# MAT1341 C : Instructor Monica Nevins Monday, March 27, 2017 : Test \#4 

Duration: 75 minutes

Family name: $\qquad$
First name: SOLUTIONS
Student number : $\qquad$
DGD Section number : $\qquad$
Please read the following instructions carefully.

- You have 75 minutes to complete this exam.
- This is a closed book exam. No notes, calculators, cell phones or related devices of any kind are permitted. All such devices, including cell phones, must be stored in your bag under your desk for the duration of the exam.
- Read each question carefully - you will save yourself time and grief later on.
- Questions 1, 2 and 3 are multiple choice, worth 1 point each. Record your answers to the multiple choice questions in the boxes provided.
- Questions 4-6 are long answer, with point values as indicated. The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.
- Question 7 is a bonus question, worth 3 points. The bonus question is more difficult; do not attempt it until you are satisfied that you have completed the rest of the test to the best of your ability.
- Where it is possible to check your work, do so.
- Good luck!

Faculté des sciences
Mathématiques et statistique

## Faculty of Science

Mathematics and Statistics
. 613-562-5864
國 613-562-5776

- www.uOttawa.ca
$\odot 585$ King Edward Ottawa ON K1N 6N5

Marker's use only:

| Question | Marks |
| :---: | :---: |
| $1,2,3(/ 3)$ |  |
| $4(/ 6)$ |  |
| $5(/ 6)$ |  |
| $6(/ 5)$ |  |
| $7(/ 3)$ <br> (bonus) |  |
| Total $(/ 20)$ |  |

1. (1 point) Let $A$ and $B$ be $n \times n$ matrices. Which of the following are always true?
(1) $A B \neq B A$,
(2) if $A B=0$ then at least one of $A$ or $B$ is the zero matrix,
(3) if $A$ is invertible and $A B=A C$ then $B=C$,
(4) if $\operatorname{Null}(A)=\{\overrightarrow{0}\}$ then $\operatorname{Row}(A)=\mathbb{R}^{n}$,
(5) $(A+B)^{2}=A^{2}+B^{2}$.
A. None are always true.
D. Only (1) and (3) are always true.
B. All are always true.
E. Only (2) and (4) are always true.
C. Only (3) and (4) are always true.
F. Only (1) and (5) are always true.


Answer: C (for explanations, please see other version)
2. (1 point) Suppose $A \vec{x}=\vec{b}$ is a linear system such that the coefficient matrix $A$ has 320 rows and 200 columns and such that $\operatorname{rank}(A)=200$. Three questions (please read carefully):

- Could the system be inconsistent?
- If $\vec{b}=\overrightarrow{0}$ does the system have infinitely many solutions?
- Are the rows of $A$ linearly independent?

The correct answers are:
A. No; No; No
C. Yes; No; No
E. No; Yes; Yes
B. Yes; No; Yes
D. Yes; Yes; No
F. No; No; Yes


Answer: C
3. (1 point) If

$$
C^{-1}=\left[\begin{array}{ccc}
1 & 2 & -1 \\
-1 & -1 & -1 \\
2 & 7 & -7
\end{array}\right]
$$

then which of the following is the second row of $C$ ?
A. $\left[\begin{array}{lll}-2 & 2 & 1\end{array}\right]$
B. $\begin{array}{lll}-1 & -1 & -1]\end{array}$
C. $\left[\begin{array}{lll}0 & 1 & 0\end{array}\right]$
D. $\left[\begin{array}{lll}1 & 1 & 0\end{array}\right]$
E. $\left.\begin{array}{lll}-9 & -5 & 2\end{array}\right]$
F. $\left.\begin{array}{lll}14 & 7 & -3\end{array}\right]$

Your answer: $\square$
Answer : E

## Université d'Ottawa | University of Ottawa

4. ( 6 points $=1$ point each) (Matrix inverses) Give characterizations of an invertible $n \times n$ matrix in terms of each of the three following concepts. (That is, for each one, complete the sentence " $A$ is invertible if and only if..." using that concept.)
(a) (Concept: columns of $A$ ):
the columns of $A$ are LI; or, the columns of $A$ form a basis for $\mathbb{R}^{n}$; or the columns of $A$ span $\mathbb{R}^{n}$.
(b) (Concept: the linear system $A x=b$ ):
the linear system $A x=b$ has a unique solution (for every $b$ ); or, the linear system $A x=b$ is consistent for every $b$
(c) (Concept: the reduced row echelon form (RREF) of $A$ ):
the RREF of $A$ is $I_{n}$
Now let $A=\left[\begin{array}{cc}1 & 4 \\ -1 & -2\end{array}\right]$.
(d) Choose one of the above characterizations (a), (b) or (c) and use it to show that this matrix $A$ is invertible.

