

Chapter 23: Eigenvalues, eigenvectors and diagonalizability.

Recall from last time:

A $n \times n$ matrix.

If $\lambda \in \mathbb{R}$ and $\vec{x} \neq \vec{0}$ and $A\vec{x} = \lambda \vec{x}$ then λ is an **eigenvalue** of A and \vec{x} is an **eigenvector** of A (associated to the eigenvalue λ).

To find the eigenvalues of A :

- compute the **characteristic polynomial** of A , which is $\det(A - \lambda I)$ a polynomial in the variable λ

- find the roots of this polynomial; these are the eigenvalues.

* The **algebraic multiplicity** of the eigenvalue λ is the number of times it is repeated as a root of the characteristic polynomial.

Ex) If $\det(A - \lambda I) = (\lambda - 3)^2(\lambda + 1)$, then the eigenvalues of A are $\lambda = 3$ and $\lambda = -1$. The algebraic multiplicity of $\lambda = 3$ is $\underline{\underline{2}}$ but the algebraic multiplicity of $\lambda = -1$ is only $\underline{1}$.

To each eigenvalue λ of A we can associate a subspace of \mathbb{R}^n , called the λ -eigenspace E_λ :

$$\begin{aligned} E_\lambda &= \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\} \\ &= \{\vec{x} \in \mathbb{R}^n \mid (A - \lambda I)\vec{x} = \vec{0}\} \\ &= \text{Null}(A - \lambda I) \end{aligned}$$

- the nonzero vectors in E_λ are all of the eigenvectors of A associated to λ .

\therefore we want a basis for each eigenspace of A .
 (method: for each λ , solve $(A - \lambda I)\vec{x} = \vec{0}$ & take basic solutions)

Theorem: $1 \leq \dim(E_\lambda) \leq$ algebraic multiplicity of λ

for each real eigenvalue λ of A .

Then $\dim(E_\lambda)$ is called the geometric multiplicity of λ .

\therefore the best case scenario is that:

- A has only real eigenvalues (as opposed to complex)
- for each eigenvalue λ , its algebraic multiplicity is equal to its geometric multiplicity (enough eigenvectors)

When this happens, we say A is **diagonalizable**.

Otherwise: A is not diagonalizable \leftarrow MAT3341.

example: $A = \begin{bmatrix} 7 & -5 & -5 \\ 0 & 2 & 0 \\ 10 & -10 & -8 \end{bmatrix}$

① find eigenvalues of A .

$$\det(A - \lambda I) = \det \begin{bmatrix} 7-\lambda & -5 & -5 \\ 0 & 2-\lambda & 0 \\ 10 & -10 & -8-\lambda \end{bmatrix} =$$

$\overset{\uparrow}{\text{cofactor}} \quad (2-\lambda) \det \begin{bmatrix} 7-\lambda & -5 \\ 10 & -8-\lambda \end{bmatrix}$

A- λI is annoying, yes.
but you can't swap the
order (unless you calculate
 $\det(\lambda I - A)$ & negate all entries
in A)

$$\begin{aligned}
 &= (2-\lambda)((7-\lambda)(-8-\lambda) - (-5)(10)) \\
 &= (2-\lambda)(\lambda^2 + 8\lambda - 7\lambda - 56 + 50) \\
 &= (2-\lambda)(\lambda^2 + \lambda - 6) \\
 &= -(\lambda-2) \cdot (\lambda-2)(\lambda+3) \\
 &= -(\lambda-2)^2(\lambda+3)
 \end{aligned}$$

So the eigenvalues of A are: $\lambda = 2$ (with algebraic multiplicity 2)
and $\lambda = -3$ (with algebraic multiplicity 1).

