

Mar 14 - Pi Day

- finish up row space & column space algorithms
- Nullspace (Ch15)

Suppose $U = \text{span}\{\vec{u}_1, \dots, \vec{u}_k\} \subseteq \mathbb{R}^n$ is a subspace of \mathbb{R}^n , where you know a spanning set but want a basis.

Last time: 2 algorithms.

① Column space algorithm.

- Put your vectors as columns of A (so $U = \text{Col}(A)$)
- Reduce to echelon form R
- A basis for U is the set of columns of A that gave leading ones in R .

* so this algorithm implements the theorem called "Reducing spanning sets".

② Row space algorithm

- Put your vectors as rows of B (so $U = \text{Row}(B)$)
- Reduce to RREF (as far as possible) R .
- A basis for U is the set of nonzero rows of R .

Let's show that this algorithm implements the other theorem, called "Enlarging LI sets":

i.e. create the matrix A

i.e. create this matrix

(eg) Suppose we are given the LI set
 $\{(1, 1, 2, 3), (2, 2, -1, 2)\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \\ 2 \end{bmatrix} \right\}$

Goal: extend this to a basis of \mathbb{R}^4 .

remember! We can write vectors both ways, so you have to choose which works for you

* Notice: this is a challenging problem! We agreed it was theoretically possible, but all we could do until now is grab some extra vectors and hope they were good, then check if it gave an LI set.

With the row space algorithm:

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & -5 & -4 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 1 & 2 & 3 \\ 0 & 0 & \textcircled{1} & 4/5 \end{bmatrix}$$

So we got leading 1s in columns 1 & 3 but columns 2 & 4 did not give a leading 1.

∴ we add vectors which give leading 1s in the missing columns. *

So our answer is: $\{(1, 1, 2, 3), (2, 2, -1, 2), (0, 1, 0, 0), (0, 0, 0, 1)\}$

Let's check that this is a basis for \mathbb{R}^4 . We could of course use the column space algorithm, but let's use the row space algorithm instead, so that we see why the trick * works.

$$\text{Let } B = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & -5 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 1 & 2 & 3 \\ 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & \textcircled{1} & 4/5 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$$

we can stop here:
4 leading 1s
⇒ rank(B) = 4

$\therefore \text{Row}(B)$ is 4-dimensional

$\Rightarrow \text{Row}(B) = \mathbb{R}^4$ (because $\text{Row}(B) \subseteq \mathbb{R}^4$)

\Rightarrow the rows of B span \mathbb{R}^4

\Rightarrow the rows of B are LI (because there are $4 = \dim \mathbb{R}^4$ of them, and they span \mathbb{R}^4 — see Chapter 16)

So this works because we added vectors that don't do anything in the row reduction except give leading 1s where leading 1s were missing.

□

* Make sure you keep these algorithms straight in your mind. Practice!

* Never let yourself be fooled by how vectors are written. Eg:

"Find a subset of $\{(1,2,3), (4,5,6), (7,8,9)\}$ which is LI" obviously means you have to use the column space algorithm so the relevant matrix is

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\left(\underline{\text{NOT}} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \right)$$

Another space associated to a matrix: the Nullspace.

Def'n: Let A be an $m \times n$ matrix. Then the **nullspace** (sometimes called kernel) of A is

$$\text{Null}(A) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \}$$

that is, $\text{Null}(A)$ is the set of all solutions to the homogeneous linear system $A\vec{x} = \vec{0}$.

FACT 1: $\text{Null}(A)$ is a subspace of \mathbb{R}^n .

Why? Let's do the subspace test. We have to check

- 3 things:
- that $\vec{0} \in \text{Null}(A)$
 - that if $\vec{u}, \vec{v} \in \text{Null}(A)$, then $\vec{u} + \vec{v} \in \text{Null}(A)$
 - that if $\vec{u} \in \text{Null}(A)$ and $c \in \mathbb{R}$ then $c\vec{u} \in \text{Null}(A)$.

Let's do that.

a) Is $\vec{0} \in \text{Null}(A)$? We need to show that it satisfies the condition. Well: $A \cdot \vec{0} = \vec{0}$ so yes, $\vec{0} \in \text{Null}(A)$.

b) Suppose $\vec{u}, \vec{v} \in \text{Null}(A)$. That means $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$. Is $\vec{u} + \vec{v} \in \text{Null}(A)$? We calculate

$$\begin{aligned} A(\vec{u} + \vec{v}) &= A\vec{u} + A\vec{v} && \text{(distributivity)} \\ &= \vec{0} + \vec{0} = \vec{0} \quad \checkmark && \text{yes.} \end{aligned}$$

c) Suppose $\vec{u} \in \text{Null}(A)$ and $c \in \mathbb{R}$. So $A\vec{u} = \vec{0}$.

