

Feb 9 MAT1341C : linear independence and span

Last time, we defined linearly dependent and linearly independent sets of vectors:

Suppose  $V$  is a vector space and  $\{v_1, \dots, v_m\}$  are vectors in  $V$ .

Solve the dependence equation

$$k_1 v_1 + k_2 v_2 + \dots + k_m v_m = 0$$

↑ unknown scalars

↑ homogeneous equation  
⇒ consistent.

There are 2 possible outcomes:

① There is a unique solution, and it is the trivial solution  $k_1 = k_2 = \dots = k_m = 0$ . That is: the ONLY solution is the TRIVIAL solution. Then  $\{v_1, \dots, v_m\}$  is **LI**. Think of this as meaning that each vector in the set is valuable.

② There are infinitely many solutions, so in particular, there are some nontrivial solutions, meaning, at least one of the  $k_i \neq 0$ . Then  $\{v_1, \dots, v_m\}$  is **LD**. Think of this as meaning that one or more vectors are redundant.

## Examples (function spaces)

1) Consider  $\{1, x, x^2\} \subseteq P_2$ . Is it LI or LD? ← the zero polynomial

We have to solve  $a(1) + b(x) + c(x^2) = 0$

$$\Leftrightarrow a + bx + cx^2 = 0 + 0x + 0x^2$$

$$\Leftrightarrow a=0 \text{ and } b=0 \text{ and } c=0.$$

The ONLY solution is the trivial solution, so this set is LI.

2) Consider  $\{1-x^2, x^2, 1+x^2\} \subseteq P_2$ . Is it LI or LD?

We have to solve

$$a(1-x^2) + b(x^2) + c(1+x^2) = 0$$

$$\Leftrightarrow a + c \overset{+0x \text{ (if you want)}}{\star} (-a+b+c)x^2 = 0 + 0x + 0x^2$$

$$\Leftrightarrow a+c=0 \quad \text{and} \quad -a+b+c=0.$$

Oh: I see a nontrivial solution:  $a=1, c=-1, b=2$ .

(since  $a+c=1+(-1)=0$  and  $-a+b+c=-1+2+(-1)=0 \checkmark$ )

Therefore this set is LD.

3) Consider  $\{1, \sin(x), \cos(x)\} \subseteq F(\mathbb{R})$ . Is it LI or LD?

We solve  $a + b\sin(x) + c\cos(x) = 0$  ← 0 function

We think this is LI.

Since this equation has to be true for every value of  $x$ , we can test it by plugging in some numbers:

$$x=0 \Rightarrow a + c = 0$$

$$x=\pi/2 \Rightarrow a + b = 0$$

$$x=\pi \Rightarrow a - c = 0$$

$$\cos(0)=1, \sin(0)=0.$$

$$\cos(\pi/2)=0, \sin(\pi/2)=1$$

$$\cos(\pi)=-1, \sin(\pi)=0.$$

A-HA! So  $(a+c) + (a-c) = 2a = 0 \Rightarrow a=0$

$\Rightarrow c=0$  and  $b=0$ .

The only solution is the trivial solution.  $\therefore$  this set is LI.

4) Consider  $\{1, \sin^2 x, \cos^2 x\} \subseteq F(\mathbb{R})$ . Is it LI or LD?

We think of the trig identity:  $\sin^2 x + \cos^2 x = 1$ , which is true for all  $x$ . Thus:  $1 - \sin^2 x - \cos^2 x = 0$  is

a dependence equation in which not all coefficients are 0.

$\therefore$  This set is LD.

If we had plugged in values, we could have gotten:  
 $a + b \sin^2 x + c \cos^2 x = 0$  for all  $x$

$$x=0 \Rightarrow a + c = 0$$

$$x=\pi/2 \Rightarrow a + b = 0$$

$$x=\pi \Rightarrow a + c = 0$$

nothing to force a unique solution yet  
 $\therefore$  keep going

$$x=\pi/4 \Rightarrow a + b\left(\frac{1}{\sqrt{2}}\right)^2 + c\left(\frac{1}{\sqrt{2}}\right)^2 = 0 \Rightarrow a + \frac{1}{2}b + \frac{1}{2}c = 0$$

$$x=\pi/6 \Rightarrow a + b\left(\frac{1}{2}\right)^2 + c\left(\frac{\sqrt{3}}{2}\right)^2 = 0 \Rightarrow a + \frac{1}{4}b + \frac{3}{4}c = 0$$

It is starting to look like  $b = c = -a$  is a likely solution!  
so we look at  $a=1$ :

$$-1 + \sin^2 x + \cos^2 x = 0$$

and say: hey! that's a trig identity, and it's true for all  $x$ !  
 $\therefore$  LD.

## Some theorems = facts about LI and LD sets.

Fact 1 If  $\{v_1, \dots, v_m\}$  is LD, then so is  $\{v_1, \dots, v_m, u_1, \dots, u_k\}$ .

