

## Chapter 22: Eigenvalues and eigenvectors

We talked a lot about getting "nice" bases for subspaces. In this final section, we will find, for each square matrix  $A$ , a basis of eigenvectors "vectors belonging to  $A$ "

that let us understand matrix multiplication geometrically.

eg)  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

multiplication by  $A$  takes

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ to } \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ to } \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ to } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Def<sub>n</sub>: Let  $A$  be an  $n \times n$  matrix.

Suppose  $\lambda \in \mathbb{R}$  "lambda"

and  $\vec{x} \in \mathbb{R}^n$ ,  $\vec{x} \neq \vec{0}$

are such that

$$A\vec{x} = \lambda\vec{x}$$

pronunciation  
eye-gan

Then  $\lambda$  is an eigenvalue of  $A$

and  $\vec{x}$  is an eigenvector of  $A$ .

We say:  $\vec{x}$  is an eigenvector of  $A$  associated to  $\lambda$ .

eg)  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  is an eigenvector of  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  with

eigenvalue  $\lambda=2$ .  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  is an eigenvector of  $A$  with eigenvalue 0

eg) If  $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$  then:

•  $\lambda=-1$  is an eigenvalue since  $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$   
(and  $(1, -1)$  is an associated eigenvector)

•  $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is an eigenvector since  $\vec{x} \neq \vec{0}$  and

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

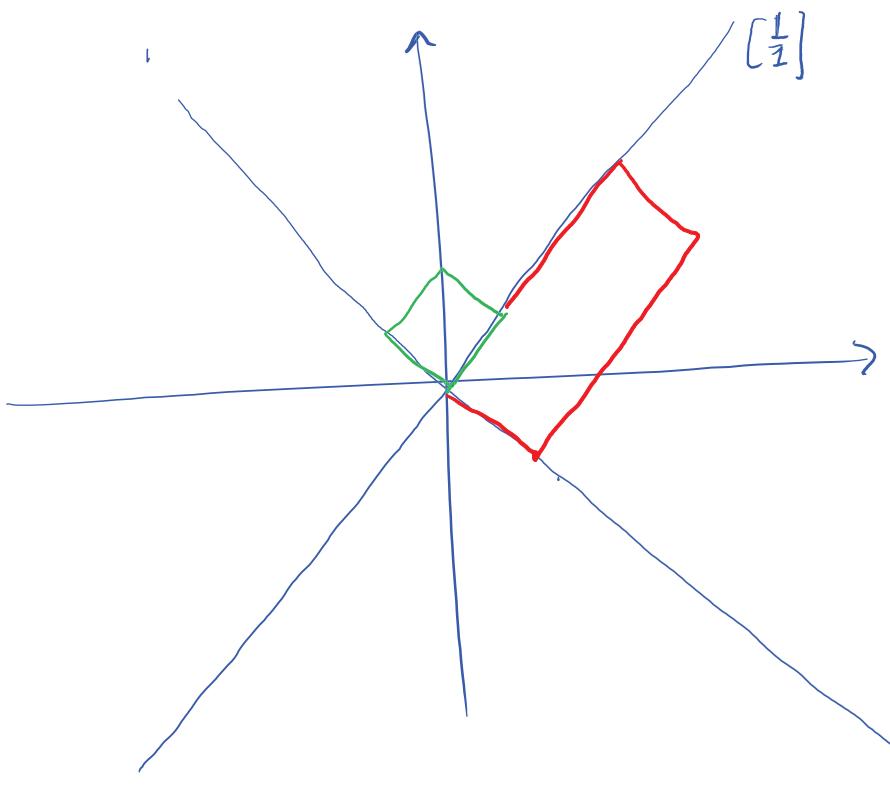
and its associated eigenvalue is 3.

•  $\begin{bmatrix} 7 \\ -7 \end{bmatrix}, \begin{bmatrix} 40 \\ -40 \end{bmatrix}, \begin{bmatrix} -19\pi \\ 19\pi \end{bmatrix}$  are also eigenvectors with eigenvalue -1.



