Summer Report

Alex McSween

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0.1 Introduction

This report serves as a summary of my work from May to August 2017. The main reference and the source of most of the exercises proven here is *Reflection Groups and Coxeter Groups* by James E. Humphreys.

Chapter 1

Finite Reflection Groups

1.1 Definitions

Reflection Let V be a Euclidean space and let $\alpha \in V$ be a nonzero vector. Define a linear operator s_α by $s_\alpha(v) = v - 2\frac{(\alpha,v)}{(\alpha,\alpha)}\alpha$. Then s_α is a reflection and it follows that

- $s_\alpha H_\alpha = H_\alpha$ where $H_\alpha = \{v \in V : (\alpha, v) = 0\}$ is the hyperplane orthogonal to α
- $s_\alpha(\alpha) = -\alpha$.

It can also be verified that s_α is an isometry.

Root System Let $\Phi \subset V$ be a finite set of nonzero vectors. If for all $\alpha \in \Phi$ we have

R1 $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$

R2 $s_\alpha \Phi = \Phi$

then Φ is a root system and W is the associated reflection group generated by all s_α.

Positive System Let $\Pi \subset \Phi$ be a set of positive vectors relative to some total ordering of V. By R1 it is clear $\Phi = \Pi \sqcup -\Pi$.

To simplify notation, for the remainder of this report, given a root $\alpha_i \in V$, we will denote the reflection defined by this root by s_i.

Simple System Let $\{\beta_1, \ldots, \beta_n\} = \Delta \subseteq \Pi \subset \Phi$ be such that

S1 Δ is a basis of Φ

S2 $\forall \alpha \in \Phi$ $\alpha = \sum_{i=1}^{n} c_i \beta_i$ where the c_i are all nonnegative or all nonpositive.

Then Δ is a simple system. In fact, we have that simple systems exist and $W = \langle s_i : \beta_i \in \Delta \rangle$. See Theorem 1.9 in *Reflection Groups and Coxeter Groups* for a proof.

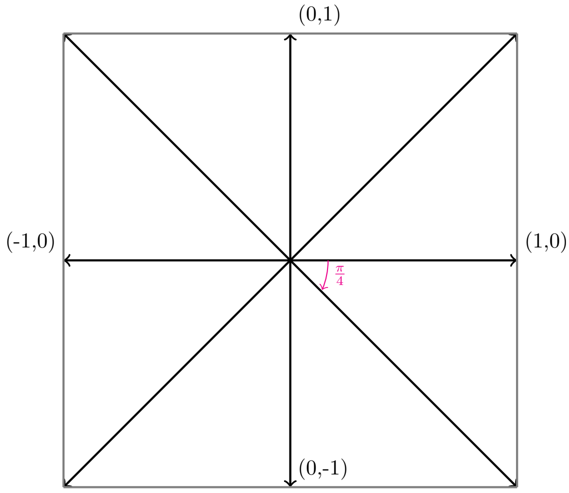
Essential Let V be a Euclidean space and W a reflection group acting on V with no nonzero fixed points. Then W is essential.

Rank Let Φ be a root system and Δ a simple system for W. The rank of Φ is the cardinality of Δ.

1.2 Some Exercises

Exercise 2, section 1.3 Find a simple system of D_4, the dihedral group with 8 elements.

Proof. To find a root system Φ of $W = D_4$ we must first find a set of invariant points of $\mathbb{R}^2$.



Recalling that D_4 is the set of symmetries of the square, it becomes clear that $\Phi = \{\pm(1,0), \pm(0,1), \pm(1,1), \pm(-1,1)\}$ is a set of invariant points. We may now check the root system axioms with our candidate for Φ.

R1 It can be clearly seen from the diagram that $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$.

R2 Consider $s_1(v) = s_{(1,0)}(v) = v - 2(v \cdot (1,0))(1,0) = (v_1, v_2) - 2(v_1, 0) = (-v_1, v_2)$. The other s_{ii} can be found as follows $s_{(0,1)}(v_1, v_2) = (v_1, -v_2)$, $s_{(1,1)}(v_1, v_2) = (v_2, v_1)$, and $s_{(-1,1)}(v_1, v_2) = (-v_2, -v_1)$.

v	$(1,0)$	$(0,1)$	$(1,1)$	$(-1,1)$	$(-1,0)$	$(0,-1)$	$-(1,1)$	$(1,-1)$
$s_{(1,0)}(v)$	$(-1,0)$	$(0,1)$	$(-1,1)$	$(1,1)$	$(1,0)$	$(0,-1)$	$(1,-1)$	$-(1,1)$
$s_{(0,1)}(v)$	$(1,0)$	$(0,-1)$	$(1,-1)$	$-(1,1)$	$(-1,0)$	$(0,1)$	$(-1,1)$	$(1,1)$
$s_{(1,1)}(v)$	$(0,1)$	$(1,0)$	$(1,1)$	$(1,-1)$	$(0,-1)$	$(-1,0)$	$-(1,1)$	$(-1,1)$
$s_{(-1,1)}(v)$	$(0,-1)$	$(-1,0)$	$-(1,1)$	$(-1,1)$	$(0,1)$	$(1,0)$	$(1,1)$	$(1,-1)$

And so $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

We have shown Φ is a root system for D_4. It remains to identify a positive system and from that a simple system. Define a total ordering on V by $(x,y) < (x',y') \iff y < y' \vee (y = y' \wedge x < x')$. Then $(0,0) < (1,0), (0,1), (1,1), (-1,1)$ so $\Pi = \{(1,0), (0,1), (1,1), (-1,1)\}$ is a positive system. Since a simple system Δ must form a basis for $\Phi \subset \mathbb{R}^2$, we suspect our simple system will consist of two vectors. A natural choice would be the standard basis vectors $\{(1,0), (0,1)\}$ which satisfy S1. However, we have that $(-1,1) = -1 \cdot (1,0) + 1 \cdot (0,1)$ where clearly -1 and 1 do not have the same sign so this system will not do. This problem indicates that we will probably need to take one vector that already contains a negative such as $(-1,1)$. Indeed $\Delta = \{(1,0), (-1,1)\}$ span Φ so it remains to verify S2.

We have that $(0,1) = 1 \cdot (-1,1) + 1 \cdot (1,0)$, $(1,1) = 2 \cdot (1,0) + 1 \cdot (-1,1)$ and clearly $(1,0) = 1 \cdot (1,0) + 0 \cdot (-1,1)$ and $(-1,1) = 0 \cdot (1,0) + 1 \cdot (-1,1)$ where $1, 2,$ and 0 are all nonnegative. Note that $-\Pi$ can be achieved by simply using $-1, -2, 0$ instead. So S2 is satisfied so Δ is a simple system for D_4. □

Exercise 1, section 1.5 Let Φ be a root system of rank n consisting of unit vectors. If $\Psi \subset \Phi$ is a set of n roots whose mutual angles agree with those between the roots in some simple system, then Ψ must be a simple system.

Proof. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a simple system. Let $\Psi = \{\beta_1, \ldots, \beta_n\}$. We know the mutual angles of Ψ agree with those of Δ so $(\alpha_i, \alpha_j) = (\beta_i, \beta_j)$ for all $i, j \in \{1, \ldots, n\}$. To show Ψ is a simple system we must show that Ψ spans Φ and that for each $\gamma \in \Phi$, $\gamma = \sum_{i=1}^{n} c_i \beta_i$ where the c_i are all nonnegative or all nonpositive.

By Corollary 1.5, we know that there exists $x \in W$ such that $x\alpha_1 = \beta_1$. If $x\alpha_2 = \beta_2$ then we are done. Otherwise, take $H = \{v \in V : (v, x\alpha_2) = (v, \beta_2)\}$ then H is the hyperplane equidistant between $x\alpha_2$ and β_2 and moreover reflection in H interchanges the two. Note that $(\beta_1, x\alpha_2) = (x\alpha_1, x\alpha_2) = (\alpha_1, \alpha_2)$ since x is an isometry. Furthermore we have that $(\alpha_1, \alpha_2) = (\beta_1, \beta_2)$ by hypothesis. So $\beta_1 = x\alpha_1 \in H$ so R_H fixes β_1. So let $f = R_H x$. Then $f(\alpha_1) = R_H(\beta_1) = \beta_1$ and $f(\alpha_2) = R_H x(\alpha_2) = \beta_2$ so f sends $\{\alpha_1, \alpha_2\}$ to $\{\beta_1, \beta_2\}$.