Is $c\vec{u} \in \text{Null}(A)$? We calculate:

$$A(c\vec{u}) = c(A\vec{u}) \quad c \text{ is a scalar - see Ch 14.}$$

$$= c(\vec{0})$$

$$= \vec{0} \quad \checkmark$$

so yes.

◦◦ $\text{Null}(A)$ passes the subspace test and is a subspace of \mathbb{R}^n .

□

Example: If $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ (a 1×3 matrix)

$$\text{then } \text{Null}(A) = \{ \vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \vec{0} \}$$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \mid \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \right\}$$

$$= \{ (x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0 \}$$

is a plane through the origin.

More generally, $\text{Null}(A)$ is an intersection of hyperplanes through the origin in \mathbb{R}^n .

Finding a basis for $\text{Null}(A)$ is easy - we've been doing it all along.

(eg) Find a basis for $\text{Null}(A)$ if $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 0 & -4 \end{bmatrix}$.

We solve $A\vec{x} = \vec{0}$ using row reduction:

$$[A \mid \vec{0}] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 2 & 4 & 0 & -4 & 0 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & -6 & -12 & 0 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array} \right] \begin{array}{l} -3R2+R1 \\ -\frac{1}{6}R2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array} \right]$$

\therefore Solution is:

$$a + 2b - 2d = 0$$

$$c + 2d = 0$$

$$b, d \text{ are free} \Rightarrow \begin{matrix} b = s \\ d = t \end{matrix} \Rightarrow \begin{matrix} a = -2s + 2t \\ c = -2t \end{matrix}$$

$$\therefore \text{Null}(A) = \left\{ \begin{bmatrix} -2s + 2t \\ s \\ -2t \\ t \end{bmatrix} \mid s, t \in \mathbb{R} \right\}$$

$$= \left\{ s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \mid s, t \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

\uparrow
LI

\uparrow

(look at the rows with leading 1s!)

$$\therefore \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} \text{ is a basis for Null}(A). \quad \square$$

\uparrow

\uparrow

these are called the **basic solutions** to $A\vec{x} = \vec{0}$:
they are obtained by setting one parameter = 1
& the rest = 0.

Theorem (Basic solutions form a basis for $\text{Null}(A)$).

The spanning set for $\text{Null}(A)$ obtained from the RREF of $[A|0]$ (in other words: the basic solutions of $A\vec{x}=\vec{0}$) is a basis for $\text{Null}(A)$.

That is: LI is automatic.

(Why? Each basic solution corresponds to one nonleading variable. That coefficient is 1, and it is zero in all other basic solutions. So if we had a dependence equation, the coefficient of each basic solution would have to be 0.)

Cool FACT: Rank-Nullity Theorem

$$* \dim \text{Null}(A) + \text{rank}(A) = n$$

\uparrow # columns of A .

$\underbrace{\dim \text{Null}(A)}_{\text{# basic solutions}} = \text{# non leading variables}$

$\underbrace{\text{rank}(A)}_{\text{# leading 1s in an RREF}} = \text{# leading variables}$ $\underbrace{n}_{\text{# variables}}$

We can restate the rank-nullity theorem in other way!

$$\Leftrightarrow \dim \text{Null}(A) + \dim \text{Col}(A) = n$$

$$\Leftrightarrow \dim \text{Null}(A) + \dim \text{Row}(A) = n$$

One big example with $\text{Col}(A)$, $\text{Row}(A)$, $\text{Null}(A)$ etc.

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Find a basis for each of the 3 spaces:

$\text{Col}(A)$, $\text{Row}(A)$, $\text{Null}(A)$,
and extend your basis of $\text{Row}(A)$ to a basis for \mathbb{R}^4 .

Solution: One row reduction does it all.

$$A = \begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 0 & -2 & -2 & 4 \end{bmatrix}$$

$$\begin{array}{l} -2R_2 + R_1 \\ 2R_2 + R_3 \end{array} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R \quad \text{RREF.}$$

a) The leading ones are in columns 1 & 2.
 \therefore a basis for $\text{Col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix} \right\}$.

b) There are 2 nonzero rows in R , so

\therefore a basis for $\text{Row}(A)$ is $\{(1, 0, 1, 1), (0, 1, 1, -2)\}$

which we can also write as $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix} \right\}$ OK ✓

c) The solution to the homogeneous system has a secret augmented columns of 0s:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \Leftrightarrow \begin{array}{l} a + c + d = 0 \Rightarrow a = -c - d \\ b + c - 2d = 0 \Rightarrow b = -c + 2d \\ c, d \text{ free: } c = s \\ d = t \end{array}$$

