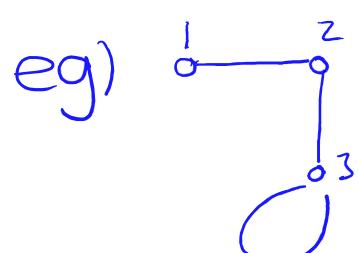


Mar 2 : matrix multiplication (Chap 14)
 and subspaces of \mathbb{R}^n and \mathbb{R}^m coming from
 $m \times n$ matrices.

Recall: To multiply 2 matrices A & B:

- # columns of A = # rows of B
- dot product of row i of A with column j of B gives the (i,j) entry of the matrix AB.

eg)  adjacency matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Then $A^2 = AA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

\uparrow # edges from i to j \uparrow # paths of length 2 from i to j.

We saw:

- $AB \neq BA$ (usually); sometimes "BA" isn't even allowed.
- it can happen that $CD = 0$ even though neither matrix is 0.
Why? because 0 just means the rows of C are orthogonal to the columns of D.

Another issue with matrix multiplication:

(eg) $A = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 7 & -1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

Then

$$AC = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \quad \text{same}$$

$$\& BC = \begin{bmatrix} 1 & 0 \\ 7 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$$

So

$$AC = BC \quad \text{You CANNOT cancel } C.$$

$$\not\Rightarrow A = B.$$

Now for some good properties of matrix multiplication.
To state them all we need 2 more ingredients.

① Remember the transpose of a matrix

"swap rows and columns"

(eg) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$

If A is $m \times n$
then
 A^T is $n \times m$.

basic facts:

$$(A + B)^T = A^T + B^T$$

$$(kA)^T = kA^T \quad \text{for any } k \in \mathbb{R}$$

$$(A^T)^T = A$$

② Identity matrix I (of size $n \times n$): $I_1 = 1$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ etc}$$

③ Zero matrix 0 (of size $m \times n$): $0_{m \times n}$

Theorem (properties of matrix multiplication)

Let A, B, C be matrices and $k \in \mathbb{R}$ a scalar.

Then, whenever defined, we have:

- 1) $A(BC) = (AB)C$ associativity
- 2) $A(B+C) = AB + AC$ distributivity on right
- 3) $(A+B)C = AC + BC$ " on left
- 4) $k(AB) = (kA)B = A(kB)$

But $AB = AC \not\Rightarrow B = C$ $\left. \begin{array}{l} \text{no cancellation} \\ \text{in general} \end{array} \right\}$
 $AB = CB \not\Rightarrow A = C$ $\left. \begin{array}{l} \text{not commutative in general} \end{array} \right\}$

- 5) $AI_n = A$ if A is $m \times n$ $\left. \begin{array}{l} \text{multiplying by identity} \\ \text{matrix is like multiplying} \\ \text{by 1.} \end{array} \right\}$
- $I_m A = A$
- $A0_{n \times p} = 0_{m \times p}$ $\left. \begin{array}{l} \text{multiplying by zero} \\ \text{matrix is like multiplying} \\ \text{by 0 - except size can change.} \end{array} \right\}$
- $0_{q \times m} A = 0_{q \times n}$

6) If AB is defined then so is $B^T A^T$

and $(AB)^T = B^T A^T$.

notice the change in order!

Why are these things true?

1) this takes work. See the textbook or try some examples to convince yourself it is true.

2) Say $A = \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_m \end{bmatrix}$ where $\vec{r}_i \in \mathbb{R}^n$ are row vectors.

say $B = [\vec{u}_1 \dots \vec{u}_k]$ and $C = [\vec{v}_1 \dots \vec{v}_k]$ where \vec{u}_i, \vec{v}_i are column vectors.

So the (i,j) entry of AB is $\vec{r}_i \cdot \vec{u}_j$

the (i,j) entry of AC is $\vec{r}_i \cdot \vec{v}_j$

and the (i,j) entry of $A(B+C)$ is $\vec{r}_i \cdot (\vec{u}_j + \vec{v}_j)$

\therefore equal, by properties of the dot product

3,4): try them yourself

5) example : $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$
(try it!)

(& if you multiply by a zero matrix, all dot products are zero.)

6) So if A is $m \times n$
 B is $n \times p$
 $\Rightarrow AB$ is $m \times p$.

Then A^T is $n \times m$ } so $B^T A^T$ is defined
 B^T is $p \times n$ } and has size $p \times m$

Now: why is $(AB)^T = B^T A^T$?

The (i,j) entry of AB is $\vec{r}_i \bullet \vec{u}_j$.

So the (j,i) entry of $(AB)^T$ is $\vec{r}_i \bullet \vec{u}_j$.

The (j,i) entry of $B^T A^T$ is $\vec{u}_j \bullet \vec{r}_i$
 jth row of B^T ↑ ith column of A^T

and so they are equal because the dot product is commutative:

$$\vec{r}_i \bullet \vec{u}_j = \vec{u}_j \bullet \vec{r}_i.$$



Simplifying expressions with matrices using this theorem.

$$(A+B)(C+D) = A(C+D) + B(C+D) \quad \text{by 3}$$

$$= AC + AD + BC + BD \quad \text{by 2}$$

So we can multiply matrices.