Now we'll assume this process holds for m vectors and will show it is true for $m+1$ vectors. Suppose we have $g \in W$ such that $g\alpha_i = \beta_i \ \forall i \in \{1, \ldots, m\}$. If we have that $g\alpha_{m+1} = \beta_{m+1}$ we are done. If not, let $K = \{v \in V : (v, g\alpha_{m+1}) = (v, \beta_{m+1})\}$ then R_K interchanges $g\alpha_{m+1}$ and β_{m+1}. It remains to verify that the $g\alpha_i$ for $i = 1, \ldots, m$ are fixed by R_K. We have that $(\beta_i, g\alpha_{m+1}) = (g\alpha_i, g\alpha_{m+1}) = (\alpha_i, \alpha_{m+1}) = (\beta_i, \beta_{m+1})$ and so for all $i \in \{1, \ldots, m\}$ we have that $g\alpha_i \in K$. Therefore we have $T(\Delta) = R_K g(\Delta) = \Psi$. So, we have shown we can always find an isometry T such that $T\Delta = \Psi$. We may now verify the properties of a simple system.

If $T \in W$ then we are done by Corollary 1.5. Otherwise, by proposition 1.2, we know for any $T \in O(V)$, $Ts_{\alpha_i}T^{-1} = s_{T(\alpha_i)}$. From the above, we then have that $Ts_{\alpha_i} = s_{T(\alpha_i)}T = s_{\beta_i}T$. Extending this inductively to any $\gamma \in \Phi$ where $\gamma = s_{\alpha_{i_1}} \ldots s_{\alpha_{i_k}}(\alpha_j)$, we have that $T(\gamma) = T(s_{\alpha_{i_1}} \ldots s_{\alpha_{i_k}}(\alpha_j)) = s_{T(\alpha_{i_1})} \ldots s_{T(\alpha_{i_k})} T(\alpha_j) = s_{\beta_{i_1}} \ldots s_{\beta_{i_k}}(\beta_j) \in \Phi$. Therefore T preserves Φ.

S1 Since we have n vectors in both Δ and $T\Delta = \Psi$, it remains to verify that those vectors are linearly independent. We know that Δ is a linearly independent set so $\sum_{i=1}^{n} c_i \alpha_i = 0 \Rightarrow c_i = 0$. It then follows that $0 = \sum_{i=1}^{n} c_i \beta_i = \sum_{i=1}^{n} c_i T\alpha_i = T \sum_{i=1}^{n} c_i \alpha_i \Rightarrow c_i = 0$ so Ψ is also a linearly independent set of n vectors since T is an isometry.

S2 From the remark above we know that for all $\gamma \in \Phi$, $T\gamma \in \Phi$ and so clearly $T^{-1}\gamma \in \Phi$. Then $T^{-1}\gamma = \sum_{i=1}^{n} c_i \alpha_i$ where c_i are all nonnegative (since otherwise take $-c_i$). Now consider $\gamma = TT^{-1}\gamma = T \sum_{i=1}^{n} c_i \alpha_i = \sum_{i=1}^{n} c_i T\alpha_i = \sum_{i=1}^{n} c_i \beta_i$ where all c_i are nonnegative. Thus S2 is satisfied.

Therefore $\Psi = T\Delta$ is a simple system.

□

Exercise Let W be a reflection group in a Euclidean space V with $\dim V = n$. Let Φ be a root system with simple system $\Delta = \{\alpha_1, \ldots, \alpha_k\}$. Prove that W is essential if and only if $\mathrm{rank}\Phi = \dim V$

Proof. Let $v \in V$ be a nonzero vector such that $s_{\alpha_i}(v) = v$ for all $\alpha_i \in \Delta$. Then, $(\alpha_i, v) = 0$ for all $i \in \{1, \ldots, k\}$. Denote $\operatorname{span} \Delta$ by V'. Then $v \in V'^{\perp}$. Note that $V = V' \oplus V'^{\perp}$ from linear algebra. Since $v \neq 0$, we know $\dim V'^{\perp} > 0$. Therefore, $\operatorname{rank}\Phi = \dim V' < n = \dim V$.

Now, suppose that $\operatorname{rank}\Phi < \dim V = n$. Consider $V' = \operatorname{span} \Delta \subset V$ again. Clearly $\dim V' < \dim V$ but from linear algebra we know $V' \oplus V'^{\perp} = V$. It then follows that $\dim V'^{\perp} > 0$ and so there exists a non zero vector $v \in V'^{\perp}$. Thus $(\alpha_i, v) = 0$ for all $\alpha_i \in \Delta$ and so v is fixed by all reflections s_{α_i} and is therefore fixed by W, so W is not essential.

□

Theorem 1.8 Let Δ be a simple system, Π the corresponding positive system. The following conditions on $w \in W$ are equivalent:

(a) $w\Pi = \Pi$

(b) $w\Delta = \Delta$

(c) $n(w) = 0$

(d) $l(w) = 0$

(e) $w = 1$

Proof. $l(1) = 0$ by convention so e) $\Rightarrow$ d). We know $n(w) = l(w)$ so d) $\Rightarrow$ c). Since $n(w)$ is the number of positive roots sent to negative roots it is clear that $w\Pi = \Pi$ if no roots change sign and vice versa so c) $\iff a$). Finally if $w = 1$ it is clear that $w\Delta = 1 \cdot \Delta = \Delta$ so e)$\Rightarrow$ b).

□

Exercise 1, section 1.7 In the Exchange Condition, suppose $l(w) = r$. Prove that the index i in the conclusion is uniquely determined.

Proof. Let $w \in W$ such that $w = s_1 \ldots s_r$ where $l(w) = r$. Suppose there exists some simple reflection s such that $ws = s_1 \ldots \hat{s_i} \ldots s_r$ and $ws = s_1 \ldots \hat{s_j} \ldots s_r$ where $i \neq j$. Then multiplying on the right by $(s_{j+1} \ldots s_r)^{-1}$ we have $s_1 \ldots \hat{s_i} \ldots s_j = s_1 \ldots s_{j-1}$. Now multiplying on the left by $(s_1 \ldots s_{i-1})^{-1}$ we get $s_{i+1} \ldots s_j = s_i \ldots s_{j-1}$. This is exactly the conditions in Theorem 1.7 part (b). Therefore, it follows that $w = s_1 \ldots \hat{s_i} \ldots \hat{s_j} \ldots s_r$ (part (c) of theorem) but then $l(w) = r - 2 \neq r$. Therefore the index in the Exchange Condition must be unique.

□

Exercise 1, section 1.13 If $s_1, \ldots, s_r$ are distinct elements of S, then $l(s_1 \ldots s_r) = r$.

Proof. Let $w = s_1 \ldots s_r$. Suppose $l(w) = n(w) < r$, then by Theorem 1.7 there exist indices such that $s_{i+1} \ldots s_j = s_i \ldots s_{j-1}$, so $s_i = s_{i+1} \ldots s_j (s_{i+1} \ldots s_{j-1})^{-1}$ so s_i is not needed as a generator of W, a contradiction. So $l(w) = r$.

□

Chapter 2

Classification of Finite Reflection Groups

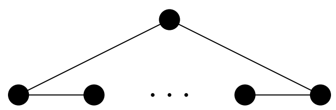
2.1 Definitions

Coxeter Graph Let Δ be a simple system for a Coxeter group W. Construct a graph Γ by associating a vertex with each simple root, and connecting those roots if the order of the product of simple reflections is 3 or greater. Denote this order by $m(\alpha, \beta)$. Unless the order is 3, label the edge with $m(\alpha, \beta)$.

Irreducible A Coxeter system is said to be irreducible if the corresponding Coxeter graph Γ is connected. A graph is connected if for every pair of vertices there exists a path between them.

Subgraph Given a graph Γ, a subgraph Γ' is obtained by omitting vertices and their adjacent edges and/or by decreasing the label on one or more edges. By convention, Γ is not a subgraph of itself.

