathfrak{p}^{n-k}$. Now for $i \in \{0, 1\}$, define a subgroup of K^n via

$$T_{i,n} = \left\{ \mathbf{t} = \begin{bmatrix} a & b \\ b\gamma_i^2 & a \end{bmatrix} : a, b \in \mathcal{O}/\mathfrak{p}^n,\ a^2 - b^2\gamma_i^2 = 1 \right\} \tag{7.5}$$

and define the character ρ_i of $T_{i,n}$ via $\rho_i(\mathbf{t}) = \chi(a + b\gamma_i)$.

Theorem 7.4 *Use the notation of Definition 7.3. Then*

$$W_n^+ \oplus W_n^- = \mathcal{D}_n\left(\rho_0, \begin{bmatrix} 0 & 1 \\ \gamma_0^2 & 0 \end{bmatrix}\right) \oplus \mathcal{D}_n\left(\rho_1, \begin{bmatrix} 0 & \varepsilon \\ \varepsilon\gamma_1^2 & 0 \end{bmatrix}\right),$$

and

$$\widetilde{W}_n^+ \oplus \widetilde{W}_n^- = \widetilde{\mathcal{D}}_n\left(\rho_0^\omega, \begin{bmatrix} 0 & \gamma_0^2\varpi^{-1} \\ \varpi & 0 \end{bmatrix}\right) \oplus \widetilde{\mathcal{D}}_n\left(\rho_1^\omega, \begin{bmatrix} 0 & \gamma_1^2\varepsilon\varpi^{-1} \\ \varepsilon\varpi & 0 \end{bmatrix}\right).$$

Proof Note that since the orbits are ramified, we have omitted the irrelevant character ξ in the notation for n odd. Let $X_0 = \left[\begin{smallmatrix} 0 & 1 \\ \gamma_0^2 & 0 \end{smallmatrix}\right]$ and $X_1 = \left[\begin{smallmatrix} 0 & \varepsilon \\ \varepsilon\gamma_1^2 & 0 \end{smallmatrix}\right]$ and x_0, x_1 their images in $\mathfrak{g}_k$. Then $T_{X_i,n} = T_{i,n}$, and ρ_i is a well-defined character of $T_{i,n}$.

Define $\Gamma_i = \left[\begin{smallmatrix} 1 & 0 \\ \gamma_i & 1 \end{smallmatrix}\right]$ and let $\mathcal{H}_i$ denote the space of intertwining operators between $V_\chi^{K_n}$ and $\mathcal{D}_n(\rho_i, x_i)$. We claim that there exists nonzero functions $\mathcal{F}_i \in \mathcal{H}_i$ such that $\mathcal{F}_i(\Gamma_i) = 1$. Establishing this claim proves the theorem.

First suppose $n = 2(k+r)$ is even, where $k = [m/2]$ as in Lemma 7.1. What needs to be shown is that whenever $\Gamma_i\mathbf{b}\Gamma_i^{-1} = \mathbf{tn}$, with $\mathbf{b} \in B_n$, $\mathbf{t} \in T_{i,n}$ and $\mathbf{n} \in N_{k+r}$, then $\chi(\mathbf{b}) = \rho_i(\mathbf{t})\eta_{x_i}(\mathbf{n})$. So let $\mathbf{b} = \left[\begin{smallmatrix} s & u \\ 0 & s^{-1} \end{smallmatrix}\right] \in B_n$, $\mathbf{t} \in T_{i,n}$ as in (7.5) and $\mathbf{n} \in N_{k+r}$ as in

(5.1) (with k replaced by $k+r$). Then $\eta_{x_0}(\mathbf{n}) = \eta(d\gamma_0^2+e)$ and $\eta_{x_1}(\mathbf{n}) = \eta(\varepsilon(d\gamma_1^2+e))$, where $\eta = \eta_{k+r}$ in (7.1). With this notation,

$$\Gamma_i \mathbf{b} \Gamma_i^{-1} = \begin{bmatrix} s - u\gamma_i & u \\ (s - s^{-1})\gamma_i - u\gamma_i^2 & s^{-1} + u\gamma_i \end{bmatrix}.$$

Setting this equal to $\mathbf{tn}$ lets one solve for u and for $s^{-1} = (a - b\gamma_i)(1 - \varpi^{k+r}(c+d\gamma_i))$. This yields the equalities $s = (a + b\gamma_i)(1 + \varpi^{k+r}(c + d\gamma_i))$ and (from the equality of $(2, 1)$ entries) $2\gamma_i(c+d\gamma_i) = d\gamma_i^2+e$. Hence $\chi(\mathbf{b}) = \chi(a+b\gamma_i)\chi(1+\varpi^{k+r}(c+d\gamma_i)) = \rho_i(\mathbf{t})\chi(1 + \varpi^k(d\gamma_i^2 + e)\varpi^r\gamma_i^{-1}/2)$. Now apply (7.3), (7.4) and Definition 7.3; note that since $n > m$, the term containing c^2 (in case m is odd) lies in the kernel of the character, and thus disappears.

Now suppose that $n = 2(k + r) + 1$ is odd. Then Shalika's inducing subgroup is $T_{i,n}I_{k+r}$; write an element $\mathbf{n} \in I_{k+r}$ as in (5.4) (with k replaced by $k+r$). The characters $\eta_{x_i,\xi}$ are defined via (5.8) by $\eta_{x_0,\xi}(\mathbf{n}) = \eta(\varpi^{-1}\gamma_0^2 d+e)$ and $\eta_{x_1,\xi}(\mathbf{n}) = \eta(\varepsilon(\varpi^{-1}\gamma_1^2 d+e))$. Setting $\Gamma_i \mathbf{b} \Gamma_i^{-1} = \mathbf{tn}$ yields again that $s = (a + b\gamma_i)(1 + \varpi^{k+r}(c + d\gamma_i))$, this time with the relation $\gamma_i(2c - c^2\varpi^k + 2d\gamma_i) = d\gamma_i^2 + e\varpi$. The necessary equality $\chi(\mathbf{b}) = \rho_i(\mathbf{t})\eta_{x_i,\xi}(\mathbf{n})$ follows from careful calculation as before.

