

April 6 : Last lecture

- Today:
- . recap diagonalization algorithm
 - . recap orthogonalization & Gram-Schmidt
 - . Google Page Rank
 - . Course summary + comments on final exam

Diagonalization

Given an $n \times n$ matrix A , the steps are:

- ① Compute $\det(A - \lambda I)$ (the characteristic polynomial)
(or $\det(\lambda I - A)$, if you are careful; see Apr 4 lecture)
- ② Factor it; the roots are the eigenvalues of A .
- ③ For each eigenvalue λ , find a basis for $\text{Null}(A - \lambda I)$.
That is: solve $(A - \lambda I) \vec{x} = \vec{0}$
$$\begin{matrix} \text{known} & \xrightarrow{\quad \text{Unknown} \quad} \\ \text{E}_\lambda & \end{matrix}$$

- ② Decide if A is diagonalizable:
- if your roots are not all in \mathbb{R} , then A is not diagonalizable over \mathbb{R} (but it could be over \mathbb{C})
MAT 2342
 - if $\dim(E_\lambda) <$ algebraic multiplicity of λ for any λ
then A is not diagonalizable (over \mathbb{R} or \mathbb{C})
- * since $\dim(E_\lambda) \geq 1$, this problem can only occur if λ is a repeated root — but you can't tell until you solve $(A - \lambda I) \vec{x} = \vec{0}$.

④ If all eigenvalues are real and $\dim E_\lambda = \text{alg. mult } \lambda$ for all λ , then A is diagonalizable.

Set $P =$ matrix whose columns are bases of each eigenspace

$D =$ diagonal matrix with eigenvalues on the diagonal, in the same order as their corresponding eigenvectors in P .

Then:

- $AP = PD$

- P is invertible

$$\therefore A = PDP^{-1}$$

* do not find P^{-1} unless asked to.

↙ check this to check your work

↙ justify this

on exam
(& in life)

example: Say A is 5×5 and

$$\det(A - \lambda I) = -\lambda^2(\lambda - 7)^2(\lambda + 6).$$

So eigenvalues are $\lambda = 0$ (alg mult 2), $\lambda = 7$ (al 2) and $\lambda = -6$.

Say we find basis for $E_0 = \{\vec{U}_1, \vec{U}_2\}$

basis for $E_7 = \{\vec{U}_3, \vec{U}_4\}$

basis for $E_{-6} = \{\vec{U}_5\}$.

So $P = [\vec{U}_1 \ \vec{U}_2 \ \vec{U}_3 \ \vec{U}_4 \ \vec{U}_5]$ & $D = \begin{bmatrix} 0 & 0 & 7 & 7 & -6 \end{bmatrix}$

(but you could have picked

$$P = [U_5 \ U_1 \ U_3 \ U_2 \ U_4] \text{ & } D = \begin{bmatrix} -6 & & & & \\ & 0 & & & \\ & & 7 & & \\ & & & 0 & \\ & & & & 7 \end{bmatrix}$$

to be funny.)

Why is P invertible?

Say we had

$$\underbrace{a_1 \vec{U}_1 + a_2 \vec{U}_2}_{\in E_0} + \underbrace{a_3 \vec{U}_3 + a_4 \vec{U}_4}_{\in E_7} + \underbrace{a_5 \vec{U}_5}_{\in E_6} = \vec{0}$$

Thm: eigenvectors from different eigenvalues are LI.

→ this equation is

$$\vec{V}_1 + \vec{V}_2 + \vec{V}_3 = \vec{0}$$

$$V_1 \in E_0$$

$$V_2 \in E_7$$

$$V_3 \in E_6$$

$\Rightarrow \vec{V}_1 = 0, \vec{V}_2 = 0, \vec{V}_3 = \vec{0}$ because if nonzero they would be LI.
∴ impossible.

$$\Rightarrow a_1 \vec{U}_1 + a_2 \vec{U}_2 = \vec{0} \Rightarrow a_1 = a_2 = 0 \text{ since } \{\vec{U}_1, \vec{U}_2\} \text{ LI}$$

$$a_3 \vec{U}_3 + a_4 \vec{U}_4 = \vec{0} \Rightarrow a_3 = a_4 = 0 \text{ since } \{U_3, U_4\} \text{ LI}$$

$$a_5 \vec{U}_5 = \vec{0} \Rightarrow a_5 = 0 \text{ since } \vec{U}_5 \neq \vec{0}.$$

Conclusion: the columns of P are LI so P is invertible.

* Notice! P is invertible so
 $AP = P\mathbf{D} \Rightarrow A = P\mathbf{D}P^{-1}$.