Circuit Let Γ be a graph. Suppose there exists a finite trail of vertices v_i and distinct edges e_i such that each e_i connects v_i and v_{i+1} where $v_1 = v_n$ for some n. Then this trail is called a circuit. Below is the graph of $\widetilde{A_n}$ which contains a circuit.



Coxeter Matrix Let Γ be a Coxeter graph. Define an $n \times n$ symmetric matrix $A = (a_{ij})$ by

$$a_{ij} = -\cos \frac{\pi}{m(\alpha_i, \alpha_j)}.$$

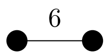
Then A is a Coxeter matrix.

Principal Minors Let A be a Coxeter matrix. The principal minors of A are the deteminants of the submatrices obtained by removing the last k rows and columns of A with $0 \leq k < n$.

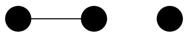
2.2 Some Exercises

Exercise 1, section 2.2 Let W be the dihedral group D_6 of order 12 with standard Coxeter generators $S = \{s, s'\}$, where S is the set of simple reflections. The Coxeter system (W, S) is irreducible. However, W has another set S' of Coxeter generators leading to a Coxeter system which is not irreducible: $S' := \{s, (s's)^3, s(s's)^2\}$.

Proof. It is clear the Coxeter system (W, S) is irreducible since the corresponding graph Γ is



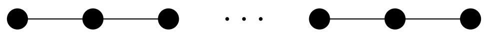
and is obviously connected. To find the corresponding graph Γ' for S' we must first find the relations between the elements. We have $(s(s's)^3)^n = (ss'ss'ss's)^n = (ss's)^n = 1 \Rightarrow n = 2$, $(ss(s's)^2)^n = ((s's)^2)^n = 1 \Rightarrow n = 3$ and $((s's)^3 s(s's)^2)^n = (s'ss'ss'sss'ss's)^n = (ss's)^n = 1 \Rightarrow n = 2$. So, for Γ' we have



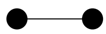
Which is clearly not a connected graph. Therefore (W, S') is not irreducible. $\square$

Principal minors of Coxeter Graphs (2.4) We will compute the principal minors of certain Coxeter graphs given in section 2.4 by calculating $\det 2A$ where A is the Coxeter matrix to show these graphs are positive definite.

Proof. Consider the Coxeter graph of A_n



Let $n = 2$ then we have

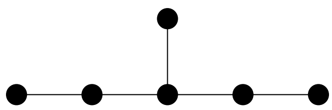


From the graph we know $m(\alpha_1, \alpha_2) = 3$ and $m(\alpha_i, \alpha_i) = 1$. It remains to calculate the cos values for our matrix. We have $-\cos(\frac{\pi}{3}) = -\frac{1}{2}$ and $-\cos(\frac{\pi}{1}) = 1$ so for our matrix A we have

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

so $\det 2A = 4 - 1 = 3 > 0$ so the graph of A_2 is positive definite.

Now consider the graph of E_6



As always, $m(\alpha_i, \alpha_i) = 1$. We can determine all of the remaining orders simply by looking at the graph. For α_1 we have α_1 we have $m(\alpha_1, \alpha_2) = 3$ and $m(\alpha_1, \alpha_3) = m(\alpha_1, \alpha_4) = m(\alpha_{,1}, \alpha_5) = m(\alpha_1, \alpha_6) = 2$. Now for s_2 we have $m(\alpha_2, \alpha_1) = m(\alpha_2, \alpha_3) = 3$, $m(\alpha_2, \alpha_4) = m(\alpha_2, \alpha_5) = m(\alpha_2, \alpha_6) = 2$. For α_3 we have $m(\alpha_3, \alpha_2) = m(\alpha_3, \alpha_4) = m(\alpha_3, \alpha_6) = 3$ and $m(\alpha_3, \alpha_1) = m(\alpha_3, \alpha_5) = 2$. For α_4 we have $m(\alpha_4, \alpha_1) = m(\alpha_4, \alpha_2) = m(\alpha_4, \alpha_6) = 2$ and $m(\alpha_4, \alpha_5) = m(\alpha_4, \alpha_3) = 3$. For α_5 we have $m(\alpha_5, \alpha_1) = m(\alpha_5, \alpha_2) = m(\alpha_5, \alpha_3) = m(\alpha_5, \alpha_6) = 2$ and $m(\alpha_5, \alpha_4) = 3$. Finally for $s\alpha_6$ we have $m(s\alpha_6, \alpha_3) = 3$ and $m(\alpha_6, \alpha_1) = m(\alpha_6, \alpha_2) = m(\alpha_6, \alpha_4) = m(\alpha_6, \alpha_5) = 2$. Using the above values we can construct our matrix $2A$.

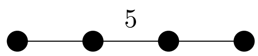
$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$$

and so $\det 2A = 3 > 0$ so the graph of E_6 is positive definite.

□

Nonpositive Coxeter Graphs (2.5) We will show that the Coxeter graph of Z_4 is not of positive type by calculating the principal minors.

Proof. Consider the graph of Z_4



We can calculate the orders and thus the matrix 2A as follows

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -\frac{1+\sqrt{5}}{2} & 0 \\ 0 & -\frac{1+\sqrt{5}}{2} & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

and so $\det 2A = 3 - 2\sqrt{5} < 0$ so Z_4 is nonpositive.

□

Section 2.12, Rotation Subgroup Let W be a finite reflection group. Then W has a normal subgroup W^+ of index 2 (the rotation subgroup, consisting of elements of determinant 1).

Proof. First we apply the subgroup test on $W^+ = \{x \in W : \det x = 1\}$. Let $x, y \in W^+$ then $\det x = 1 = \det y$. Consider $\det xy^{-1} = \det x \det y^{-1} = \det x(\det y)^{-1} = 1 \cdot 1 = 1$ so W^+ is a subgroup of W. It remains to verify it is normal and has index two.Let $x \in W^+$ and $w \in W$. Consider $\det wxw^{-1} = \det w \det x \det w^{-1} = \det w \cdot 1 \cdot (\det w)^{-1} = 1$ so $wW^+w^{-1} \subseteq W^+$ so W^+ is a normal subgroup of W. Since det is a homomorphism, we can apply the First Isomorphism Theorem. We have that $\ker \det = \{x \in W : \det x = 1\} = W^+$ and $\operatorname{Im} \det = \{1, -1\}$ so $W/W^+ \simeq \{1, -1\}$. Since we know W^+ is a normal subgroup, we can conclude that $2 = |W/W^+| = |W : W^+|$.

□

Chapter 3

Weyl Groups

3.1 Definitions

Crystallographic Group Let L be a lattice in V and $G \subseteq GL(V)$ be a subgroup. If G stabilizes L, meaning $gL \subset L$ for all $g \in G$, then G is said to be crystallographic. It is clear in this case $gL = L$ since G is a group.

Crystallographic root system Suppose that for all $\alpha_i, \alpha_j \in \Phi$ we have $\frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)} \in \mathbb{Z}$. Then Φ is crystallographic. These ratios are called Cartan integers.

If Φ is crystallographic, then W is crystallographic and preserves the lattice $L = \operatorname{span}_{\mathbb{Z}} \Delta$. In particular, $\Phi \subseteq \operatorname{span}_{\mathbb{Z}} \Delta$. Therefore we have $W \operatorname{span}_{\mathbb{Z}} \Delta = \operatorname{span}_{\mathbb{Z}} \Delta$. Therefore, it is enough that the above condition hold only for simple roots because it ensures $\operatorname{span}_{\mathbb{Z}} \Delta$ in V is $W-$stable.

Fundamental Domain Let $D \subseteq V$. If $\forall v \in V\ \exists w \in W$ such that wv is conjugate to exactly one point in D then D is a fundamental domain for the action of W. Our standard choice for D is $\{\lambda \in V : (\lambda, \alpha) \geq 0 \text{ for all } \alpha \in \Delta\}$.

Facet Let D be a fundamental domain. Let $I \subseteq \Delta$ and define $C_I = \{\lambda \in D : (\lambda, \alpha) = 0\ \forall \alpha \in I, (\lambda, \alpha) > 0\ \forall \alpha \in \Delta \setminus I\}$. Then the C_I partition D and are called facets of type I.