The statement for $\widetilde{K}$ now follows from Proposition 6.1. ■

Remark 7.5 While the extensions ρ_i of the characters η_{x_i} (Definition 7.3) depend on the choice of γ_i modulo $\mathfrak{p}^m$, and are uniquely defined by λ_χ only modulo $\mathfrak{p}^{n-k}$, the inducing data for the Shalika representations occuring in Theorem 7.4 depend only on γ_i modulo at most $\mathfrak{p}^{k+r+1}$. We deduce that the orbit parameters of the decomposition into irreducibles are determined by Lemma 7.1. Further, note that when $n \geq 2m$, one may without loss of generality choose $\gamma_0 = \gamma_1 = 0$ in Definition 7.3 and in Theorem 7.4. This implies that the choice of additive character η does not matter here and moreover that ρ_i depends only on the value of $\chi(-1)$, as expected (Proposition 4.5). In this case, the orbit representatives X_i (for K) are simply the nilpotent elements $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & \varepsilon \\ 0 & 0 \end{smallmatrix}\right]$, and it is the identity double coset which supports the nonzero intertwining operator of the proof of the theorem. This simplification will be used in the proof of Theorem 9.2.

Example 1

$$\operatorname{Res}_K V_{\mathbf{1}} = \mathbf{1} \oplus \mathrm{St} \oplus \bigoplus_{n>1} \left[\mathcal{D}_n \left(\mathbf{1}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \oplus \mathcal{D}_n \left(\mathbf{1}, \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \end{bmatrix} \right) \right]$$

and, applying Corollary 6.4,

$$\operatorname{Res}_{\widetilde{K}} V_{\mathbf{1}} = \mathbf{1} \oplus \mathrm{St} \oplus \bigoplus_{n>1} \left[\widetilde{\mathcal{D}}_n \left(\mathbf{1}, \begin{bmatrix} 0 & \varpi^{-1} \\ 0 & 0 \end{bmatrix} \right) \oplus \widetilde{\mathcal{D}}_n \left(\mathbf{1}, \begin{bmatrix} 0 & \varepsilon\varpi^{-1} \\ 0 & 0 \end{bmatrix} \right) \right].$$

Similarly for V_ϑ, where the orbits are the same but the inducing character is ϑ in place of $\mathbf{1}$.

8 The Reducible Principal Series

The principal series representations $V_\chi = \mathrm{Ind}_B^G \chi$ of $G = SL(2,k)$, with normalization as in (3.1), are reducible exactly when $\chi = \mathrm{sgn}_\tau$, for τ nontrivial in $k^\times/k^{\times 2}$. In each of these cases, V_χ decomposes into two irreducible subrepresentations. In this Section, we recall the results of [GGPS], in which the authors explicitly describe the irreducible constituents of $\mathrm{Ind}_B^G \mathrm{sgn}_\tau$ in the "χ-realization" (hereafter refered to as the F-realization, to avoid clash of notation) of the principal series.

We first begin by realizing the principal series V_χ, for χ a unitary character of B, on the space W of complex-valued L^2-functions on k. In our original realization (3.1), we consider $\mathrm{Ind}_B^G \chi$ as acting on the space

$$V_\chi = \{f\colon G \to \mathbb{C} \mid f \in C^\infty \text{ and } \forall b \in B,\ f(bg) = \chi(b)|b|f(g)\}$$

by right translation. Given a function $f \in V_\chi$, define $\phi \in W$ via

$$\phi(x) = f\left(\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\right). \tag{8.1}$$

(That this function is in L^2 follows from the unitarity of χ and the decomposition $G = BK$.) The corresponding action of $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in G$, defined a.e., is (*c.f.* [GGPS, Ch. 2 §3.1])

$$(g \cdot \phi)(x) = \phi\left(\frac{ax+c}{bx+d}\right)\chi(bx+d)^{-1}|bx+d|^{-1}, \tag{8.2}$$

since, whenever $bx + d \neq 0$, one has

$$\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (bx+d)^{-1} & b \\ 0 & bx+d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ (ax+c)(bx+d)^{-1} & 1 \end{bmatrix}.$$

Choose an additive character δ of k and define the Fourier transform of a function $\phi \in W$ via (*c.f.* [GGPS, Ch. 2 §3.2])

$$\widetilde{\phi}(u) = \int_k \phi(x)\delta(-ux)\,dx \tag{8.3}$$

for all $u \in k$. The action of $g \in G$ on a function $\widetilde{\phi}$ in this realization can be explicitly expressed as integration with a kernel function. This defines the F-realization of $\mathrm{Ind}_B^G \chi$ as a representation of G on a space of complex-valued L^2 functions of k.

Now suppose that $\chi = \mathrm{sgn}_\tau$, for $\tau \in \{\varepsilon, \varpi, \varepsilon\varpi\}$. Then $\mathrm{Ind}_B^G \mathrm{sgn}_\tau$ is reducible and in the F-realization its two invariant irreducible subspaces are $H^\pm$, where H^+ consists of all functions $\widetilde{\phi}$ supported on the set $\{u \in k^* \mid \mathrm{sgn}_\tau(u) = 1\}$, and H^- of those supported on its complement [GGPS, Ch. 2 §3.5]. Where no confusion can result, we use $H^\pm$ to denote the corresponding irreducible constituents in every other realization.

