

Mar 28

Last time: complex numbers ( $\mathbb{C}$ ).

Two ways to write a complex number:

1) rectangular form  $z = a + ib$

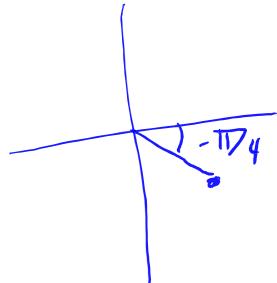
2) polar form  $z = re^{i\theta}$ .

① is all-purpose, ② is fabulous for multiplication.

(In EE & Physics,  
you might use  
 $j = \sqrt{-1}$  instead of  $i$ .  
We use  $i$ .)

eg) Convert  $z = 2e^{-i\pi/4}$  to rectangular coordinates.

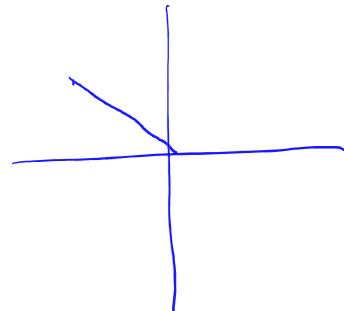
$$\begin{aligned}\text{Sol: } z &= 2e^{-i\pi/4} = 2 \cos(-\pi/4) + i \cdot 2 \sin(-\pi/4) \\ &= 2\left(\frac{1}{\sqrt{2}}\right) - 2\left(\frac{1}{\sqrt{2}}\right)i \\ &= \sqrt{2} - \sqrt{2}i \quad \square\end{aligned}$$



eg) Convert  $z = -5\sqrt{3} + 5i$  to polar form.

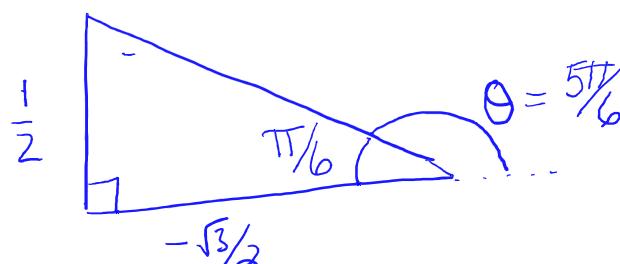
$$\begin{aligned}\text{Sol: } r &= |z| = \sqrt{(-5\sqrt{3})^2 + 5^2} \quad \leftarrow \text{no "i"!} \\ &= \sqrt{25 \cdot 3 + 25} \\ &= \sqrt{100} \\ &= 10\end{aligned}$$

$$\begin{aligned}r &= |z| \\ &= \sqrt{a^2 + b^2} \\ \cos\theta &= \frac{a}{r} \\ \sin\theta &= \frac{b}{r} \\ \tan\theta &= \frac{b}{a}\end{aligned} \quad \left\{ \text{find } \theta \right.$$



$$\cos\theta = -\frac{5\sqrt{3}}{10} = -\frac{\sqrt{3}}{2}$$

$$\sin\theta = \frac{5}{10} = \frac{1}{2}$$



$$\text{so } \theta = 5\pi/6.$$

(we needed both  $\sin\theta$  &  $\cos\theta$  to solve this!)

$$\therefore z = 10e^{i\frac{5\pi}{6}} \quad \square$$

The final major topic of the term is eigenvalues, eigenvectors and diagonalization.

We need a little bit of complex numbers,  
+ a lot of determinants  
to explain what is going on.

## Chap 21: Determinants.

(only for square matrices)

The determinant of a square matrix  $A$  is a number.

It tells us two things:

- ① algebraically:  $\det(A) \neq 0 \Leftrightarrow A$  is invertible.
- ② geometrically:  $|\det(A)|$  is the  $n$ -dimensional volume of the "parallelepiped" spanned by the rows of  $A$ .