Conjugacy Let $x, y \in V$ be vectors. We say x is conjugate to y if and only if there exists $w \in W$ such that $wy = x$. The set $\{x \in V : \exists w \in W \text{such that } wy = x\}$ is the W-orbit of y.

Minimal Linear Space Let S be a subset of a Euclidean space V. If $X \supseteq S$ is a subspace of V and we have that for all subspaces $U \subseteq V$ containing S, $X \subseteq U$ then X is the minimal linear space of S and we write $X = \mathcal{L}(S)$.

Associativity Let $F_1, F_2 \subseteq V$ be facets. F_1 is associate to F_2 if and only if there exists $w \in W$ such that $w\mathcal{L}(F_1) = \mathcal{L}(F_2)$. We write $F_1 \sim F_2$.

3.2 Some Exercises

Section 1.15 The facets C_I partition the fundamental domain D.

Proof. It is clear from the definition these C_I are nonempty since Δ is a basis. Let $I \neq J \subseteq \Delta$ and if one is contained in the other let $I \subseteq J$. Then there exists $\alpha \in I$ such that $\alpha \notin J$. Let $\lambda \in C_I$ then $(\lambda, \alpha) = 0$ and so clearly $\lambda \notin C_J$ since we do not have $(\lambda, \alpha) > 0$ therefore $C_I \cap C_J = \emptyset$. Let $x \in \cup C_I$. Then, there exists some $I \subseteq \Delta$ such that $x \in C_I$. By the definition of D it is clear that $C_I \subseteq D$ and thus $x \in D$. Conversely, let $x \in D$. So we have that $(x, \alpha) \geq 0$ for all $\alpha \in \Delta$. Take $I = \{\alpha \in \Delta \colon (x, \alpha) = 0\}$ then $x \in C_I$ and so $x \in \cup C_I$. Thus $D = \cup C_I$. Therefore the facets C_I partition our fundamental domain D.

□

Associativity is an equivalence relation

Proof. Let $F_1, F_2, F_3 \subseteq V$ and let W be a group acting on V.

- Reflexive: Clearly if we take $w = 1$ then $w\mathcal{L}(F_1) = \mathcal{L}(F_1)$ so $F_1 \sim F_1$.
- Transitive: If $F_1 \sim F_2$ and $F_2 \sim F_3$ then there exist $w, w' \in W$ such that $w\mathcal{L}(F_1) = \mathcal{L}(F_2)$ and $w'\mathcal{L}(F_2) = \mathcal{L}(F_3)$ therefore $ww'\mathcal{L}(F_1) = \mathcal{L}(F_3)$ where $ww' \in W$ so $F_1 \sim F_3$.
- Symmetric: If $F_1 \sim F_2$ then $w\mathcal{L}(F_1) = \mathcal{L}(F_2)$. Apply $w^{-1} \in W$ to both sides to obtain $w^{-1}w\mathcal{L}(F_1) = \mathcal{L}(F_1) = w^{-1}\mathcal{L}(F_2)$ so $F_2 \sim F_1$.

Then $\sim$ is an equivalence relation. □

Vector representation of a facet Let C_I be a facet. Define the set P_I to be the set of indices of the simple roots in I. Consider the set $P = \{1, \ldots, n+1\} \setminus P_I = \{i_1, \ldots, i_\ell\}$ with $i_1 < \ldots < i_\ell$. Define $i_0 = 0$ and note that $i_\ell = n + 1$. Finally, construct a set of vectors $S_I = \{v_k = \epsilon_{i_{k-1}+1} + \ldots + \epsilon_{i_k} \colon k \in \{1, \ldots, \ell\}\}$.

Conversely, if given the set $S_I = \{v_k = \epsilon_{i_{k-1}+1} + \ldots + \epsilon_{i_k} \colon k \in \{1, \ldots, \ell\}\}$ of vectors formed from a partition of $\{1, \ldots, \ell\}$, we can reconstruct I. First, find the set $P = \{i_1, \ldots, i_\ell\}$ where $i_\ell = n + 1$ and note $P = \{1, \ldots, n+1\} \setminus P_I$ where P_I is the indexing set of I. From there, we can construct $I = \{\alpha_j \colon j \in P_I\}$.

Lemma 3.2.1. *Let C_I be a facet and let S_I as defined above, then $C_I = \{a_1 v_1 + \ldots + a_\ell v_\ell \colon a_1 > \ldots > a_\ell\}$.*

Proof. Let $A = \{a_1 v_1 + \ldots + a_\ell v_\ell \colon a_i > a_{i+1}\}$. Let $x \in A$. Then $x = a_1 v_1 + \ldots + a_\ell v_\ell = a_1(\epsilon_{i_0+1} + \ldots + \epsilon_{i_1}) + \ldots + a_\ell(\epsilon_{i_{\ell-1}+1} + \ldots + \epsilon_{i_\ell})$ with $a_1 > \ldots > a_\ell$. Now, take $\alpha_t \in I$ and compute $(x, \alpha_t) = (a_1(\epsilon_1 + \ldots + \epsilon_{i_1}) + \ldots + a_\ell(\epsilon_{i_{\ell-1}+1} + \ldots + \epsilon_{i_\ell}), \epsilon_t - \epsilon_{t+1})$. Since $t \notin P$, we know the coefficients in the tth and $t+1th$ positions will be the same. Thus, $(x, \alpha_t) = 0$. Now, let $\alpha_{i_j} \in \Delta \setminus I$. So $(x, \alpha_{i_j}) = (a_1(\epsilon_1 + \ldots + \epsilon_{i_1}) + \ldots + a_\ell(\epsilon_{i_{\ell-1}+1} + \ldots + \epsilon_{i_\ell}), \epsilon_{i_j} - \epsilon_{i_{j+1}})$. Due to the decreasing construction of these coefficients we have $(x, \alpha_{i_j}) = a_j - a_{j+1} > 0$. Therefore $x \in C_I$ and so $A \subseteq C_I$.

Now let $y \in C_I$. Then for all $\alpha_{r_t} \in I$, $(y, \alpha_{r_t}) = 0$, thus $a_{r_t} = a_{r_{t+1}}$. Using our construction of the S_I above we have that $y = a_1(\epsilon_1 + \ldots + \epsilon_{i_1}) + \ldots + a_\ell(\epsilon_{i_{\ell-1}+1} + \ldots + \epsilon_{i_\ell})$. It remains to verify that $a_1 > \ldots >_\ell$. Since for all $\alpha_{i_j} \in \Delta \setminus I$ we have $(y, \alpha_{i_j}) > 0$ and so $a_{i_j} > a_{i_{j+1}}$, thus the coefficients are decreasing. Therefore $y \in A$ and so $C_I \subseteq A$.

So we have $C_I = A$. □

Lemma 3.2.2. *Given a subset S of a Euclidean space V, we have that the minimal linear space, $\mathcal{L}(S)$, is* $\operatorname{span} S$.

Proof. Let $U = \operatorname{span} S$. Note that U is a vector space. Let $u \in U$. Then $u = a_1v_1 + \ldots + a_nv_n$ for some $n \geq 1$, $v_i \in S$ and $a_i \in \mathbb{R}$. Since $\mathcal{L}(S) \supseteq S$, we know that $v_i \in \mathcal{L}(S)$ for all i. Furthermore, since $\mathcal{L}(S)$ is a vector space, it must contain all linear combinations of its elements. It follows that $u \in \mathcal{L}(S)$ and so $U \subseteq \mathcal{L}(S)$. However, from the definition of minimal linear space, this forces $\mathcal{L}(S) = U$. $\square$

Lemma 3.2.3. *Let $C = \{a_1v_1 + \ldots + a_nv_n : a_i > a_{i+1}, a_i \in \mathbb{R}\}$ be a facet. The minimal linear space of C, $\mathcal{L}(C)$, is $\{a_1v_1 + \ldots + a_nv_n : a_i \in \mathbb{R}\}$. Note that $\mathcal{L}(C)$ has in fact been obtained by dropping the inequalities in the facet C.*

Proof. Let $S = \{a_1v_1 + \ldots + a_nv_n : a_i \in \mathbb{R}\}$. Note $S = \operatorname{span}\{v_1, \ldots, v_n\}$. Clearly, if we can conclude that each $v_i \in \mathcal{L}(C)$ we can conclude $S \subseteq \mathcal{L}(C)$. Furthermore, by minimality of $\mathcal{L}(C)$ we can then conclude that they are in fact equal.