9 Representations of K Occuring in the Irreducible Constituents of $\mathrm{Ind}_B^G \mathrm{sgn}_\tau$

Our goal now is, starting with an irreducible representation of K occuring in

$$\mathrm{Res}_K \mathrm{Ind}_B^G \mathrm{sgn}_\tau,$$

to compute the support of its elements in the F-realization (Section 8), and thus deduce to which irreducible constituent of $\mathrm{Ind}_B^G \mathrm{sgn}_\tau$ it belongs.

We first consider V^{K_1}, which, by Lemma 4.3, itself decomposes into two irreducibles:

- when $\tau = \varepsilon$, $V^{K_1}_{\mathrm{sgn}_\tau} \simeq V^{K_1}_{\mathbf{1}} \simeq \mathbf{1} \oplus \mathrm{St}$;
- when $\tau \in \{\varpi, \varepsilon\varpi\}$, $V^{K_1}_{\mathrm{sgn}_\tau} \simeq V^{K_1}_{\mathrm{sgn}} \simeq \Xi^+_{\mathrm{sgn}} \oplus \Xi^-_{\mathrm{sgn}}$.

The trivial and Steinberg representations of the first case are easy to distinguish and identify; whereas to distinguish the two equidimensional irreducible constituents of the latter case, we consult [DM, Ch. 15, Table 2]. They reveal that the characters $\Xi^\pm_{\mathrm{sgn}}$ of these two irreducible representations of $SL(2,\kappa)$ are distinguished by their values on $g = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, as follows. Let η be a fixed choice of additive character of κ (nontrivial on $\mathbb{F}_p \subseteq \kappa$) and define

$$\sigma_1 = \sum_{a \in \kappa^{*2}} \eta(a) \quad \text{and} \quad \sigma_\varepsilon = \sum_{a \in \kappa^{*2}} \eta(\varepsilon a). \tag{9.1}$$

Then $\Xi^+_{\mathrm{sgn}}(g) = -\sigma_1$ and $\Xi^-_{\mathrm{sgn}}(g) = -\sigma_\varepsilon$ (where the characters $\Xi^\pm_{\mathrm{sgn}}$ have been defined relative to this choice of η as well).

Extend η to a character of $\mathcal{O}$ by setting $\eta(\mathfrak{p}) = 1$. Let δ be an additive character of k satisfying $\delta(a) = \eta(a)$ for all $a \in \mathcal{O}$. Thus δ is trivial on $\mathfrak{p}$, and nontrivial on $\mathcal{O}$. We will define the F-realization of our representation relative to this choice of δ (both here and in Theorem 9.2).

Theorem 9.1 *Let $V = \mathrm{Ind}_B^G \mathrm{sgn}_\tau$ and consider the decomposition of V^{K_1} into irreducible K-representations identified with representations of $K^1 \simeq SL(2,\kappa)$, as above.*

- *For $\tau = \varepsilon$, we have $\mathbf{1} \subseteq H^-$ and $\mathrm{St} \subseteq H^+$.*
- *For $\tau \neq \varepsilon$, and $-1 \notin k^2$, we have $\Xi^+_{\mathrm{sgn}} \subseteq H^+$ and $\Xi^-_{\mathrm{sgn}} \subseteq H^-$.*
- *For $\tau \neq \varepsilon$, and $-1 \in k^2$, we have $\Xi^+_{\mathrm{sgn}} \subseteq H^-$ and $\Xi^-_{\mathrm{sgn}} \subseteq H^+$.*

Proof Let us begin by making explicit the identification of principal series representations of the finite group of Lie type $SL(2,\kappa)$ with subspaces of V_{sgn_τ} invariant under K. Write $\mathcal{B}$ for the image of $B \cap K$ in $SL(2,\kappa)$, and sgn_τ (equal to either $\mathbf{1}$ or sgn) for the corresponding character on $\mathcal{B}$.

The space $V = \mathrm{Ind}_{\mathcal{B}}^{SL(2,\kappa)} \mathrm{sgn}_\tau$ consists of functions from $SL(2,\kappa)$ to $\mathbb{C}$ transforming on the left by sgn_τ under $\mathcal{B}$. A basis for V is as follows. For each $n \in \kappa$, let f_n be the function in V supported on the right coset $\mathcal{B}\left[\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right]$ and satisfying

$f_n\left(\begin{smallmatrix}1&0\\n&1\end{smallmatrix}\right)=1$. Similarly, let $f_{\mathfrak{s}}$ be the function in V supported on $\mathcal{B}\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]$ and satisfying $f\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)=1$.

These can be lifted to functions in $\operatorname{Ind}_{B\cap K}^{K}\operatorname{sgn}_\tau$ by letting K_1 act trivially. To extend these further to functions in $\operatorname{Ind}_B^G\operatorname{sgn}_\tau$, define $F_n(bk)=\operatorname{sgn}_\tau(b)|b|f_n(k)$ for $b\in B$, $k\in K$, and $n\in\kappa\cup\{\mathfrak{s}\}$. The corresponding functions ϕ_n in the L^2-realization (see (8.1)) are $\phi_n(x)=F_n\left(\left[\begin{smallmatrix}1&0\\x&1\end{smallmatrix}\right]\right)$ for all $x\in k$. Explicitly, for $n\in\kappa$,

$$\phi_n(x)=\begin{cases}1 & \text{if } x\in n+\mathfrak{p},\\ 0 & \text{otherwise,}\end{cases}$$

whereas for $n=\mathfrak{s}$, if $x\notin\mathcal{O}$, then $\left[\begin{smallmatrix}1&0\\x&1\end{smallmatrix}\right]=\left[\begin{smallmatrix}x^{-1}&1\\0&x\end{smallmatrix}\right]\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x^{-1}\\0&1\end{smallmatrix}\right]\in B\mathfrak{s}K_1$, so

$$\phi_{\mathfrak{s}}(x)=\begin{cases}\operatorname{sgn}_\tau(x^{-1})|x|^{-1} & \text{if } x\notin\mathcal{O},\\ 0 & \text{otherwise.}\end{cases}$$