② Find a basis for each eigenspace

$$\lambda = -3: \quad A - \lambda I = A - (-3I) = A + 3I = \begin{bmatrix} 10 & -5 & -5 \\ 0 & 5 & 0 \\ 10 & -10 & -5 \end{bmatrix}$$

Row reduce:

$$\left[\begin{array}{ccc|c} A+3I & | & 0 \\ \hline 10 & -5 & -5 & 0 \\ 0 & 5 & 0 & 0 \\ 10 & -10 & -5 & 0 \end{array} \right] \xrightarrow{\frac{1}{10}R_1} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \hline 0 & 1 & 0 & 0 \\ 10 & -10 & -5 & 0 \end{array} \right] \xrightarrow{-10R_1} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & -5 & 0 & 0 \end{array} \right] \xrightarrow{+R_3} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{array}{l} \frac{1}{2}R_2+R_1 \\ 5R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \text{ in RREF.}$$

solution: $x - \frac{1}{2}z = 0 \Rightarrow x = \frac{1}{2}t$
 $y = 0 \Rightarrow y = 0$
 $z = \text{free: } z = t$

basic solution: $t=1$
gives $\left\{ \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \right\}$ as a
basis for E_{-3} .

We need a basis for E_{-3} . $\left\{ \left(\frac{1}{2}, 0, 1 \right) \right\}$ is a basis.

so is $\left\{ (1, 0, 2) \right\}$. (\vec{v}) basis if
and only if $2\vec{v}$
is a basis

Check: $A \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 & -5 & -5 \\ 0 & 2 & 0 \\ 10 & -10 & -8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -6 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \checkmark$

or use $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ - same ans.

$$\lambda=2: [A - 2I \mid \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}] = \left[\begin{array}{ccc|c} 5 & -5 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -10 & -10 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \text{ in RREF.}$$

Soln: $x-y-z=0$, y, z free, $y=s, z=t$.

$$E_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s+t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \mid s, t \in \mathbb{R} \right\}$$

basis for E_2 : $\{(1, 1, 0), (1, 0, 1)\}$.

check: $A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ $A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

: geometric multiplicity
= algebraic multiplicity

True & all evs are
 \checkmark A is diagonalizable

new

③ Diagonalize A :

$$\text{Let } P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Since (theorem) eigenvectors from different eigenvalues are LI,
the columns of P are LI.

Proof: If we had $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$

$$\underbrace{a\vec{u}}_{E_1} + \underbrace{b\vec{v}}_{E_2} + \underbrace{c\vec{w}}_{E_3} = \vec{0}$$

then $\vec{u} \Rightarrow a=0$ and $b\vec{v} + c\vec{w} = \vec{0}$ (since $b\vec{v} + c\vec{w} \in E_2$)
 $\{\vec{v}, \vec{w}\}$ basis for $E_2 \Rightarrow b=c=0$. \therefore total set is LI \square

$\therefore P$ is invertible.

$$A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

Moreover:

$$AP = A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \\ -6 & 0 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_P \underbrace{\begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_D.$$

where D is a diagonal matrix of eigenvalues of A listed in the same order as the eigenvectors.

$$\Rightarrow AP = PD \Rightarrow P^{-1}AP = D \text{ or } A = PDP^{-1}$$

* since P is invertible.

Thus we have diagonalized A .

This is TRUE.
 You don't need to calculate P^{-1} to prove it.

Application: Say $A = PDP^{-1}$ with D diagonal.
We can find powers of A without doing $\underbrace{A \cdots A}_{m \text{ times}}$.

$$A = PDP^{-1} \text{ so } A^4 = \underbrace{(PDP^{-1})}_{I} \underbrace{(PDP^{-1})}_{I} \underbrace{(PDP^{-1})}_{I} = P D^4 P^{-1}$$

eg $P = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$ $P^{-1} = \frac{1}{1} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ $D = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \Rightarrow D^4 = \begin{bmatrix} (-1)^4 & 0 \\ 0 & (\frac{1}{2})^4 \end{bmatrix}$

$$A^4 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^4 & 0 \\ 0 & (\frac{1}{2})^4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2(\frac{1}{2})^4 \\ -1 & (\frac{1}{2})^4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 - 2(\frac{1}{2})^4 & 6 - 6(\frac{1}{2})^4 \\ -1 + (\frac{1}{2})^4 & -2 + 3(\frac{1}{2})^4 \end{bmatrix}$$

so for m very large:

$$A^m = P D^m P^{-1}$$

eg here: A^{2n} $\underset{\substack{\approx \\ \text{almost equal}}}{\sim}$ $\begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$ since $\lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^{2n} = 0$. \square

