A NEW METHOD OF CLASSIFICATION OF p-ADIC QUADRATIC FORMS

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Abstract. In this paper, we will first present some theory about quadratic forms and the p-adic numbers. We then present the classification of quadratic forms over $\mathbb{Q}_p$ for all p, both finite and infinite. We then define the Witt group and Witt ring: algebraic structures on the set of quadratic forms. Finally, we present the Hasse-Minkowski theorem, which relates p-adic quadratic forms to rational ones.

Introduction

Quadratic forms play a major role in many different fields. They can be used in order to generate groups and spaces; their applications range from number theory all the way to linear regressions. As such, it is important to know about the classification of quadratic forms over $\mathbb{Q}$. Often, rational numbers are not easy to work with. As a result, we often prefer to work in the p-adic numbers. The Hasse-Minkowski theorem is the pathway from the local p-adic case to the global rational case.

In this paper, we introduce a new classification method of p-adic quadratic forms. While such a classification is well-known, established in texts such as [Ser93], our main theorem is, to the best of our knowledge, a novel contribution.

We begin by reviewing the theory of quadratic forms in Section 1, including defining quadratic forms, the notion of equivalence between quadratic forms, and the diagonalization of quadratic forms. We continue by introducing isotropic and anisotropic quadratic forms; we review the following standard decomposition of quadratic forms.

Corollary 1.3.7. A quadratic form may be decomposed into the direct sum of copies of H and a unique anisotropic part, called the anisotropic kernel.

We then define the Value set (Definition 1.4.1), which for an anisotropic quadratic form is its image mod squares. The Value set will serve to be the heart of our classification.

In Section 2, we introduce the p-adic numbers, alternate completions of $\mathbb{Q}$. The results about square classes in Subsection 2.5 will serve to be very important in our classification; specifically, Corollary 2.5.3, which gives the square class representatives over $\mathbb{Q}_p$.

In Section 3, we present the classification of p-adic quadratic forms, for p odd and finite. We rely on Corollary 1.3.7 and Corollary 2.5.3 in order to greatly reduce the problem to that of classification of anisotropic quadratic forms with entries from a preferred set of square class representatives. Our main result is the following.

Main Theorem. Two anisotropic quadratic forms over $\mathbb{Q}_p$ are equivalent if and only if their images are the same, for p finite.

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The complete classification of p-adic quadratic forms is characterized in Theorem 3.2.1, Lemma 3.2.2, and Proposition 3.2.3. This classification is summarized by the following Proposition.

Proposition. The dimension and anisotropic kernel are a complete set of invariants of a quadratic form over $\mathbb{Q}_p$, for p finite.

For p odd, we show that these results hold for any finite degree field extension of $\mathbb{Q}_p$.

We then move onto Section 4, where we look at the Witt Ring: an algebraic structure on the set of quadratic forms. We show that for p odd of finite, the Witt ring, $W(F)$, is isomorphic to $\mathbb{Z}_2(x,y)/\langle x^2-1, y^2-1\rangle$ when $-1 \in F^{\times 2}$ and $\mathbb{Z}_4(x)/\langle x^2-1\rangle$ when $-1 \notin F^{\times 2}$.

Finally, in Section 5, we complete the classification for the outlying cases, noting that the only thing that changes is the square classes. For $p = 2$, we show that the Theorem and Proposition stated above still hold. Using this complete classification for p both finite and infinite, we present the Hasse-Minkowski theorem, which we show to imply that two quadratic forms over $\mathbb{Q}$ are equivalent if and only if they are equivalent over all p-adic fields. With this, we have provided a classification of not only p-adic quadratic forms but also rational ones.

Notation Used. Most notation presented here is explored in more detail, in order to help the reader who is not familiar with these concepts, throughout this paper.

We use F to denote a field and V to denote a vector space over F. Sometimes, when the dimension is important we will write F^n to denote $F \times F \times \ldots \times F$. To denote the invertible elements of F, we will write $F^\times$. Denote by $char F$ the characteristic of a field F.

We denote a quadratic form as $Q(x)$, or simply Q. It might be the case, that in order to emphasize the relation between a quadratic form and a bilinear form, we write $Q(x,x)$ or $A(x,x)$ to mean a quadratic form. We will denote the associated bilinear form to a quadratic form as $A(x,y)$ or simply abbreviate to $\langle x, y\rangle$ or A when the choice is clear from the context. The matrix representation of a quadratic form will be denoted as M, or $M_{(f)}$, where $\{f_1, \ldots, f_n\}$ is a basis of V. The subscript chosen for M will correspond to the basis; usually either e or f will be chosen. Often we will use the words, "let M be a quadratic form," where M is the associated matrix to a quadratic form Q; it is expected the reader views this to mean $Q(x) = x^T M x$, as stated after Definition 1.1.6. If two quadratic forms Q and Q' are equivalent, we will write $Q \sim Q'$.

We use $\oplus$ to denote the direct sum between two quadratic spaces and $\otimes$ to denote the Kronecker product between two quadratic spaces. Moreover, we use u, v, w to denote vectors and *span* to denote the spanning set of vectors. Also, we let H represent a hyperbolic plane and Q_{aniso} denote the anisotropic kernel of a given quadratic form.

In this paper, we use $\text{Val}(Q)$ to denote the value set of a quadratic form Q (Definition 1.4.1). The value set will serve to be the heart of our classification.

We will use $\mathbb{Z}$ to denote the set of integers, $\mathbb{Q}$ to denote the field of rational numbers, $\mathbb{Q}_p$ to denote the field of p-adic numbers, and $\mathbb{R}$ to denote the real numbers. We will sometimes use $\mathbb{Q}_\infty$ to denote the real numbers in order to keep the notation tighter. In general, when writing $p = \infty$, we will be referring to the real numbers or usual absolute value. We write $\mathbb{Z}_p$ to denote the ring of p-adic integers. To refer to a sequence of numbers, we write $(a_n)_{n \geq 1}$. We denote the p-adic valuation as $v_p(n)$ and the p-adic norm as $|x|_p := p^{-v_p(x)}$.

To denote a quadratic field extension, we write $E = F[\sqrt{\alpha}]$, where is α is a non-square element of F. The associated norm map is denoted by $N_{[E/F]} : E \to F$. We write $F^{\times 2}$ when referring to square elements in a given field, and so $F^\times / F^{\times 2}$ will be the square classes of F.

Let F be a finite degree field extension of $\mathbb{Q}_p$, for p odd. We write $\epsilon \in F$ to denote a non-square element of F, with valuation 0. Furthermore, ϖ is an element of F with minimal positive valuation, formally called a uniformizer of F. Often, to denote arbitrary elements of $F^\times / F^{\times 2}$, we will use $\alpha, \beta, \gamma, \delta$.

If two quadratic forms Q, Q' are Witt equivalent (Definition 4.1.1), we will write $Q \approx Q'$. Furthermore, we denote by $W(F)$ the Witt group or ring of a field F. When presenting the Hasse-Minkowski theorem, given a quadratic form Q over $\mathbb{Q}$, we will write Q_p to denote the quadratic form over the field $\mathbb{Q}_p$.

1. Introduction to Quadratic Forms

The study of quadratic forms is both vibrant and widely influential. In this section, we will present some common facts about quadratic forms, such as defining equivalence and introducing the concept of diagonalization of quadratic forms. Through this section, F is a field with $char F \neq 2$, A is a bilinear form, and $M_{(e)}$ is the representation of A over a basis $\{e_1, \ldots, e_n\} \in F^n$. In subsequent sections, when the choice of form is clear from context, we may abbreviate $A(x, y)$ to $\langle x, y \rangle$. For more details on this section, one may consult [Zie19].

1.1. **Equivalence of Bilinear Forms.**

Bilinear forms act as a stepping stone into quadratic forms. In understanding the theory of bilinear forms, the theory of quadratic forms will become more apparent.

Definition 1.1.1. A function $A : V \times V \to F$, where V is a vector space over F, of two vectors x and y is a *bilinear form* if A is linear in x and y, that is, if

$$A(\sum_{i=1}^{k} \alpha_i x_i, \sum_{j=1}^{m} \beta_j y_j) = \sum_{i=1}^{k} \sum_{j=1}^{m} \alpha_i \beta_j A(x_i, x_j),$$

for all $\alpha_i, \beta_j \in F$, and $x_i, y_j \in V$.

Definition 1.1.2. Given a bilinear form $A : V \times V \to F$, the matrix representation of A over a basis $\{e_1, \ldots, e_n\}$ of V is the matrix $M = (a_{ij})$, where $a_{ij} = A(e_i, e_j)$ under that basis.

As stated, we wish to have a strong notion of equivalence between bilinear forms. We proceed as follows. Suppose $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ are bases of V, and consider $M_{(e)} = (A(e_i, e_j))$ and $M_{(f)} = (A(f_i, f_j))$. Assume that

$$f_i = \sum_{j=1}^{n} p_j^{(i)} e_j,$$

with $P = (p_j^{(i)})$. Then one can calculate

$$\begin{aligned} b_{ik} = A(f_i, f_k) &= A(\sum_{j=1}^{n} p_j^{(i)} e_j, \sum_{j'=1}^{n} p_{j'}^{(k)} e_{j'}) \\ &= \sum_{j,j'=1}^{n} p_j^{(i)} p_{j'}^{(k)} A(e_j, e_{j'}) \\ &= \sum_{j,j'=1}^{n} p_j^{(i)} p_{j'}^{(k)} a_{jj'}. \end{aligned}$$

It is clear that $M_{(f)} = P^T M_{(e)} P$. Furthermore, since P and P^T are nonsingular, it follows that the rank of the matrix representation is independent of the choice of basis. This leads us to define equivalence between two bilinear forms, which will be very useful.

Definition 1.1.3. Two matrices M and M' are said to be *congruent* if there exists an invertible matrix P such that $P^T M P = M'$.

Definition 1.1.4. Two bilinear forms A and A' are said to be equivalent if they admit matrix representations M and M' such that M and M' are congruent. In this case, we write $A \sim A'$.

This will be well defined since all this definition is really saying is that they have the same matrix up to some change of basis.

Proposition 1.1.5. *Suppose A and A' are equivalent bilinear forms. Then, for every pair of bases $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ of V, if $M_{(e)}$ is the matrix representation of A with respect to $\{e_1, \ldots, e_n\}$ and $M'_{(f)}$ is the matrix representation of A' with respect to $\{f_1, \ldots, f_n\}$, then $M_{(e)}$ and $M'_{(f)}$ are congruent. Moreover, $\det M_{(e)} = x^2 \det M_{(f)}$, for some $x \in F^\times$.*

Proof. The first statement is a simple fact of linear algebra. Using standard determinant rules, one immediately finds $\det M' = (\det P)^2 \cdot \det M = x^2 \det M$. ☐

We are now ready to define quadratic forms, our object of interest.

Definition 1.1.6. A *quadratic form* is a function $Q : V \to F$ defined by $Q(x) = A(x, x)$, for a given non-degenerate, symmetric bilinear form A. We sometimes denote Q by $A(x, x)$ to emphasize this relationship.

We say a bilinear form A is non-degenerate if one (equivalently all) of its matrix representations is non-degenerate. We say A is symmetric if $A(x, y) = A(y, x)$ for all $x, y \in V$, so that its matrix representations are symmetric matrices. Using this definition, we can say that if M is the matrix representation of Q, then $Q(x) = x^T Q x$. We will add one fact that is pretty obvious but still extremely beneficial to our cause of classification, reducing the work needed immensely.

Proposition 1.1.7. *For any $\sigma \in S_n$, we have*

$$\begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{bmatrix} \sim \begin{bmatrix} \alpha_{\sigma(1)} & & \\ & \ddots & \\ & & \alpha_{\sigma(n)} \end{bmatrix}.$$

Proof. If we let $P = [e_{\sigma(1)}|e_{\sigma(2)}|\dots|e_{\sigma(n)}]$, then

$$P^T \begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{bmatrix} P = \begin{bmatrix} \alpha_{\sigma(1)} & & \\ & \ddots & \\ & & \alpha_{\sigma(n)} \end{bmatrix}.$$

□

1.2. **Diagonalization of Quadratic Forms.** In this section, we show an interesting fact: any quadratic form over F^n may be represented as a diagonal matrix over some basis. The matrix of a quadratic form relative to this basis, say $\{f_1, \dots, f_k\}$, will be diagonal with

$$a_{ik} = A(f_i, f_k) = 0 \text{ for } i \neq k.$$

We will call this a diagonal quadratic form. Essentially, all that we are doing is completing the square many times. It should be noted that even quadratic forms with wildly different diagonal forms might be equivalent to one another (this will be shown through examples later in this section).

Proposition 1.2.1. *Let $A(x, x)$ be a quadratic form in F^n. Then there exists a basis $\{f_1, \dots, f_n\}$ of F^n there exist $l_i \in F^\times$, such that for any vector $x = \sum_{k=1}^{n} \tau_k f_k$ over this basis, the value $A(x, x) = l_1\tau_1^2 + \cdots + l_n\tau_n^2$. The matrix of A with respect to $\{f_1, \dots, f_n\}$ is thus diagonal.*

Proof. Take $A(x, x)$ to be a quadratic form in F^n. We wish to find a basis $\{f_1, \dots, f_n\} \in F^n$ such that for $x = \sum_{k=1}^{n} \tau_k f_k$ we get that $A(x, x) = l_1\tau_1^2 + \cdots + l_n\tau_n^2$ for some fixed $l_1, \cdots, l_n \in F^\times$. That is, we wish to find a basis for which the matrix representation of the quadratic form $A(x, x)$ is diagonal.

Let $\{e_1, \dots, e_n\}$ be an arbitrary basis of F^n and $b_{ik} = A(e_i, e_k)$. Since (b_{ik}) is symmetric, given $x = \sum_{k=1}^{n} \xi_k e_k$, we have that $A(x, x) = \sum_{k=1}^{n} \sum_{i \leq k} b_{ik}\xi_i\xi_k$. We thus wish to find ρ_{ij} such that

$$\begin{cases} \tau_1 &= \rho_{11}\xi_1 + \cdots + \rho_{1n}\xi_n \\ \tau_2 &= \rho_{21}\xi_1 + \cdots + \rho_{2n}\xi_n \\ &\vdots \\ \tau_n &= \rho_{n1}\xi_1 + \cdots + \rho_{nn}\xi_n \end{cases} \tag{1}$$

gives $A(x, x) = \sum_{k=1}^{n} l_k\tau_k^2$, for some l_i.

