

## Ch 18 matrix inverses

Mon 7 : + spaces associated with matrices.

Last time: matrix inverses (for square matrices ONLY)

$A^{-1}$  is a matrix such that  $A^{-1}A = I$  &  $AA^{-1} = I$ .

\* it doesn't always exist.

\* matrices that have an inverse are called invertible

Usefulness: solve  $AX = b$

$$\Leftrightarrow A^{-1}AX = A^{-1}b$$

$$\Leftrightarrow IX = A^{-1}b$$

$$\Leftrightarrow X = A^{-1}b$$

\* careful:  
order  
matters  
a LOT

e.g) if  $A$  is  $2 \times 2$  then  $A$  is invertible iff  
 $\det(A) := \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} := da - bc \neq 0$

in which case  $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .

Last time: for general  $A$ , row reduce

$$[A | I_n] \sim \dots \sim [I_n | B].$$

if you get  $I_n$  here

↑ then  $\underbrace{AB = I_n}$ .

because each column of  $B$  solves the linear system  
 $AX = e_i$  for a column  $e_i$  of  $I_n$ .

$$[A | \vec{b}] \sim [I_n | \vec{x}]$$

$$\Leftrightarrow A\vec{x} = \vec{b}.$$

## Properties of matrix inverses

Thm: Suppose  $A$  is an invertible  $n \times n$  matrix.

Then:

$(A^{-1})^{-1} = A$ .

①  $A^{-1}$  is invertible, and  $(A^{-1})^{-1} = A$ .

②  $A^P$  is invertible for any positive integer  $P$ ,

$$\text{and } (A^P)^{-1} = (A^{-1})^P$$

③  $A^T$  (transpose) is invertible, and

$$(A^T)^{-1} = (A^{-1})^T$$

④ for any scalar  $c$ ,  $cA$  is invertible  
and  $(cA)^{-1} = \frac{1}{c} A^{-1}$ .

⑤ if  $B$  is any other invertible matrix  
then  $AB$  is invertible and

$$(AB)^{-1} = B^{-1}A^{-1} \quad \text{note: swap order (just like w/ transpose!)}$$

Proof: Remember: to show a matrix is invertible, you just have to find a matrix that does the job of being the inverse.

So for example:

$$\textcircled{3} \quad \text{Try } A^T \cdot (A^{-1})^T = (A^{-1} \cdot A)^T \quad \text{Property of transpose}$$

$$= I^T = I \checkmark$$

$$(A^{-1})^T \cdot A^T = (A A^{-1})^T = I^T = I \checkmark$$

$$\textcircled{5} \quad (AB)(B^{-1}A^{-1}) = A B B^{-1} A^{-1} \quad \text{associativity}$$

$$= A I A^{-1}$$

$$= AA^{-1} = I \checkmark$$

$$\& (B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB$$

$$= B^{-1}IB = B^{-1}B = I \checkmark.$$

Try the rest!  $\square$

eg) Simplify  $((BA)^{-1})^T A^T$

$$= (A^{-1} B^{-1})^T A^T$$

$$= (B^{-1})^T (A^{-1})^T A^T$$

$$= (B^{-1})^T I$$

$$= (B^{-1})^T$$

$$(BA)^{-1} = A^{-1}B^{-1}$$

$$(CD)^T = D^T C^T$$

$$(A^{-1})^T A^T = (AA^{-1})^T = I$$

eg) Simplify  $(A+B)^{-1}$ .

Tough luck. This doesn't even simplify for real numbers!

Theorem: The following conditions on an  $n \times n$  square matrix  $A$  are equivalent  
i.e. if one is true, all the rest are true;  
if one is false, all the rest are false.

- ①  $\text{rank}(A) = n$
- ② the linear system  $Ax = b$  is consistent for EVERY choice of  $b \in \mathbb{R}^n$
- ③ the linear system  $Ax = 0$  has a unique solution
- ④ the linear system  $Ax = b$  has a unique solution for every choice of  $b \in \mathbb{R}^n$
- ⑤ the RREF of  $A$  is  $I_n$
- .

