Université d'Ottawa | University of Ottawa

MAT1341 C : Instructor Monica Nevins
Monday, January 23, 2017

Test #1 : diagnostic test

DGD #: ________

Duration: 75 minutes

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Please read the following instructions carefully.

- You have 75 minutes to complete this exam.
- This is a closed book exam. No notes, calculators, cell phones or related devices of any kind are permitted. All such devices, including cell phones must be stored in your bag under your desk for the duration of the exam.
- Read each question carefully — you will save yourself time and grief later on.
- The first 10 questions are multiple-choice, worth 1 point each, with no partial credit towards an incorrect answer. **Record your answers to the multiple choice questions into the table below.**
- The final question is long answer and worth 3 marks. **You must provide a detailed solution, written clearly, with appropriate justification, to obtain full marks.**
- Where it is possible to check your work, do so.
- Good luck!

θ	$\sin(\theta)$	$\cos(\theta)$
0	0	1
$\pi/6$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$
$\pi/4$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$
$\pi/3$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$
$\pi/2$	1	0

Question	Your answer
1	C
2	B
3	E
4	D
5	A
6	B
7	C
8	D
9	F
10	A

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1. What is the projection of the vector $\vec{v} = (-2, 4)$ onto the vector $\vec{u} = (3, 1)$?
Answer: $\text{proj}_{\vec{u}}(\vec{v}) =$

A. $-\frac{1}{5}(-2, 4)$

B. $-\frac{1}{10}(3, 1)$

C. $-\frac{1}{5}(3, 1)$

D. $\frac{2}{5}(-1, 2)$

E. $(-15, -5)$

F. $-\frac{1}{5}(-1, 2)$

$$\vec{v} \cdot \vec{u} = (-2,4) \cdot (3,1) = -6 + 4 = -2$$

$$\|\vec{u}\|^2 = \left(\sqrt{(3,1) \cdot (3,1)}\right)^2 = \left(\sqrt{9+1}\right)^2 = 10$$

$$\therefore \text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2} \vec{u} = -\frac{2}{10}(3,1) = -\frac{1}{5}(3,1)$$

2. What is the angle between the two vectors $(\sqrt{3}, 0, 1)$ and $(3, 2, \sqrt{3})$?

A. 0

B. $\pi/6$

C. $\pi/4$

D. $\pi/3$

E. $\pi/2$

F. π

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} = \frac{(\sqrt{3}, 0, 1) \cdot (3, 2, \sqrt{3})}{\sqrt{3+0+1}\sqrt{9+4+3}} = \frac{3\sqrt{3} + \sqrt{3}}{\sqrt{4}\sqrt{16}}$$

$$= \frac{4\sqrt{3}}{2 \times 4} = \frac{\sqrt{3}}{2}$$

$$\therefore \theta = \pi/6$$

3. Which of the following is a set of parametric equations for the line passing through the two points $(-1, 3, 2)$ and $(1, 7, 0)$?

A. $x = -1 + t, y = 3 + 7t, z = 2 \quad t \in \mathbb{R}$

B. $x = -s + t, y = 3s + 7t, z = 2s \quad s, t \in \mathbb{R}$

C. $x = -1 + t, y = 3 + 10t, z = 2 + 2t \quad t \in \mathbb{R}$

D. $x = 2t, y = 4t, z = -2t \quad t \in \mathbb{R}$

E. $x = 1 + t, y = 7 + 2t, z = -t \quad t \in \mathbb{R}$

F. $x = -1 + 2t, y = 3 + 4t, z = 2 + 5t \quad t \in \mathbb{R}$

direction vector: $(-1,3,2) - (1,7,0) = (-2,-4,2)$
(or any multiple thereof)

This eliminates A, B, C, F, which do not have a direction vector in the correct direction.

D does not pass through $(1,7,0)$.

E: $\left\{ \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} : t \in \mathbb{R} \right\}$ is correct.

4. Which one of the following statements about the two lines

$$L_1 = \left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + s \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \mid s \in \mathbb{R} \right\} \quad \text{and} \quad L_2 = \left\{ \begin{bmatrix} 2 \\ -5 \\ 8 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

is true?

A. L_1 and L_2 intersect at the point $(1, -1, -2)$.

B. L_1 and L_2 intersect at the point $(10, 1, 4)$.

C. L_1 and L_2 intersect at the point $(10, -1, -4)$.

D. L_1 and L_2 intersect at the point $(10, -1, 4)$.

E. L_1 and L_2 are orthogonal.

F. L_1 and L_2 are parallel.

We try to solve for s,t making the corresponding points equal:

x: $1 + 3s = 2 + 2t$

y: $-1 = -5 + t \Rightarrow \underbrace{t = 4}$

z: $-2 + 2s = 8 - t$

$-2 + 2s = 8 - 4$

$2s = 4 + 2 = 6$

$s = 3$

When $t = 4$ & $s = 3$ we have:

$\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \\ 4 \end{bmatrix}$

and

$\begin{bmatrix} 2 \\ -5 \\ 8 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \\ 4 \end{bmatrix}$

which is a point of intersection!