Let $i \in \{1, \cdots, n\}$ and let $k \in \mathbb{R}$. We can define two elements $y, z \in C$ such that

$$y = (k+1)v_1 + (k)v_2 + \ldots + (k-i+2)v_{i+1} + (k-i+1)v_i + (k-i-1)v_{i-1} + \ldots + (k-n)v_n$$
$$z = (k+1)v_1 + (k)v_2 + \ldots + (k-i+2)v_{i+1} + (k-i)v_i + (k-i-1)v_{i-1} + \ldots + (k-n)v_n$$

Then, we have that $y - z = v_i$. Since the coefficients are strictly decreasing we have that both $y, z \in C$, and so $y, z \in \mathcal{L}(C)$ as well. Therefore we can conclude $y - z = v_i \in \mathcal{L}(C)$ for each $i \in \{1, \ldots, n\}$.

$\square$

Chapter 4

Groups of type A_n

S_{n+1} acts on $\mathbb{R}^{n+1}$ in the following way. The transposition (i, j) sends $\epsilon_i - \epsilon_j$ to its negative and fixes the orthogonal complement, the set of all vectors in $\mathbb{R}^{n+1}$ with equal *ith* and *jth* coordinates. Furthermore, S_{n+1} stabilizes the hyperplane of vectors whose coordinates add up to 0. As shown in the following section, the Weyl group, W, of type A_n is the permutation group S_{n+1}. Thus, groups of type A_n can be thought of as acting on this $n-$dimensional space through permutations.

4.1 Construction of A_n

Lemma 4.1.1 (Root System for A_n)**.** $\Phi = \{\epsilon_i - \epsilon_j : 1 \leq i \neq j \leq n+1\}$ *is a root system*

Proof.

R1 Let $\alpha \in \Phi$ then $\alpha = \epsilon_i - \epsilon_j$ for some $i \neq j \in \{1, \ldots, n+1\}$. Clearly $-\alpha = \epsilon_j - \epsilon_i$ where again $j \neq i \in \{1, \ldots, n+1\}$ so $-\alpha \in \Phi$.

R2 Let $\alpha \in \Phi$ then again $\alpha = \epsilon_i - \epsilon_j$ with $i \neq j \in \{1, \ldots, n+1\}$. Let $v \in V$ then using the formula in chapter 1 we have

$$\begin{aligned}
s_\alpha(v) &= v - 2\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha \\
&= (v_1, \ldots, v_{n+1}) - (v_i - v_j)(\epsilon_i - \epsilon_j) \\
&= (v_1, \ldots, v_i, \ldots, v_j, \ldots, v_{n+1} + (v_j - v_i)\epsilon_i + (v_i - v_j)\epsilon_j \\
&= (v_1, \ldots, v_j, \ldots, v_i, \ldots, v_{n+1})
\end{aligned}$$

so s_α is the transposition (i, j). Then applying s_α to any element of Φ will yield yet another element in Φ. Thus $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

We have verified the axioms of a root system from chapter 1. Therefore Φ is the root system for groups of type A_n. □

Lemma 4.1.2 (Simple System for A_n)**.** $\Delta = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq n+1\}$ *is a simple system for* A_n.

Proof.

S1 Linear independence is clear, it remains to show that $\Phi \subseteq \operatorname{span}\Delta$. Let $\Phi \subseteq V$ be a root system for a group W of type A_n. Let $\alpha = \epsilon_i - \epsilon_j \in \Phi$. Then we have that $\epsilon_i - \epsilon_j = (\epsilon_i - \epsilon_{i+1}) + (\epsilon_{i+1} - \epsilon_{i+2}) + \ldots + (\epsilon_{j-2} - \epsilon_{j-1}) + (\epsilon_{j-1} - \epsilon_j)$ wherein each term is an element of Δ. Therefor Δ forms a basis for Φ.

S2 Let $\alpha \in \Phi$. Then $\alpha = \epsilon_i - \epsilon_j$. If $i < j$ then see the construction in S1 where the coefficient on each element of Δ is 1 and thus they all have the same sign. Otherwise, both sides of the equation in S1 can be multiplied by -1 in which case all coefficients will again have the same sign.

We have verified the axioms of a simple system from chapter 1. Therefore Δ is a simple system for groups of type A_n. □

Theorem 4.1.3. *The Weyl group W of type A_n is the permutation group S_{n+1}.*

Proof. Given the simple system $\Delta = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \colon i \in \{1, \ldots, n+1\}\}$, we consider the simple reflections s_i. Let $x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$, by the reflection formula defined in chapter 1 we have

$$\begin{aligned}
s_i(x) &= x - 2\frac{(x, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i \\
&= x - (x \cdot \alpha_i)\alpha_i \\
&= (x_1, \ldots, x_{n+1}) - (x_i - x_{i+1})(\epsilon_i - \epsilon_{i+1}) \\
&= (x_1, \ldots, x_{n+1}) - (0, \ldots, 0, x_i - x_{i+1}, x_{i+1} - x_i, 0, \ldots, 0) \\
&= (x_1, \ldots, x_i - (x_i - x_{i+1}), x_{i+1} - (x_{i+1} - x_i), \ldots, x_{n+1}) \\
&= (x_1, \ldots, x_{i+1}, x_i, \ldots, x_{n+1})
\end{aligned}$$

and so we have that the action of s_i is equivalent to the transposition $(i, i+1)$. Thus we have that our group W is generated by transpositions. And so W is the full permutation group S_{n+1}. □

Remark Since every facet is conjugate under the Weyl group to a facet of the fundamental domain, it suffices to determine which facets of the form C_I where $I \subseteq \Delta$ are associate.

4.2 Associativity Classes of A_2

A_2 From the previous section we have that $\Delta = \{\alpha_1, \alpha_2\} = \{(1, -1, 0), (0, 1, -1)\}$ forms a simple system for A_2.

Theorem 4.2.1 (Associativity classes of A_2)**.** *The representatives of the distinct associativity classes of A_2 are given by the following facets of the fundamental domain D $\{C_\emptyset, C_{\{\alpha_1\}}, C_\Delta\}$.*

Proof. We will begin by finding these C_I. In the following table (and in the remaining sections) $a > b > c$ represent arbitary real numbers.

I	C_I	Representative
$\emptyset$	$\{(x, y, z) \in \mathbb{R}^3 \colon x > y > z\}$	(a, b, c)
$\{\alpha_1\}$	$\{(x, y, z) \in \mathbb{R}^3 \colon x = y > z\}$	(a, a, b)
$\{\alpha_2\}$	$\{(x, y, z) \in \mathbb{R}^3 \colon x > y = z\}$	(a, b, b)
$\{\alpha_1, \alpha_2\}$	$\{(x, y, z) \in \mathbb{R}^3 \colon x = y = z\}$	(a, a, a)

When the condition that A_2 acts on the plane $x+y+z=0$ is considered, we see that $C_\emptyset$ is one of the 2 dimensional chambers and is clearly associate to all chambers of A_2 since they all share the same minimal linear space, namely the plane where the coordinates add to 0. It's also clear that in (a,a,a), we have that $a=0$ and so $C_\Delta = \{0\}$. To find the associativity classes of the remaining facets we first apply a permuation to (a,b,b) in the following way

$$s_2 s_1(a,b,b) = (b,b,a)$$

By definition, this representative is conjugate to our original facet and so is associate. Then, we consider the minimal linear space of this new facet. First note $(b,b,a) = b(1,1,0) + a(0,0,1)$

$$\mathcal{L}(b,b,a) = \text{span}\,\{(1,1,0),(0,0,1)\} = \mathcal{L}(a,a,b)$$

Since in the minimal linear space the inequalities do not matter, we can rename this facet as follows

$$b \mapsto a, a \mapsto b$$

to obtain

$$(b,b,a) \mapsto (a,a,b)$$

Then, clearly this facet shares a minimal linear space with our facet $C_{\{\alpha_1\}}$ and so is associate. By transitivity of associativity we then have that $C_{\{\alpha_1\}} \sim C_{\{\alpha_2\}}$. And thus our associativity classes are $\{C_\emptyset, C_\Delta, C_{\{\alpha_1\}}\}$. ☐

4.3 Associativity Classes of A_3

A_3 We have that $\Delta = \{(1,-1,0,0),(0,1,-1,0),(0,0,1,-1)\}$ is a simple system for A_3.