Recall: The rank of  $A$  is the number of leading ones in an RREF of  $A$ .  
The leading ones in an RREF are all in different rows and different columns:

$$\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{So: } ⑤ \Leftrightarrow ①$$

Next: remember that you get a unique solution  $\Leftrightarrow$  you have no nonleading variables (because those are the ones that give you parameters in your solution)

- $\Leftrightarrow$  all variables are leading variables
- $\Leftrightarrow$  all columns have a leading 1.

$$\therefore ③ \Leftrightarrow ①$$

Next: remember your system has the possibility of being inconsistent iff the RREF has a degenerate equation = zero row. So ALWAYS CONSISTENT  $\Leftrightarrow$  no zero rows in the RREF  $\neq A$ .

$\Leftrightarrow$  each row has a leading 1

$$\text{So: } ② \Leftrightarrow ①$$

So if ② is true, ③ is true, so ④ is true.  
(and if ④ is true, ② & ③ are true)

$$\therefore ④ \Leftrightarrow ①$$



In fact we're going to show that all those things are equivalent to  $A$  being invertible. To show that, we need one more fact.

FACT: If  $A$  and  $B$  are  $n \times n$  matrices such that  $AB = I_n$ , then  $BA = I_n$ .  
*(One-sided inverses are 2-sided inverses)*  
Recall: this is not obvious because in general  $AB \neq BA$ .

Why? Say  $AB = I_n$ . Let's show that ③ is true for  $B$ . If  $Bx = 0$  then

$$ABx = A \cdot 0$$

$$\text{so } IX = 0$$

$$\Rightarrow X = 0.$$

$\therefore Bx = 0$  has only the trivial solution.

$\therefore$  The RREF of  $B$  is  $I_n$  by our theorem.

So we can reduce  $[B | I_n] \sim \sim [I_n | C]$ .

By our previous work  $\therefore BC = I_n$ .

But then look:

$$A(BC) = AIn = A$$

$$(AB)C = InC = C$$

$\therefore A = C$ ; they're the same.  $\therefore BA = In$ .

□

Theorem: Let  $A$  be a square  $n \times n$  matrix.

Then  $A$  is invertible if and only if  
the RREF of  $A \in In$ , ( $\Leftrightarrow ① \Leftrightarrow ② \Leftrightarrow ③ \Leftrightarrow ④$ )  
in which case  $A^{-1}$  is what appears on  
the right we we reduce  
 $[A | In] \sim \dots \sim [In | A^{-1}]$ .

Proof: If the RREF of  $A \in In$ , then

$$[A | In] \sim \dots \sim [In | B]$$

for some matrix  $B$ . This means  $AB = In$ .  
By the Fact  $BA = In$ .  $\therefore A$  is invertible  
and  $B = A^{-1}$ .

Conversely, if  $A$  is invertible then  $A^{-1}$  exists  
so  $Ax = b \Leftrightarrow x = A^{-1}b$ .  $\therefore ④$  holds  
 $\therefore ⑤$  holds  $\therefore$  the RREF of  $A \in In$ . □

eg) Is  $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}$  invertible?

no: this matrix has no pivot in column 2  
 $\therefore \text{rank}(A) < 3 \therefore \text{not invertible.}$

(eg) Suppose we need to solve similar systems over & over. It's then worth it to do row reduction once to find the inverse, then never row reduce again.

Say you need to solve  $Ax_1 = b_1$

then  $Ax_2 = b_2$   
⋮

Just find  $A^{-1}$  by

$$[A | I_n] \sim \dots \sim [I_n | A^{-1}]$$

then:  $x_1 = A^{-1}b_1$   
 $x_2 = A^{-1}b_2$   
⋮

(matrix multiplication  
is a lot faster &  
easier than row  
reduction)

But remember that these great things only work  
for the special case of square invertible matrices.