But here, A is not invertible, because

$$\det(A) = \det(A - 0I) = 0 \quad \text{because } 0 \text{ is an eigenvalue}$$

□

Orthogonalization & Gram-Schmidt

Key point: when we have an orthogonal set of vectors, we have lots of shortcuts.

① If $\{\vec{U}_1, \dots, \vec{U}_n\}$ is an orthogonal basis of U
 then for any $\vec{v} \in U$,

$$\vec{v} = \frac{\vec{U}_1 \cdot \vec{v}}{\|\vec{U}_1\|^2} \vec{U}_1 + \dots + \frac{\vec{U}_n \cdot \vec{v}}{\|\vec{U}_n\|^2} \vec{U}_n. \quad \left. \begin{array}{l} \text{super-easy} \\ \text{to find} \\ \text{coordinates.} \end{array} \right\}$$

② If $\{u_1, \dots, \vec{U}_n\}$ is an orthogonal basis for $U \subseteq \mathbb{R}^m$
 then for any $\vec{v} \in \mathbb{R}^m$,

$$\text{proj}_U(\vec{v}) = \frac{u_1 \cdot \vec{v}}{\|u_1\|^2} \vec{U}_1 + \dots + \frac{u_n \cdot \vec{v}}{\|u_n\|^2} \vec{U}_n$$

(this gives the point in U closest to \vec{v} .)

To create an orthogonal basis, starting from a non-orthogonal one, we do the Gram-Schmidt algorithm:

Say we start with $\{\vec{v}_1, \dots, \vec{v}_n\}$.

Set

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{u_1 \cdot v_2}{\|u_1\|^2} u_1$$

$$u_3 = v_3 - \frac{u_1 \cdot v_3}{\|u_1\|^2} u_1 - \frac{u_2 \cdot v_3}{\|u_2\|^2} u_2$$

etc
:

subtraction
because
We want the perpendicular
part, not the projection.

these are always the
new vectors you made
in previous steps.



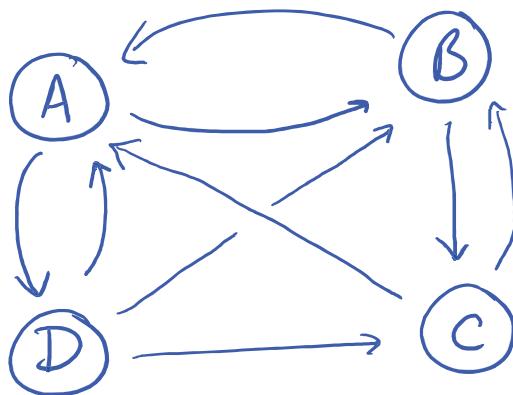
Application of eigenvalues & eigenvectors. Google Page Rank algorithm

- Search engines index websites based on the words they contain.
 - But there are too many sites with your search term; how can Google decide which ones are the most important?
- early ideas:
- # times the search term appears?
(but this is easy for the owner of the site to manipulate)
 - how many websites link to it?
(again, you can manipulate this — we only care if important sites link to it)

Key

Think of the internet as a graph and write down its adjacency matrix... with a couple of tweaks.

an
internet
with
4 sites.
(scale to
 4×10^9)



Tweak #1: directed edges representing where there is a link from one page to another.

Tweak #2: assign a probability to each link:

$$G = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$$

"hypermatrix"

So column 1 says: if you click randomly on \textcircled{A} then you have a 50% chance to end up on \textcircled{B} & 50% chance to end up on \textcircled{D} .

We saw: if A is the adjacency matrix of a graph,
then A counts # paths of length 1 from one node to another
 A^2 counts # paths of length 2 from one node to another
 A^3 ... length 3 ...
⋮

Here: G, G^2, G^3, \dots count the probability that you will end up at a given node after so many steps.

* Key point: as $k \rightarrow \infty$, G^k is figuring out where thousands of users would end up if they just randomly followed links.

How is this meaningful? Users clicking randomly?
If you own a website, you link to sites you think are important.
Billions of websites \Rightarrow the ultimate in crowd-sourcing
"what is important?"

Perron - Frobenius theorem

Let G be a matrix as above. Then one of the eigenvalues of G is $\lambda = 1$, and it is larger than all other eigenvalues of G , in absolute value.

(But how to find the eigenvector?)

Consequence: $\lim_{n \rightarrow \infty} A^n \vec{x} = c\vec{v}$ for some $c \in \mathbb{R}$.

for any \vec{x}
 \uparrow
 starting position

and

$$A\vec{v} = \vec{v}$$

\vec{v} is an eigenvector.