Application 1: population dynamics. (Markov chains)

Suppose we have an island with foxes and rabbits.
(Foxes eat rabbits).

How does their population evolve over time?

Say: f_t = population of foxes at time t

r_t = population of rabbits at time t .

We observe the population next week is:

$$f_{t+1} = 0.6f_t + 0.5r_t$$

$$r_{t+1} = -0.1f_t + 1.2r_t$$

where $0.6f_t$ is the # foxes left if no rabbits
 $1.2r_t$ is the # rabbits next week if no foxes
- $0.1f_t$ is the # rabbits each fox eats per week
 $0.5r_t$ is the net contribution to survival of foxes by rabbits.

So $\begin{bmatrix} f_{t+1} \\ r_{t+1} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.5 \\ -0.1 & 1.2 \end{bmatrix} \begin{bmatrix} f_t \\ r_t \end{bmatrix}$ for any t .

This is a state machine: give me population \vec{v}_t (=state) at time t , and output $A\vec{v}_t$ is population at time $t+1$.

In particular, if initial population is \vec{v}_0 , then:

$$\vec{v}_1 = A\vec{v}_0, \quad \vec{v}_2 = A\vec{v}_1 = A(A\vec{v}_0) = A^2\vec{v}_0$$

$$\vec{v}_3 = A\vec{v}_2 = A(A^2\vec{v}_0) = A^3\vec{v}_0$$

and $\vec{v}_t = A^t \vec{v}_0$.

Relations to eigenvalues and eigenvectors:

Notice $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1.1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $\lambda = 1.1$ is an eigenvalue with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

\therefore if our population $\vec{v} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ $A\vec{v} = 1.1\vec{v} \quad A(1.1\vec{v}) = (1.1)(1.1)\vec{v}, \dots$

then $A^t \vec{v} = (1.1)^t \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ perfect balance population grows at same rate.

Also: $A \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 0.7 \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ so $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 0.7$

\therefore if population starts $\begin{bmatrix} 500 \\ 100 \end{bmatrix}$ then $A^t \vec{v}_0 = (0.7)^t \begin{bmatrix} 500 \\ 100 \end{bmatrix}$ population dies out because $(0.7)^t \rightarrow 0$ as $t \rightarrow \infty$.

More generally:

$$\vec{v}_0 = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad \text{for some } a, b$$