Wonderful! But what does this have to do with vector spaces, linear independence, spanning sets and bases? What about  $M \times N$  matrices?

Three ways of looking at a linear system:

① As numbers: 
$$\begin{array}{l} x+2y+3z = 4 \\ 2x-y+z = 2 \\ x + 2z = 0 \end{array} \quad \left. \begin{array}{l} x+2y+3z = 4 \\ 2x-y+z = 2 \\ x + 2z = 0 \end{array} \right\}$$
 a system of linear equations

② As vectors: 
$$x \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$
 a linear combination of vectors

③ As matrices: 
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$
 a matrix equation

We solve all of these by row reducing the same augmented matrix  $[A | \vec{b}]$ .

Consider ②  $\leftrightarrow$  ③: so the set of all linear combinations of the columns of  $A$  is the set of all vectors  $A\vec{x}$ , for various choices of  $\vec{x}$ .

We now define some key subspaces arising from matrices. We do this for arbitrary matrices, they don't have to be square.

Def.: Let  $A$  be an  $m \times n$  matrix, with columns  $\{\vec{u}_1, \dots, \vec{u}_n\} \subseteq \mathbb{R}^m$ .

Then

$$\text{Col}(A) = \text{span}\{\vec{u}_1, \dots, \vec{u}_n\}$$

is a subspace of  $\mathbb{R}^m$ , called the **column space** of  $A$ . We can also write

$$\text{Col}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}.$$

(eg)  $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}$   $2 \times 3$  matrix.

$$\begin{aligned} \text{Col}(A) &= \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\right\} \\ &= \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} \quad \text{since the rest are multiples of } \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &\subseteq \mathbb{R}^2 \end{aligned}$$

$$\begin{aligned} \{A\vec{x} \mid \vec{x} \in \mathbb{R}^3\} &= \left\{ \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \\ &= \{a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 3 \\ 6 \end{bmatrix} + c\begin{bmatrix} 4 \\ 8 \end{bmatrix} \mid a, b, c \in \mathbb{R}\} = \text{Col}(A)! \end{aligned}$$

$$(eg) A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{Col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$$

↑ ↑  
L1 L2 ... basis for  $\mathbb{R}^2$

$$\therefore \text{Col}(A) = \mathbb{R}^2.$$

Fact 1:  $A\vec{x} = \vec{b}$  is consistent (i.e. has at least one solution)

$\Leftrightarrow \vec{b}$  is a linear combination of the columns of  $A$  (by (b))

$\Leftrightarrow \vec{b} \in \text{Col}(A).$

eg)  $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \quad A\vec{x} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$  has a solution

$\hookrightarrow$  we can solve  $\begin{array}{l} x + 3y + 4z = b_1 \\ 2x + 6y + 8z = b_2 \end{array} \leftarrow$

$\Rightarrow b_2 = 2b_1 \quad \sim \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \text{Col}(A).$

eg)  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{Col}(A) = \mathbb{R}^2$

$\Rightarrow A\vec{x} = \vec{b}$  is consistent for EVERY  $\vec{b} \in \mathbb{R}^2.$

FACT 2:  $A\vec{x} = \vec{0}$  has a unique solution

$\Leftrightarrow$  there is a unique linear combination of the columns of  $A$  that gives the  $0$  vector

$\Leftrightarrow$  the columns of  $A$  are LI.

eg)  $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}$   $A \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  &  $A \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$  many solutions to  $Ax = 0$

$\Leftrightarrow$  columns of  $A$  are LD.

eg)  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$   $[A|0] = \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array} \right]$

leading 1 in each column

$\Rightarrow$  unique solution to  $Ax = 0$ .

$\therefore$  columns of  $A$  are LI.

Consequence 1: if  $A\vec{x} = \vec{0}$  has a unique solution then the columns of  $A$  form a basis for  $\text{Col}(A)$ . (LI + span( $\text{Col}(A)$  by definition))

Consequence 2: the column space algorithm (Ch 16)