Theorem 4.3.1 (Associativity classes of A_3). *The set of associativity classes of A_3 is represented by $\{C_\emptyset, C_{\{\alpha_1\}}, C_{\{\alpha_1,\alpha_2\}}, C_{\{\alpha_1,\alpha_3\}}, C_\Delta\}$.*

Proof. Let us again find the C_I and their associativity classes. The cases for $C_\emptyset$ and C_Δ are consistent with A_2 for all n and will be omitted. Furthermore, explicitly calculating the sets C_I will also be omitted and we will instead use the construction of the S_I in section 3.2 to find a representative.

I	Representative
$\{\alpha_1\}$	(a,a,b,c)
$\{\alpha_2\}$	(a,b,b,c)
$\{\alpha_3\}$	(a,b,c,c)
$\{\alpha_1,\alpha_2\}$	(a,a,a,b)
$\{\alpha_1,\alpha_3\}$	(a,a,b,b)
$\{\alpha_2,\alpha_3\}$	(a,b,b,b)

Following a similar pattern as in the $n=2$ case we can find an element of our Weyl group W to permute the coordinates of our facet representatives.

$$s_2 s_1(a,b,b,c) = (b,b,a,c)$$

$$s_2s_3s_1s_2(a,b,c,c) = (c,c,a,b)$$

and so these representatives and their permutations are associate. Now we consider the minimal linear spaces of their permutations and do renamings to get

$$(b,b,a,c) \mapsto (a,a,b,c)$$

$$(c,c,a,b) \mapsto (a,a,b,c)$$

and so these facets share a minimal linear space and are associate. It then follows that $C_{\{\alpha_1\}} \sim C_{\{\alpha_1\}} \sim C_{\{\alpha_3\}}$.

Similarly for the 2 dimensional cases we can apply permutations

$$s_3s_2s_1(a,b,b,b) = (b,b,b,a)$$

and then consider the minimal linear spaces, apply a renaming to get that $C_{\{\alpha_1,\alpha_2\}} \sim C_{\{\alpha_2,\alpha_3\}}$. Clearly, there is no permutation to map $C_{\{\alpha_1,\alpha_3\}}$ into another facet. Furthermore, its minimal linear space is spanned by $\{(1,1,0,0),(0,0,1,1)\}$ which is different from any other facet. Therefore we have $\{C_\emptyset, C_\Delta, C_{\{\alpha_1\}}, C_{\{\alpha_1,\alpha_2\}}, C_{\{\alpha_1,\alpha_3\}}\}$ is our set of associativity classes. □

For the remaining dimensions, we will skip finding the permutation of the Weyl group and the renaming process. The following lemma allows us to do so.

Lemma 4.3.2. *Two facets C_I, C_J are associate if and only if their representatives have the same numbers of repeated coordinates.*

Proof. Let C_I, C_J be associate facets. Then, there exists $w \in W$ such that $w\mathcal{L}(C_I) = \mathcal{L}(C_J)$. Taking a representative from each of these linear spaces, we can see that since groups of type A_n act by permutation, we cannot possibly change the numbers of repeated coordinates and so they must be the same, although in different orders, for C_I and C_J. Conversely, if two facet representatives have the same numbers of repeated coordinates, we can find a permutation that maps one into the other. By Lemma 4.1.1, we can conclude this permutation is an element of our Weyl group W. Furthermore, using Lemma 3.2.3 to think of $\mathcal{L}(C_I)$ in terms of the repeated coordinates of C_I, we can see that this permutation will actually send the basis elements of $\mathcal{L}(C_I)$ to the basis elements of $\mathcal{L}(C_J)$. Therefore these two facets are associate. See Lemma 4.4.3.

□

4.4 Associativity Classes of A_4

A_4 We have that $\Delta = \{(1,-1,0,0,0),(0,1,-1,0,0),(0,0,1,-1,0),(0,0,0,1,-1)\}$ is a simple system for A_4.

Theorem 4.4.1 (Associativity classes of A_4)**.** *The associativity classes of A_4 are $\{C_\emptyset, C_{\{\alpha_1\}}, C_{\{\alpha_1,\alpha_2\}}, C_{\{\alpha_1,\alpha_3\}}, C_{\{\alpha_1,\alpha_2,\alpha_3\}}, C_{\{\alpha_1,\alpha_2,\alpha_4\}}, C_\Delta\}$.*

Proof.

I	Representative
$\{\alpha_1\}$	(a,a,b,c,d)
$\{\alpha_2\}$	(a,b,b,c,d)
$\{\alpha_3\}$	(a,b,c,c,d)
$\{\alpha_4\}$	(a,b,c,d,d)
$\{\alpha_1,\alpha_2\}$	(a,a,a,b,c)
$\{\alpha_1,\alpha_3\}$	(a,a,b,b,c)
$\{\alpha_1,\alpha_4\}$	(a,a,b,c,c)
$\{\alpha_2,\alpha_3\}$	(a,b,b,b,c)
$\{\alpha_2,\alpha_4\}$	(a,b,b,c,c)
$\{\alpha_3,\alpha_4\}$	(a,b,c,c,c)
$\{\alpha_1,\alpha_2,\alpha_3\}$	(a,a,a,a,b)
$\{\alpha_1,\alpha_2,\alpha_4\}$	(a,a,a,b,b)
$\{\alpha_1,\alpha_3,\alpha_4\}$	(a,a,b,b,b)
$\{\alpha_2,\alpha_3,\alpha_4\}$	(a,b,b,b,b)

Following Lemma 4.3.2, we can see $C_{\{\alpha_1\}} \sim C_{\{\alpha_2\}} \sim C_{\{\alpha_3\}} \sim C_{\{\alpha_4\}}$. For the $3-$dimensional facets we have $C_{\{\alpha_1,\alpha_2\}} \sim C_{\{\alpha_2,\alpha_3\}} \sim C_{\{\alpha_3,\alpha_4\}}$ and $C_{\{\alpha_1,\alpha_3\}} \sim C_{\{\alpha_1,\alpha_4\}} \sim C_{\{\alpha_2,\alpha_4\}}$. And for the $2-$dimensional we have $C_{\{\alpha_1,\alpha_2,\alpha_3\}} \sim C_{\{\alpha_2,\alpha_3,\alpha_4\}}$ and $C_{\{\alpha_1,\alpha_2,\alpha_4\}} \sim C_{\{\alpha_1,\alpha_3,\alpha_4\}}$. Thus, our associativity classes are $\{C_\emptyset, C_{\{\alpha_1\}}, C_{\{\alpha_1,\alpha_2\}}, C_{\{\alpha_1,\alpha_3\}}, C_{\{\alpha_1,\alpha_2,\alpha_3\}}, C_{\{\alpha_1,\alpha_2,\alpha_4\}}, C_\Delta\}$. $\square$

Conjecture 4.4.2 (False)**.** *The number of associativity classes for each dimension is equal to the dimension of the space minus the dimension of the facets.*

Proof. For the first set of facets, $C_{\{\alpha_1\}}$ and so on, there is one associativity class which is $5-4 = \dim V - \dim F$. However, for the $2-$dimensional case we have that $C_{\{\alpha_1,\alpha_2,\alpha_3\}} \sim C_{\{\alpha_2,\alpha_3,\alpha_4\}}$ and $C_{\{\alpha_1,\alpha_2,\alpha_4\}} \sim C_{\{\alpha_1,\alpha_3,\alpha_4\}}$. Wherein the relation between the dimension of the facets and number of classes fails. $\square$

We note that in the $n = 2,3,4$ cases we had that $C_{\{\alpha_i\}} \sim C_{\{\alpha_j\}}$ for all $i \neq j$. We may expect this pattern to continue into higher dimensions. Furthermore, in $n = 3,4$ we had that $C_{\{\alpha_1,\alpha_2\}} \sim C_{\{\alpha_2,\alpha_3\}}$.