Can we divide?

"NO":

- ① how could we "undo" the dot product?
- ② We saw $AB = AC \not\Rightarrow B = C$
so we couldn't divide both sides by A , at least for $A \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.

New perspective on ①:

"divide by 2" = "multiply by $\frac{1}{2}$ ".

so we don't need a "division formula" if we have the inverse.

$\frac{1}{2}$ is the inverse of 2
because $\frac{1}{2} \times 2 = 2 \times \frac{1}{2} = 1$.

Def'n: Suppose A is an $n \times n$ matrix.

If you can find an $n \times n$ matrix B

such that $AB = I_n$ and $BA = I_n$

then A is called **invertible**

and B is called the inverse of A

and we write $B = A^{-1}$ (never $\frac{1}{A}$)

e.g.) If $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ then set $B = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \text{ & } BA = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $B = A^{-1}$. □

eg) $I_n^{-1} = I_n$ since $I_n I_n = I_n$.
 So the identity matrix is its own inverse (just like $1 = \frac{1}{1}$).

Finding A^{-1} , when it exists

Theorem (only for 2×2 matrices!)

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then A is invertible if and only if $ad - bc \neq 0$,

in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Check: $AA^{-1} = I_2$?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (\text{& other directions same}).$$

switch the diagonal
 & negate the off-diagonal.

Why do we need $ad - bc \neq 0$?

Well: say $ad - bc = 0$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d \\ -c \end{bmatrix} = \begin{bmatrix} ad - bc \\ cd - cd \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A \vec{x} = \vec{0}$$

If we had an inverse we could multiply
and get

$$A^{-1} A \vec{x} = A^{-1} \vec{0} = \vec{0}$$

$$\overset{\parallel}{I} \vec{x}$$

$$\overset{\parallel}{\vec{x}}$$

$$\Rightarrow \vec{x} = \vec{0} \Rightarrow c = d = 0.$$

but if $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

then $AB = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$ no matter how
you choose B
 \therefore impossible to find an inverse of A .

So how can we find A^{-1} if A is bigger than 2×2 ?

Let's think about what we want.

A say a 3×3 matrix.

We want B (3×3) such that

$$AB = I_3$$

Write B as columns:

$$A[\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Leftrightarrow A\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A\vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Here, $\vec{u}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is an unknown vector.

What is " $A\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ "?

$$\text{eg } A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 2 & 5 \end{bmatrix} \therefore A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} x + y + 2z \\ x + 2y + 5z \\ 2x + 2y + 5z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow x + y + 2z = 1$$

$$x + 2y + 5z = 0$$

$$2x + 2y + 5z = 0$$

a linear system, which we solve
using the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & 5 & 0 \\ 2 & 2 & 5 & 0 \end{array} \right]$$

\therefore To find \bar{u}_1 , row reduce

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & 5 & 0 \\ 2 & 2 & 5 & 0 \end{array} \right]$$

To find \bar{u}_2 , row reduce

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 5 & 1 \\ 2 & 2 & 5 & 0 \end{array} \right]$$

To find \bar{U}_3 , raw reduce $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 5 & 0 \\ 2 & 2 & 5 & 1 \end{array} \right]$.

But wait! When you do raw reduction all the hard work, and all the steps you do, are dictated by the coefficient matrix. So we could do exactly the same operations 3 times — what a waste.

Instead: augment the matrix with 3 columns:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 5 & 0 & 1 & 0 \\ 2 & 2 & 5 & 0 & 0 & 1 \end{array} \right] = [A | I_n]$$

In raw reduction, columns never interact, so you could cover up the last two columns to see the raw reduction of the first system, for example.

We do the raw reduction:

$$\sim \begin{array}{l} -R_1 + R_2 \\ -2R_1 + R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right]$$

In REF — we see at this point that all 3 systems are consistent \therefore there will be a solution! ✓

Moreover: no nonleading variables
so the solution will be unique ✓

∴ continue:

$$\begin{array}{l} -2R_3 + R_1 \\ -3R_1 + R_2 \\ \sim \end{array} \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 5 & 0 & -2 \\ 0 & 1 & 0 & 5 & 1 & -3 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 5 & 1 & -3 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right]$$

so: the solution to $A\tilde{U}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the first column: $\tilde{U}_1 = \begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix}$
 & the solution to $A\tilde{U}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the second column: $\tilde{U}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$
 & ... $\tilde{U}_3 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$

$$\therefore A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 5 & 1 & -3 \\ -2 & 0 & 1 \end{bmatrix}$$

Check!

$$AA^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 5 & 1 & -3 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5-4 & 0 & 0 \\ 0 & -1+2 & 0 \\ 0 & 0 & 2-6+5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Even better, it also works on the left:

$$A^{-1}A = \begin{bmatrix} 0 & -1 & 1 \\ 5 & 1 & -3 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} -1+2 & 0 & 0 \\ 0 & 5+2-6 & 0 \\ 0 & 0 & -4+5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So yes row reducing $[A | I_n] \sim \dots \sim [I_n | A^{-1}]$
 gives the inverse of A as long as A reduces to I_n
 (if not, A is not invertible).