Using (a): the columns of $A$ are the vectors $(1,-1),(4,-2)$ which are two vectors that are not scalar multiples of each other therefore LI, therefore form a basis for $\mathbb{R}^{2}$; thus $A$ is invertible.
Or using (b):

$$
\begin{aligned}
{\left[\begin{array}{cc|c}
1 & 4 & x \\
-1 & -2 & y
\end{array}\right] } & \sim(R 1+R 2)\left[\begin{array}{cc|c}
1 & 4 & x \\
0 & 2 & x+y
\end{array}\right] \\
& \sim \frac{1}{2} R 2\left[\begin{array}{cc|c}
1 & 4 & x \\
0 & 1 & \frac{1}{2}(x+y)
\end{array}\right] \\
& \sim(-4 R 2+R 1)\left[\begin{array}{cc|c}
1 & 0 & x-2(x+y) \\
0 & 1 & \frac{1}{2}(x+y)
\end{array}\right]
\end{aligned}
$$

so indeed, each linear system admits a unique solution, so $A$ is invertible.
Or using (c): we saw above that RREF of $A$ is $I_{2}$, whence $A$ is invertible.
(e) Give $A^{-1}$.

We can row reduce $\left[A \mid I_{2}\right]$ until $\left[I_{2} \mid A^{-1}\right]$, or use the formula

$$
A^{-1}=\frac{1}{-2-(-4)}\left[\begin{array}{cc}
-2 & -4 \\
1 & 1
\end{array}\right]=\frac{1}{2}\left[\begin{array}{cc}
-2 & -4 \\
1 & 1
\end{array}\right]=\left[\begin{array}{cc}
-1 & -2 \\
\frac{1}{2} & \frac{1}{2}
\end{array}\right]
$$

(f) Show that your answer is correct, by doing one of the matrix multiplications that defines the inverse.

We multiply $A A^{-1}$ (or $A^{-1} A$ ) and show the answer is $I$.
5. $(6=1+1+1+2+1$ points $)$ Consider the matrix

$$
B=\left[\begin{array}{lllll}
2 & 2 & 4 & 4 & 0 \\
1 & 1 & 3 & 4 & 3 \\
0 & 0 & 1 & 2 & 3
\end{array}\right]
$$

Be sure to justify the correctness of your answers to each of the following questions, by referring to results, algorithms and theorems from class.
(a) Find the reduced row echelon form (RREF) of $B$.
(b) Give a basis for $\operatorname{Col}(B)$.
(c) Give a basis for $\operatorname{Row}(B)$.
(d) Find a basis for $\operatorname{Null}(B)$.
(e) Extend your basis of $\operatorname{Row}(B)$ to a basis for $\mathbb{R}^{5}$.

We have

$$
\begin{aligned}
& B=\left[\begin{array}{lllll}
2 & 2 & 4 & 4 & 0 \\
1 & 1 & 3 & 4 & 3 \\
0 & 0 & 1 & 2 & 3
\end{array}\right] \underset{\frac{1}{2} R 1}{\sim}\left[\begin{array}{lllll}
1 & 1 & 2 & 2 & 0 \\
1 & 1 & 3 & 4 & 3 \\
0 & 0 & 1 & 2 & 3
\end{array}\right] \\
& \underset{-R 1+R 2}{\sim}\left[\begin{array}{lllll}
1 & 1 & 2 & 2 & 0 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 & 3
\end{array}\right] \underset{-2 R 2+R 1}{\underset{-2}{\sim}+R 3}\left[\begin{array}{ccccc}
1 & 1 & 0 & -2 & -6 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{aligned}
$$

and this last matrix is the RREF of $B$. By the columnspace algorithm, a basis for $\operatorname{Col}(B)$ consists of the columns of $B$ giving leading 1s in the RREF of $B$, hence a basis for $\operatorname{Col}(B)$ is