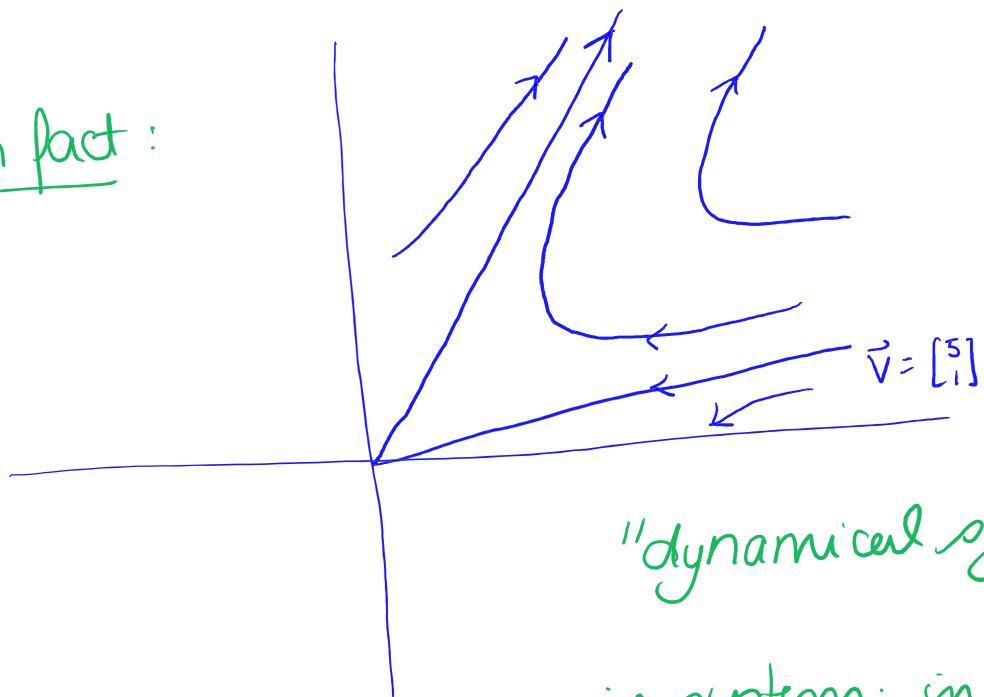
because $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right\}$ is a basis for \mathbb{R}^2

$$\begin{aligned}\Rightarrow A^t \vec{v}_0 &= A^t (a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 5 \\ 1 \end{bmatrix}) \\ &= a (A^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}) + b (A^t \begin{bmatrix} 5 \\ 1 \end{bmatrix}) \\ &= a (1.1)^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b (0.7)^t \begin{bmatrix} 5 \\ 1 \end{bmatrix}.\end{aligned}$$

\Rightarrow can model population growth for any t , any initial population.

$$\vec{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

in fact :



"dynamical system"

This was a discrete dynamical system; in MAT2844/2724 you look at continuous dynamical systems using differential equations — eigenvalues & eigenvectors are still the key!

$$P = \begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix} \quad P^{-1} = -\frac{1}{4} \begin{bmatrix} 1 & -5 \\ -1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1.1 & \\ & 0.7 \end{bmatrix}$$

$$\text{Fact: } A = PDP^{-1} \Rightarrow A^k = PD^k P^{-1} = -\frac{1}{4} \begin{bmatrix} 1 & -5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1.1^k & 0.7^k \\ 0.7^k & 1.1^k \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} 1 & -5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1.1^k & 5(0.7)^k \\ 5(1.1)^k & 1.1^k + 0.7^k \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} 1.1^k - 5(0.7)^k & 5(1.1)^k - 5(0.7)^k \\ -(1.1)^k + (0.7)^k & -5(1.1)^k + 0.7^k \end{bmatrix}$$

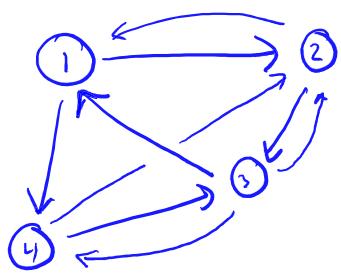
if you wanted an actual formula.

Application #2: Google PageRank (See link website)

PageRank : decides the relative importance of a webpage based on how many links go to it.

(so that Google can rank the pages that contain your keyword)

Basic idea: Internet = graph



vertices = pages

edges = links

the adjacency matrix of a graph records the edges.

Here: more sophisticated: assign a probability to each link.

$$A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

"hypermatrix"

Then if you start with any vector \vec{v} , $A\vec{v}$, $A^2\vec{v}$, ... give the probabilities of where you will end up, if you follow those links w/ those probabilities.

Fact: such a matrix has all eigenvalues ≤ 1 ; eigenvector for eigenvalue 1: steady state. This state represents $\lim_{t \rightarrow \infty} A^t \vec{v}_0$, for any \vec{v}_0 .

More important pages have more links to them.

The steady state has a probability for each page; the higher the probability, the more important the page.