For $n\in\kappa$, we have $\widetilde{\phi_n}(u)=\int_k\phi_n(x)\delta(-ux)\,dx=\int_{n+\mathfrak{p}}\delta(-ux)\,dx$. If $u\in\mathfrak{p}$, then $\delta(-ux)\equiv 1$ and so the integral evaluates to the volume q^{-1}. If $u\notin\mathfrak{p}$, write $u=m\varpi^l$, with $l\le 0$, and make the change of variable $-ux=-mn\varpi^l+\sum_{i=l+1}^{0}a_i\varpi^i+r$, with $r\in\mathfrak{p}$ and the coefficients a_i varying over representatives of κ in k (hereafter abbreviated $a_i\in\kappa$). Then we have

$$\begin{aligned}\widetilde{\phi_n}(u)&=q^l\sum_{a_{l+1},\dots,a_0\in\kappa}\int_{\mathfrak{p}}\delta(-mn\varpi^l+\sum_{i=l+1}^{0}a_i\varpi^i+r)\,dr\\&=q^l\delta(-mn\varpi^l)\prod_{i=l+1}^{0}\sum_{a_i\in\kappa}\delta(a_i\varpi^i)\int_{\mathfrak{p}}dr\\&=\begin{cases}0 & \text{if } l\le -1,\quad(\text{since }\sum_{a_0\in\kappa}\delta(a_0)=0)\\ q^{-1}\delta(-mn) & \text{if } l=0.\end{cases}\end{aligned}$$

Thus one has

$$\widetilde{\phi}_n(u)=\begin{cases}q^{-1}\delta(-un) & \text{if } u\in\mathcal{O},\\ 0 & \text{otherwise.}\end{cases}\tag{9.2}$$

The computation of $\widetilde{\phi}_{\mathfrak{s}}$ is a bit more involved. From this point, assume $u=m\varpi^l$ with $m\in\mathcal{O}^\times$. We compute using (8.3),

$$\begin{aligned}\widetilde{\phi}_{\mathfrak{s}}(u)&=\int_{k\setminus\mathcal{O}}\operatorname{sgn}_\tau(x^{-1})|x|^{-1}\delta(-ux)\,dx\\&=\sum_{k<l}q^k\operatorname{sgn}_\tau(-u^{-1})\int_{\varpi^k\mathcal{O}^\times}\operatorname{sgn}_\tau(x^{-1})\delta(x)\,dx.\end{aligned}$$

For $\tau = \varepsilon$, $\operatorname{sgn}_\tau(x^{-1}) = (-1)^k$ on $\varpi^k\mathcal{O}^\times$. Thus the summand for each k in the range $1 \le k < l$ becomes $q^k(-1)^l(-1)^k(q^{-k} - q^{-k-1}) = (-1)^{k+l}(1 - q^{-1})$.

On the other hand, for $\tau \in \{\varpi, \varepsilon\varpi\}$, the character sgn_τ takes each of the values ± 1 half of the time, and so the integral for $k \ge 1$ is zero.

For those terms with $k \le 0$, we make the change of variables $x = a_k\varpi^k + \cdots + a_0 + r$, with $r \in \mathfrak{p}$, $a_i \in \kappa$, and the first coefficient $a_k \ne 0$. Then $\operatorname{sgn}_\tau(x) = \operatorname{sgn}_\tau(a_k\varpi^k)$, and $\delta(x) = \prod_{i=k}^0 \delta(a_i\varpi^i)$.

If $k \le -1$, one has, as before, a factor equal to $\sum_{a_0\in\kappa} \delta(a_0) = 0$, and the summand is identically zero. When $k = 0$, however, the corresponding summand is $\operatorname{sgn}_\tau(-u^{-1}) \sum_{a_0\in\kappa^\times} \operatorname{sgn}_\tau(a_0)\delta(a_0) \int_{\mathfrak{p}} dr$. If $\tau = \varepsilon$, $\operatorname{sgn}_\tau(a_0) = 1$, so the summand equals $(-1)^l(-1)q^{-1} = (-1)^{l+1}q^{-1}$. When $\tau \in \{\varpi, \varepsilon\varpi\}$, the summand may be rewritten as $q^{-1}\operatorname{sgn}_\tau(-u)(\sigma_1 - \sigma_\varepsilon)$ (see (9.1)).

In summary (with $u = m\varpi^l$) we have for $\operatorname{sgn}_\varepsilon$:

$$\widetilde{\phi}_{\mathfrak{s}}(u) = \begin{cases} 0 & \text{if } l \le 0, \\ q^{-1} & \text{if } l \ge 1, l \text{ odd}, \\ -1 & \text{if } l \ge 1, l \text{ even}. \end{cases} \tag{9.3}$$

For sgn_τ with $\tau \ne \varepsilon$, we have

$$\widetilde{\phi}_{\mathfrak{s}}(u) = \begin{cases} 0 & \text{if } l \le 0, \\ q^{-1}\operatorname{sgn}_\tau(-u)(\sigma_1 - \sigma_\varepsilon) & \text{if } l \ge 1. \end{cases} \tag{9.4}$$

We are now ready to prove the theorem. Let $\tau = \varepsilon$; then $\operatorname{sgn}_\varepsilon \equiv \mathbf{1}$ on $\mathcal{O}^\times$. The trivial subrepresentation in $\operatorname{Ind}_{\mathcal{B}}^{SL(2,\kappa)} \mathbf{1}$ has as basis the function $\sum_n f_n + f_{\mathfrak{s}}$ taking value 1 everywhere. Set $\phi_{\mathbf{1}} = \sum_n \phi_n + \phi_{\mathfrak{s}}$. By (9.2) and (9.3), we have $\widetilde{\phi_{\mathbf{1}}}(u) = \sum_n \widetilde{\phi_n}(u) + \widetilde{\phi}_{\mathfrak{s}}(u) = 1 + q^{-1}$ if $l \ge 1$ is odd and equals zero otherwise. Hence $\widetilde{\phi_{\mathbf{1}}} \in H^-$.