The formula is horrible, and it is recursive.

$$\bullet A = [a], \ 1 \times 1 \text{ matrix:} \quad \det A = a.$$

$$\bullet A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ 2 \times 2 \text{ matrix:} \quad \det A = ad - bc \\ = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\bullet A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \ 3 \times 3 \text{ matrix:} \quad \text{straight lines mean det.}$$

$$\det A = a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg).$$

In general:

Say  $A$  is  $n \times n$ .

$a_{ij}$  is the entry in row  $i$ , column  $j$  of  $A$

$A_{ij}$  is the matrix you get by removing row  $i$  and column  $j$  from  $A$ .

Then

$$\det A = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + \dots + (-1)^{1+n} a_{1n} \det(A_{1n})$$

"cofactor expansion on first row of  $A$ ".

eg)  $A = \begin{bmatrix} 2 & 5 & -4 & -3 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$$\begin{aligned} \det A &= 2 \det \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - 5 \det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + (-4) \det \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad - (-3) \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\cdot \det \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = 0( ) - 0( ) + 1 \underbrace{\det \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{-1} = -1$$

$$\cdot \det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = 1 \underbrace{\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{-1} - 0 + 1 \underbrace{\det \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_0 = -1$$

$$\cdot \det \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 1 \det \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} - 0 + \det \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = 0$$

$$\cdot \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 \underbrace{\det \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{-1} - 0 + 0 = -1$$

$$\therefore \det A = 2(-1) - 5(-1) - 4(0) + 3(-1) = 0.$$

Notice:  $R1 = 2R2 - 5R3 - 4R4$

so rows are LD  
A is not invertible.

□

Calculating determinants is HARD:

for an  $n \times n$  matrix, you have

$n$   $(n-1) \times (n-1)$  determinants

$\therefore n(n-1)$   $(n-2) \times (n-2)$  determinants

$\therefore n!$  determinants to compute.

$$n! = n(n-1)\dots(3 \times 2 \times 1)$$

$$3! = 3 \times 2 \times 1 = 6$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$$5! = 5 \times 4! = 120$$

eg)  $10! \sim 3$  million so you're not going to calculate  $\det A$  for a  $10 \times 10$  matrix this way.

First shortcut: cofactor expansion on any row or column.

Thm: If  $A$  is an  $n \times n$  matrix

with entries  $a_{ij}$

& where  $A_{ij}$  means "remove row  $i$  & column  $j$ ".

Then for any row  $i$ :

$$\det A = (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + \dots + (-1)^{i+n} a_{in} \det A_{in}$$

and for any column  $j$ :

$$\det A = (-1)^{1+j} a_{1j} \det(A_{1j}) + (-1)^{2+j} a_{2j} \det(A_{2j}) + \dots + (-1)^{n+j} a_{nj} \det(A_{nj}).$$

(So you can use whatever formula will be easiest.)

Tip: The signs  $(-1)^{i+j}$  are just

$$\left[ \begin{array}{cccccc} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array} \right]$$

eg)  $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ 6 & 0 & 7 \end{bmatrix}$ :

on first row:  $1 \begin{vmatrix} 0 & 5 \\ 0 & 7 \end{vmatrix} - 2 \begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} + 3 \begin{vmatrix} 4 & 6 \\ 6 & 0 \end{vmatrix} = 1(0) - 2(-2) + 3(0) = 4.$

on second column:

$$\begin{bmatrix} + & (-) & + \\ - & (+) & - \\ + & (-) & + \end{bmatrix} : -2 \det \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} + 0 \det \begin{bmatrix} 1 & 3 \\ 6 & 7 \end{bmatrix} = 0 \det \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$$

$$= -2(-2) + 0 - 0$$

$$= 4$$

Tip: expand along the row or column with the most zeros.

eg)  $\det \begin{bmatrix} 2 & 5 & -4 & -3 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

~~choose row 4~~  $\rightarrow$   $\begin{bmatrix} + & - & + & - \\ - & + & - & + \end{bmatrix}$

$$= -0 + 0 - 1 \det \begin{bmatrix} 2 & 5 & -3 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} + 0$$

$\xrightarrow{\quad \quad \quad \quad \quad}$

$$= -1 \left( -1 \det \underbrace{\begin{bmatrix} 5 & -3 \\ -1 & 1 \end{bmatrix}}_{5 - (-3) = 2} + 0 - 1 \det \begin{bmatrix} 2 & 5 \\ 0 & -1 \end{bmatrix} \right)$$

$$= -1((-1)(2) + (-1)(-2))$$

$$= -(-2 + 2)$$

$$= 0$$

less work.