Examples

$$\textcircled{1} \quad A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ 5 & -5 & -3 \end{bmatrix}$$

Tip: say you don't like $\det(A - \lambda I)$. Can we do $\det(\lambda I - A)$ instead?

Recall: multiply a row by $c \Rightarrow$ change determinant by factor C^c

$$\det \begin{bmatrix} 3 & 3 \\ 4 & 5 \end{bmatrix} = 3 \det \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$$

check:

$$3 \times 5 - 3 \times 4 = 3(1 \times 5 - 1 \times 4) \quad \checkmark$$

To get from $A - \lambda I$ to $\lambda I - A$ we multiply the whole matrix by -1 , not just one row.

$$\det \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = (-1)(-1) \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

↑ ↑
first row second row

$$\therefore \boxed{\det(\lambda I - A) = (-1)^n \det(A - \lambda I)}$$

This is OK:
 $p(x)$ & $-p(x)$
 have the same
 roots \therefore calculation
 of eigenvalues OK.

Back to $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ 5 & -5 & -3 \end{bmatrix}$.

* notice all elements of A are subtracted, not just diagonal!

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -1 & 0 \\ 1 & \lambda - 3 & 0 \\ -5 & 5 & \lambda + 3 \end{bmatrix} = (\lambda + 3) \det \begin{bmatrix} \lambda - 1 & -1 \\ 1 & \lambda - 3 \end{bmatrix}$$

$$= (\lambda + 3)((\lambda - 1)(\lambda - 3) - (-1)(1))$$

$$= (\lambda+3)(\lambda^2 - 4\lambda + 3 + 1)$$

$$= (\lambda+3)(\lambda-2)^2$$

\therefore the characteristic polynomial of A is

$$-(\lambda+3)(\lambda-2)^2$$

\uparrow
this is $(-1)^3$ ← a factor of -1 for each row.

OK: so A has 2 eigenvalues:

- $\lambda = -3$ with algebraic multiplicity 1
(and therefore geometric multiplicity 1,
since $1 \leq \dim E_1 \leq \text{alg mult}$)
- $\lambda = 2$ with algebraic multiplicity 2.
Need to find its geometric multiplicity
 \Leftrightarrow basis for E_2 .

$$E_2: A - 2I = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 5 & -5 & -5 \end{bmatrix}$$

$$\therefore [A - 2I | \begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}] \approx \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{aligned} x-y &= 0 \\ z &= 0 \\ y &\text{ anything} \\ y &= s \end{aligned}$$

$$\therefore \text{basis is } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ so } \dim E_2 = 1 < \text{alg mult} \quad \Rightarrow x=s, z=0$$

\therefore "not enough eigenvectors" = "geometric multiplicity of 2 < algebraic multiplicity of 2"

$\therefore A$ is not diagonalizable.

What's the problem? - 3: eigenvector $(0, 0, 1)$

$$P = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

not square

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

not invertible

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

Invertible but

$AP \neq PD$.

∴ just no way to get the equation

$$AP = PD \quad \text{w/ } P \text{ invertible}$$

∴ just no way to get $A = PDP^{-1}$.
⇒ can't diagonalize.

□

Tips: every eigenvalues gives $\dim E_\lambda \geq 1$.

∴ if you get $\{\vec{0}\}$ as your only solution
to $(A - \lambda I)\vec{x} = \vec{0}$, you made a mistake:
- either wrong eigenvalue
- or bad row reduction.

Check by multiplying $A\vec{v}$ & comparing with $\lambda\vec{v}$.

Don't calculate P^{-1} unless you are asked to.
To KNOW that $A = PDP^{-1}$ all you need is:
① $AP = PD$ (i.e. column eigenvectors) and

② P invertible

Theorem: eigenvectors from different eigenspaces are LI
∴ if we take the Union of bases of all our eigenspaces,
→ this set is LI.

columns of P
are the vectors
in all the
bases of the
eigenspaces.



∴ P invertible \Leftrightarrow there are n eigenvectors in this basis.