On the other hand, a basis for the Steinberg representation is given by $\{\psi_n = \phi_n - \phi_{\mathfrak{s}} \mid n \in \kappa\}$. Evaluating each $\widetilde{\psi_n}$ at $u \in k^\times$ yields, by (9.2) and (9.3),

$$\widetilde{\psi_n}(u) = \begin{cases} q^{-1}\delta(-un) & \text{if } u \in \mathcal{O}^\times, \\ q^{-1} + 1 & \text{if } u \in \mathfrak{p}^{2l}, l \ge 1, \\ 0 & \text{otherwise}. \end{cases}$$

Hence St corresponds to a representation in H^+.

Now suppose τ is equal to either ϖ or $\varepsilon\varpi$. Then $V_{\operatorname{sgn}_\tau}^{K_1} = V_{\operatorname{sgn}}^{K_1}$, which decomposes into two inequivalent subrepresentations under $SL(2,\kappa)$. We construct explicit bases for these subrepresentations of $\operatorname{Res}_K \operatorname{Ind}_B^G \operatorname{sgn}_\tau$ and compute the values of their characters on $g = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ to distinguish them.

Using (9.2) and (9.4), we construct bases for the K-invariant subspaces of V^{K_1} (in the L^2 realization) by considering their intersections with the inverse images of $H^\pm$

under Fourier transform. For $m \in \kappa^*$, define $h^m = \sum_{n\in\kappa} \delta(mn)\phi_n$. Define also $h^\pm = c\sum_{n\in\kappa} \phi_n \pm \phi_{\mathfrak{s}}$, where (notation of (9.1))

$$c = q^{-1}\operatorname{sgn}_\tau(-1)(\sigma_1 - \sigma_\varepsilon). \tag{9.5}$$

One can easily see that $\widetilde{h^\pm} \in H^\pm$ and $\widetilde{h^m} \in H^{\operatorname{sgn}_\tau(m)}$. These clearly form bases for the corresponding irreducible subspaces of V^{K_1} (or equivalently of $\operatorname{Ind}_{\mathcal{B}}^{SL(2,\kappa)} \operatorname{sgn}_\tau$); denote these subspaces $\widetilde{H}^+$ and $\widetilde{H}^-$.

We need to determine the character of, say, $\widetilde{H}^+$, that is, the trace of g on this subspace. Compute this as follows. First note, using (8.2), that

$$\begin{aligned} g\cdot\phi_1 &= \phi_{\mathfrak{s}},\\ g\cdot\phi_{\mathfrak{s}} &= \operatorname{sgn}_\tau(-1)\phi_{-1},\\ g\cdot\phi_n &= \operatorname{sgn}_\tau(1-n)\phi_{n(1-n)^{-1}} \quad \text{for } n \neq 1. \end{aligned}$$

The "reverse" change of variables is given by

$$\phi_n = q^{-1}\sum_{m\in\kappa^\times} \delta(-mn)h^m + (2cq)^{-1}(h^+ + h^-)$$

and $\phi_{\mathfrak{s}} = \frac{1}{2}(h^+ - h^-)$. This yields, with some effort, that the trace of g on $\widetilde{H}^+$ is given by

$$\begin{aligned} \operatorname{tr}(g)\big|_{\widetilde{H}^+} = q^{-1}\sum_{m\in\kappa^{\times 2}}\sum_{n\in\kappa\setminus\{1\}} &\operatorname{sgn}_\tau(1-n)\delta(-mn^2(1-n)^{-1})\\ &+ 2^{-1}c + (2cq)^{-1}\operatorname{sgn}_\tau(-1). \end{aligned}$$

We evaluate this in several stages. First note that since $(\sigma_1 - \sigma_\varepsilon)^2 = \operatorname{sgn}(-1)q$ [DM, Ch. 15.9], the sum of the last two terms simplifies to $q^{-1}\operatorname{sgn}_\tau(-1)(\sigma_1 - \sigma_\varepsilon) = c$ (see (9.5)). For the rest, let us compute the sum over m for each n.

When $n = 0$, we have $q^{-1}\sum_{m\in\kappa^{\times 2}} \delta(0) = (q+1)(2q)^{-1}$. For the $(q-3)/2$ values of n such that $\operatorname{sgn}_\tau(1-n) = 1$ and $n \neq 0$, we have $q^{-1}\sum_{m\in\kappa^{\times 2}} \delta(-mn^2(1-n)^{-1}) = q^{-1}\sigma_{-1}$, where $\sigma_{-1} = \sigma_1$ if $\operatorname{sgn}_\tau(-1) = 1$ and $\sigma_{-1} = \sigma_\varepsilon$ otherwise. Similarly, for the $(q-1)/2$ values of n such that $\operatorname{sgn}_\tau(1-n) = -1$, the sum evaluates to $-q^{-1}\sigma_{-\varepsilon}$. Hence

$$\operatorname{tr}(g)\big|_{\widetilde{H}^+} = q^{-1}\Big(\frac{q-1}{2} + \frac{q-3}{2}\sigma_{-1} - \frac{q-1}{2}\sigma_{-\varepsilon} + \operatorname{sgn}_\tau(-1)(\sigma_1 - \sigma_\varepsilon)\Big).$$