## Quick properties of the determinant:

1) If A has a row or column of 0s, then  
 $\det A = 0$ .

because if you expand on that row,  
each term is 0.

2)  $\det A = \det(A^T)$

because expanding on the first column of  
A gives the same calculation as expanding  
on the first row of  $A^T$ .

3) If A is triangular then  $\det A$  is the product  
of the diagonal entries.

A matrix  $A = (a_{ij})$  is triangular if  $a_{ij} = 0$  for  $i > j$ .

i.e.  $A = \begin{bmatrix} * & * & * & 0 & 0 \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$  is triangular.

$$\text{eg } \det \begin{bmatrix} 3 & 4 & 5 & 7 \\ 0 & 7 & -2 & -4 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix} = +3 \det \begin{bmatrix} 7 & -2 & -4 \\ 0 & 1 & 9 \\ 0 & 0 & -1 \end{bmatrix} - 0 + 0 - 0$$

$$= 3 \left( 7 \cdot \det \begin{bmatrix} 1 & 9 \\ 0 & -1 \end{bmatrix} - 0 + 0 \right)$$

$$= 3 \cdot 7 \cdot (1 \cdot -1 - 0)$$

$$= 3 \cdot 7 \cdot 1 \cdot -1 = -21 \quad \square$$

Can we use row reduction to find the determinant?  
Not exactly:

Say  $R$  is a matrix in RREF.  
Then  $R$  is triangular.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

or  $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  or  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ or } 0$

Either  $R$  is invertible  $\Leftrightarrow R = I \Leftrightarrow \det R = 1$   
or  $R$  is not invertible  $\Leftrightarrow$  a row of 0s  $\Leftrightarrow \det R = 0$ .

So if  $R$  is the RREF of  $A$ : usually  $\underbrace{\det A}_{\in R} \neq \underbrace{\det R}_{1 \text{ or } 0}$ .

Second shortcut: *keeping track of row operations to calculate the determinant.*

Theorem (effect of row operations on the determinant)  
Say you start with an  $n \times n$  matrix  $A$ , and do a row operation, and the result is the matrix  $B$ .

Then:

- 1) If the row operation was: interchange 2 rows  
then  $\det A = -\det B$ .

2) If the row operation was: multiply a row by c  
then  $\det B = c \det A$ .

3) If the row operation was: add a multiple  
of one row TO another row  
then  $\det A = \det B$ .

(eg)  $A = \begin{bmatrix} 3 & 5 & 7 & 9 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 3 & 7 \end{bmatrix}$  do row reduction &  
KEEP TRACK of row operations.

swap  
 $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 5 & 7 & 9 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 3 & 13 \end{bmatrix} - 3R_1 + R_2 \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & 4 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 13 \end{bmatrix}$$

$$\frac{1}{2}R_3 \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & 4 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

$-3R_3 + R_4$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & 4 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 7 \end{bmatrix} = B$$

$\det B = -7$   
(triangular matrix)

$\therefore \det B = (-1) \cdot (1) \cdot \left(\frac{1}{2}\right) \cdot (1) \det A$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$   
 $R_1 \leftrightarrow R_2 \quad -3R_1 + R_2 \quad \frac{1}{2}R_3 \quad -3R_3 + R_4$

← you multiplied row 1  
A by  $\frac{1}{2}$  so you  
have to multiply  
 $\det A$  by  $\frac{1}{2}$ .