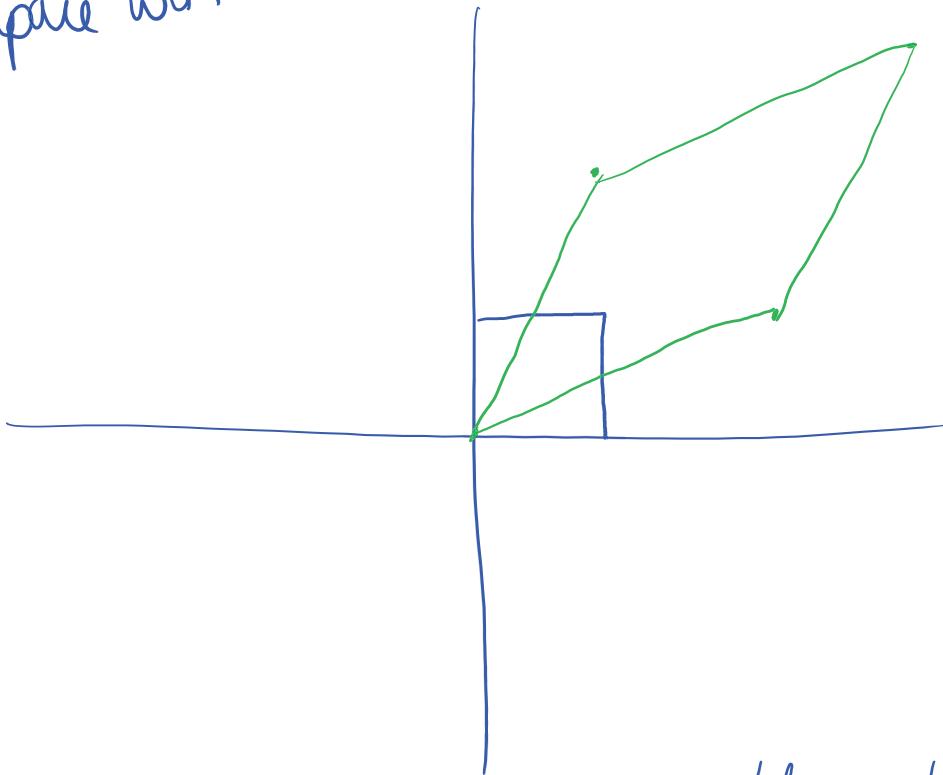
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$  is NEVER an eigenvector.

$$\boxed{\begin{aligned} A\vec{0} &= \lambda\vec{0} \\ \text{for every } \lambda, \\ \text{so too boring} \end{aligned}}$$



multiplication by  
 $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$  takes the  
box on axes  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$   
to the box  
on same axes  
but scaled by  
 $\lambda$  in each axis.

compare with



box on axes  
 $(1,0), (0,1)$   
goes to  
parallelogram  
on axes  
 $(1,2), (2,1)$ .

If 0 is an eigenvalue then there is a (nonzero) eigenvector  $x$  associated to  $\lambda=0$ :  
so  $A\vec{x} = 0\vec{x} = \vec{0} \Rightarrow x$  is a nontrivial soln  
to  $Ax=0 \rightarrow A$  not inv.

May 30

eg) Let  $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ . Then  $B \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ .

so 0 is an eigenvalue of B.

① How do we find eigenvalues?

We notice that 0 is an eigenvalue if and only if A is not invertible, because  $AX=0$  has a nontrivial solution.

So here is what we can say in general:

$\lambda$  is an eigenvalue of A

$\Leftrightarrow$  there is a nonzero vector  $x$  such that

$$Ax = \lambda \vec{x}$$

$$\Leftrightarrow Ax - \lambda \vec{x} = \vec{0}$$

$$\Leftrightarrow Ax - \lambda I \vec{x} = \vec{0}$$

$$\Leftrightarrow (A - \lambda I) \vec{x} = \vec{0}.$$

$\Leftrightarrow$  there is a nontrivial solution to

$$(A - \lambda I) \vec{x} = \vec{0}$$

$\Leftrightarrow A - \lambda I$  is not invertible.

$\lambda$  is an unknown number we want to solve for

Recall:  $B$  is invertible if and only if  $\det B \neq 0$ .  
 $\therefore B$  not invertible if and only if  $\det B = 0$ .