$$
\{(2,1,0),(4,3,1)\}
$$

By the rowspace algorithm, a basis for $\operatorname{Row}(B)$ consists of the nonzero rows of the RREF of $B$, which are

$$
\{(1,1,0,-2,-6),(0,0,1,2,3)\}
$$

To find a basis for the nullspace, we augment the above reduction with a column of 0 s , and read off the solution:

$$
x_{1}+x_{2}-2 x_{4}-6 x_{5}=0 ; \quad x_{3}+2 x_{4}+3 x_{5}=0
$$

so if $x_{2}=r, x_{4}=s, x_{5}=t$ then

$$
x_{1}=-r+2 s+6 t ; \quad x_{3}=-2 s-3 t .
$$

Thus $\operatorname{Null}(B)=\{r(-1,1,0,0,0)+s(2,0,-2,1,0)+t(6,0,-3,0,1)$ midr, $s, t \in$ $\mathbb{R}\}$ so a basis is the set of basic solutions

$$
\{(-1,1,0,0,0),(2,0,-2,1,0),(6,0,-3,0,1)\} .
$$

Finally, using the rowspace algorithm, we see that there are missing leading 1 s in columns 2, 4 and 5 , so that we can extend our basis by adding those standard basis vectors that give these missing leading 1 s :

$$
\{(1,1,0,-2,-6),(0,0,1,2,3),(0,1,0,0,0),(0,0,0,1,0),(0,0,0,0,1)\} .
$$

6. $(1+2+2=5$ points)
(a) Give the general formula for the Fourier expansion (coordinates) of a vector $\vec{v}$ with respect to an orthogonal basis $\left\{\vec{u}_{1}, \vec{u}_{2}, \vec{u}_{3}\right\}$ of $\mathbb{R}^{3}$. (That is, state the theorem from class.)

We have

$$
\vec{v}=\frac{\overrightarrow{u_{1}} \cdot \vec{v}}{\overrightarrow{u_{1}} \cdot \overrightarrow{u_{1}}} \overrightarrow{u_{1}}+\frac{\overrightarrow{u_{2}} \cdot \vec{v}}{\overrightarrow{u_{2}} \cdot \overrightarrow{u_{2}}} \overrightarrow{u_{2}}+\frac{\overrightarrow{u_{3}} \cdot \vec{v}}{\overrightarrow{u_{3}} \cdot \overrightarrow{u_{3}}} \overrightarrow{u_{3}}
$$

(b) Consider the following vectors in $\mathbb{R}^{3}$ :

$$
\vec{u}_{1}=\left[\begin{array}{c}
-2 \\
2 \\
1
\end{array}\right], \quad \vec{u}_{2}=\left[\begin{array}{l}
2 \\
1 \\
2
\end{array}\right], \quad \vec{u}_{3}=\left[\begin{array}{c}
1 \\
2 \\
-2
\end{array}\right] .
$$

Prove that $\left\{\vec{u}_{1}, \vec{u}_{2}, \vec{u}_{3}\right\}$ is an orthogonal basis of $\mathbb{R}^{3}$. Justify your answer clearly.
We check that they are orthogonal by computing the dot products of each pair:

$$
\begin{aligned}
& \vec{u}_{2} \cdot \overrightarrow{u_{1}}=(2,1,2) \cdot(-2,2,1)=-4+2+2=0 \\
& \overrightarrow{u_{2}} \cdot \overrightarrow{u_{3}}=(2,1,2) \cdot(1,2,-2)=2+2-4=0 \\
& \overrightarrow{u_{1}} \cdot \overrightarrow{u_{3}}=(-2,2,1) \cdot(1,2,-2)=-2+4-2=0
\end{aligned}
$$

Therefore $\left\{\overrightarrow{u_{1}}, \overrightarrow{u_{2}}, \overrightarrow{u_{3}}\right\}$ is an orthogonal set. By a theorem from class, this means that the set is linearly independent. Since $\operatorname{dim}\left(\mathbb{R}^{3}\right)=3$, any linearly independent set in $\mathbb{R}^{3}$ with 3 vectors is a basis for $\mathbb{R}^{3}$. Therefore $\left\{\overrightarrow{u_{1}}, \overrightarrow{u_{2}}, \overrightarrow{u_{3}}\right\}$ is an orthogonal basis for $\mathbb{R}^{3}$.
(c) Find the coordinates of $\vec{v}=(15,9,-6)$ with respect to the orthogonal basis $\left\{\vec{u}_{1}, \vec{u}_{2}, \vec{u}_{3}\right\}$ from (b).