Lemma 4.4.3. *Let C_I, C_J be facets and S_I, S_J be their corresponding vector sets defined above. $w\mathcal{L}(C_I) = \mathcal{L}(C_J)$ if and only if $wS_I = S_J$.*

Proof. Since S_I, S_J form bases for $\mathcal{L}(C_I)$ and $\mathcal{L}(C_J)$ respectively, it is clear that $wS_I = S_J$ implies $w\mathcal{L}(C_I) = \mathcal{L}(C_J)$. For the proof of the converse see Lemma 4.3.2. $\square$

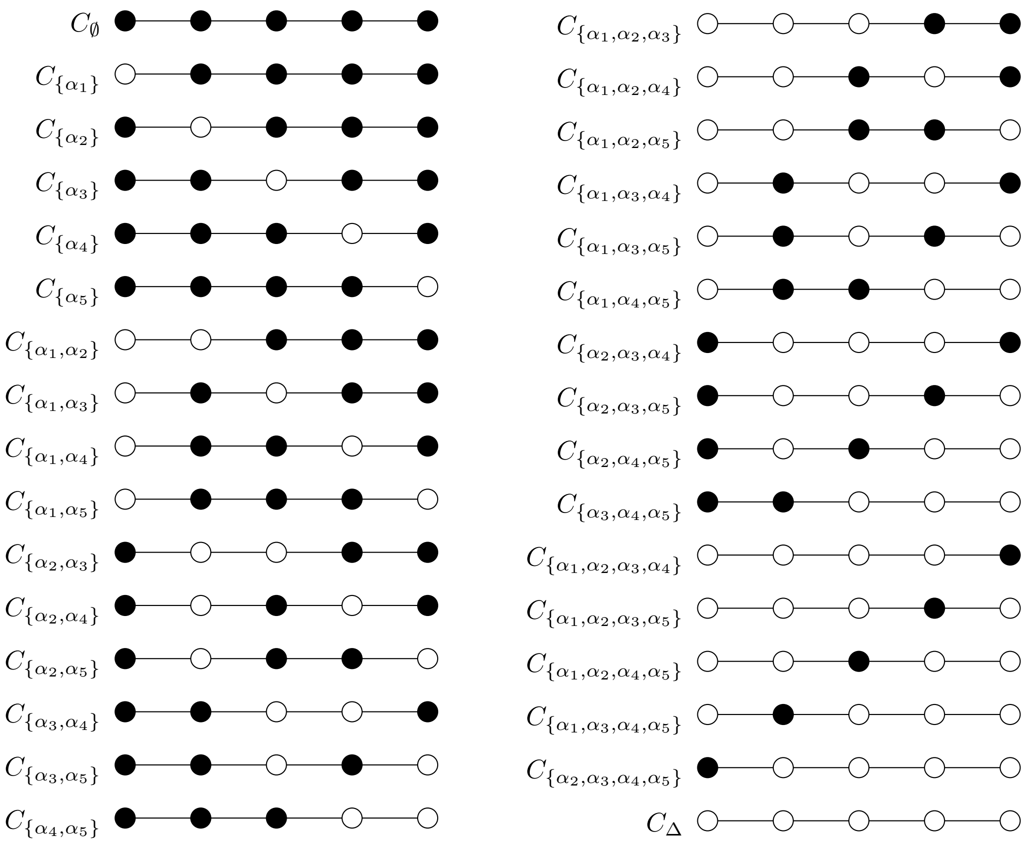
Theorem 4.4.4. *Given two associate facets $C_I^n \sim C_J^n$ in A_n we can conclude $C_I^{n+1} \sim C_J^{n+1}$ in A_{n+1} as well.*

Proof. Let W_n be a group of type A_n. Let C_I^n, C_J^n be two facets such that $C_I^n \sim C_J^n$. Then, there exists $w \in W_n$ such that $w\mathcal{L}(C_I^n) = \mathcal{L}(C_J^n)$. It follows that $wS_I^n = S_J^n$ by the previous Lemma. Since $w = s_m \dots s_k$ for some simple reflections, by abuse of notation we can conclude that $\alpha_m, \dots \alpha_k \in \Delta_n \subseteq \Delta_{n+1}$ and so $w \in W_{n+1}$ as well. Construct an embedding, φ, of S_I^n into S_I^{n+1} by $\varphi\colon (a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, 0)$. Then, using this embedding we can see that we essentially have $S_I^n = S_I^{n+1}$. Therefore we can conclude that $wS_I^{n+1} = S_J^{n+1}$ and so $w\mathcal{L}(C_I^{n+1}) = \mathcal{L}(C_J^{n+1})$ as desired. So $C_I^{n+1} \sim C_J^{n+1}$.

$\square$

4.5 Associativity Classes of A_5 Using Dynkin Diagrams

A_5 We have that $\Delta = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_4 - \epsilon_5, \epsilon_5 - \epsilon_6\}$. We will construct and colour the Dynkin diagrams of each facet C_I. A node will be coloured white if it is in I and black otherwise.



It becomes clear visually that both $C_\emptyset$ and C_Δ stand alone as expected. For the remaining classes, we can compare the groups of adjacent white vertices. It makes sense then, that the singleton facets would form one class. For the 4−dimensional facets, we see that $C_{\{\alpha_1,\alpha_2\}}, C_{\{\alpha_2,\alpha_3\}}, C_{\{\alpha_3,\alpha_4\}}, C_{\{\alpha_4,\alpha_5\}}$ all contain two white vertices in a row. Note that these two vertices in a row correspond to the Dynkin diagram of A_2. It can be verified in the usual way that these facets are all associate. The remaining 4−dimensional facets also form one class and each contain two single white vertices, or $A_1 \times A_1$. The 3−dimensional facets either contain A_3, $A_2 \times A_1$, or $A_1 \times A_1 \times A_1$, the classes $C_{\{\alpha_1,\alpha_2,\alpha_3\}} \sim C_{\{\alpha_2,\alpha_3,\alpha_4\}} \sim C_{\{\alpha_3,\alpha_4,\alpha_5\}}$, $C_{\{\alpha_1,\alpha_2,\alpha_4\}} \sim C_{\{\alpha_1,\alpha_2,\alpha_5\}} \sim C_{\{\alpha_1,\alpha_3,\alpha_4\}} \sim C_{\{\alpha_1,\alpha_4,\alpha_5\}} \sim C_{\{\alpha_2,\alpha_3,\alpha_5\}} \sim C_{\{\alpha_2,\alpha_4,\alpha_5\}}$, and $C_{\{\alpha_1,\alpha_3,\alpha_5\}}$ respectively. Again, this can be verified using the permutation-renaming process on the representatives. Finally, for the 2−dimensional facets we have diagrams containing those of A_4, $A_3 \times A_1$, or $A_2 \times A_2$. The respective classes are $C_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}} \sim C_{\{\alpha_2,\alpha_3,\alpha_4,\alpha_5\}}$,

$C_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_5\}} \sim C_{\{\alpha_1,\alpha_3,\alpha_4,\alpha_5\}}$, and $C_{\{\alpha_1,\alpha_2,\alpha_4,\alpha_5\}}$.

4.6 The General Case

A_n

Theorem 4.6.1 (Associativity classes of A_n)**.** *Two facets C_I, C_J in the fundamental domain D are associate if and only if there exists a permutation sending S_I to S_J.*

The proof of this theorem will follow the discussion below.

Conjecture 4.6.2 (False)**.** *Before computing the classes of larger groups, it was believed the classes formed by whether all the contents of I were consecutive or not for each dimension. So for example $C_{\{\alpha_1,\alpha_3,\alpha_5\}} \sim C_{\{\alpha_1,\alpha_2,\alpha_5\}}$.*

Proof. Consider A_6. The simple system is given by $\Delta = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_4 - \epsilon_5, \epsilon_5 - \epsilon_6, \epsilon_6 - \epsilon_7\}$. The associativity classes of A_6 are in fact $\{C_\emptyset, C_{\{\alpha_1\}}, C_{\{\alpha_1,\alpha_2\}}, C_{\{\alpha_1,\alpha_3\}}, C_{\{\alpha_1,\alpha_2,\alpha_3\}}, C_{\{\alpha_1,\alpha_2,\alpha_4\}}, C_{\{\alpha_1,\alpha_3,\alpha_5\}}, C_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}}, C_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_5\}}, C_{\{\alpha_1,\alpha_2,\alpha_4,\alpha_5\}}, C_{\{\alpha_1,\alpha_2,\alpha_4,\alpha_6\}}, C_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\}}, C_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_6\}}, C_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_6\}}, C_\Delta\}$. Which is left to be verified by the reader using the following table.