5. Which of the following is a Cartesian equation for the plane passing through the three points $(-1, 2, 0)$, $(1, 4, 1)$, and $(1, 1, 7)$?

A.	$5x - 4y - 2z = -13$	D.	$2x + 2y + z = 13$
B.	$13x - 10y - 6z = -33$	E.	$x + y + 7z = 1$
C.	$-x + 2y = 7$	F.	$5x + 4y - 2z = 3$

We find 2 direction vectors:

$(-1,2,0) - (1,4,1) = (-2,-2,-1)$

$(-1,2,0) - (1,1,7) = (-2,1,-7)$

Therefor a normal vector is

$$(-2,-2,-1) \times (-2,1,-7) = (15,-12,-6)$$

So the Cartesian equation is $15x - 12y - 6z = d$ for some d. Plug in $(-1,2,0)$ to find

$d = -15 - 12(2) - 6(0) = -39$

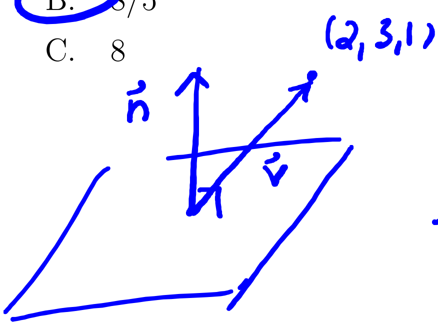
$\therefore 15x - 12y - 6z = -39$

$\Leftrightarrow 5x - 4y - 2z = -13$

6. What is the distance from the point $(2, 3, 1)$ to the plane with equation $-3y + 4z = 3$?

$\vec{n} = (0,-3,4)$

A.	3	D.	5
B.	8/5	E.	0
C.	8	F.	The distance does not exist.



A point on the plane is $(0,-1,0)$, for example. So we could take

$\vec{v} = (2,3,1) - (0,-1,0) = (2,4,1)$.

We want $\|\text{proj}_{\vec{n}} \vec{v}\|$:

$$\text{proj}_{\vec{n}}(\vec{v}) = \frac{\vec{n} \cdot \vec{v}}{\|\vec{n}\|^2}\vec{n} = \frac{(0,-3,4)\cdot(2,4,1)}{0^2+(-3)^2+4^2}\vec{n} = \frac{-8}{25}\vec{n}$$

$$\therefore \|\text{proj}_{\vec{n}}(\vec{v})\| = \frac{8}{25}\sqrt{0^2+(-3)^2+4^2} = \frac{8}{25}\cdot 5 = \frac{8}{5}$$

7. Which of the following is a set of parametric equations for the plane with Cartesian equation $x - y + 2z = 3$?

~~A.~~ $x = 1 + s, y = -1 - s, z = 2 + 2s \quad s \in \mathbb{R}$ ← a line

~~B.~~ $x = 2 + s - t, y = -2 + s + t, z = 1 + s + t \quad s, t \in \mathbb{R}$ $(2,-2,1) \notin$ plane

C. $x = 2 + s + t, y = -1 + s - t, z = -t \quad s, t \in \mathbb{R}$

D. $x = 1 + 2t, y = -s - t, z = 1 + s \quad s, t \in \mathbb{R}$

~~E.~~ $x = 2 + 2s + t, y = 2 - t, z = -s - t \quad s, t \in \mathbb{R}$ ← $(2,2,0) \notin$ plane

F. $x = 1 + s + t, y = -s + t, z = 1 + 2s \quad s, t \in \mathbb{R}$

X D is $\begin{bmatrix}1\\0\\1\end{bmatrix} + s\begin{bmatrix}0\\-1\\1\end{bmatrix} + t\begin{bmatrix}2\\-1\\0\end{bmatrix}$ but these direction vectors are not orthogonal to $\vec{n} = (1,-1,2)$.

X F is $\begin{bmatrix}1\\0\\1\end{bmatrix} + s\begin{bmatrix}1\\-1\\2\end{bmatrix} + t\begin{bmatrix}1\\1\\0\end{bmatrix}$ but $\begin{bmatrix}1\\-1\\2\end{bmatrix} \cdot \vec{n} \neq 0$.