We will proceed by induction on the number of ξ_i's in (1). $P(1)$: If (1) has only one variable, then clearly $A(x, x) = b_{11}\xi_i^2$. So, for $\rho_{11} \neq 0$, the induction hypothesis holds.
$P(k-1) \implies P(k)$: Assume that one of the b_{ii}'s is nonzero for $i = \{1, 2, 3, \cdots, k\}$. Then, by grouping, and taking $b_{kk} \neq 0$, we get

$$\begin{aligned} &b_{1k}\xi_1\xi_k + b_{2k}\xi_2\xi_k + \cdots + b_{k-1,k}\xi_{k-1}\xi_k + b_{kk}\xi_k^2 \\ &\quad = b_{kk}\big[\frac{b_{1k}}{2b_{kk}}\xi_1 + \frac{b_{2k}}{2b_{kk}}\xi_2 + \cdots + \xi_k\big]^2 + A'(x, x), \end{aligned}$$

where $A'(x,x)$ is a quadratic form relying only on $\xi_1, \cdots, \xi_{k-1}$. Define now, this transformation from τ_i to τ_i' as

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ \frac{b_{1k}}{2b_{kk}} & \frac{b_{2k}}{2b_{kk}} & \cdots & 1 \end{pmatrix}. \tag{2}$$

The determinant of the matrix of transformation has a determinant equal to one. So, it is nonsingular and we get a new system

$$A(x,x) = A''(x,x) + b_{kk}\tau_k'^2,$$

where $A''(x,x)$ relies only on $\tau_1', \ldots, \tau_{k-1}'$. By the induction hypothesis, there must be a transformation

$$\begin{cases} \eta_1 & = \rho_{11}\tau_1' + \cdots + \rho_{1,k-1}\tau_{k-1}' \\ & \vdots \\ \eta_{k-1} & = \rho_{k-1,1}\tau_1' + \cdots + \rho_{k-1,k-1}\tau_{k-1}' \end{cases} \tag{3}$$

which gives $A''(x,x) = l_1\tau_1^2 + \cdots + l_{k-1}\tau_{k-1}^2$. If to (3) we add $\eta_k = \tau_k'$, then we are left with

$$A(x,x) = A''(x,x) + b_{kk}\tau_k^2 = l_1\tau_1^2 + \cdots + l_k\tau_k^2.$$

We note that the matrix of transformation obtained by adding $\eta_k = \tau_k'$ is the product of two nonsingular matrices (namely those in (1) and (2)) and so, it is nonsingular.

We are now left to consider the case when $b_{11} = \ldots = b_{kk} = 0$. For the quadratic form to be nonzero, we must have some term $b_{ik}\xi_i\xi_k$ with a nonzero coefficient. Assume $b_{12} \neq 0$. Thus, we require a new transformation. Namely,

$$\left(\begin{array}{cc|c} 1 & 1 & \\ 1 & -1 & \\ \hline & & I_{k-2} \end{array}\right),$$

and it has det $= -2$, thus it is nonsingular. Our transformation changes $b_{12}\xi_1\xi_2$ to $b_{12}\xi_1'^2 + b_{12}\xi_2'^2$, so that the previous case applies. As a result, we conclude that any quadratic form may be diagonalized. $\square$

Example 1.2.2. Consider the following quadratic forms over $\mathbb{Q}$: $4x^2 - 3y^2$ and $x^2 - 12y^2$. These two forms are equivalent since

$$4x^2 - 3y^2 \sim 4(4x^2 - 3y^2) \sim (4x)^2 - 12y^2 \sim k^2 - 12y^2.$$

Example 1.2.3. Let F be a finite field with $char F \neq 2$, then $x^2 - 6xy + 8y^2 \sim aX^2 - aY^2$, for all $a = \pm 1$. Notice that the second form is already in diagonal form. One observes that $x^2 - 6xy + 8y^2 = (x - 3y)^2 - y^2$.

As a side note, using this factoring in other fields will yield different results. For example, in $\mathbb{R}$ or $\mathbb{C}$ the forms $x^2 - 6xy + 8y^2$ and $ax^2 - ay^2$ will be equivalent for all a. In $\mathbb{Q}$, they are equivalent when $\sqrt{a}$ is rational.

Proposition 1.2.4. *Suppose A', A'' are quadratic forms over F such that $A' \sim A''$. Let $M_{(e)}$ be a matrix representation of A' and $M_{(f)}$ that of A''. Then* $\det M_{(e)} = \det M_{(f)}$, *in* $F^\times / F^{\times 2}$.

Proof. Since there exists a matrix P such that $M_{(f)} = P^T M_{(e)} P$, we get that $\det M_{(f)} = (\det P)^2 \cdot \det M_{(e)} = \det M_{(e)}$, in $F^\times / F^{\times 2}$. $\square$

Definition 1.2.5. Let V' and V'' be two spaces over a field F, where V' is equipped with the bilinear form A' and V'' is equipped with A''. We say V' and V'' are A-isomorphic if

(1) There exists an isomorphism ω between V' and V'';
(2) $A'(x, y) = A''(\omega x, \omega y)$ for all $x, y \in V'$.

Theorem 1.2.6. *Given two finite dimensional linear spaces V' and V'' over F, equipped with Q' and Q'' respectively. Then, there exists an Q-isomorphism between V' and V'' if and only if:*

(1) $\dim V' = \dim V''$;
(2) There exists a basis $\{e'_1, \ldots, e'_n\}$ in V' and a basis $\{e''_1, \ldots, e''_n\}$ in V'' such that the matrix representations of Q' and Q'' are identical, with respect to their given bases.

Proof. Suppose $(V', Q') \cong (V'', Q'')$. Then, since V' and V'' are isomorphic linear spaces, this means that their dimension is also the same.

If $\{e'_1, \ldots, e'_n\}$ is a basis for which Q' is diagonal in V', then

$$Q(e'_i, e'_j) = \begin{cases} 0, & i \neq j \\ \xi_i, & \text{else.} \end{cases}$$

Similarly, if $\{e''_1, \ldots, e''_n\} \in V''$ corresponds to $\{e'_1, \ldots, e'_n\}$ such that $e''_i = \omega e'_i$, where ω is the isomorphism between V' and V'', then

$$Q(\omega e'_i, \omega e'_j) = Q(e''_i, e''_j) = \begin{cases} 0, & \text{if } i \neq j \\ \xi_i, & \text{else.} \end{cases}$$

So, we have that Q' and Q'' have the same matrix representations with respect to these bases.

Conversely, suppose that V' and V'' have the same dimension, and it is equal to n. Let $\{e'_1, \ldots, e'_n\} \in V'$ and $\{e''_1, \ldots, e''_n\} \in V''$ be bases for which the quadratic forms are diagonal with coefficients $\xi_1, \cdots, \xi_n$ such that

$$Q' = Q(e'_i, e'_j) = Q(e''_i, e''_j) = Q'' = \begin{cases} 0, & \text{if } i \neq j \\ \xi_i, & \text{else.} \end{cases}$$

Thus we can define an isomorphism by sending $\{e'_1, \ldots, e'_n\}$ to $\{e''_1, \ldots, e''_n\}$. $\square$

We now proceed to a well-known major result in the classification of quadratic forms: Witt's cancellation theorem will serve as a useful tool in moving up dimensions of quadratic forms as it will help relate lower dimensional cases to higher dimensional ones. The proof of this well-known theorem may be found in [Zie19].

Theorem 1.2.7 (Witt's Cancellation Theorem)**.** *Let F be a field. Suppose Q_1, Q_2, Q_3 are three quadratic forms over F. If $Q_1 \oplus Q_2 \sim Q_1 \oplus Q_3$, then $Q_2 \sim Q_3$.*

1.3. **Isotropic and Anisotropic Quadratic Forms.** In order to fully understand quadratic forms, it is important to introduce the notion of isotropic and anisotropic quadratic forms.

Definition 1.3.1. A quadratic form Q over a field F is *isotropic* if there exists a $w \neq 0$ in V such that $Q(w) = 0$.

A significant isotropic quadratic form is the hyperbolic plane, which is defined as follows.

Definition 1.3.2. A hyperbolic plane is a 2-dimensional quadratic form such that, with respect to some basis $\{u, v\}$, we have $H = span\{u, v\}$ such that $\langle u, u\rangle = \langle v, v\rangle = 0$ and $\langle u, v\rangle = \langle v, u\rangle = 1$.

It is immediately clear that this is isotropic. In fact, every 2-dimensional isotropic quadratic form is equivalent. We see that $Q(x, y) = xy$ is a hyperbolic plane. Suppose H is a hyperbolic plane, and pick some $u \in H$ such that $Q(u) = 0$. Suppose $v \in H$ is chosen such that $\{u, v\}$ forms a basis for H. If $\langle u, v\rangle = 0$ then, since for any $w \in H$ we can write $w = \alpha u + \beta v$, this would mean that $\langle u, x\rangle = 0$, for all $w \in H$, which means that Q is degenerate. Thus, $\langle u, v\rangle \neq 0$. We may assume that $\langle u, v\rangle = 1$, by scaling. We immediately see that the matrix associated to Q over $\{u, v\}$, is

$$M = \begin{bmatrix} 0 & 1 \\ 1 & \langle v, v\rangle \end{bmatrix}.$$

If $\langle v, v\rangle = 0$, then we are done.

Suppose now that $\langle v, v\rangle \neq 0$, then $Q(-\frac{1}{2}\langle v, v\rangle u + v) = 0$, so that

$$\begin{bmatrix} 1 & -\frac{1}{2}\langle v, v\rangle \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & \langle v, v\rangle \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2}\langle v, v\rangle & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, we have shown that Q and H are equivalent as quadratic spaces and we are done. As a result, any hyperbolic plane H will be equivalent to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Starting now, we will write

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

for *any* hyperbolic plane and refer to it as *the* hyperbolic plane.

Proposition 1.3.3. *The hyperbolic plane is preserved by multiplication by any non-zero scalar. That is, $H \sim \lambda \otimes H$ for any $\lambda \in F^\times$.*

Proof. If we take $B' = \{u' = \frac{1}{\lambda}u, v\}$ we immediately have $\langle u', u'\rangle = \langle v, v\rangle = 0$ and $\langle u', v\rangle = \langle v, u'\rangle = 1$. Which means that

$$[H]_{B'} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

□

Proposition 1.3.4. *The image of H is F.*

Proof. If $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $Q(x, y) = 2xy$. Thus the values H takes non-trivially are $\{2xy \mid (x, y) \in F^2 \setminus \{(0, 0)\}\} = F$. □

Theorem 1.3.5. *Let Q be an isotropic quadratic form. Then there exists a quadratic form Q' such that*

$$Q \sim H \oplus Q'.$$

Proof. Suppose that $Q(u) = 0$ for some $u \neq 0$. Since Q is non degenerate, there exists a w such that $\langle u, w\rangle = 1$. We will show that there exists a unique α such that $Q(\alpha u + w) = 0$.

We compute

$$Q(\alpha u + w) = \alpha^2 Q(u) + Q(w) + 2\langle u, w\rangle = Q(w) + 2\alpha,$$

which will be zero exactly when $\alpha = -Q(w)/2$.

Set $v = \alpha u + w$; then $\{u, v\}$ span a hyperbolic plane, which means that $\langle u, v\rangle = \alpha Q(u) + \langle u, w\rangle = 0 + 1 = 1$. This implies that

$$Q(xu + yv) = xy.$$

Let $Q' = Q^{\perp}$, it is clear that $Q \cap Q' = \{0\}$. So $Q = H \oplus Q'$. □

Definition 1.3.6. A quadratic form Q over a field F is *anisotropic* if for all $w \neq 0$ in V one has $Q(w) \neq 0$.

It is immediately clear that if Q is an anisotropic quadratic form, then there does not exist a quadratic form Q' such that $Q = nH \oplus Q'$, for $n \in \mathbb{N}_{\geq 1}$. That is, an anisotropic quadratic form does not contain a hyperbolic plane. Otherwise, Q would have to evaluate to 0 for a nonzero vector. We present a Corollary of Theorem 1.3.5.

Corollary 1.3.7. *Let Q be an isotropic quadratic form. Then Q can be written as $Q = \ell H \oplus Q_{aniso}$ for some $\ell \in \mathbb{N}_{\geq 1}$ where Q_{aniso} is anisotropic. We call Q_{aniso} the anisotropic kernel of Q. When Q is anisotropic, $Q = Q_{aniso}$.*

1.4. **The Value Set of a Quadratic Form.** The value set of a quadratic function will allow us to distinguish between quadratic forms. Though it is not necessarily true that if two quadratic forms have the same value set then they are equivalent, it is true that forms with different value sets are certainly different from one another. Note that $Q(ax) = a^2 Q(x)$, so the image of Q is always closed under scaling by squares.

Definition 1.4.1. If Q is a quadratic form over a field F, we define its value set to be

$$\operatorname{Val}(Q) = V_Q := \begin{cases} \{Q(x) \mid x \in V \setminus \{0\}\}/F^{\times 2} & \text{if } Q \text{ is anisotropic,} \\ \emptyset & \text{if } Q = 0, \\ F & \text{if } Q \text{ is isotropic and nonzero.} \end{cases}$$

Based on the definition of equivalence between quadratic forms, the following follows.

Proposition 1.4.2. *If Q and Q' are equivalent quadratic forms, then Val$(Q) =$ Val(Q').*

Proof. If Q and Q' are equivalent quadratic forms, then by Theorem 1.2.6 the matrices representing them are congruent. It follows that the images of Q and Q' in F are equal. □

As mentioned at the start of this section, the converse is not true. We present some examples to outline this fact.

Example 1.4.3. The following is for the analogous notion over a ring $\mathbb{Z}$, but a worthwhile example nonetheless. Suppose $Q = x^2 + xy + y^2$ and $Q' = x^2 + 3y^2$ over $\mathbb{Z}$. It should be pretty clear that these quadratic forms are not equivalent. Yet, over $\mathbb{Z}$, suppose that $\alpha = x^2 + xy + y^2$, for $x, y \in \mathbb{Z}$, then we wish to show that there exist $x', y' \in \mathbb{Z}$ such that $x^2 + xy + y^2 = x'^2 + 3y'^2$.

This is an ellipse, and we wish to show that for set x, y there always exists a lattice point (x', y') on the ellipse. Suppose $x = y' + x'$ and $y = y' - x'$, then $x^2 + xy + y^2 = (y' + x')^2 + (y' + x')(y' - x') + (y' - x')^2 = x'^2 + 3y'^2$.

Proving in the other direction is casework on if x' and y' are even and odd. Thus, these have the same value set over $\mathbb{Z}$ but, are not equivalent.

We present an example over a field, namely $\mathbb{R}$, that also highlights the notion that $\mathrm{Val}(Q) = \mathrm{Val}(Q') \not\Rightarrow Q \sim Q'$.

Example 1.4.4. Over $\mathbb{R}$, for any dimension, n, there exists an anisotropic form Q that represents all positive numbers. Take the form $Q(x) = x_1^2 + \ldots + x_n^2$, where $x \in V \subseteq \mathbb{R}^n$ for example. Clearly, quadratic forms of different dimensions are not equivalent.

In Chapter 3, we will see that in the p-adic numbers, the value set gives a lot of information about a given anisotropic form: it gives the dimension, and actually uniquely defines the anisotropic form (Proposition 3.1.8, Lemma 5.1.17), up to equivalence.

Proposition 1.4.5. *If Q is anisotropic, then $F^{\times 2} \subset \mathit{Val}(Q) \subset F^{\times}$.*

1.5. **The Kronecker Product.** Before introducing classification and specific methods, it is important to have a good understanding of how to move between quadratic forms. We have seen one way of combining quadratic forms on two spaces: the direct sum, whose matrix is given in block diagonal form. Another way is to take the tensor product of the spaces, and the resulting operation on matrices is called the Kronecker product. Of course, it is important to consider the dimension and new basis of this product. As such, we formalize as follows. Without the loss of generality, we consider diagonal quadratic forms.