We compute the Fourier coefficients of the formula in (a):

$$
\begin{aligned}
& \frac{\overrightarrow{u_{1}} \cdot \vec{v}}{\overrightarrow{u_{1}} \cdot \overrightarrow{u_{1}}}=\frac{-2,2,1) \cdot(15,9,-6)}{(-2,2,1) \cdot(-2,2,1)}=\frac{-30+18-6}{9}=\frac{-18}{9}=-2 \\
& \frac{\overrightarrow{u_{2}} \cdot \vec{v}}{\overrightarrow{u_{2}} \cdot \overrightarrow{u_{2}}}=\frac{(2,1,2) \cdot(15,9,-6)}{(2,1,2) \cdot(2,1,2)}=\frac{30+9-12}{4+1+4}=\frac{27}{3}=3 \\
& \frac{\overrightarrow{u_{3}} \cdot \vec{v}}{\overrightarrow{u_{3}} \cdot \overrightarrow{u_{3}}}=\frac{(1,2,-2) \cdot(15,9,-6)}{(1,2,-2) \cdot(1,2,-2)}=\frac{15+18+12}{9}=\frac{45}{9}=5 .
\end{aligned}
$$

Therefore

$$
\vec{v}=-2 \overrightarrow{u_{1}}+3 \overrightarrow{u_{2}}+5 \overrightarrow{u_{3}}
$$

and we can check directly that this is true. Thus the coordinates of $\vec{v}$ with respect to this basis are $(-2,3,5)$.
7. (Bonus, max 3 points) Let $\{f, g, h\} \subset F(\mathbb{R})$ be a set of linearly independent real-valued functions. Let $V=\operatorname{span}\{f, g, h\}$. Let $T$ be an invertible $3 \times 3$ matrix whose $(i, j)$ entry is the number $t_{i j}$. Define three new functions by

$$
\begin{gathered}
a(x)=t_{11} f(x)+t_{21} g(x)+t_{31} h(x), \quad b(x)=t_{12} f(x)+t_{22} g(x)+t_{32} h(x), \\
c(x)=t_{13} f(x)+t_{23} g(x)+t_{33} h(x)
\end{gathered}
$$

Prove that $\{a, b, c\}$ is a basis for $V$.
To test linear independence, we write down the dependence relation with unknown $q, r, s$ :

$$
\begin{aligned}
0 & =q a(x)+r b(x)+s c(x) \\
& =q\left(t_{11} f(x)+t_{21} g(x)+t_{31} h(x)\right)+r\left(t_{12} f(x)+t_{22} g(x)+t_{32} h(x)\right)+s\left(t_{13} f(x)+t_{23} g(x)+t_{33} h(x)\right) \\
& =\left(q t_{11}+r t_{12}+s t_{13}\right) f(x)+\left(q t_{21}+r t_{22}+s t_{23}\right) g(x)+\left(q t_{31}+r t_{32}+s t_{33}\right) h(x)
\end{aligned}
$$

Since $\{f, g, h\}$ are LI, these last coefficients must be zero. We can rewrite these three coefficients as the matrix product

$$
\left[\begin{array}{lll}
t_{11} & t_{12} & t_{13} \\
t_{21} & t_{22} & t_{23} \\
t_{31} & t_{32} & t_{33}
\end{array}\right]\left[\begin{array}{l}
q \\
r \\
s
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right] .
$$

Since this first matrix is $T$ and $T$ is invertible, we can multiply both sides by $T^{-1}$; since $T^{-1} 0=0$ this gives

$$
\left[\begin{array}{l}
q \\
r \\
s
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right] .
$$

Therefore the only solution to the dependence relation is the trivial solution, so the set $\{a, b, c\}$ is a linearly independent set in $V$. Since $\operatorname{dim}(V)=3$ and we have 3 LI vectors, $\{a, b, c\}$ is a basis for $V$ (by the theorem from class).