We simplify, using that $\sigma_1 + \sigma_\varepsilon = -1$, to obtain

$$\operatorname{tr}(g)\big|_{\widetilde{H}^+} = \begin{cases} -\sigma_\varepsilon = \Xi^-_{\operatorname{sgn}}(g) & \text{if } \operatorname{sgn}_\tau(-1) = 1,\\ -\sigma_1 = \Xi^+_{\operatorname{sgn}}(g) & \text{if } \operatorname{sgn}_\tau(-1) = -1, \end{cases}$$

as was required to show. The case of $\widetilde{H}^-$ is similar. ∎

The representations $\mathrm{Res}_K \mathrm{Ind}_B^G \chi$ decompose into a direct sum of their K_1-fixed vectors and pairs of irreducible K-representations by Proposition 4.4. These K-representations are explicitly determined in Theorem 7.4. Let $\chi = \mathrm{sgn}_\tau$; then $m = 1$. As discussed in Remark 7.5, we may without loss of generality choose the parameters γ_i to equal zero. Then

$$T_n = T_{0,n} = T_{1,n} = \left\{ \mathbf{t} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a \in \pm 1, b \in \mathcal{O}/\mathfrak{p}^n \right\},$$

and the characters ρ_i of T_n are given simply by $\rho_i(\mathbf{t}) = \mathrm{sgn}_\tau(a)$ ($= \mathrm{sgn}_\tau(\mathbf{t})$, upon identification of T_n with a subgroup of B^n). Let us write $X_1 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $X_\varepsilon = \left[\begin{smallmatrix} 0 & \varepsilon \\ 0 & 0 \end{smallmatrix}\right]$ for representatives of these nilpotent K-orbits in $\mathfrak{g}_{2k+1}$, and x_1, x_ε for the corresponding elements in $\mathfrak{g}_k$. It follows that the irreducible K-representations occuring in $\mathrm{Ind}_B^G \mathrm{sgn}_\tau$ are for all $n > 1$, $\mathcal{D}_n(\mathrm{sgn}_\tau, x_1)$ and $\mathcal{D}_n(\mathrm{sgn}_\tau, x_\varepsilon)$. As they are in fact independent of the choice of additive character η (primitive modulo $\mathfrak{p}^k$) defining η_x and $\eta_{x,\xi}$, we may without loss of generality define η via $\eta(a) = \delta(\varpi^{-k+1}a)$ for all $a \in \mathcal{O}/\mathfrak{p}^k$.

The following theorem identifies the irreducible component of $\mathrm{Ind}_B^G \mathrm{sgn}_\tau$ in which each of these irreducible K-representations lies.

Theorem 9.2 *Given the decomposition* $\mathrm{Ind}_B^G \mathrm{sgn}_\tau = H^+ \oplus H^-$ (Section 8) *and the decomposition of* $\mathrm{Ind}_B^G \mathrm{sgn}_\tau$ *into irreducible representations of K as above, we have the following:*

- *for $\tau = \varepsilon$, we have $\mathcal{D}_{2k}(\mathrm{sgn}_\varepsilon, x_t) \subseteq H^-$ and $\mathcal{D}_{2k+1}(\mathrm{sgn}_\varepsilon, x_t) \subseteq H^+$ for all $k \geq 1$;*
- *for $\tau \neq \varepsilon$ such that $\mathrm{sgn}_\tau(\varpi) = 1$, we have that $\mathcal{D}_n(\mathrm{sgn}_\tau, x_1)$ is in H^+ and that $\mathcal{D}_n(\mathrm{sgn}_\tau, x_\varepsilon)$ is in H^- for all $n \geq 2$; and*
- *for $\tau \neq \varepsilon$ such that $\mathrm{sgn}_\tau(\varpi) = -1$, we have that $\mathcal{D}_{2k}(\mathrm{sgn}_\tau, x_\varepsilon)$ and $\mathcal{D}_{2k+1}(\mathrm{sgn}_\tau, x_1)$ are in H^+, whereas $\mathcal{D}_{2k}(\mathrm{sgn}_\tau, x_1)$ and $\mathcal{D}_{2k+1}(\mathrm{sgn}_\tau, x_\varepsilon)$ are in H^-, for all $k \geq 1$.*

Proof We first outline our proof. Let h be an element of $\mathcal{D}_n(\mathrm{sgn}_\tau, x_t)$ for some $n > 1$, $t \in \{1, \varepsilon\}$. Thus h can be viewed as a function from K to $\mathbb{C}$, trivial on K_n, and transforming on the left by the character $\mathrm{sgn}_\tau \otimes \eta_{x_t}$ (respectively, $\mathrm{sgn}_\tau \otimes \eta_{x_t,\xi}$) of the subgroup $T_{2k}N_k$ (as in (5.3)) (respectively, $T_{2k+1}I_k$ (as in (5.9))). Let us write $J = N_k$ or I_k, as the case may be. (More precisely, it is $T_n J K_n$, not $T_n J$, which is the subgroup of K; let us write $T_n J$ for both and abuse notation by writing η_{x_t} for $\eta_{x_t,\xi}$, even though the latter does depend on ξ.)

The proof of Theorem 7.4 (and Remark 7.5) gave an explicit intertwining operator from $\mathcal{D}_n(\mathrm{sgn}_\tau, x_t)$ to $\mathrm{Ind}_{B\cap K}^K \mathrm{sgn}_\tau$. Specifically, set

$$\mathcal{F} \in \mathcal{H} = \mathcal{H}(B \cap K \backslash K / T_n J, \mathrm{sgn}_\tau, \mathrm{sgn}_\tau \otimes \eta_{x_t})$$

to be a function with support on the identity double coset. Then the intertwining operator L is given by $Lh = \mathcal{F} * h$, where $*$ denotes convolution. As in the proof of Theorem 9.1, one can extend Lh to an element of $\mathrm{Ind}_B^G \mathrm{sgn}_\tau$ and then identify it with an element of the L^2 realization. Applying the Fourier transform then yields a function which by invariance lies exactly in one of H^+ or H^-.