We are looking for a way to represent $(V, Q) \otimes (V', Q')$, and so, if $\{x_1, \ldots, x_n\} \subseteq F$ is a basis for V for which Q is diagonal and $\{y_1, \ldots, y_n\} \subseteq F$ is a basis for V' for which Q' is diagonal, then the basis of their product should be every combination of vectors in $\{x_i\}$ with those in $\{y_i\}$. We are now ready to present the Kronecker product, a special case of the tensor product.

Definition 1.5.1 (The Kronecker Product)**.** Given two diagonal quadratic forms (V, Q) and (V', Q'), where $V \subseteq F^n, V' \subseteq F^m$, their Kronecker product is defined to be

$$(V, Q) \otimes (V', Q') := (V \otimes V', Q \otimes Q'),$$

where $V \otimes V'$ is the tensor product of the spaces and

$$\begin{aligned} Q \otimes Q' &= (\lambda_1 x_1^2 + \ldots + \lambda_n x_n^2) \otimes (\tau_1 y_1^2 + \ldots + \tau_m y_m^2) \\ &:= \lambda_1\tau_1(x_1y_1)^2 + \ldots + \lambda_1\tau_m(x_1y_m)^2 \\ &+ \lambda_2\tau_1(x_2y_1)^2 + \ldots + \lambda_2\tau_m(x_2y_m)^2 \\ &+ \quad \vdots \\ &+ \lambda_n\tau_1(x_ny_1)^2 + \ldots + \lambda_n\tau_m(x_ny_m)^2. \end{aligned}$$

It is evident that the Kronecker product of two forms with dimensions m and n will be of dimension mn.

Example 1.5.2. Suppose $Q(x) = ax^2$, then for any quadratic form Q', $Q \otimes Q' = aQ'$.

2. The p-adic numbers

The real numbers are derived from completing $\mathbb{Q}$ with respect to the absolute value. While completing $\mathbb{Q}$, a real number may be defined to be a set of Cauchy sequences that are equivalent. Two Cauchy sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are in this case said to be equivalent if $\lim_{n\to\infty}(a_n - b_n) = 0$. This process intuitively "fills in the gaps" of rational numbers by introducing Cauchy sequences. We will proceed in a similar way in Section 2.3.

Of course, the notion of completing $\mathbb{Q}$ stems from the notion of closeness which we have assigned on the set. Since a Cauchy sequence relies heavily on a notion of closeness, when this changes, drastically different things might occur. Thus, we must look into how we may define closeness in order to build the p-adic numbers. The answer lies in the norm chosen. The following argument is our take on a very classical one; the curious reader can see [Gou20, Chapter 3] for more details.

2.1. **Norms.**

Definition 2.1.1. Let F be a field. A norm on F is a map $|\cdot,\cdot| : F \to [0,\infty)$ that satisfies the following three properties

(A1). If $|x| = 0$ then $x = 0$ (Positive Definiteness);
(A2). $|xy| = |x| \cdot |y|$ (Multiplicativity);
(A3). $|x + y| \leq |x| + |y|$ (The Triangle Inequality).

A norm is particularly useful as it can define the distance between two elements of a set. That is, any norm induces a metric onto a space.

Definition 2.1.2. Let F be a field. A metric on F is a map $d : F \times F \to \mathbb{R}$ that satisfies, for all $x, y, z \in F$,

(M1). The distance between a point to itself is 0, $d(x, x) = 0$;
(M2). If $x \neq y$, then $d(x, y) > 0$ (Positivity);
(M3). $d(x, y) = d(y, x)$ (Symmetry);
(M4). $d(x, y) \leq d(x, z) + d(z, y)$ (The Triangle Inequality),

A norm will induce a metric of F as follows:

$$d(x, y) = |x - y|,$$

for any given norm on F. It is easily verified that d is a metric.

Definition 2.1.3. A norm is said to be non-archimedean if for all $x, y \in F$ one has $|x+y| \leq \max\{|x|, |y|\}$ (Ultrametric Triangle Inequality). A norm is archimedean if it does not satisfy the ultrametric triangle inequality.

Intuitively this is saying if a norm is non-archimedean, all triangles are isosceles.

Example 2.1.4. The absolute value on $\mathbb{Q}$ is archimedean since $|1+1| = 2 > \max\{|1, |1|\} = 1$. In fact, this is the only archimedean norm on $\mathbb{Q}$, up to equivalence (see Theorem 2.2.7). The archimedean norm on $\mathbb{Q}$ will be denoted by $|\cdot|_\infty$, this notation will be explained after Theorem 2.2.7.

2.2. The p-adic valuation.

Definition 2.2.1. Let p be a prime number. The p-adic valuation is a map $v_p : \mathbb{Z} \to \mathbb{R}$ defined on $n \in \mathbb{N}$ to be

$$v_p(n) := \max\{k \in \mathbb{N} \mid \frac{n}{p^k} \in \mathbb{N}\}.$$

Furthermore, we set $v_p(0) := \infty$ and $v_p(-n) := v_p(n)$.

Since we are aiming to complete $\mathbb{Q}$, we wish to extend this definition to $\mathbb{Q}$. We will do so in the obvious way. Let $\frac{a}{b} \in \mathbb{Q}$, then we will define

$$v_p(\frac{a}{b}) := v_p(a) - v_p(b).$$

One may calculate that $v_p(ab) = v_p(a) + v_p(b)$, for $a, b \in \mathbb{Q}$. We want the p-adic norm to be symmetric as we want it to define a norm. We must verify that this extension is well-defined. We will proceed with a computation. Suppose in $\mathbb{Q}$, $\dfrac{a}{b} \sim \dfrac{a'}{b'}$. Then, $ab' = a'b$. Thus, we have that $v_p(ab') = v_p(a) + v_p(b') = v_p(a') + v_p(b) = v_p(a'b)$. Rearranging we get that $v_p(a) - v_p(b) = v_p(a') - v_p(b')$. By our definition of the extension of v_p we get that $v_p(\dfrac{a}{b}) = v_p(\dfrac{a'}{b'})$, as desired.

We will now define a p-adic norm. This will be done in terms on the p-adic valuation of any number in $\mathbb{Q}$.

Definition 2.2.2. The p-adic norm is a map, $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ such that given $x \in \mathbb{Q}$

$$|x|_p := p^{-v_p(x)}.$$

We are now concerned with the question: is the p-adic norm different from the standard absolute value, $|\cdot|_\infty$? The first step to answering this is to say what it means for two norms to be the same.

Definition 2.2.3. Two norms are equivalent if they induce the same metric topology on F.

An equivalent formulation to this definition is if $|\cdot|_1$ and $|\cdot|_2$ are norms, they are equivalent if every open set with respect to $|\cdot|_1$ is also open with respect to $|\cdot|_2$. This condition is not friendly to check, so we introduce a new criterion. Before doing this, we introduce one definition and a fact to aid with clarity.

Definition 2.2.4. The trivial norm is a map, $|\cdot|_0 : \mathbb{Q} \to \{0, 1\}$ such that $|x|_0 = 1$ if $x \neq 0$, and $|0|_0 = 0$.

Proposition 2.2.5. *$\mathbb{Q}$ is complete with respect to the trivial norm.*

Since the trivial norm has no more merit to study, when we refer to a norm on $\mathbb{Q}$, we will be assuming every norm is nontrivial. We now introduce the alternate criterion.

Proposition 2.2.6. *Let $|\cdot|_1$ and $|\cdot|_2$ be nontrivial norms on a field F. The following are equivalent:*

(a) *$|\cdot|_1$ is equivalent to $|\cdot|_2$;*
(b) *for all $x \in F$, if $|x|_1 < 1$ then $|x|_2 < 1$;*
(c) *there exists $\alpha \in \mathbb{R}_{>0}$ such that $|x|_1 = |x|_2^\alpha$ for all $x \in F$.*

Proof. Suppose that $|\cdot|_1$ is equivalent to $|\cdot|_2$. This means if $(x_n)_{n\geq 1}$ converges in $(F, |\cdot|_1)$ it must converge in $(F, |\cdot|_2)$ and vice versa. Given some $x \in F$, we have $\lim_{n\to\infty} x^n = 0$ if and only if $|x| < 1$ for either norm. So, for all $x \in F$, if $|x|_1 < 1$ then $|x|_2 < 1$.

To prove that $(b) \implies (c)$, there really is no other way to show it other than the following difficult method. We begin with the assumption, that for all $x \in F$ we have $|x|_1 < 1 \implies |x|_2 < 1$. Note that for all norms $|0| = 0$ and $|1| = |1| \cdot |1| = 1$ (since it cannot be zero). Suppose $x_0 \in F$ is such that $x_0 \neq 0$ and $|x_0|_1 < 1$. This implies that $|x_0|_2 < 1$. So, there exists some $\alpha > 0$ with $|x_0|_1 = |x_0|_2^\alpha$.

Choose some other $x \in F$. We will proceed by cases. If $|x|_1 = |x_0|_1$, this means that $|x|_2 = |x_0|_2$ since otherwise we have that (b) is violated while considering either $|x/x_0|_2$ or $|x_0/x|_2$. Thus, $|x|_1 = |x|_2^\alpha$.

If $|x|_1 = 1$ if we apply (b) to either x or $1/x$ we get $|x|_2 = 1$. Thus $|x|_1 = |x|_2^\alpha$, trivially.

This leaves us to consider only the case of $|x|_i \neq 1$ and $|x|_i \neq |x_0|_i$ for $i = 1, 2$. Let us choose β such that $|x|_1 = |x|_2^\beta$. This means that for any $n \in \mathbb{Z}_+$ one has $|x^n|_1 = |x^n|_2^\beta$. Furthermore, we may assume without loss of generality that $|x|_1 < 1$ and $|x_0| < 1$ since otherwise we operate with $1/x$ instead. Let $m, n \in \mathbb{Z}_+$. Then

$$|x|_1^n < |x_0|_1^m \iff \Big|\frac{x^n}{x_0^m}\Big|_1 < 1 \iff \Big|\frac{x^n}{x_0^m}\Big|_2 < 1 \iff |x|_2^n < |x_0|_2^m,$$

where the first and last inequalities come by rearranging and the second and third from (b). We may then use the first and last statements to get

$$n \log |x|_1 < m \log |x_0|_1 \iff n \log |x|_2 < m \log |x_0|_2.$$

Equivalently, for all $n, m \in \mathbb{N}$

$$\frac{n}{m} > \frac{\log |x_0|_1}{\log |x|_1} \iff \frac{n}{m} > \frac{\log |x_0|_2}{\log |x|_2}.$$

Thus, we clearly see that

$$\frac{\log |x_0|_1}{\log |x|_1} = \frac{\log |x_0|_2}{\log |x|_2}.$$

If we substitute in $|x_0|_1 = |x_0|_2^\alpha$ and $|x|_1 = |x|_2^\beta$, then it is evident that $\alpha = \beta$.

Finally, suppose that there exists an $\alpha \in \mathbb{R}$ such that $|x|_1 = |x|_2^\alpha$ for all $x \in F$. Then it is immediately clear that $|x - a|_1 < r$ if and only if $|x - a|_2 < r^{1/\alpha}$. So, any open ball in $(F, |\cdot|_1)$ is also open in $(F, |\cdot|_2)$.

We have shown that $(a) \implies (b) \implies (c) \implies (a)$. Thus, all three statements are equivalent, as desired. $\square$

We are now ready to present the main theorem concerning what kinds of norms can be put on $\mathbb{Q}$.

Theorem 2.2.7 (Ostrowski's Theorem)**.** *Every non-trivial norm on $\mathbb{Q}$ is equivalent to one of the norms $|\cdot|_p$ where p is prime or $p = \infty$.*

Sketch of proof. There are two cases to check: when the norm is archimedean and when it is not. In the first case, one should first show that if $n = a_0 + a_1 n_0 + \ldots + a_k n_0^k$ is the expansion of n in n_0, then $|n| \leq n_0^{k\alpha} \frac{n_0^\alpha}{n_0^\alpha - 1}$, where α is the unique integer such that $|n_0| = n_0^\alpha$.

Set $c = n_0^\alpha/(n_0^\alpha - 1)$. This is a constant, which implies that for an integer of the form n^N

$$|n^N| \leq cn^{N\alpha}.$$

Replicating the computation done above will yield that $|\cdot|$ is equivalent to $|\cdot|_\infty$.

When $|\cdot|$ is non-archimedean, the trick lies in observing that $|n| \leq 1$ for all integers. Since $|\cdot|$ is non-trivial, it will then follow that there is a smallest integer with $|n_0| < 1$. One should then show that n_0 is going to be prime by considering $n_0 = a \cdot b$, for a, b integers smaller than n_0 and conclude quickly that $|a| = |b| = 1$ but since $|n_0| < 1$ this is an absurdity. Next, one should prove that if n_0 does not divide n then $|n| = 1$. Then it should be clear that $|\cdot|$ is equivalent to $|\cdot|_{n_0}$. □

Ostrowki's Theorem is the main reason for using the notation $|\cdot|_\infty$ for the usual absolute value. The point is that then every norm on $\mathbb{Q}$ comes from a prime (whether it be finite or infinite). There are many times when it is helpful to work with all of the primes, such as the Hasse-Minkowski theorem (see Section 5.1). We will present an example of the use of working with all the primes before constructing the field $\mathbb{Q}_p$.

Proposition 2.2.8. *For any $x \in \mathbb{Q}^\times$, we have*

$$\prod_{p \text{ prime } \leq \infty} |x|_p = 1.$$

Proof. Let x be a positive integer, which has factorization $x = p_1^{a_1} \cdot \ldots \cdot p_n^{a_n}$, for p_i distinct primes, and $a_i \in \mathbb{N}$. One computes

$$|x|_p = \begin{cases} 1, & \text{if } p \text{ is not a factor of } x, \\ p_i^{-a_i}, & \text{if } p = p_i \text{ for some } i, \\ x, & \text{if } p = \infty. \end{cases}$$

The result follows immediately since $|x|_p = |-x|_p$. □

This formula allows us to compute $|x|_p$, given all other valuations of x. It establishes a close relationship between all the norms on $\mathbb{Q}$.

2.3. **Construction of $\mathbb{Q}_p$.** We have now covered the tools required in order to build the p-adic fields. In building these fields, we will highlight the idea that all norms of $\mathbb{Q}$ are equally important and should be treated as such. Much like the reals were built using Cauchy sequences, we will proceed in the same way.

Definition 2.3.1. Let F be a field and let $|\cdot|$ be a norm on F.

(1) A sequence $(a_n)_{n\geq 1}$ is said to be Cauchy if for all $\varepsilon > 0$ there exists an N_0 such that for all $m, n > N_0$ one has $|a_m - a_n| < \varepsilon$.
(2) A field F is complete with respect to a norm $|\cdot|$ if every Cauchy sequence in F has a limit.
(3) A subset $S \subseteq F$ is dense if for all $a \in F$ and all $\varepsilon > 0$ we have that $B(a, \varepsilon) \cap S \neq \emptyset$, where $B(a, \varepsilon) := \{x \mid |x - a| < \varepsilon\}$.