$\Rightarrow -7 = -\frac{1}{2} \det A$

$\Rightarrow \det A = 14$ .  $\square$

\* Keep track of your row operations; then you can  
get  $\det A$  for free

## More properties of determinants

4)  $\det(cA) = c^n \det A$  if  $A$  is  $n \times n$ .  
 $c$  is a scalar.  
 because you multiplied each of  $n$  rows by  $c$ .

Geometrically: multiply each vector by  $c \Rightarrow$  scale  $n$ -dimensional volume by  $c^n$ .

5)  $\det(AB) = \det A \det B$   
 (fabulous property — if multiplication by  $B$   
 scales volumes by a factor of  $\det B$  & multiplication  
 by  $A$  scales volumes by a factor of  $\det A$   
 then  $AB$  scales volume by  $\det A \det B$ .)

6)  $\det A = 0$  if and only if  $A$  is not invertible.

Proof:  $A$  is invertible iff the RREF of  $A$  is  $I_n$ ,  
 in which case  $\det A = r \cdot 1$  where  $r$  comes  
 from keeping track of row operation effects.  
 $A$  not invertible iff the RREF of  $A$  has  
 a zero row  $\Leftrightarrow \det R = 0 \Leftrightarrow \det A = 0$ .  $\square$   
 by row ops.

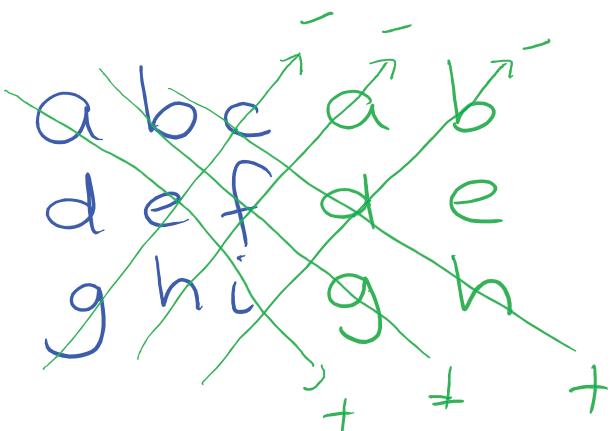
7) If  $A$  is invertible then  
 $\det(A^{-1}) = \frac{1}{\det A}$

} If  $A$  scales volumes  
 by  $\det A$ ;  $A^{-1}$   
 scales by  $\frac{1}{\det A}$ .

Pf:  $A \cdot A^{-1} = I \Rightarrow \det(A \cdot A^{-1}) = 1$   
 $\Rightarrow \det A \cdot \det(A^{-1}) = 1 \quad \checkmark$

Some final tricks & thoughts on determinants:

- 3x3 shortcut DOES NOT WORK on 4x4 or higher!!!



$$\det A =$$

$$aei + bfg + cdh$$

$$- ceg - afh - bdi$$

- n-dimensional volume: special case:

$A = \begin{bmatrix} m_1 & & & 0 \\ & m_2 & & \\ 0 & & \ddots & \\ & & & m_n \end{bmatrix}$  diagonal matrix.

← scales each basis vector by  $m_i$

∴ scales volume by  $|m_1 m_2 \dots m_n|$

- Multivariable Calculus:

Say you're doing a multiple integral and want to make the change of variables from  $(x,y)$  to  $(u,v)$ .

The Jacobian is  $\det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$  and it measures the scaling factor you need on the integral to

compensate for the change in volume.

## Chapter 22: Eigenvalues and eigenvectors

We talked a lot about getting "nice" bases for subspaces. In this final section, we will find, for each square matrix  $A$ , a basis of eigenvectors "vectors belonging to  $A$ "

that let us understand matrix multiplication geometrically.

eg)  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

multiplication by  $A$  takes  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  to  $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$   
 $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  to  $\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$   
 $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  to  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

So takes the unit cube to the rectangular prism with sides  $(2,0,0)$ ,  $(0,3,0)$ ,  $(0,0,1)$ .  
(&  $\det A = \text{volume of this prism}$ )