Let us begin. For $h \in \mathcal{D}_n(\mathrm{sgn}_\tau, X_t)$, and $\mathcal{F}$ as above with $\mathcal{F}(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}) = 1$, $Lh(k) = (\mathcal{F} * h)(k) = \int_K \mathcal{F}(g)h(g^{-1}k)\,dg$. Since $\mathcal{F}$ is supported on $B \cap K \cdot T_n J = (B \cap K)J$, there exists some nonzero constants c, c' such that

$$\begin{aligned} Lh(k) &= c \int_{b \in B\cap K} \int_{n \in J} \mathcal{F}(bn)h(n^{-1}b^{-1}k)\,dn\,db \\ &= c \int_{B\cap K} \int_J \mathrm{sgn}_\tau(b)\eta_{x_t}(n)\eta_{x_t}(n^{-1})h(b^{-1}k)\,dn\,db \\ &= c' \int_{B\cap K} \mathrm{sgn}_\tau(b)h(b^{-1}k)\,db. \end{aligned}$$

The function $TLh\colon k \to \mathbb{C}$ in the L^2 realization of $\mathrm{Ind}_B^G \mathrm{sgn}_\tau$ corresponding to Lh is given by (*cf.* (8.1))

$$TLh(x) = \begin{cases} Lh\left(\begin{smallmatrix}1&0\\x&1\end{smallmatrix}\right) & \text{if } x \in \mathcal{O}, \text{ and} \\ \mathrm{sgn}_\tau(x)|x|^{-1}Lh\left(\begin{smallmatrix}0&-1\\1&x^{-1}\end{smallmatrix}\right) & \text{if } x \notin \mathcal{O}. \end{cases}$$

Let us now fix a choice of h; namely, choose $h \in \mathcal{D}_n(\mathrm{sgn}_\tau, x_t)$ such that $h(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}) = 1$ and such that h is supported on the coset $T_n J$. We wish to take the Fourier transform (8.3) of TLh for this h.

Since, for $x \notin \mathcal{O}$,

$$b^{-1}k = \begin{bmatrix} a^{-1} & -c \\ 0 & a \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & x^{-1} \end{bmatrix} = \begin{bmatrix} c & -a + cx^{-1} \\ a^{-1} & a^{-1}x^{-1} \end{bmatrix} \notin T_n J,$$

it follows that $h(b^{-1}k) = 0$ for all $b \in B \cap K$ for this k. Hence $TLh(x) = 0$ for $x \notin \mathcal{O}$ and we have

$$\begin{aligned} \widetilde{TLh}(u) &= \int_{\mathcal{O}} TLh(x)\delta(-ux)\,dx \\ &= c' \int_{\mathcal{O}} \int_{B\cap K} \mathrm{sgn}_\tau(b)h\left(b^{-1}\begin{bmatrix}1&0\\x&1\end{bmatrix}\right)\delta(-ux)\,db\,dx \\ &= c'' \int_{x\in\mathcal{O}} \int_{a\in\mathcal{O}^\times} \int_{c\in\mathcal{O}} \mathrm{sgn}_\tau(a)h\left(\begin{bmatrix} a^{-1} & -c \\ 0 & a \end{bmatrix}\begin{bmatrix}1&0\\x&1\end{bmatrix}\right)\delta(-ux)\,dc\,da\,dx \\ &= c'' \int_{\mathcal{O}} \int_{\mathcal{O}^\times} \int_{\mathcal{O}} \mathrm{sgn}_\tau(a)h\begin{pmatrix} a^{-1} - cx & -c \\ ax & a \end{pmatrix}\delta(-ux)\,dc\,da\,dx. \end{aligned}$$

By the choice of h, this last integrand is nonzero only if $a \in \pm 1 + \mathfrak{p}^k$ and $x \in \mathfrak{p}^k$ (when $n = 2k$) or $x \in \mathfrak{p}^{k+1}$ (when $n = 2k+1$). Writing $l = k$ if $n = 2k$ and $l = k+1$ if $n = 2k+1$, we evaluate (for $a \in \alpha + \mathfrak{p}^k$ and $x \in \mathfrak{p}^l/\mathfrak{p}^n$):

$$h\begin{pmatrix} a^{-1} - cx & -c \\ ax & a \end{pmatrix} = h\left(\begin{bmatrix}\alpha & *\\0&\alpha\end{bmatrix}\begin{bmatrix}*&*\\x&*\end{bmatrix}\right) = \mathrm{sgn}_\tau(a)\delta(xt\varpi^{-n+1}).$$

Thus we have

$$\widetilde{TLh}(u) = c'' \int_{\mathfrak{p}^l} \int_{\pm 1+\mathfrak{p}^k} \int_{\mathcal{O}} \operatorname{sgn}_\tau(a)^2 \delta(xt\varpi^{-l-k+1})\delta(-ux)\, dc\, da\, dx$$
$$= c''' \int_{\mathfrak{p}^l} \delta(x(t\varpi^{-l-k+1} - u))\, dx.$$

This integral is nonzero if and only if $u \equiv t\varpi^{1-l-k} \mod \mathfrak{p}^{1-l}$. Thus $\widetilde{TLh}$ is supported on the set $\{u \mid \operatorname{sgn}_\tau(u) = \operatorname{sgn}_\tau(t\varpi^{l+k-1})\}$ (with $t \in \{1, \varepsilon\}$ and $l \in \{k, k+1\}$ as fixed above). The theorem follows. ■

Corollary 9.3 *Given the decomposition* $\operatorname{Ind}_B^G \operatorname{sgn}_\tau = H^+ \oplus H^-$ *and the decomposition into* $\widetilde{K}$*-irreducibles of* $\operatorname{Res}_{\widetilde{K}} \operatorname{Ind}_B^G \operatorname{sgn}_\tau$ *as in Theorems 7.2 and 7.4, we have the following:*