Intuitively, a Cauchy sequence is something that should have a limit since its terms are crowded into smaller and smaller balls. So, a field is complete if the sequences that should have a limit do.

The archimedean norm is different than the rest since there exists an inclusion of $\mathbb{Q}$ into $\mathbb{R}$ such that $|\cdot|_\infty$ extends to $\mathbb{R}$, $\mathbb{R}$ is complete with respect to this metric, and $\mathbb{Q}$ is dense in

$\mathbb{R}$. We have explained how to build $\mathbb{R}$ at the start of this section. Our goal is to now build a completion, with respect to each of the other norms of $\mathbb{Q}$ that is analogous to $\mathbb{R}$. That is, we want to find a field that extends a p-adic norm, which is complete, and in which $\mathbb{Q}$ is dense. The existence of such an object is a general theorem about metric spaces.

Proposition 2.3.2. *$\mathbb{Q}$ is not complete with respect to any non-trivial norm.*

Proof. [Gou20, Section 3.1]. □

Since $\mathbb{Q}$ is not complete, we must construct a completion. The simplest way to do this is to add all the values that the Cauchy sequences should converge to into $\mathbb{Q}$, as what was explained for $\mathbb{R}$. Since they do not literally exist, we will let the set of all Cauchy sequences that should converge to a value be the value itself.

Definition 2.3.3. Let $|\cdot|_p$ be a non-archimedean norm on $\mathbb{Q}$. Define

$$C = C_p(\mathbb{Q}) := \{(a_n)_{n\geq 1} \mid (a_n)_{n\geq 1} \text{ is Cauchy with respect to } |\cdot|_p\}.$$

Proposition 2.3.4. *C is a commutative ring with unity if we define*

$$(a_n)_{n\geq 1} + (b_n)_{n\geq 1} = (a_n + b_n)_{n\geq 1}$$
$$(a_n)_{n\geq 1} \cdot (b_n)_{n\geq 1} = (a_n b_n)_{n\geq 1}$$

Proof. Since the sequences on the right are Cauchy, this immediately follows. □

The issue with C is that different Cauchy sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ might have $\lim_{n\to\infty}(a_n - b_n) = 0$. As such, they should converge to the same value. Despite this, they are different objects in C. We thus look for a way to identify two sequences that should have the same limit. It is here that the structure of a ring helps us out.

Definition 2.3.5. Let $\mathscr{N} \subset C$ be the ideal

$$\mathscr{N} := \{(a_n)_{n\geq 1} \mid \lim_{n\to\infty} |a_n|_p = 0\}.$$

It can be shown that $\mathscr{N}$ is a maximal ideal of C.

Definition 2.3.6. The field of p-adic numbers is defined to be the quotient

$$\mathbb{Q}_p := C/\mathscr{N}.$$

Since two constant sequences never differ by an element of $\mathscr{N}$ their difference will be another constant sequence. And so, we have an inclusion of $\mathbb{Q}$ in $\mathbb{Q}_p$ as we can send a to the equivalence class of sequences converging to a. We may construct this class as the equivalence class to $(a)_{n\geq 1}$. We are now left to check the other two properties. The first of which is that $|\cdot|_p$ extends to $\mathbb{Q}_p$.

We introduce the following lemma to make sense of the definition of the extension of $|\cdot|_p$ to $\mathbb{Q}_p$ that will follow.

Lemma 2.3.7. *Let $(a_n)_{n\geq 1} \in C$ such that $(a_n)_{n\geq 1} \notin \mathscr{N}$. There exists some N such for all $m, n > N$, $|a_m|_p = |a_n|_p$.*

Proof. Since $(a_n)_{n\geq 1}$ is Cauchy such that $a_n \not\to 0$, there must exist some $\varepsilon > 0$ and n_0 such that for all $n \geq n_0$ $|a_n|_p \geq \varepsilon$. Furthermore, since this is a Cauchy sequence, there exists some n_0' such for all $m, n \geq n_0'$, $|a_n - a_m|_p < \varepsilon$. Thus, for $N = \max\{n_0, n_0'\}$ every term must have $|a_n|_p = |a_m|_p$, by the non archimedean property. □

Definition 2.3.8. If $\lambda \in \mathbb{Q}_p$ is an element represented by a Cauchy sequence $(a_n)_{n\geq 1}$, we define

$$|\lambda|_p = \lim_{n\to\infty} |a_n|_p.$$

Proposition 2.3.9. *$\mathbb{Q}$ is a dense subset of $\mathbb{Q}_p$.*

This is a very standard proof in which one shows any open ball around a given $\lambda \in \mathbb{Q}_p$ contains an element (of the image) of $\mathbb{Q}$.

Proposition 2.3.10. *$\mathbb{Q}_p$ is complete with respect to $|\cdot|_p$.*

Proof. Let $(\lambda_n)_{n\geq 1}$ be a Cauchy sequence of elements of $\mathbb{Q}_p$. Since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, it immediately follows that there exist rational numbers $(a_n)_{n\geq 1}$ such that

$$\lim_{n\to\infty} |\lambda_n - a_n|_p = 0.$$

The sequence $(a_n)_{n\geq 1}$ must be Cauchy due to the above. So, we let λ denote the element of $\mathbb{Q}_p$ corresponding to this sequence. It immediately follows that

$$\lim_{n\to\infty} \lambda_n = \lambda,$$

which means that $\mathbb{Q}_p$ is complete. $\square$

Proposition 2.3.11. *For every prime $p \in \mathbb{Z}$ the field $\mathbb{Q}_p$ is unique up to isomorphisms preserving norms.*

2.4. **Field Extensions of $\mathbb{Q}_p$.** We are interested in extending $\mathbb{Q}_p$. That is, we want a field F containing $\mathbb{Q}_p$. Suppose β is a non-square element of $\mathbb{Q}_p$, then we would want to extend F by adjoining the root of some irreducible polynomial, such as $x^2 - \beta$. We could also consider the field $\mathbb{Q}_p(x) := \{f(x)/g(x) \mid f, g \in \mathbb{Q}_p[x]\}$; but, for our purposes, we need only consider finite field extensions.

Definition 2.4.1. Let F be a field containing $\mathbb{Q}_p$. Then F is a vector space over $\mathbb{Q}_p$. We say that the degree of F, $[F : \mathbb{Q}_p] = \dim_{\mathbb{Q}_p} F$, is finite if the dimension of F over $\mathbb{Q}_p$ is finite.

We would like to consider norms of F, but in order to keep things interesting, we want it to act as an extension of the p-adic norm. Thus, we are looking for a norm $|\cdot| : F \to [0, \infty)$ such that $|\lambda| = |\lambda|_p$ for $\lambda \in \mathbb{Q}_p$.

Any such norm will be a norm on F as a vector space over $\mathbb{Q}_p$. Furthermore, it will need to be non-archimedean since this truly only depends on values of $\mathbb{Z}$ which are in $\mathbb{Q}_p$. This norm has really nice properties that we will be highlighting, and then we will show that what we are looking for is unique and does exist.

Proposition 2.4.2. *Let F be a finite extension of $\mathbb{Q}_p$. Suppose that there exists $|\cdot|$ that extends the p-adic norm to F. Then*

(a) *F is complete with respect to $|\cdot|$ and,*
(b) *we can take limits of a sequence in F by taking limits of coefficients with respect to a given basis $\{x_1, \ldots, x_n\}$ of F as a $\mathbb{Q}_p-$vector space.*

In particular the topology of F induced by $|\cdot|$ is independent of the choice of $|\cdot|$.

Proof. Since all norms of a finite-dimensional vector space are equivalent (a) follows. (b) is saying that any norm is equivalent to the sup-norm for any given basis. And the final point is true since the topology is simply the unique topology of F as a normed $\mathbb{Q}_p$-vector space. $\square$

An important, and probably pretty clear, in light of Proposition 2.2.6, corollary follows.

Corollary 2.4.3. *There is at most one norm on F extending the absolute value the p-adic norm on $\mathbb{Q}_p$.*

We know now that there is at most one extension of $|\cdot|_p$ to F, and that F is complete with respect to that completion. However, we do not know that such an extension must exist. We will give a construction. One consequence of the above is that the norm does not depend on the context.

Let E and F be fields, and assume that $[E : F]$ is finite. We will say that E/F is a finite extension. Normal extensions are a nice thing: given any extension E/F, there exists a normal extension of F containing E. The smallest such extension is the normal closure E/F. The crucial fact is that there exists a function

$$N_{[E/F]} : E \to F,$$

which is called the norm from E to F (it is certainly unfortunate that this is called a norm too, but it is separate from norms as above). This will allow us to "go down" from elements of the larger field E to the smaller field F. The norm map can be defined in many ways; the following will be beneficial in our exploration.

Definition 2.4.4. Let $a \in E$. If E is a finite-dimensional F-vector space, consider the F-linear map from E to E by multiplication by x. Let M be the matrix corresponding to this map. Define $N_{[E/F]}(\alpha)$ to be the determinant of this matrix.

For a quadratic extension, this definition will serve to be very useful since $N_{[E/F]}(\alpha) = \alpha\overline{\alpha}$. Concretely, $N_{[E/F]}(x + y\sqrt{\beta}) = x^2 - y^2\beta$, for $\beta \in F^\times \setminus F^{\times 2}$, where $E = F[\sqrt{\beta}]$.

Proposition 2.4.5. $F^{\times 2} \subseteq N_{[E/F]}(E^\times) \subseteq F^\times$.

2.5. **Squares in $\mathbb{Q}_p$.** We are looking to tell a story about non-square elements in $\mathbb{Q}_p$. They are important both when talking about field extensions and for the study of quadratic forms. Thus, a strong understanding of them is important. We are interested not in $F^\times$ but rather in equivalence classes of a field by its group of squares. So, we investigate $F^\times/F^{\times 2}$. We proceed by looking at isomorphisms of $\mathbb{Q}_p^\times$. We adapt F from Section 2.4. To start, we introduce the following results.

Proposition 2.5.1. *For $p \neq 2$, $(\mathbb{Q}_p^\times, \cdot) \cong (\mathbb{Z}, +) \times (\mathbb{Z}/(p-1)\mathbb{Z}, \cdot) \times (\mathbb{Z}_p, +)$, where this is a group isomorphism.*

This isomorphism comes from the fact that $\mathbb{Q}_p$ contains μ_{p-1}, the group of $p-1$th roots of unity $\{\xi^n\}$. We can obtain the last term by sending $(m, n, x) \to p^m\xi^n\exp(px)$.

Proposition 2.5.2. *Suppose F has residual characteristic not equal to 2. Then $(F^\times, \cdot) \cong (\mathbb{Z}, +) \times (F, \cdot) \times \cup_1$, where $\cup_1 := \{1 + x \mid x \in P\}$.*

Hensel's Lemma (Theorem 5.1.1) guarantees that elements of $\cup_1$ are squares.

Corollary 2.5.3. *Suppose F has residual characteristic $p \neq 2$. Let $\epsilon \in F$ be a non-square element with valuation 0 and ϖ be an element with minimal, positive valuation, formally called a uniformizer of F. Then $F^\times/F^{\times 2} = \{1, \epsilon, \varpi, \epsilon\varpi\}$.*

Proof. We will show a proof for $F = \mathbb{Q}_p$; a similar argument holds for any F, using Proposition 2.5.2 instead. Since $char(\mathbb{Q}_p) = 0$, we may fruitfully apply Proposition 2.5.1. By the proposition,

$$(\mathbb{Q}_p^\times, \cdot) \cong (\mathbb{Z}, +) \times (\mathbb{Z}/(p-1)\mathbb{Z}, +) \times (\mathbb{Z}_p, +).$$

So,

$$\mathbb{Q}_p^{\times 2} \cong 2(\mathbb{Z}, +) \times (\mathbb{Z}/(p-1)\mathbb{Z}, +)^2 \times (\mathbb{Z}_p, +)^2.$$

If we quotient these groups it is clear that

$$\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2} \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

It immediately follows that $\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2} = \{1, \epsilon, \varpi, \epsilon\varpi\}$. □

3. Classifying Quadratic forms in $\mathbb{Q}_p$, p odd.

Throughout this section, F is a finite degree algebraic field extension of $\mathbb{Q}_p$ with p odd, and all quadratic forms are over F.

Corollary 2.5.3 will serve as the foundation for our classification of quadratic forms in F. We will note that when $-1 \notin F^{\times 2}$, we may freely choose ϵ to be -1. Notice that because every quadratic form is equivalent to a diagonal quadratic form, it is true that for the purposes of classification, we need only consider the diagonal quadratic forms with entries from our preferred list of square class representatives $\{1, \epsilon, \varpi, \epsilon\varpi\}$. Our end goal would be to have a set of invariants that completely classify all quadratic forms of any dimension up to equivalence. We will show that such a set of invariants exists in F. Recall, we work with nondegenerate quadratic forms.

Depending on whether or not $-1 \in F^{\times 2}$, the diagonalization on H becomes very different. When $-1 \in F^{\times 2}$, we have that $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, while when $-1 \notin F^{\times 2}$, we have that $H = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}$.

Even though the hyperbolic plane is very different in each case, we may still recover a joint way to write it, independent of whether or not -1 is a square. This comes from the fact that we are using -1 as ϵ when $-1 \notin F^{\times 2}$.

We will proceed by referring to the following table.

	-1	$-\epsilon$	$-\varpi$	$-\epsilon\varpi$
$-1 \in F^{\times 2}$	1	ϵ	ϖ	$\epsilon\varpi$
$-1 \notin F^{\times 2}$	ϵ	1	$\epsilon\varpi$	ϖ

3.1. The Norm Map and Value Set as a Classification Method.

Throughout this section, we will only be considering the anisotropic quadratic forms. We can do this thanks to Corollary 1.3.7, which asserts if we have found every anisotropic kernel up to equivalence, then we may construct every quadratic form up to equivalence, using this decomposition.

We will be using norm maps and value sets of quadratic forms as a way to distinguish anisotropic kernels. Thus, we want to build machinery to allow us to simply look at a quadratic form and say if it is anisotropic or not and if it is equivalent to another given quadratic form.

We are really interested in the question: in $\mathbb{Q}_p$ if two anisotropic quadratic forms of the same dimension have the same value set, are they equivalent? We will present an answer to this and see how it is possible to view the value set of a two-dimensional quadratic form as the image of a norm map. And so, we begin with some basic properties about the value set of a quadratic form.