∴ basic solutions are:

$$\begin{array}{l} c=1 \\ d=0 \end{array} \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right. \quad \begin{array}{l} c=0 \\ d=1 \end{array} \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right.$$

∴ a basis for $\text{Null}(A)$ is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

d) To extend the LI set $\{(1, 0, 1, 1), (0, 1, 1, -2)\}$ to a basis, we put it as the rows of a matrix:

$$\begin{bmatrix} \textcircled{1} & 0 & 1 & 1 \\ 0 & \textcircled{1} & 1 & -2 \\ & & ? & ? \end{bmatrix}$$

(the nonzero rows of R ,
so it's already in RREF)
(yay!)

& see which leading ones are missing.

$$\begin{bmatrix} 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \left\{ \text{add these vectors.} \right.$$

∴ $\{(1, 0, 1, 1), (0, 1, 1, -2), (0, 0, 1, 0), (0, 0, 0, 1)\}$
is a basis of \mathbb{R}^4 which includes the LI set
 $\{(1, 0, 1, 1), (0, 1, 1, -2)\}$.

(A suitable justification for each answer:

(a) by column space algorithm

(b) by row space algorithm

(c) by null space basis theorem

(d) by row space algorithm / extending LI sets algorithm)

Cool fact:

A basis for $\text{Null}(A)$ was

$$B_1 = \{ (-1, -1, 1, 0), (-1, 2, 0, 1) \}$$

and a basis for $\text{Row}(A)$ was

$$B_2 = \{ (1, 0, 1, 1), (0, 1, 1, -2) \}.$$

The dot product of any element of B_1 with any element of B_2 is 0. They are orthogonal sets.

Even better:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ -1 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -1 & 2 & 1 \\ 0 & 2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

swap $R_3 \leftrightarrow R_4$
-1 R_3 , $5R_4 + R_3$

So their union is a basis for \mathbb{R}^4 .

(More on this in the next chapter: orthogonality)

What about the solution set to an inhomogeneous system $A\vec{x} = \vec{b}$?

Is it a subspace? No: $A\vec{0} = \vec{0} \neq \vec{b}$ so the $\vec{0}$ never solves an inhomogeneous system.

But we've noticed that the answers are quite similar nonetheless.

eg) Find the general solution to $Ax = b$ where
 $A = \begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ & $\vec{b} = \begin{bmatrix} 10 \\ 3 \\ 4 \end{bmatrix}$.

Sol'n: we solve
 $[A | \vec{b}] \cong \begin{bmatrix} 1 & 2 & 3 & -3 & | & 10 \\ 0 & 1 & 1 & -2 & | & 3 \\ 1 & 0 & 1 & 1 & | & 4 \end{bmatrix} \sim \sim \begin{bmatrix} 1 & 0 & 1 & 1 & | & 4 \\ 0 & 1 & 1 & -2 & | & 3 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$

So the system is consistent and it has infinitely many solutions

$$\left. \begin{array}{l} x + z + w = 4 \\ y + z - 2w = 3 \\ z, w \text{ free} \end{array} \right\} \begin{array}{l} z = s \\ w = t \\ x = 4 - s - t \\ y = 3 - s + 2t \\ z = s \\ w = t \end{array}$$

$$\text{solution} = \left\{ \begin{bmatrix} 4 \\ 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \mid s, t \in \mathbb{R} \right\}$$

\vec{x}_0

\vec{u}_1
basic solutions!

\vec{u}_2

this is NOT span $\left\{ \begin{bmatrix} 4 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ because $\begin{bmatrix} 4 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ is CONSTANT.

Notice that $A\vec{x}_0 = \begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 4 \end{bmatrix} = \vec{b}$

(thank goodness, since $\vec{x}_0 \in S$, it should be a solution to $A\vec{x} = \vec{b}$).

whereas $A\vec{u}_1 = \begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

& $A\vec{u}_2 = \vec{0}$

(thank goodness: basic solutions are supposed to be in the null space!)

So: $S = \{ \vec{x}_0 + \vec{x} \mid \vec{x} \in \text{Null}(A) \}$

\leftarrow a particular solution to $A\vec{x} = \vec{b}$.

Thm Suppose $Ax = b$ is a consistent linear system. Then if \vec{x}_1, \vec{x}_2 are 2 solutions, $\vec{x}_1 - \vec{x}_2 \in \text{Null}(A)$. If \vec{x}_0 is a solution and $\vec{x} \in \text{Null}(A)$ then $\vec{x}_0 + \vec{x}$ solves $A(\vec{x}_0 + \vec{x}) = \vec{b}$.