Check C is correct:

$(2+s+t) - (-1+s-t) + 2(-t) = 3 + s(1-1) + t(1+1-2) = 3$

for all s and t. ✓

8. Suppose $\vec{u}, \vec{v}$ and $\vec{w}$ are three nonzero vectors in $\mathbb{R}^3$. Which of the following statements is/are true?

(1) $(\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$ is a scalar. X this is a vector

(2) The expression $(\vec{u} \times \vec{v}) \times (\vec{u} \cdot \vec{v})$ is invalid. ✓ $\vec{u} \cdot \vec{v}$ is a number; can't take its cross product.

(3) $\vec{u} \cdot (\vec{v} \times \vec{w})$ is a vector. X it's a scalar

(4) $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \cos(\theta)$, where θ is the angle between $\vec{u}$ and $\vec{v}$. X $\underline{\sin}\,\theta$

(5) If $(\vec{u} \times \vec{v}) \cdot \vec{w} = 0$, then the three vectors are coplanar. ✓ area of parallelepiped $= 0$.

(6) If $\vec{u} \times \vec{v} \neq \vec{0}$, then the vectors $\vec{u}$ and $\vec{u} \times \vec{v}$ are parallel. X no, they would be $\perp$.

Answer:

A. (4) and (5)	C. (2) only	E. (1) and (6)
B. (3) only	D. (2) and (5)	F. (1) and (5)

9. What is the volume of the parallelepiped determined by the three vectors $\vec{u} = (-1, 0, 1)$, $\vec{v} = (0, 1, 2)$ and $\vec{w} = (1, -1, 1)$?

A. 2

B. 3

C. -4

D. 1

E. 0

F. 4

We use the scalar triple product.

$\vec{v} \times \vec{w} = \begin{matrix}(0,1,2)\\ \times (1,-1,1)\end{matrix} = (3, -(-2), -1) = (3,2,-1)$

$|\vec{u} \cdot (\vec{v} \times \vec{w})| = |(-1,0,1) \cdot (3,2,-1)| = |-3+0-1| = 4$

10. Which of the following is a parametric form for the line which goes through the point $(1, 2, 3)$ and is parallel to the planes with Cartesian equations $\underbrace{2x - 3y + 4z = 1}$ and $\underbrace{x - y + z = 2}$?

A. $\{(1, 2, 3) + t(1, 2, 1) \mid t \in \mathbb{R}\}$

B. $\{(2, 3, 0) + t(2, 1, 0) \mid t \in \mathbb{R}\}$

C. $\{(-1, 2, 4) + t(-2, 0, 1) \mid t \in \mathbb{R}\}$

D. $\{(1, 2, 3) + s(2, -3, 4) + t(1, -1, 1) \mid s, t \in \mathbb{R}\}$

E. $\{(1, 2, 3) + t(1, 1, 1) \mid t \in \mathbb{R}\}$

F. $\{(1, 1, -3) + t(2, 4, 2) \mid t \in \mathbb{R}\}$

$\vec{n}_1 = (2,-3,4)$ $\quad \vec{n}_2 = (1,-1,1)$

To be parallel to both planes is to be orthogonal to both normal vectors.

A vector $\perp \vec{n}_1$ & $\vec{n}_2$ is a multiple of $\vec{n}_1 \times \vec{n}_2$.

$\vec{n}_1 \times \vec{n}_2 = \begin{matrix}(2,-3,4)\\ \times (1,-1,1)\end{matrix} = (1, -(-2), 1) = (1,2,1)$

A point on the line is $(1,2,3)$ so $L = \left\{ \begin{bmatrix}1\\2\\3\end{bmatrix} + t\begin{bmatrix}1\\2\\1\end{bmatrix} \middle| t \in \mathbb{R} \right\}$

11. (3 points) Long answer question with two parts (a), (b)

(a) The following is the augmented matrix of a system of linear equations. Using Gaussian elimination, and writing down each of the row operations that you use (in the form "$3R_2+R_1 \to R_1$", for example, as done in class), reduce the following system to reduced row echelon form (RREF).

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 4 & 3 & -9 & -2 \\ 2 & 4 & -2 & -6 \end{array}\right]$$

$$\underset{\substack{-4R1+R2 \\ -2R1+R3}}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & -5 & -5 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{-\frac{1}{5}R2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{-2R2+R1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

Solution:

$$x - 3z = 1 \qquad \therefore\ x = 1 + 3z = 1 + 3t$$
$$y + z = -2 \qquad y = -2 - z = -2 - t$$
$$z = t$$

a parameter

(b) Letting the variables of the linear system be called x, y and z, as usual, write down the general solution to the linear system in vector parametric form, by filling in the following set appropriately:

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+3t \\ -2-t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$