Proposition 3.1.1. *Let $Q = [\alpha]$ and Q' be quadratic forms, where $\alpha \in \{1, \epsilon, \varpi, \epsilon\varpi\}$. Then $Val(Q \otimes Q') = \alpha \times Val(Q')$.*

Proof. Suppose that $Q' = \{\alpha_1 x_1^2 + \ldots + \alpha_n x_n^2 \mid (x_1, \ldots, x_n) \in V\}$. When Q' is anisotropic, applying the Kronecker product shows that

$$\begin{aligned}\mathrm{Val}([\alpha] \otimes Q') &= \{\alpha \cdot \alpha_1 x_1^2 + \ldots + \alpha \cdot \alpha_n x_n^2 : x \in V \setminus \{0\}\}/F^{\times 2} \\ &= \alpha \times \{\alpha_1 x_1^2 + \ldots + \alpha_n x_n^2 : x \in V \setminus \{0\}\}/F^{\times 2} = \alpha \times \mathrm{Val}(Q'),\end{aligned}$$

as desired. The case that Q' is isotropic is trivial since $\alpha \times F = F$ and H is preserved under multiplication by $\alpha \in F^\times$. □

We may now use Proposition 3.1.1 to show a very important result.

Proposition 3.1.2. *Let $\alpha, \beta \in \{1, \epsilon, \varpi, \epsilon\varpi\} = F^\times / F^{\times 2}$. Then,*

$$Val(\begin{bmatrix} \alpha & \\ & -\beta \end{bmatrix}) = \begin{cases} \{\alpha, -\beta\} & \text{if } \alpha\beta \neq 1 \\ F & \text{if } \alpha\beta = 1. \end{cases}.$$

Proof. It can be observed that if $\alpha\beta = 1$, then by Propositon 1.3.3 it follows that $\begin{bmatrix} \alpha & \\ & -\beta \end{bmatrix}$ is the hyperbolic plane. It is also true that $\mathrm{Val}\begin{bmatrix} 1 & \\ & \alpha \end{bmatrix} = \{x^2 + \alpha y^2 \mid (x, y) \neq (0, 0)\}/F^{\times 2} = N_{E/F}(x + y\sqrt{-\alpha})/F^{\times 2}$. That is, the value set of a 2-dimensional diagonal quadratic form with a 1 in its diagonal will be the image of a norm map will. We now compute

$$\begin{aligned}\mathrm{Val}(\begin{bmatrix} \alpha & \\ & -\beta \end{bmatrix}) &= \mathrm{Val}([\alpha] \otimes \begin{bmatrix} 1 & \\ & -\alpha\beta \end{bmatrix}) \\ &= \mathrm{Val}([\alpha]) \times \mathrm{Val}(\begin{bmatrix} 1 & \\ & -\alpha\beta \end{bmatrix}) \\ &= \alpha \times \begin{cases} \{1, -\alpha\beta\} & \text{if } \alpha \neq \beta \\ F & \text{if } \alpha = -\beta. \end{cases} = \begin{cases} \{\alpha, -\beta\} & \text{if } \alpha \neq \beta \\ F & \text{if } \alpha = -\beta. \end{cases}\end{aligned}$$

□

Proposition 3.1.3. *When $-1 \in F^{\times 2}$, an anisotropic quadratic form cannot have repeated value in its diagonal. When $-1 \notin F^{\times 2}$ an anisotropic quadratic form cannot have $\{1, \epsilon\}$ or $\{\varpi, \epsilon\varpi\}$ in its diagonal simultaneously. Moreover, in this case, the diagonal cannot have more than two repetitions of values.*

Example 3.1.4. Suppose we are in the case that $-1 \in F^{\times 2}$ and that we are looking for a quadratic form Q whose value set is $\{1, -\epsilon\}$. Then, we can link the norm map with the value set in the following way.

$$\begin{aligned}\{1,-\epsilon\} = \{1,\epsilon\} &= \{N_{E/F}(x+y\sqrt{\epsilon}) \mid x,y \in F^\times\} \\ &= \{x^2+\epsilon y^2 \mid x,y \in F^\times\} \\ &= \mathrm{Val}(\begin{bmatrix} 1 & \\ & \epsilon \end{bmatrix})\end{aligned}$$

We are led to a lemma.

Lemma 3.1.5. *For any anisotropic quadratic form Q, the value set of Q cannot be smaller than the dimension of Q.*

Proof. Evidently if $Q = \begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix}$ then $\mathrm{Val}(Q)$ contains $\{\alpha,\beta,\gamma\}$, but we have seen it can be larger by now. If $-1 \in F^{\times 2}$, and the value set is smaller than the dimension of Q, this implies there are repeated values on the diagonal, which contradict Proposition 3.1.3. If $-1 \notin F^{\times 2}$, we have already seen that $\mathrm{Val}\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = \{1,\epsilon\}$, which implies by Proposition 3.1.1 that $\mathrm{Val}\begin{bmatrix} \varpi & \\ & \epsilon\varpi \end{bmatrix} = \{\varpi,\epsilon\varpi\}$. The result follows. ☐

Corollary 3.1.6. *There are no anisotropic quadratic forms of dimension* 5 *or higher.*

Proof. This immediately follows from Lemma 3.1.5 ☐

We have one more proposition to get to until we will are able to develop the machinery behind the classification of quadratic forms. This proposition will allow us to see the dimension of a quadratic form based solely on its value set, given that it is anisotropic.

Proposition 3.1.7. *For $n \le 4$, if Q is anisotropic and $|Val(Q)| = n$, then $|Val(Q)| = \dim Q$.*

Proof. We know that a quadratic form of dimension 1 has at most $\mathrm{Val}([\alpha]) = \{\alpha x^2 \mid x \in F^\times\}/F^{\times 2} = \{\alpha\}$. If we were to add another element to the diagonal, by Proposition 3.1.2, we will have either the image of a norm map or the coset of the image of a norm map both of which have only 2 elements. A quadratic form, Q' of dimension 4 must have $\mathrm{Val}(Q') = \{1,\epsilon,\varpi,\epsilon\varpi\}$ by Lemma 3.1.5.

Suppose now we have $Q = \begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix}$. We are interested in the values that $\alpha x^2+\beta y^2+\gamma z^2$ can take, where $\alpha,\beta,\gamma \in \{1,\epsilon,\varpi,\epsilon\varpi\}$. Suppose, without the loss of generality, that $|\alpha|_p = |\beta|_p$. Then, if $|\alpha|_p = |\beta|_p = 1$, then one must have that $\alpha \neq -\beta$, since otherwise $\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \sim H$. As a result, $1,\epsilon \in \mathrm{Val}(Q)$. Furthermore, the value that γ is not mustn't be in $\mathrm{Val}(Q)$ since there are no other components of the same norm to produce it. Since all that can be made with $x^2+\epsilon y^2$ is again, 1 and ϵ, in $F^\times/F^{\times 2}$.

☐

It turns out that in F all of our problems are solved: if two anisotropic forms have the same value set, they are equivalent. This is outlined in the next result.

Proposition 3.1.8. *Let Q and Q' be anisotropic quadratic forms over F. Then*

$$Q \sim Q' \iff \mathit{Val}(Q) = \mathit{Val}(Q').$$

Proof. ($\Longrightarrow$). This direction is obvious, it has been shown in Proposition 1.4.2.

($\Longleftarrow$). This is where the real battle begins since this does not come for free. Suppose Q and Q' are anisotropic quadratic forms of dimension less than 5 and that $\mathrm{Val}(Q) = \mathrm{Val}(Q')$. We automatically get the $\dim Q = \dim Q'$ by Proposition 3.1.7. and so, the cases of dimensions $1, 2$, and 4 come for free as in dimension 1, the value set is the one square class representative in the form, and in dimension 2, since each quadratic extension has a unique image, this too holds. Finally, in dimension 4 there is only one anisotropic form, up to equivalence, so this too comes for free.

We are now left to puzzle over the case of dimension 3. One can immediately notice that if we find a unique family with a value set $\{1, \epsilon, \varpi\}$, then we have shown this for every possible subset since

$$\begin{aligned}
1 \times \{1, \epsilon, \varpi\} &= \{1, \epsilon, \varpi\} \\
\epsilon \times \{1, \epsilon, \varpi\} &= \{\epsilon, 1, \epsilon\varpi\} \\
\varpi \times \{1, \epsilon, \varpi\} &= \{\varpi, \epsilon\varpi, 1\} \\
\epsilon\varpi \times \{1, \epsilon, \varpi\} &= \{\epsilon\varpi, \varpi, \epsilon\}.
\end{aligned}$$

Thus, if $P := \{Q \mid Q \text{ is a quadratic form with } \mathrm{Val}(Q) = \{1, \epsilon, \varpi\}\}/\sim$, then you may generate the other sets by taking $[\alpha] \otimes P := \{[\alpha] \otimes Q \mid Q \text{ is a quadratic form with } \mathrm{Val}(Q) = \{1, \epsilon, \varpi\}\}/\sim$, by Proposition 3.1.1. As such, we split this problem into cases. Suppose $-1 \in F^{\times 2}$, then we may compute P since $-\alpha = \alpha$ as far as square classes go and we need values in $F^{\times 2} \cup \epsilon F^{\times 2} \cup \varpi F^{\times 2}$ we get than

$$P = \left\{ \begin{bmatrix} \pm 1 & & \\ & \pm\epsilon & \\ & & \pm\varpi \end{bmatrix} \right\} \Big/ \sim \; = \left\{ \begin{bmatrix} 1 & & \\ & \epsilon & \\ & & \varpi \end{bmatrix} \right\},$$

up to permutations of the diagonal.

The case when $-1 \notin F^{\times 2}$ is slightly more complicated. Suppose that $\{\alpha x^2 + \beta y^2 + \gamma z^2 : x, y, z \in F^{\times}\} = \{1, \epsilon, \varpi\}$, without the loss of generality we can assume that $\alpha = 1$ (the case when $\alpha = \epsilon$ is almost identical). We note that this means that $\beta, \gamma \neq -1 = \epsilon$. We compute

$$\begin{aligned}
x^2 + \beta y^2 + \gamma z^2 &\in F^{\times 2} \cup \epsilon F^{\times 2} \cup \varpi F^{\times 2} \\
&= F^{\times 2} \cup -F^{\times 2} \cup \varpi F^{\times 2} \\
&= N_{(E/F)}(Ex) \cup \varpi F^{\times 2}
\end{aligned}$$

so one gets that

$$P = \left\{ \begin{bmatrix} 1 & & \\ & 1 & \\ & & \varpi \end{bmatrix}, \begin{bmatrix} \epsilon & & \\ & \epsilon & \\ & & \varpi \end{bmatrix} \right\} \Big/ \sim .$$

Since $\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ and $\begin{bmatrix} \epsilon & \\ & \epsilon \end{bmatrix}$ have the same value set, they are equivalent, and it follows that the two three-dimensional forms we have found are equivalent. Which means $P = \left\{ \begin{bmatrix} 1 & & \\ & 1 & \\ & & \varpi \end{bmatrix} \right\}$.

Thus, the proposition is proven. $\square$

One can notice that ϵ can be written as a sum of squares so, by using $1, 1$ or ϖ, ϖ, respectively in the diagonal we managed to overcome the issue of $\{1, \epsilon\}, \{\varpi, \epsilon\varpi\} \subseteq \text{Val}(Q)$, when $-1 \notin F^{\times 2}$.

The proof of Proposition 3.1.8 gives an algorithm for the construction of a set of representatives of the distinct classes of quadratic forms. The general method that we will follow is as follows.

- Take the set of square class representatives, $\{1, \epsilon, \varpi, \epsilon\varpi\}$.
- Take all subsets of size 0 through 4 of $\{1, \epsilon, \varpi, \epsilon\varpi\}$, since $\text{Val}(Q) \subseteq \{1, \epsilon, \varpi, \epsilon\varpi\}$.
- Given a subset, compute all quadratic forms that are anisotropic and whose value set is your subset. (This will not prove to be difficult since we know the dimension based on the size of the subset already)
  - If $-1 \in F^{\times 2}$, one can immediately take the entries of the diagonal to be the square class representatives.
  - If $-1 \notin F^{\times 2}$, once $\{1, \epsilon\} \subseteq \text{Val}(Q)$ or $\{\varpi, \epsilon\varpi\} \subseteq \text{Val}(Q)$ we can use $1, 1$ or ϖ, ϖ, respectively in the diagonal to solve this issue. Otherwise, take entries of Q from $\text{Val}(Q)$ directly.
- Repeat for all subsets and you will get 1 anisotropic quadratic form of dimension 0, 4 of dimension 1, 6 of dimension 2, 4 of dimension 3, and 1 of dimension 4.
- From here, reversing the decomposition that $Q = nH \oplus Q_{\text{aniso}}$, you can generate every quadratic form, up to equivalence.

3.2. **Classification of Quadratic forms.** We will follow the algorithm listed above. The computations are not all that riveting, as such, I will not present any more than what has been seen in Section 3.1. Suppose we are looking for all subsets of $\{1, \epsilon, \varpi, \epsilon\varpi\}$, this is encoded in the following table.

Dimension	Subsets	Number of Forms
0	$\emptyset$	1
1	$\{1\}, \{\epsilon\}, \{\varpi\}, \{\epsilon\varpi\}$	4
2	$\{1, \epsilon\}, \{1, \varpi\}, \{1, \epsilon\varpi\}, \{\epsilon, \varpi\}, \{\epsilon, \epsilon\varpi\}, \{\varpi, \epsilon\varpi\}$	6
3	$\{1, \epsilon, \varpi\}, \{1, \epsilon, \epsilon\varpi\}, \{1, \varpi, \epsilon\varpi\}, \{\epsilon, \varpi, \epsilon\varpi\}$	4
4	$\{1, \epsilon, \varpi, \epsilon\varpi\}$	1

With this in mind, we may now present the complete classification of anisotropic forms over a non-archimedean field F with $charF \neq 2$. The following theorem is a summary of the results in Section 3.1.