I	Representative	I	Representative	I	Representative
$\{\alpha_1\}$	(a,a,b,c,d,e,f)	$\{\alpha_1,\alpha_2,\alpha_3\}$	(a,a,a,a,b,c,d)	$\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$	(a,a,a,a,a,b,c)
$\{\alpha_2\}$	(a,b,b,c,d,e,f)	$\{\alpha_1,\alpha_2,\alpha_4\}$	(a,a,a,b,b,c,d)	$\{\alpha_1,\alpha_2,\alpha_3,\alpha_5\}$	(a,a,a,a,b,b,c)
$\{\alpha_3\}$	(a,b,c,c,d,e,f)	$\{\alpha_1,\alpha_2,\alpha_5\}$	(a,a,a,b,c,c,d)	$\{\alpha_1,\alpha_2,\alpha_3,\alpha_6\}$	(a,a,a,a,b,c,c)
$\{\alpha_4\}$	(a,b,c,d,d,e,f)	$\{\alpha_1,\alpha_2,\alpha_6\}$	(a,a,a,b,c,d,d)	$\{\alpha_1,\alpha_2,\alpha_4,\alpha_5\}$	(a,a,a,b,b,a,c)
$\{\alpha_5\}$	(a,b,c,d,e,e,f)	$\{\alpha_1,\alpha_3,\alpha_4\}$	(a,a,b,b,b,c,d)	$\{\alpha_1,\alpha_2,\alpha_4,\alpha_6\}$	(a,a,a,b,b,c,c)
$\{\alpha_6\}$	(a,b,c,d,e,f,f)	$\{\alpha_1,\alpha_3,\alpha_5\}$	(a,a,b,b,c,c,d)	$\{\alpha_1,\alpha_3,\alpha_4,\alpha_5\}$	(a,a,b,b,b,b,c)
$\{\alpha_1,\alpha_2\}$	(a,a,a,b,c,d,e)	$\{\alpha_1,\alpha_3,\alpha_6\}$	(a,a,b,b,c,d,d)	$\{\alpha_1,\alpha_3,\alpha_4,\alpha_6\}$	(a,a,b,b,b,c,c)
$\{\alpha_1,\alpha_3\}$	(a,a,b,b,c,d,e)	$\{\alpha_1,\alpha_4,\alpha_5\}$	(a,a,b,c,c,c,d)	$\{\alpha_1,\alpha_3,\alpha_5,\alpha_6\}$	(a,a,b,b,c,c,c)
$\{\alpha_1,\alpha_4\}$	(a,a,b,c,c,d,e)	$\{\alpha_1,\alpha_4,\alpha_6\}$	(a,a,b,c,c,d,d)	$\{\alpha_1,\alpha_4,\alpha_5,\alpha_6\}$	(a,a,b,c,c,c,c)
$\{\alpha_1,\alpha_5\}$	(a,a,b,c,d,d,e)	$\{\alpha_1,\alpha_5,\alpha_6\}$	(a,a,b,c,d,d,d)	$\{\alpha_2,\alpha_3,\alpha_4,\alpha_5\}$	(a,b,b,b,b,b,c)
$\{\alpha_1,\alpha_6\}$	(a,a,b,c,d,e,e)	$\{\alpha_2,\alpha_3,\alpha_4\}$	(a,b,b,b,b,c,d)	$\{\alpha_2,\alpha_3,\alpha_4,\alpha_6\}$	(a,b,b,b,b,c,c)
$\{\alpha_2,\alpha_3\}$	(a,b,b,b,c,d,e)	$\{\alpha_2,\alpha_3,\alpha_5\}$	(a,b,b,b,c,c,d)	$\{\alpha_2,\alpha_3,\alpha_5,\alpha_6\}$	(a,b,b,b,c,c,c)
$\{\alpha_2,\alpha_4\}$	(a,b,b,c,c,d,e)	$\{\alpha_2,\alpha_3,\alpha_6\}$	(a,b,b,b,c,d,d)	$\{\alpha_2,\alpha_4,\alpha_5,\alpha_6\}$	(a,b,b,c,c,c,c)
$\{\alpha_2,\alpha_5\}$	(a,b,b,c,d,d,e)	$\{\alpha_2,\alpha_4,\alpha_5\}$	(a,b,b,c,c,c,d)	$\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\}$	(a,a,a,a,a,a,b)
$\{\alpha_2,\alpha_6\}$	(a,b,b,c,d,e,e)	$\{\alpha_2,\alpha_4,\alpha_6\}$	(a,b,b,c,c,d,d)	$\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_6\}$	(a,a,a,a,a,b,b)
$\{\alpha_3,\alpha_4\}$	(a,b,c,c,c,d,e)	$\{\alpha_2,\alpha_5,\alpha_6\}$	(a,b,b,c,d,d,d)	$\{\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_6\}$	(a,a,a,a,b,b,b)
$\{\alpha_3,\alpha_5\}$	(a,b,c,c,d,d,e)	$\{\alpha_3,\alpha_4,\alpha_5\}$	(a,b,c,c,c,c,d)	$\{\alpha_1,\alpha_2,\alpha_4,\alpha_5,\alpha_6\}$	(a,a,a,b,b,b,b)
$\{\alpha_3,\alpha_6\}$	(a,b,c,c,d,e,e)	$\{\alpha_3,\alpha_4,\alpha_6\}$	(a,b,c,c,c,d,d)	$\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}$	(a,a,b,b,b,b,b)
$\{\alpha_4,\alpha_5\}$	(a,b,c,d,d,d,e)	$\{\alpha_4,\alpha_5,\alpha_6\}$	(a,b,c,d,d,d,d)	$\{\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}$	(a,b,b,b,b,b,b)
$\{\alpha_4,\alpha_6\}$	(a,b,c,d,d,e,e)				

and so we have that in particular $C_{\{\alpha_1,\alpha_3,\alpha_5\}} \not\sim C_{\{\alpha_1,\alpha_2,\alpha_5\}}$.

□

Remark By the construction of the S_I in section 3.2, we know if there is a coefficient of 1 on ϵ_i in some vector of S_I, then it has a coefficient of 0 in all other vectors of S_I. In other words, for each $i \in \{1, hdots, n\}$, there is exactly one vector $v \in S_I$ where ϵ_i has a nonzero coefficient.

Proof of Theorem 4.6.2. If $wS_I = S_J$ then clearly $w\mathcal{L}(C_I) = \mathcal{L}(C_J)$ and we are done. Suppose $C_I \sim C_J$. Then there exists $w \in W$ such that $w\mathcal{L}(C_I) = \mathcal{L}(C_J)$. Then we have that wS_I is a basis for $\mathcal{L}(C_J)$, as is S_J. Suppose $wS_I \neq S_J$. We know $|S_I| = |wS_I| = |S_J| = k$, so there must be some $v \in wS_I$ which not in S_J. Since $v \in wS_I$, we know $v \in \mathcal{L}(C_J)$ so v can be expressed as a linear combination of elements in S_J. Then, by the remark above we know that all coefficients of the linear combination must be 1. So there are vectors $u_1, \ldots u_t \in S_J$ with $t > 1$ such that $v = u_1 + \ldots + u_t$. Again by the remark above, there are no other vectors in S_J with nonzero coefficients on the standard basis vectors that the u_j consist of. Therefore, the remaining $k-1$

elements of wS_I are linear combinations of the remaining $k-t$ elements of S_J, implying they are linearly dependent, a contradiction. So $wS_I = S_J$. $\square$

Alternatively, for a more visual approach, the facets can be compared using their coloured Dynkin diagrams.

Corollary 4.6.3 (Associativity Classes of A_n using Dynkin Diagrams)**.** *Two facets C_I, C_J are associate if and only if their coloured Dynkin diagrams consist of the same subdiagrams.*