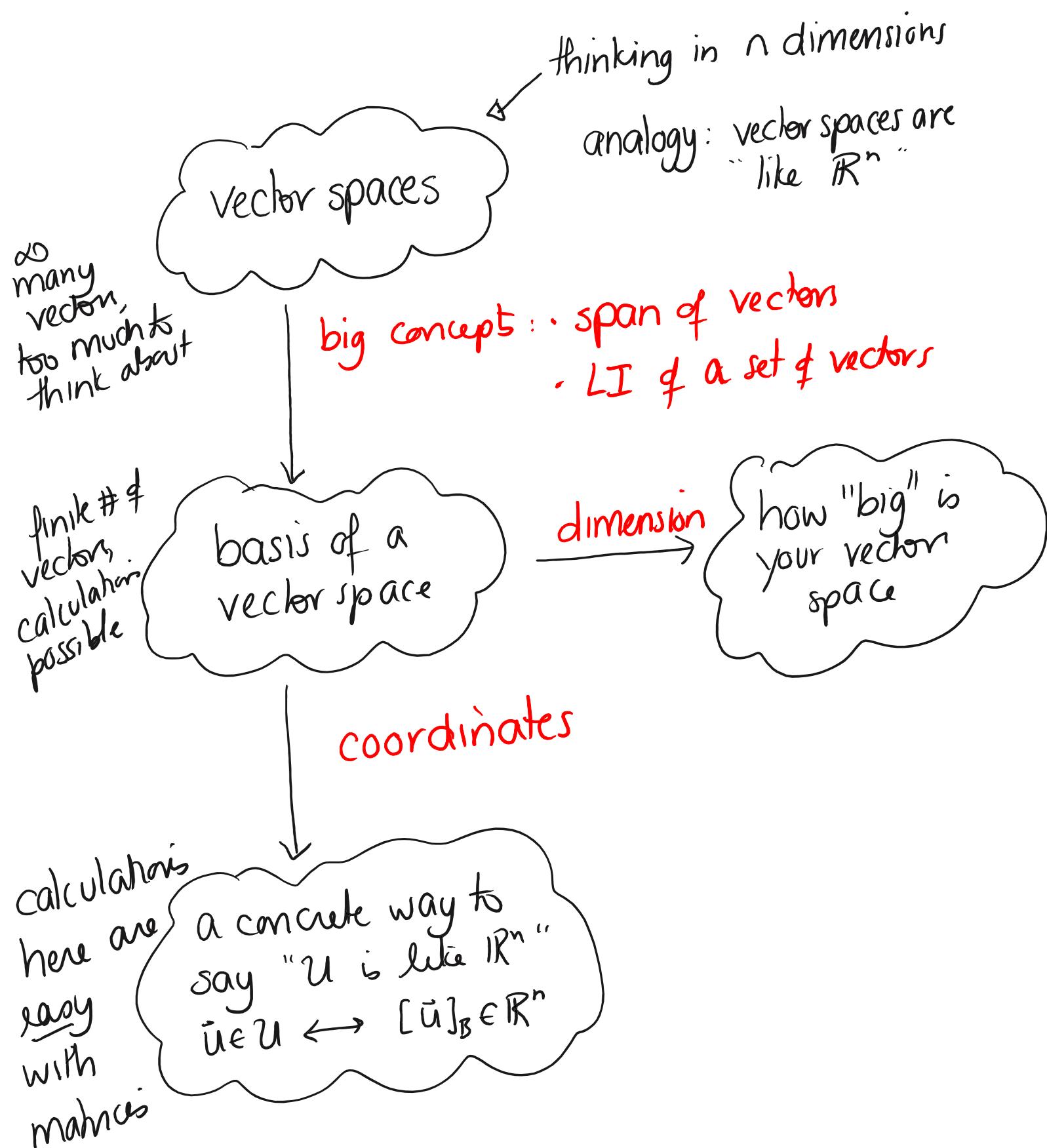
In fact \vec{v} is special: its coordinates are ≥ 0 and give the probability of arriving at each website.

= relative importance of each website.

= Google PageRank.

in our example: $\vec{v} = \begin{bmatrix} 32\% \\ 32\% \\ 21\% \\ 15\% \end{bmatrix}$

Wrap-up: what MAT 1341 was about
& what to expect on the final exam



tools for solving a linear system of equations efficiently (row reduction)

finding linear combinations of vectors that add up to \vec{v}



as a collection of vectors (rows or columns) to give efficient algorithms

as "generalized numbers"; objects you can do math with:

$$A^2 + B, \quad A^{-1}, \quad I, \quad O, \dots$$

$$A^k$$

as state machines of dynamical systems (rabbits & foxes, Google PageRank)

MAT 2384: systems of linear differential equations

(next courses)
as functions (linear transformations)

from one vector space to another

in MAT 2322: the derivative of a multivariable function.

for math majors:

- over other fields
- beyond vector spaces: groups, rings, algebras MAT 2143 & beyond
- in the design of error-correcting codes MAT 3343