Theorem 3.2.1. *The complete characterization of every anisotropic form over F over is*

Val(Q)	$-1 \in F^{\times 2}$	$-1 \notin F^{\times 2}$	*Joint*
$\emptyset$	$[0]$	$[0]$	$[0]$

$\{1\}$	$[1]$	$[1]$	$[1]$
$\{\epsilon\}$	$[\epsilon]$	$[\epsilon]$	$[\epsilon]$
$\{\varpi\}$	$[\varpi]$	$[\varpi]$	$[\varpi]$
$\{\epsilon\varpi\}$	$[\epsilon\varpi]$	$[\epsilon\varpi]$	$[\epsilon\varpi]$
$\{1, \epsilon\}$	$\begin{bmatrix} 1 & \\ & \epsilon \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & -\epsilon \end{bmatrix}$
$\{1, \varpi\}$	$\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}$
$\{1, \epsilon\varpi\}$	$\begin{bmatrix} 1 & \\ & \epsilon\varpi \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & \epsilon\varpi \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & \epsilon\varpi \end{bmatrix}$
$\{\epsilon, \varpi\}$	$\begin{bmatrix} \epsilon & \\ & \varpi \end{bmatrix}$	$\begin{bmatrix} \epsilon & \\ & \varpi \end{bmatrix}$	$\begin{bmatrix} \epsilon & \\ & \varpi \end{bmatrix}$
$\{\epsilon, \epsilon\varpi\}$	$\begin{bmatrix} \epsilon & \\ & \epsilon\varpi \end{bmatrix}$	$\begin{bmatrix} \epsilon & \\ & \epsilon\varpi \end{bmatrix}$	$\begin{bmatrix} \epsilon & \\ & \epsilon\varpi \end{bmatrix}$
$\{\varpi, \epsilon\varpi\}$	$\begin{bmatrix} \varpi & \\ & \epsilon\varpi \end{bmatrix}$	$\begin{bmatrix} \varpi & \\ & \varpi \end{bmatrix}$	$\begin{bmatrix} \varpi & \\ & -\varpi \end{bmatrix}$
$\{1, \epsilon, \varpi\}$	$\begin{bmatrix} 1 & & \\ & \epsilon & \\ & & \varpi \end{bmatrix}$	$\begin{bmatrix} 1 & & \\ & 1 & \\ & & \varpi \end{bmatrix}$	$\begin{bmatrix} 1 & & \\ & -\epsilon & \\ & & \varpi \end{bmatrix}$
$\{1, \epsilon, \epsilon\varpi\}$	$\begin{bmatrix} 1 & & \\ & \epsilon & \\ & & \epsilon\varpi \end{bmatrix}$	$\begin{bmatrix} 1 & & \\ & 1 & \\ & & \epsilon\varpi \end{bmatrix}$	$\begin{bmatrix} 1 & & \\ & -\epsilon & \\ & & \epsilon\varpi \end{bmatrix}$
$\{1, \varpi, \epsilon\varpi\}$	$\begin{bmatrix} 1 & & \\ & \varpi & \\ & & \epsilon\varpi \end{bmatrix}$	$\begin{bmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{bmatrix}$	$\begin{bmatrix} 1 & & \\ & \varpi & \\ & & -\epsilon\varpi \end{bmatrix}$
$\{\epsilon, \varpi, \epsilon\varpi\}$	$\begin{bmatrix} \epsilon & & \\ & \varpi & \\ & & \epsilon\varpi \end{bmatrix}$	$\begin{bmatrix} \epsilon & & \\ & \varpi & \\ & & \varpi \end{bmatrix}$	$\begin{bmatrix} \epsilon & & \\ & \varpi & \\ & & -\epsilon\varpi \end{bmatrix}$
$\{1, \epsilon, \varpi, \epsilon\varpi\}$	$\begin{bmatrix} 1 & & & \\ & \epsilon & & \\ & & \varpi & \\ & & & \epsilon\varpi \end{bmatrix}$	$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix}$	$\begin{bmatrix} 1 & & & \\ & -\epsilon & & \\ & & \varpi & \\ & & & -\epsilon\varpi \end{bmatrix}$.

Notice that $Q = 0$ is anisotropic since the only vector it can take in is the zero vector.

With this, we have characterized the quadratic forms over F with $char F \neq 2$. Two corollaries arise due to this classification and completely capture the results from Theorem 3.2.1.

Lemma 3.2.2. *The map Val is a bijection*

$$Val \colon \{Q \mid Q \text{ is an anisotropic quadratic form}\}/\sim \quad \xrightarrow{\cong} \quad \mathcal{P}(\{1, \epsilon, \varpi, \epsilon\varpi\}),$$

where $\mathcal{P}(\{1, \epsilon, \varpi, \epsilon\varpi\})$ is the power set of the set of square class representatives. This bijection preserves dimension: $\dim\ Val(S) = |S|$.

In fact, something stronger can be said.

Proposition 3.2.3. *The dimension of a quadratic form and the value set of its anisotropic kernel is a complete set of invariants.*

Proof. This result follows. □

4. The Witt Ring

Now that we have a complete classification of quadratic forms, finding a structure on them would be beneficial. This can be done in both a group and ring structure. It would make sense that any group of quadratic forms would be Abelian since $Q \oplus Q' = Q' \oplus Q$. As a result, one would hope that some ring structure, equipped with a tensor product, would in fact be possible. Throughout this section, we will explore these and show how the hyperbolic plane being different in each case affects how the anisotropic forms, which act as representatives, interact with each other. Throughout this section, F is a finite degree algebraic extension of $\mathbb{Q}_p$, for p odd and finite.

4.1. **The Witt Group.**

The direct sum of two quadratic forms is again a quadratic form, but this only gives us the structure of a commutative monoid. It turns out that it has a natural quotient group, called the Witt group, whose elements are represented by the distinct anisotropic forms.

Definition 4.1.1. Two quadratic forms Q and Q' are Witt equivalent if and only if their anisotropic kernels are equivalent. In this case, we write $Q \approx Q'$.

Now, it should be clear that is it in fact possible to create a finite group, with equivalence classes characterized by our preferred set of anisotropic representatives.

Proposition 4.1.2 (The Witt Group)**.** *The Witt Group, defined to be*

$$W(F) := \{Q : Q \textit{ is a quadratic form over } F\}/\approx,$$

equipped with the direct sum is an abelian group.

Proof. We begin by checking the group axioms. Let Q be a quadratic form, then $Q \oplus H \approx Q$. As a result, the hyperbolic plane represents the identity element of the Witt Group. It is true that $(Q \oplus Q') \oplus Q'' = Q \oplus (Q' \oplus Q'')$, by the direct sum. Finally, if $-1 \in F^{\times 2}$, then $Q \oplus Q \approx H$, since $\begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \sim H \sim \lambda H$. If $-1 \notin F^{\times 2}$, then $Q \oplus \epsilon Q \approx H$, since $\begin{bmatrix} \alpha & \\ & \alpha\epsilon \end{bmatrix} \sim H \sim \lambda H$, for all α in $F^{\times}$. As a result, every element has an inverse, which means that $W(F)$ is an abelian group. □

We are now able to present the structure of the Witt group for F.

Proposition 4.1.3. *The structure of the Witt group of F is*

$$W(F) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^4 & \textit{if } -1 \in F^{\times 2} \\ (\mathbb{Z}/4\mathbb{Z})^2 & \textit{if } -1 \notin F^{\times 2}. \end{cases}$$

Proof. One might notice that $W(F)$ can always be generated by $\langle [1], [\epsilon], [\varpi], [\epsilon\varpi] \rangle$, and from Theorem 3.2.1, it has order 16. When $-1 \in F^{\times 2}$, we have seen that $Q \oplus Q \approx H$, with 4 generators of order 2. Evidently, in this case, $W(F) \cong (\mathbb{Z}/2\mathbb{Z})^4$.

Suppose the $-1 \notin F^{\times 2}$. Then, $[1] \oplus [1] \oplus [1] \oplus \sim \begin{bmatrix} 1 & & \\ & \epsilon & \\ & & \epsilon \end{bmatrix} \approx [\epsilon]$. Thus, our set of generators, in this case, can be reduced to $\langle [1], [\varpi] \rangle$. In fact, $[1] \oplus [1] = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \not\approx H$. But, $[1] \oplus [1] \oplus [1] \oplus [1] \approx H$. Thus, we have have 2 generators of order 4, which implies $W(F) \cong (\mathbb{Z}/4\mathbb{Z})^2$. $\square$

4.2. **The Witt Ring.** Our goal is to induce the Kronecker product onto the Witt groups which we already have in order to create a ring. Of course, we will follow through with the routine of separating into the standard two cases. Before this, we will show that such a ring actually does exist. The Witt groups have a natural ring structure, but it is not that of the Witt ring.

Proposition 4.2.1. *$W(F) = (\{Q \mid Q$ is a quadratic form over $F\}/\approx, \oplus, \otimes)$ is a ring.*

Proof. We already know the quadratic forms mod Witt equivalence form an Abelian group, thus we check ring axioms. First, there is an identity for the tensor product since $[1] \otimes Q = Q$. Since we can use solely diagonal quadratic forms, the tensor product is the analog of multiplication, which means it preserves finite sums. All that is left to verify is that the tensor product is associative. Since we are using the Kronecker product, this is too not hard to see since multiplication is associative. $\square$

4.2.1. *The Witt Ring if $-1 \in F^{\times 2}$.* We begin by looking at idempotent elements.

Proposition 4.2.2. *There are no non-trivial idempotent elements in the Witt ring when $-1 \in F^{\times 2}$. In fact, if x is an element of the Witt ring, then $x^2 = 0$ or $x^2 = 1$, where $x^2 = x \otimes x$, $0 = H$, and $1 = [1]$.*

Proof. Since the hyperbolic plane is $\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$, it follows that if we have two quadratic forms, Q and Q', of dimension m and n respectively if mn is even then since the definition of this is symmetric, there will be $mn/2$ pairs of equal values in the diagonal. If mn is odd, there will be $(mn-1)/2$ pairs of equal terms in the diagonal and a 1 left over. $\square$

Since there are no idempotents that are not 0 or 1, it is not the case that the ring structure is $(\mathbb{Z}/2\mathbb{Z})^4$. Before figuring out the structure of the Witt group, we will simplify the case that only involves ϵ. If $x = [\epsilon]$, then in fact the subgroup generated by $[1]$ and $[\epsilon]$ is closed under the Kronecker product and we have the following multiplication table.

$\otimes$	0	1	x	$1+x$
0	0	0	0	0
1	0	1	x	$1+x$
x	0	x	1	$1+x$
$1+x$	0	$1+x$	$1+x$	0

Proposition 4.2.3. *When $-1 \in F^{\times 2}$, $\phi : \mathbb{Z}_2[x,y]/\langle x^2-1, y^2-1\rangle \to W(F)$ is a ring isomorphism given by*

$$\phi(a+bx+cy+dxy) = \begin{bmatrix} a & & & \\ & b\epsilon & & \\ & & c\varpi & \\ & & & d\epsilon\varpi \end{bmatrix},$$

where a zero means that row and column are omitted.

Proof. The main idea of this isomorphism is that we are taking advantage of the fact that any quadratic form, Q, can be expressed $Q = Q' + \varpi Q''$, for Q', Q'' in $\mathbb{F}_q$. For example, if $a = 0 \in (\mathbb{Z}/2\mathbb{Z})^4$, then $[a]$ means H in the Witt group. Furthermore, $[1]+[1] = [0] \approx H$.

By Theorem 3.2.1, this is a bijection of sets. Now, we must check the structure. We compute

$$(a+bx+cy+dxy)+(a'+b'x+c'y+d'xy) = (a+a')+(b+b')x+(c+c')y+(d+d')xy$$

$$\begin{bmatrix} a & & & \\ & b\epsilon & & \\ & & c\varpi & \\ & & & d\epsilon\varpi \end{bmatrix} \oplus \begin{bmatrix} a' & & & \\ & b'\epsilon & & \\ & & c'\varpi & \\ & & & d\epsilon\varpi \end{bmatrix} = \begin{bmatrix} a+a' & & & \\ & (b+b')\epsilon & & \\ & & (c+c')\varpi & \\ & & & (d+d')\epsilon\varpi \end{bmatrix}.$$

Where the last equality holds since working with the direct sum over $F^{\times 2}$ is equivalent to addition in $\mathbb{Z}/2\mathbb{Z}$. We proceed to multiplication. This will hold since we have already seen that the Kronecker product for diagonal matrices is the analogue of polynomial multiplication. As we have stated before, working over $F^{\times 2}$ is the same as working over $\mathbb{Z}/2\mathbb{Z}$, so we conclude that this is a ring isomorphism.

□

4.2.2. *The Witt Ring if $-1 \notin F^{\times 2}$.* If $-1 \notin F^{\times 2}$, the square class representatives are $\{1, -1, \varpi, -\varpi\}$, which means that the ring structure should not be as complicated as that of when $-1 \in F^{\times 2}$. In fact, in this case, there is an element that does not square to 0 or 1.

Proposition 4.2.4. *When $-1 \notin F^{\times 2}$,* $\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}^2 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix}.$

In fact, since our coset representatives are $\{1, -1, \varpi, -\varpi\}$, the Witt ring will not be isomorphic to a polynomial ring of two variables, but rather, one.

Proposition 4.2.5. *When $-1 \notin F^{\times 2}$, $\phi : \mathbb{Z}_4[x]/\langle x^2-1\rangle \to W(F)$ is a ring isomorphism given by*

$$\phi(a+bx+cx^2+dx^3) = \begin{bmatrix} a & & & \\ & b\varpi & & \\ & & c\varpi^2 & \\ & & & d\varpi^3 \end{bmatrix} = \begin{bmatrix} a & & & \\ & b\varpi & & \\ & & c & \\ & & & d\varpi \end{bmatrix},$$

where again a zero means that a row and column ought to be omitted.

Proof. Once again, it is clear that we have a bijection from sets. We now need to check the operations, which hold by similar logic to Proposition 4.2.3. We know that $[x]$ is a generator since $\begin{bmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{bmatrix} \sim \begin{bmatrix} -1 & & \\ & -\varpi & \\ & & \varpi \end{bmatrix} \approx [-1]$. □

The Witt ring is a peculiar thing. The structure of the Witt ring is not what is expected from a ring with such a simple abelian group. The reason why $-1 \in F^{\times 2}$ was more simple in the classification of anisotropic quadratic forms is due to the Witt ring already separating $\{1, \epsilon\}$ and $\{\varpi, \epsilon\varpi\}$, while $-1 \notin F^{\times 2}$ does not, so we had to manually divide them.

5. Quadratic Forms over $\mathbb{Q}$

In this chapter, we will be using a local-global property to give a classification of quadratic forms over $\mathbb{Q}$. In Sections 5.1 and 5.2 we complete the classification over p-adic fields in order to be able to use a local-global property, the Hasse-Minkowski Theorem, presented in 5.3 which gives a classification over $\mathbb{Q}$.

5.1. **Quadratic forms over $\mathbb{Q}_2$.** When we completed the classification of quadratic forms over $\mathbb{Q}_p$, for p odd and finite, the only result that we relied on was the fact $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2} \cong (\mathbb{Z}/2\mathbb{Z})^2$. This means that the theory we have about diagonal forms, isotropic forms, the hyperbolic plane, and anisotropic forms are all independent of this fact. As a result, we will explore how different the square classes are in this case and how this affects the results we have seen when p is odd and finite. Before we may say what exactly goes wrong, it will be important to see why the other case was so much nicer. Throughout this section, we will use results from Section 3 in the case that $F = \mathbb{Q}_p$, and we will write $\epsilon = -1, \epsilon' = 5, \varpi = 2$.

Theorem 5.1.1 (Hensel's Lemma.)**.** *Let $f(x) \in \mathbb{Z}_p[x]$ and $a \in \mathbb{Z}$ satisfy $|f(a)|_p < |f'(a)|_p^2$. Then there exists a unique $\alpha \in \mathbb{Z}_p$ such that $f(\alpha) \equiv 0 \mod p$ and $|\alpha - a|_p < |f'(a)|_p$.*

Proof. [Gou20, Section 3.4]. □

Hensel's Lemma tells us that if we can find an approximate root satisfying the desirable conditions, then there is an actual root congruent to the approximate root modulo the conditions.

Proposition 5.1.2. *Let $b \in \mathbb{Z}_2$. Then b is a square if and only if $b \equiv 1$ mod 8.*

Proof. Notice that for all $x \in \mathbb{Z}_2^\times$ we have that $x = 1 + 2y$, where $y \in \mathbb{Z}_2$, so $x + x^2 \in 2\mathbb{Z}_2$. Suppose $b = a^2$ is a unit and a square. Then $a = 1 + 2x$, for some $x \in \mathbb{Z}_2$, and so $a^2 = 1 + 4x + 4x^2$. Since $x + x^2 \in 2\mathbb{Z}_2$, it follows that $a^2 \equiv 1 \mod 8$.