$\therefore \lambda$  is an eigenvalue of  $A$  if and only if  
 $\det(A - \lambda I) = 0$ .  $\leftarrow$  take this equation and solve for  $\lambda$ .

eg)  $A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{bmatrix}$$

where  $\lambda$  is an unknown we want to solve for.

$$\begin{aligned}\det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)(1-\lambda) - 4(4) \\ &= \lambda^2 - 2\lambda + 1 - 16 \\ &= \lambda^2 - 2\lambda - 15 \rightarrow \text{a quadratic polynomial.}\end{aligned}$$

So  $\det(A - \lambda I) = 0$

$$\Leftrightarrow \lambda^2 - 2\lambda - 15 = 0$$

$$\Leftrightarrow (\lambda - 5)(\lambda + 3) = 0$$

$$\Leftrightarrow \lambda = 5 \text{ or } \lambda = -3$$

$\therefore$  the only eigenvalues of  $A$  are  $\lambda = -3$  &  $\lambda = 5$ .

Let's check:

$$A - 5I = \begin{bmatrix} 1-5 & 4 \\ 4 & 1-5 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \text{ which is not invertible.}$$

$$A - (-3I) = \begin{bmatrix} 1+3 & 4 \\ 4 & 1+3 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \text{ which is not invertible.}$$

good!

Def'n: If  $A$  is  $n \times n$  then

$$\det(A - \lambda I)$$

is called the characteristic polynomial of  $A$ . It is a polynomial of degree  $n$ .

eg)  $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & -7 \end{bmatrix}$  Find the characteristic polynomial of  $A$  and find all eigenvalues of  $A$

Soln:  $\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 3 & 4 \\ 0 & 5-\lambda & 6 \\ 0 & 0 & -7-\lambda \end{bmatrix}$

$$= (2-\lambda)(5-\lambda)(-7-\lambda)$$

the characteristic polynomial of  $A$

since  $\det$  of a triangular matrix is the product of the diagonals.

The eigenvalues of  $A$  are the roots of the characteristic polynomial:

$$(2-\lambda)(5-\lambda)(-7-\lambda) = 0$$

$$\Leftrightarrow -(\lambda-2)(\lambda-5)(\lambda+7) = 0$$

$$\Leftrightarrow \underbrace{\lambda=2 \text{ or } \lambda=5 \text{ or } \lambda=-7}_{\text{the eigenvalues of } A}.$$

□

\*the eigenvalues of a triangular matrix are the numbers on the diagonal

② How do we find eigenvectors?

Once you know an eigenvalue  $\lambda$ , an eigenvector associated to  $\lambda$  is a solution to

$$(A - \lambda I) \vec{x} = \vec{0}$$

i.e.  $\vec{x}$  is a nonzero vector in  $\text{Null}(A - \lambda I)$ .

Def.: If  $\lambda$  is an eigenvalue of  $A$  then the subspace

$$\begin{aligned} E_\lambda &= \text{Null}(A - \lambda I) \\ &= \{ \vec{x} \mid (A - \lambda I) \vec{x} = \vec{0} \} \\ &= \{ \vec{x} \mid A \vec{x} = \lambda \vec{x} \} \end{aligned}$$

is called the  $\lambda$ -eigenspace of  $A$  and any nonzero vector in  $E_\lambda$  is an eigenvector of  $A$  (with eigenvalue  $\lambda$ ).

\* So instead of asking "what are the eigenvectors of  $A$ ?" (there are  $\infty$  many!) we can ask "What is a basis for each eigenspace of  $A$ ?"

eg)  $A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$  eigenvalues were 5 & -3.

$$\lambda=5: E_5 = \left\{ \vec{x} \mid (A-5I)\vec{x} = \vec{0} \right\} = \text{Null}(A-5I) = \text{Null}\left(\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\right).$$

solve:  $\begin{bmatrix} -4 & 4 & | & 0 \\ 4 & -4 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x-y=0 \\ 0=0 \end{array}$

$$\text{so } \text{Null}(A-5I) = \left\{ \begin{bmatrix} y \\ y \end{bmatrix} \mid y \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$\therefore$  a basis for  $E_5$  is  $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ .

$$\lambda=-3: [A-(-3I) \mid 0] = \begin{bmatrix} 4 & 4 & | & 0 \\ 4 & 4 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \quad x+y=0 \Rightarrow x=-y$$

$$\text{so } \text{Null}(A+3I) = \left\{ \begin{bmatrix} -t \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

$\therefore$  a basis for  $E_{-3}$  is  $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

Notice:  $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$  is a basis for  $\mathbb{R}^2$ ...

eg) Find a basis for each eigenspace of

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & -7 \end{bmatrix}$$

$$\lambda=2: A-2I = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & -9 \end{bmatrix}$$

$$\therefore [A-2I | \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}] = \begin{bmatrix} 0 & 3 & 4 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & -9 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 0 & 3 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} y=0 \\ z=0 \\ x=\text{anythn} \end{array}$$

$$E_2 = \{(r, 0, 0) | r \in \mathbb{R}\} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

so a basis for  $E_2$  is  $\{(1, 0, 0)\}$ .

*check!*

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \checkmark$$

similarly (ex):

$$\begin{aligned} \cdot \text{ a basis for } E_5 \text{ is } & \{(1, 1, 0)\} & A \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 0 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \checkmark \\ \cdot \text{ a basis for } E_{-7} \text{ is } & \{(-5, -9, 18)\} & A \begin{pmatrix} -5 \\ -9 \\ 18 \end{pmatrix} = \begin{pmatrix} 35 \\ 63 \\ -7 \times 18 \end{pmatrix} = -7 \begin{pmatrix} 5 \\ 9 \\ 18 \end{pmatrix} \checkmark \end{aligned}$$

ex: check that

$\{(1, 0, 0), (1, 1, 0), (-5, -9, 18)\}$  is a basis of  $\mathbb{R}^3$   
(consisting entirely of eigenvectors of  $A$ )

Theorem: Let  $A$  be an  $n \times n$  matrix.

Then any set of eigenvectors of  $A$  corresponding to distinct eigenvalues is LI.

So: start with an  $n \times n$  matrix.

The characteristic polynomial  $\det(A - \lambda I)$

has degree  $n$

$\therefore$  it has at most  $n$  roots.

Each root is an eigenvalue.

Each eigenspace has dimension  $\geq 1$ .

So if we had  $n$  distinct <sup>real</sup> roots, then we would have an eigenvector basis of  $\mathbb{R}^n$ .

(and sometimes we would have an eigenvector basis of  $\mathbb{R}^n$  even if some roots are repeated).

Def'n: A matrix  $A$  is called **diagonalizable** if there is a basis of  $\mathbb{R}^n$  consisting of eigenvectors of  $A$ .

Not every matrix is diagonalizable.  
TWO problem cases:

① complex roots to characteristic polynomial.

Consider  $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ .  $\theta \in \mathbb{R}$

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix} \\ &= (\cos\theta - \lambda)(\cos\theta - \lambda) + \sin^2\theta \\ &= \lambda^2 - 2\cos\theta\lambda + \underbrace{\cos^2\theta + \sin^2\theta}_1 \end{aligned}$$

Roots:

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

$$= \cos\theta \pm \sqrt{-\sin^2\theta}$$

$$= \cos\theta \pm \sin\theta i \quad \in \mathbb{C} \quad \text{not real.}$$

\* so we can solve this problem if we look for eigenvectors in the complex vector space  $\mathbb{C}^2$ .  
(not this year)

If your matrix has complex eigenvalues, it's not diagonalizable

## ② not enough eigenvectors

eg)  $A = \begin{bmatrix} 4 & 3 \\ 0 & 4 \end{bmatrix}$  Triangular matrix, so eigenvalue is 4 with algebraic multiplicity 2.

$$A - 4I = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \text{ which has rank 1.}$$

$$\begin{aligned}\therefore \dim E_4 &= \dim \text{Null}(A - 4I) \\ &= 2 - \text{rank}(A - I) \\ &= 2 - 1 = 1.\end{aligned}$$

But this is the only eigenspace and it's only 1-dimensional — so we can't get 2 LI eigenvectors for this matrix.  
\* so this matrix is not diagonalizable.

If your characteristic polynomial has repeated roots, it CAN happen that A is not diagonalizable (but not always!)