- *for* $\tau = \varepsilon$, $\mathbf{1} \subseteq H^+$, $\mathrm{St} \subseteq H^-$, $\widetilde{\mathcal{D}}_{2k}(\operatorname{sgn}_\varepsilon, x_{t\varpi^{-1}}) \subseteq H^+$ *and* $\widetilde{\mathcal{D}}_{2k+1}(\operatorname{sgn}_\varepsilon, x_{t\varpi^{-1}}, \xi) \subseteq H^-$ *for all* $k \geq 1$, $t \in \{1, \varepsilon\}$*;*
- *for* $\tau \neq \varepsilon$*: if* $\operatorname{sgn}_\tau(\varpi) = 1$*, then* $\Xi^\pm \subseteq H^\mp$,

$$\widetilde{\mathcal{D}}_{2k}(\operatorname{sgn}_\tau, x_{\varpi^{-1}}) \quad \textit{and} \quad \widetilde{\mathcal{D}}_{2k+1}(\operatorname{sgn}_\tau, x_{\varpi^{-1}}, \xi)$$

  are in H^+ *and* $\widetilde{\mathcal{D}}_{2k}(\operatorname{sgn}_\tau, x_{\varepsilon\varpi^{-1}})$ *and* $\widetilde{\mathcal{D}}_{2k+1}(\operatorname{sgn}_\tau, x_{\varepsilon\varpi^{-1}}, \xi)$ *are in* H^-*, for all* $k \geq 1$*;*
- *for* $\tau \neq \varepsilon$*: if* $\operatorname{sgn}_\tau(\varpi) = -1$*, then*

$$\Xi^\pm \subseteq H^\pm \quad \textit{and} \quad \widetilde{\mathcal{D}}_{2k}(\operatorname{sgn}_\tau, x_{\varpi^{-1}}) \quad \textit{and} \quad \widetilde{\mathcal{D}}_{2k+1}(\operatorname{sgn}_\tau, x_{\varepsilon\varpi^{-1}}, \xi)$$

  are in H^+*, whereas* $\widetilde{\mathcal{D}}_{2k}(\operatorname{sgn}_\tau, x_{\varepsilon\varpi^{-1}})$ *and* $\widetilde{\mathcal{D}}_{2k+1}(\operatorname{sgn}_\tau, x_{\varpi^{-1}}, \xi)$ *are in* H^-*, for all* $k \geq 1$.

Proof Recall from Lemma 3.1 the automorphism Υ of $\operatorname{Ind}_B^G \chi$, which intertwines π_χ and π_χ^ω. Since the subspaces H^+ and H^- are invariant under both actions of G, to determine the decomposition of H^+ and H^- under $\widetilde{K}$, it suffices by Lemma 3.1 and Theorems 9.1 and 9.2 to determine when $\Upsilon H^+ = H^+$ and when $\Upsilon H^+ = H^-$. Then the explicit identification follows from Proposition 6.1 and Corollary 6.4.

We use the notation of the proof of Theorem 9.1. For $n \in \kappa \cup \{\mathfrak{s}\}$, we compute

$$\Upsilon\phi_n(x) = \Upsilon F_n \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = F_n \begin{pmatrix} 0 & -1 \\ 1 & x\varpi^{-1} \end{pmatrix}$$
$$= \begin{cases} f_n \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) & \text{if } x \in \mathfrak{p}^2, \\ \operatorname{sgn}_\tau(x^{-1}\varpi)|x^{-1}\varpi| f_n \left(\begin{smallmatrix} 1 & 0 \\ x^{-1}\varpi & 1 \end{smallmatrix}\right) & \text{if } x \notin \mathfrak{p}^2. \end{cases}$$

Suppose first that $\tau = \varepsilon$. Then $\phi_{\mathbf{1}} = \sum_{n \in \kappa \cup \{\mathfrak{s}\}} \phi_n \in H^-$ and $\operatorname{sgn}_\tau(m\varpi^k) = (-1)^k$ if $m \in \mathcal{O}^\times$. Thus $\Upsilon\phi_{\mathbf{1}}(x) = 1$ if $x \in \mathfrak{p}^2$ and for any $x \in m\varpi^k + \mathfrak{p}^{k+1}$ with

$k \leq 1$, $\Upsilon\phi_{\mathbf{1}}(x) = (-1)^{k-1}q^{k-1}$. Consequently,

$$\begin{aligned}\widetilde{\Upsilon\phi_{\mathbf{1}}}(u) &= \int_k \Upsilon\phi_{\mathbf{1}}(x)\delta(-ux)\,dx \\ &= \int_{\mathfrak{p}^2} \delta(-ux)\,dx + \sum_{k\leq 1}(-1)^{k-1}q^{k-1}\int_{\varpi^k\mathcal{O}^\times}\delta(-ux)\,dx.\end{aligned}$$

We evaluate this as before and see that $\widetilde{\Upsilon\phi_{\mathbf{1}}}(u)$ is nonzero only for $\mathrm{sgn}_\varepsilon(u) = 1$. Hence $\Upsilon\phi_{\mathbf{1}} \in H^+$, and so necessarily $\Upsilon H^- = H^+$ for $\tau = \varepsilon$.

Now suppose that $\tau \neq \varepsilon$. Then $h^+ = c\sum_{n\in\kappa}\phi_n + \phi_{\mathfrak{s}} \in H^+$. A similar calculation yields that $\widetilde{\Upsilon h^+}(u)$ is nonzero only when $\mathrm{sgn}_\tau(-u\varpi) = 1$. Thus $\Upsilon H^+ = H^+$ when $\mathrm{sgn}_\tau(-\varpi) = 1$, and $\Upsilon H^+ = H^-$ otherwise. ■

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