Conversely, if $b \equiv 1 \mod 8$, then set $f(x) = x^2 - b$. For each $a \equiv \pm 1 \mod 4$, the conditions for Hensel's Lemma are satisfied, hence there is a unique root b satisfying the condition. □

Proposition 5.1.3. $\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2} \cong \{1, -1, 2, -2, 5, -5, 10, -10\}$.

This means that we have a group generated by $\epsilon = -1, \varpi = 2$ but also $\epsilon' = 5$. As such, we need to rethink how exactly it is that we should proceed with the classification. To begin, we will look at what possible value sets we can get. We will consider the two-dimensional quadratic forms $x^2 + \alpha y^2$, where $\alpha \in \{1, -1, 2, -2, 5, -5, 10, -10\}$. Immediately, it is clear that $x^2 - y^2 \sim H$, so there are really 7 options to check. In fact, there are 7 order two subgroups of $(\mathbb{Z}/2\mathbb{Z})^3$, each of which corresponds to one of these as can be seen in the following Lemma.

Lemma 5.1.4. *The Value sets of 2-dimensional quadratic forms* $\begin{bmatrix} 1 & \\ & \alpha \end{bmatrix}$ *are as follows.*

α	*Val*$(\begin{bmatrix} 1 & \\ & \alpha \end{bmatrix})$
1	$\{1, 2, 5, 10\}$
-1	*isotropic*
2	$\{1, 2, -5, -10\}$
-2	$\{1, -1, 2, -2\}$
5	$\{1, -2, 5, -10\}$
-5	$\{1, -1, 5, -5\}$
10	$\{1, -2, -5, 10\}$
-10	$\{1, -1, 10, -10\}$.

How to Compute this table. The computations of this table come from Hensel's Lemma. Suppose we wish to compute Val$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = \{x^2 + y^2 \mid (x, y) \neq (0, 0)\}/\mathbb{Q}_2^2$. Immediately it is clear, all $1, 2, 5, 10$ are the sum of two squares. Thus, the value set will contain $\{1, 2, 5, 10\}$. One can show that other values are not obtainable with Hensel's Lemma. □

Suppose that we have a quadratic form Q for which $\mathrm{Val}(Q) = \{1, \alpha, \beta, \alpha\beta\}$, then if we'd like $\gamma \otimes Q \sim Q$, then it ought to be true that $\gamma \cdot \{1, \alpha, \beta, \alpha\beta\} \cong \{1, \alpha, \beta, \alpha\beta\}$. That is, $\mathrm{Val}(Q) = \mathrm{Val}(\gamma \otimes Q)$. As a result, if our hope that an analog of Lemma 3.2.2 would hold, the following should hold.

Proposition 5.1.5. *Let Q be a quadratic form of dimension two. Then for any γ that permutes Val(Q) upon multiplication, $\gamma \otimes Q \sim Q$.*

Proof. We may assume that $\mathrm{Val}(Q)$ is a group, since otherwise $\mathrm{Val}(\gamma \otimes Q)$ will be a group, and this case is exactly similar. Suppose that $\mathrm{Val}(Q) = \{1, \alpha, \beta, \alpha\beta\}$, such that $Q = \begin{bmatrix} 1 & \\ & \alpha \end{bmatrix}$. Then, by Proposition 3.1.1 it follows that $\mathrm{Val}(\gamma \otimes Q) = \{\gamma, \alpha\gamma, \beta\gamma, \alpha\beta\gamma\}$. Since $\mathrm{Val}(Q)$ is a group, the only γ that satisfy our hypothesis are those in $\mathrm{Val}(Q)$. The result is obvious for $\gamma \in \{1, \alpha\}$, consider $\beta \otimes Q$. As such, we wish to check whether or not

$$\begin{bmatrix} 1 & \\ & \alpha \end{bmatrix} \sim \begin{bmatrix} \beta & \\ & \alpha\beta \end{bmatrix}.$$

We search for a, b, c, d such that

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & \\ & \alpha \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} &= \begin{bmatrix} a^2 + b^2\alpha & ac + bd\alpha \\ ac + bd\alpha & c^2 + d^2\alpha \end{bmatrix} \\ &= \begin{bmatrix} \beta & \\ & \alpha\beta \end{bmatrix}. \end{aligned}$$

By our assumption, there exists a vector (a, b) such that $Q(a, b) = \beta$, then $Q(-b\alpha, a) = b^2\alpha^2 + \alpha a^2 = \alpha(a^2 + \alpha b^2) = \alpha\beta$, which would imply that there exists a matrix such that this is true. □

Thus, we are sure that when Q is of dimension 2 and anisotropic, $Q \sim Q' \iff \mathrm{Val}(Q) = \mathrm{Val}(Q')$. We may conclude that there are 14 anisotropic, and 1 isotropic quadratic forms of dimension 2.

Proposition 5.1.6. *The complete classification of 2-dimensional anisotropic quadratic forms over $\mathbb{Q}_2$ is*

Value Set	*Anisotropic Quadratic Forms*
$\{1,2,5,10\}$	$\begin{bmatrix}1 & \\ & 1\end{bmatrix} \sim \begin{bmatrix}2 & \\ & 2\end{bmatrix} \sim \begin{bmatrix}5 & \\ & 5\end{bmatrix} \sim \begin{bmatrix}10 & \\ & 10\end{bmatrix}$
$\{1,2,-5,-10\}$	$\begin{bmatrix}1 & \\ & 2\end{bmatrix} \sim \begin{bmatrix}2 & \\ & 1\end{bmatrix} \sim \begin{bmatrix}-5 & \\ & -10\end{bmatrix} \sim \begin{bmatrix}-10 & \\ & -5\end{bmatrix}$
$\{1,-2,5,-10\}$	$\begin{bmatrix}1 & \\ & 5\end{bmatrix} \sim \begin{bmatrix}-2 & \\ & -10\end{bmatrix} \sim \begin{bmatrix}5 & \\ & 1\end{bmatrix} \sim \begin{bmatrix}-10 & \\ & -2\end{bmatrix}$
$\{1,-2,-5,10\}$	$\begin{bmatrix}1 & \\ & 10\end{bmatrix} \sim \begin{bmatrix}-2 & \\ & -5\end{bmatrix} \sim \begin{bmatrix}-5 & \\ & -2\end{bmatrix} \sim \begin{bmatrix}10 & \\ & 1\end{bmatrix}$
$\{1,2,-1,-2\}$	$\begin{bmatrix}1 & \\ & -2\end{bmatrix} \sim \begin{bmatrix}2 & \\ & -1\end{bmatrix} \sim \begin{bmatrix}-1 & \\ & 2\end{bmatrix} \sim \begin{bmatrix}-2 & \\ & 1\end{bmatrix}$
$\{1,5,-1,-5\}$	$\begin{bmatrix}1 & \\ & -5\end{bmatrix} \sim \begin{bmatrix}-5 & \\ & 1\end{bmatrix} \sim \begin{bmatrix}-1 & \\ & 5\end{bmatrix} \sim \begin{bmatrix}5 & \\ & -1\end{bmatrix}$
$\{1,10,-1,-10\}$	$\begin{bmatrix}1 & \\ & -10\end{bmatrix} \sim \begin{bmatrix}-10 & \\ & 1\end{bmatrix} \sim \begin{bmatrix}-1 & \\ & 10\end{bmatrix} \sim \begin{bmatrix}10 & \\ & -1\end{bmatrix}$
$\{-1,-2,-5,-10\}$	$\begin{bmatrix}-1 & \\ & -1\end{bmatrix} \sim \begin{bmatrix}-2 & \\ & -2\end{bmatrix} \sim \begin{bmatrix}-5 & \\ & -5\end{bmatrix} \sim \begin{bmatrix}-10 & \\ & -10\end{bmatrix}$
$\{-1,-2,5,10\}$	$\begin{bmatrix}-1 & \\ & -2\end{bmatrix} \sim \begin{bmatrix}-2 & \\ & -1\end{bmatrix} \sim \begin{bmatrix}5 & \\ & 10\end{bmatrix} \sim \begin{bmatrix}10 & \\ & 5\end{bmatrix}$
$\{-1,2,-5,10\}$	$\begin{bmatrix}-1 & \\ & -5\end{bmatrix} \sim \begin{bmatrix}-5 & \\ & -1\end{bmatrix} \sim \begin{bmatrix}2 & \\ & 10\end{bmatrix} \sim \begin{bmatrix}10 & \\ & 2\end{bmatrix}$
$\{-1,2,5,-10\}$	$\begin{bmatrix}-1 & \\ & -10\end{bmatrix} \sim \begin{bmatrix}-10 & \\ & -1\end{bmatrix} \sim \begin{bmatrix}2 & \\ & 5\end{bmatrix} \sim \begin{bmatrix}5 & \\ & 2\end{bmatrix}$
$\{5,10,-5,-10\}$	$\begin{bmatrix}5 & \\ & -10\end{bmatrix} \sim \begin{bmatrix}10 & \\ & -5\end{bmatrix} \sim \begin{bmatrix}-5 & \\ & 10\end{bmatrix} \sim \begin{bmatrix}-10 & \\ & 5\end{bmatrix}$
$\{2,10,-2,-10\}$	$\begin{bmatrix}2 & \\ & -10\end{bmatrix} \sim \begin{bmatrix}-10 & \\ & 2\end{bmatrix} \sim \begin{bmatrix}-2 & \\ & 10\end{bmatrix} \sim \begin{bmatrix}10 & \\ & -2\end{bmatrix}$
$\{2,5,-2,-5\}$	$\begin{bmatrix}2 & \\ & -5\end{bmatrix} \sim \begin{bmatrix}-5 & \\ & 2\end{bmatrix} \sim \begin{bmatrix}-2 & \\ & 5\end{bmatrix} \sim \begin{bmatrix}5 & \\ & -2\end{bmatrix}.$

When considering dimension three, we must be more careful than when p was odd. Since every two-dimensional quadratic form is equivalent to a quadratic form that is not a permutation of its entries, there will be many opportunities for the hyperbolic plane to hide. For example, $\begin{bmatrix}1 & & \\ & 2 & \\ & & 5\end{bmatrix}$ is isotropic since $1+2+5=8\equiv 0 \mod 8$. The underlying issue is that since -5 is in the value set of $\begin{bmatrix}1 & \\ & 2\end{bmatrix}$, once one adds 5, it is possible to get 0 non-trivially. In general, once $\alpha \in \mathrm{Val}(Q)$, one has that $Q \oplus [-\alpha]$ is isotropic. This turns out to be helpful since there are clear patterns that can be seen in the value sets of two-dimensional anisotropic forms. In fact, the converse of this is true

Proposition 5.1.7. *$Q + [-\alpha]$ is isotropic if and only if $\alpha \in Val(Q)$.*

Proof. ($\implies$) Suppose $Q + [-\alpha]$ is isotropic. Then, there exists a vector $x = (x_0, x_1)$ such that $(Q + [-\alpha])(x) = Q(x_0) - \alpha x_1^2 = 0$. This would mean that $Q(x_0) = \alpha x_1^2$, which means that $\alpha \in \mathrm{Val}(Q)$.

($\Longleftarrow$) If $\alpha \in \mathrm{Val}(Q)$, obviously $Q + [-\alpha]$ is isotropic. $\square$

Example 5.1.8. $\mathrm{Val}(\begin{bmatrix} 1 & & \\ & 2 & \\ & & -5 \end{bmatrix}) = \{1, -1, 2, -2, 5, -5, -10\}$. One may compute this by taking

$$\bigcup_{1 \leq i \leq 3} \bigcup_{\substack{\alpha \in \mathrm{Val}(\begin{smallmatrix} \xi_j & 0 \\ 0 & \xi_k \end{smallmatrix}) \\ [i,j,k]=[1,2,3]}} \mathrm{Val}(\begin{bmatrix} \xi_i & \\ & \alpha \end{bmatrix}).$$

It is shocking as to why there is no 10 in this value set.

Proposition 5.1.9. *Let $Q \sim \begin{bmatrix} \xi_1 & & \\ & \xi_2 & \\ & & \xi_3 \end{bmatrix}$. Then*

$$Val(Q) = \bigcup_{\substack{\alpha \in Val(\begin{smallmatrix} \xi_j & 0 \\ 0 & \xi_k \end{smallmatrix}) \\ [i,j,k]=[1,2,3]}} Val(\begin{bmatrix} \xi_1 & \\ & \alpha \end{bmatrix}).$$

Proof. If $\alpha \in \mathrm{Val}(\begin{bmatrix} \xi_2 & \\ & \xi_3 \end{bmatrix})$, then $\alpha = \xi_2 v^2 + \xi_3 w^2$, for $(v, w) \neq 0$. So, the union of $\mathrm{Val}(\begin{bmatrix} \xi_1 & \\ & \alpha \end{bmatrix})$, for all such α will be the set.

$$\bigcup_{\substack{\alpha \in \mathrm{Val}(\begin{smallmatrix} \xi_i & 0 \\ 0 & \xi_j \end{smallmatrix}) \\ [i,j,k]=[1,2,3]}} \mathrm{Val}(\begin{bmatrix} \xi_1 & \\ & \alpha \end{bmatrix}) = \{\xi_1 x^2 + \alpha y^2 : (x, y) \in V \setminus \{0\}\}/F^{\times 2}$$

$$\begin{aligned} &= \{\xi_1 x^2 + (\xi_2 v^2 + \xi_3 w^2) y^2 : (x, yv, yw) \in V \setminus \{0\}\}/F^{\times 2} \\ &= \{\xi_1 x^2 + \xi_2 (yv)^2 + \xi_3 (yw)^2 : (x, yv, yw) \in V \setminus \{0\}\}/F^{\times 2} \\ &= \mathrm{Val}(Q). \end{aligned}$$

$\square$

Proposition 5.1.10. *If $\beta, \gamma \neq 1$ and $\beta, \gamma \in \{2, 5, 10\}$, then $Q = \begin{bmatrix} 1 & \\ & \beta \end{bmatrix} \sim \begin{bmatrix} -\gamma & \\ & -\beta\gamma \end{bmatrix}$.*

Proof. We know that $\mathrm{Val}(Q) = \{1, \beta, -\gamma, -\beta\gamma\}$. Thus, there exists a vector v such that $Q(v) = -\gamma$. If we look at $w \in \{v\}^{\perp}$ then $Q(w) \in \mathrm{Val}(Q)$. We then consider the following quadratic forms for equivalence.

$$\begin{bmatrix} 1 & \\ & -\gamma \end{bmatrix}, \begin{bmatrix} \beta & \\ & -\gamma \end{bmatrix}, \begin{bmatrix} -\gamma & \\ & -\gamma \end{bmatrix}, \begin{bmatrix} -\gamma & \\ & -\beta\gamma \end{bmatrix}.$$

Witt's cancellation theorem rejects the first two candidates. We have that $-1 \in \mathrm{Val}(\begin{bmatrix} -\gamma & \\ & -\gamma \end{bmatrix})$, but not in $\mathrm{Val}(Q)$, which rejects it as a candidate for equivalence. As a result, $Q = \begin{bmatrix} 1 & \\ & \beta \end{bmatrix} \sim \begin{bmatrix} -\gamma & \\ & -\beta\gamma \end{bmatrix}$. □

Lemma 5.1.11. *Let $\{\alpha, \beta, \gamma\} \subset \{1, 2, 5, 10\}$ or $\{\alpha, \beta, \gamma\} \subset \{-1, -2, -5, -10\}$ be distinct square class representatives. Then $\begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix}$ is isotropic.*

Proof. We consider $\begin{bmatrix} 1 & & \\ & \beta' & \\ & & \gamma' \end{bmatrix}$, since we can simply tensor by α and recover the original form as a form remains isotropic under scaling. But, we also have that $\begin{bmatrix} \beta' & \\ & \gamma' \end{bmatrix} \sim \begin{bmatrix} -1 & \\ & \alpha' \end{bmatrix}$, by Proposition 5.1.10. Thus, we have shown what we desire. The case of $\alpha, \beta, \gamma \subset \{-1, -2, -5, -10\}$ follows since a form remains isotropic under scaling. □

Proposition 5.1.12. *There are 8 anisotropic quadratic forms of dimension 3 over $\mathbb{Q}_2$, up to equivalence. Their classification, up to representatives, is as follows*

Value Set	*Anisotropic Quadratic Form*
$\{1, 2, 5, 10, -2, -5, -10\}$	$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$
$\{1, 2, 5, 10, -1, -5, -10\}$	$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}$
$\{1, 2, 5, 10, -1, -2, -10\}$	$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 5 \end{bmatrix}$
$\{1, 2, 5, 10, -1, -2, -5\}$	$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 10 \end{bmatrix}$
$\{2, 5, 10, -1, -2, -5, -10\}$	$\begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix}$
$\{1, 5, 10, -1, -2, -5, -10\}$	$\begin{bmatrix} -1 & & \\ & -1 & \\ & & -2 \end{bmatrix}$
$\{1, 2, 10, -1, -2, -5, -10\}$	$\begin{bmatrix} -1 & & \\ & -1 & \\ & & -5 \end{bmatrix}$

$\{1,2,5,-1,-2,-5,-10\}$	$\begin{bmatrix}-1 & & \\ & -1 & \\ & & -10\end{bmatrix}$

Proof. To generate the eight 3-dimensional forms, we used Proposition 5.1.7 and added to $\begin{bmatrix}1 & \\ & 1\end{bmatrix}$ and $\begin{bmatrix}-1 & \\ & -1\end{bmatrix}$, elements whose negatives are not in the respective value sets. We are sure that each of these is non-equivalent since they have distinct value sets. What is left to show is that any other 3-dimensional anisotropic quadratic form over $\mathbb{Q}_2$, is equivalent to one of these 8 forms.

We will show this combinatorially. Suppose that $\alpha, \beta, \gamma \in \{1,2,5,10\}$. Then we have the following three cases to consider.

(i) **Two are equal.** Immediately, Q will either be isotropic or equivalent to one of the above, depending on whether or not the signs agree.

(ii) **All distinct, but signs are the same.** This is Lemma 5.1.11.

(iii) **All distinct, with two of the same sign and one of the opposite.** In this case, we want to consider $Q = \begin{bmatrix}\alpha & & \\ & \beta & \\ & & -\gamma\end{bmatrix}$. Note that $Q = [\alpha] \otimes Q' = [\alpha] \otimes \begin{bmatrix}1 & & \\ & \alpha\beta & \\ & & -\alpha\gamma\end{bmatrix}$. By Proposition 5.1.10, we have that $Q' = \begin{bmatrix}1 & & \\ & \alpha\beta & \\ & & -\alpha\gamma\end{bmatrix} \sim \begin{bmatrix}-\alpha\gamma & & \\ & -\beta & \\ & & -\alpha\gamma\end{bmatrix} \sim \begin{bmatrix}-1 & & \\ & -1 & \\ & & -\alpha\beta\end{bmatrix} \sim \begin{bmatrix}-\alpha & & \\ & -\alpha & \\ & & -\alpha\beta\end{bmatrix}$; thus $Q = \alpha \otimes Q'$ is on our list, as desired.

□

Corollary 5.1.13. *Every 3-dimensional anisotropic quadratic form Q has $|Val(Q)| = 7$. Moreover,*

$$Val(Q) = Val(\begin{bmatrix}\alpha & & \\ & \beta & \\ & & \gamma\end{bmatrix}) = \{1,-1,2,-2,5,-5,10,-10\} \setminus \{-\alpha\beta\gamma\}.$$

Proof. This follows from Proposition 5.1.12 as tensoring will scale the determinant, up to squares, the same as the value set. □

Proposition 5.1.14. *There is one 4-dimensional anisotropic quadratic form, up to equivalence.*

Proof. We consider $Q = \begin{bmatrix}\alpha & & & \\ & \beta & & \\ & & \gamma & \\ & & & \delta\end{bmatrix}$. We must have that $\delta = 1$, (all -1's will be equivalent to this form, by the table in Proposition 5.1.12, so we omit that case) otherwise we run out of representatives that keep Q anisotropic. As a result, we consider $\begin{bmatrix} 1 & & & \\ & \alpha & & \\ & & \beta & \\ & & & \gamma \end{bmatrix}$. Since $\begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix}$ is anisotropic, we can write

$$Q = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \delta \end{bmatrix},$$

as any other form when choosing from our preferred list from Proposition 5.1.12 will lead to a copy of the hyperbolic plane. Furthermore, we know that $\text{Val} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = \{1, 2, 5, 10, -2, -5, -10\}$, which implies, by Proposition 5.1.7, that $\delta = 1$. ☐

With this, we can now conclude the final results about the classification over a finite degree field extension of $\mathbb{Q}_2$.

Proposition 5.1.15. *There are no anisotropic quadratic forms of dimension* 5 *or higher over* $\mathbb{Q}_2$

Proof. Suppose such a form, Q exists. Then it must be of the form $Q_4 + [\alpha]$, where Q_4 is the 4-dimentional anisotropic quadratic form. Since

$$\text{Val}(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}) = \{1, -1, 2, -2, 5, -5, 10, -10\},$$

any α such that $Q = Q' \oplus [\alpha]$ will be isotropic by Proposition 5.1.7, where Q' is the 4-dimensional anisotropic form. ☐

Finally, we may give a characterization over any p, not just odd or even. The following two results will hold for any finite p.

Proposition 5.1.16. *The size of the value set of an anisotropic quadratic form uniquely defines its dimension over a p-adic field.*

Lemma 5.1.17 (The Classification of Anisotropic Forms over $\mathbb{Q}_p$)**.** *Let $F = \mathbb{Q}_p$ for a finite prime p. Suppose that Q and Q' are anisotropic quadratic forms over F. Then*

$$Q \sim Q' \iff \mathit{Val}(Q) = \mathit{Val}(Q').$$

5.2. **Quadratic forms over $\mathbb{R}$.** The theory of quadratic forms over $\mathbb{R}$ is more rigid than that of $\mathbb{Q}_p$, for finite p. This is largely due to how positive and negative numbers play out in this case. In Proposition 1.2.1, we showed that any non-degenerate quadratic form $Q(x)$ can be reduced to the diagonal form

$$Q(x) = l_1\tau_1^2 + \ldots + l_n\tau_n^2,$$

with $l_i \in F^{\times}$. It turns out that changing that basis of $Q(x)$ does not affect the number of positive and negative coefficients.

Theorem 5.2.1 (Sylvester's Law of Inertia)**.** *If a quadratic form $Q(x)$ in a real space is written in diagonal form, the number of positive and negative entries is an invariant of the form.*

In fact, due to this theorem and one more result, it will be immediately apparent how every anisotropic quadratic form ought to be classified. Since we are once again concerned with square classes, $\mathbb{R}^{\times}/\mathbb{R}^{\times 2}$ will serve to be very useful.

Proposition 5.2.2. $\mathbb{R}^{\times}/\mathbb{R}^{\times 2} \cong \{1, -1\}$.

Since the only entries in the diagonal we really have to care about are 1 and -1 and the number of positives and negatives is an invariant for any quadratic form, it is immediately apparent what the anisotropic quadratic forms over $\mathbb{R}$ are.

Proposition 5.2.3. *Over $\mathbb{R}$, there are two anisotropic forms of dimension $n \geq 1$, for each n. They are*

$$I_n = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \quad \textit{and} \quad -I_n = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix}.$$

Proof. We begin by showing these two are anisotropic (it is clear they are not equivalent by Theorem 5.2.1). Suppose that $x \in F^{\times n}$ is a real vector. Then $I_n(x) = x_1^2 + \ldots + x_n^2 > 0$ since we are excluding the zero vector. Similarly, $-I_n(x) = -(x_1^2 + \ldots + x_n^2) < 0$. Thus, there are no nonzero vectors, x, for which these forms are 0.

Suppose Q is a quadratic form of n dimensions and that its diagonal form has $l_i = 1$ and $l_j = -1$ for $i \neq j \in \{1, \ldots, n\}$, then $Q(e_i + e_j) = 0$, which means that Q is isotropic. □

In fact, every isotropic form can be characterized by the number of positive and negative entries in its diagonal. In order to aid us we define the following.

Definition 5.2.4. Let $Q_{k,n}$ with $0 \leq k \leq n$, be the non degenerate n-dimensional quadratic form

$$Q_{k,n} := I_k \oplus -I_{(n-k)}.$$

Then, by Proposition 5.2.3, the anisotropic forms are all of the forms $Q_{0,n}$ or $Q_{n,n}$ for $n \in \mathbb{N}$, and all the other quadratic forms over a real linear space will be anisotropic. In fact,

$$Q_{k,n} = \begin{cases} kH \oplus Q_{0,n-k} & \text{if } k < n/2, \\ (n-k)H \oplus Q_{k,0} & \text{if } k > n/2, \\ kH & \text{if } k = n/2, \end{cases} \tag{4}$$

which mimics the decomposition discussed in Corollary 1.3.7. This formula comes from the fact that since $H = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$, as soon as we can "find a pair $(1, -1)$" in the diagonal of Q, it can be replaced with a copy of H. The interesting thing about $\mathbb{R}$ is that we now have found an explicit way to compute the anisotropic kernel of any quadratic form, as seen in (4).

Definition 5.2.5. We define the signum for a real quadratic form Q to be $sgn(Q) = Tr(Q_{\text{aniso}})$, where Q_{aniso} is the anisotropic kernel of Q.

Proposition 5.2.6. *$sgn : W(\mathbb{R}) \to \mathbb{Z}$ is an isomorphism from the Witt group over the real numbers to the integers.*

5.3. **Classification over $\mathbb{Q}$: The Hasse-Minkowski Theorem.** Since the p-adic numbers (for p finite of infinite) are derived from $\mathbb{Q}$, it is only natural that there exists a local-global property between them: a property which connects every $\mathbb{Q}_p$ to $\mathbb{Q}$. In this case, such a property is called the Hasse-Minkowski theorem. In this section, we briefly introduce the theorem and the uses we have for it.

The heart of the Hasse-Minkowski theorem lies in the notion that, given a quadratic form Q, over $\mathbb{Q}$, by looking at all $\mathbb{Q}_p$, one can conclude whether or not Q is isotropic over $\mathbb{Q}$. To learn about this mysterious theorem, the interested reader may consult [Gam06].

Theorem 5.3.1 (Hasse-Minkowski 1)**.** *A quadratic form Q is isotropic over $\mathbb{Q}$ if and only if it is isotropic over all $\mathbb{Q}_p$ for p finite and infinite.*

Proof. [Gam06, Section 4] □

Since we have the classification of quadratic forms over $\mathbb{Q}_p$, using the Hasse-Minkowski theorem, we may immediately determine if a quadratic form is isotropic or not. In fact, with a few more results other forms of this theorem will serve as our classification over $\mathbb{Q}$.

Theorem 5.3.2 (Hasse-Minkowski 2)**.** *Let $\alpha \in \mathbb{Q}$ and Q be a quadratic form over $\mathbb{Q}$. Then $\alpha \in Val(Q)$ if and only if $\alpha \in Val(Q)$ over $\mathbb{Q}_p$ for all p both finite and infinite.*

Proof. By Proposition 5.1.7, $Q + [-\alpha]$ is isotropic if and only if $\alpha \in \text{Val}(Q)$. If $Q + [-\alpha]$ is isotropic over all $\mathbb{Q}_p$, then by Theorem 5.3.1, $Q + [-\alpha]$ is isotropic. Thus, $\alpha \in \text{Val}(Q)$ over $\mathbb{Q}$. □

The most powerful version of the Hasse-Minkowski theorem is yet to come. It will truly give us a classification over $\mathbb{Q}$ for free, now that we have found the classifications over $\mathbb{Q}_2$ and $\mathbb{R}$, as well as $\mathbb{Q}_p$, p odd.

Theorem 5.3.3 (Hasse-Minkowski 3)**.** *Let Q and W be quadratic forms over $\mathbb{Q}$. Then $Q \sim W$ if and only if $Q \sim W$ over $\mathbb{Q}_p$ for all p both finite and infinite.*

Proof. ($\implies$) Suppose there exists a matrix P such that $P^T Q P = W$, then the p-adic matrix of P will satisfy $P_p^T Q_p P_p = W_p$, for all p, as $\mathbb{Q}$ is contained in $\mathbb{Q}_p$.

($\impliedby$) Suppose that $Q_p \sim W_p$ for all p and that Q over $\mathbb{Q}$ has $\alpha \in \text{Val}(Q)$. Then,

$$Q + [-\alpha] = H + Q'.$$

Since $Q + [-\alpha]$ is isotropic and $Q_p \sim W_p$ if follows that $W + [-\alpha]$ is also isotropic. By Hasse-Minkowski 1, it then follows that $W + [-\alpha]$ is isotropic over $\mathbb{Q}$. Which means that $W + [-\alpha] = H + W'$.

We finish the proof by induction on dimension. Suppose that $\dim Q - \dim Q' = 1$ and $\dim W - \dim W' = 1$. Since $Q_p \sim W_p$ and $Q_p + [-\alpha] \sim W_p + [-\alpha]$, it is clear by the induction hypothesis that $Q'_p \sim W'_p$, by Witt's cancellation theorem. But this would mean $Q' \sim W'$ over $\mathbb{Q}$. Hence, by Witt's cancellation theorem, $Q + [-\alpha] \sim H + Q' \sim H + W' \sim W + [-\alpha]$, which would mean $Q \sim W$ over $\mathbb{Q}$. □

Example 5.3.4. The Hasse-Minkowski theorem really only works for quadratic forms. The most famous example is the Selmer example that shows $3x^3 + 4y^3 + 5z^3$ represents zero non-trivially over every $\mathbb{Q}_p$, for p both finite and infinite, but not over $\mathbb{Q}$.

With this, we have a complete classification of not only the p-adic quadratic forms but also the ones over $\mathbb{Q}$. This concludes our classification of p-adic and rational quadratic forms.

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