You can't make an LD set LI by adding vectors.

example: We saw  $\{1-x^2, x^2, 1+x^2\}$  is LD, because

$$1(1-x^2) + 2(x^2) - 1(1+x^2) = 0$$

Then  $\{1-x^2, x^2, 1+x^2, x\}$  is also LD, because

$$1(1-x^2) + 2(x^2) - 1(1+x^2) + 0(x) = 0$$

$\mathbb{R}$  not all coefficients are zero!

### Proof of Fact 1:

Since  $\{v_1, \dots, v_m\}$  is LD, there are scalars  $k_1, \dots, k_m$ , not all zero, such that  $k_1 v_1 + \dots + k_m v_m = 0$ .

Then it is also true that  $k_1 v_1 + \dots + k_m v_m + 0 u_1 + \dots + 0 u_k = 0$

and this is a dependence equation on the bigger set  $\{v_1, \dots, v_m, u_1, \dots, u_k\}$  with not all coefficients equal to 0.  $\therefore$  The bigger set is LD  $\square$

Application: We know that  $\{\vec{0}\}$  is LD. Therefore any set containing  $\vec{0}$  is LD.

eg:  $\{(1,0), (0,0), (0,1)\}$  is LD.

In sets, the order doesn't matter.

Fact 2: If  $\{v_1, \dots, v_m\}$  is LI, then any subset  $\{v_1, \dots, v_k\}$  ( $k < m$ ) is also LI.

Example: We showed that  $\{1, \sin(x), \cos(x)\}$  is LI.  
Therefore  $\{\sin(x), \cos(x)\}$  is LI.

Proof of Fact 2: So the subset  $\{v_1, \dots, v_k\}$  is either LD or LI.

If it were LD, then by fact 1,  $\{v_1, \dots, v_k, v_{k+1}, \dots, v_m\}$  would be LD. But it's LI, so that's impossible.  $\{v_1, \dots, v_m\}$  is LI.  $\square$

Application: If  $\{u, v, w\}$  is LI, then no two of these vectors are collinear.

Caution: If  $\{u, v, w\}$  is a set of vectors and no two of them are collinear, it could still happen that  $\{u, v, w\}$  is LD!  
eg:  $\{(1,0), (1,1), (0,1)\}$ .

Shortcuts we have learned:

• if  $\vec{0}$  is in your set, your set is LD.

• if  $\vec{u}$  and  $\vec{v}$  are multiples of each other, then any set containing  $\{\vec{u}, \vec{v}\}$  is LD. (In particular,  $\{\vec{u}, \vec{v}\}$  is LD.)

Why? If  $\vec{u} = c\vec{v}$  for some  $c \in \mathbb{R}$  then

$$1\vec{u} - c\vec{v} = \vec{0}$$

$\Leftarrow$  a nonzero coefficient  $\therefore$  this is a nontrivial dependence relation

See the caution!

Recall: We have defined the span of a set of vectors of  $V$ :

$$\text{span}\{v_1, \dots, v_m\} = \{c_1 v_1 + \dots + c_m v_m \mid c_i \in \mathbb{R}\}$$

It is a subspace of  $V$ , and consists of all linear combinations of  $u_1, \dots, u_m$ .

Fact 3: A set  $\{v_1, \dots, v_m\}$  is LD if and only if there is at least one vector  $v_i$  which is in the span of the rest.

This is, in fact, how we got to the definition of LD in the first place!

Proof:  $[ \Rightarrow ]$  If  $\{v_1, \dots, v_m\}$  is LD, then there is a nontrivial dependence equation

$$k_1 v_1 + \dots + k_m v_m = 0 \quad \text{not all } k_i \text{ are } 0.$$

Suppose  $k_i \neq 0$ . (a specific  $i$ )

Then

$$k_i v_i = -k_1 v_1 - \dots - k_{i-1} v_{i-1} - k_{i+1} v_{i+1} - \dots - k_m v_m$$

(move everything else to the other side)

divide by  $k_i$ , since  $k_i \neq 0$ .

$$\Rightarrow v_i = -\frac{k_1}{k_i} v_1 + \dots - \frac{k_{i-1}}{k_i} v_{i-1} + \frac{k_{i+1}}{k_i} v_{i+1} + \dots + \frac{k_m}{k_i} v_m.$$

$$\Rightarrow v_i \in \text{span}\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_m\}.$$

Note: often, we have many choices for  $v_i$ .

$[ \Leftarrow ]$  Suppose  $v_i = c_1 v_1 + \dots + c_{i-1} v_{i-1} + c_{i+1} v_{i+1} + \dots + c_m v_m.$

Then  $c_1 v_1 + \dots + c_{i-1} v_{i-1} - \underline{\underline{1}} v_i + c_{i+1} v_{i+1} + \dots + c_m v_m = 0$

is a nontrivial dependence equation, so the set is LD.

□

Caution: LD does NOT mean that EVERY vector is in the span of the rest.

eg:  $\{(1,1), (2,2), (1,2)\}$



only each of these is in the span of the rest

Application: Reducing spanning sets that are "too big".

eg) Consider  $W = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ .

The spanning set is LD because

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, for example,  $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ .

Since  $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$  are in  $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ , this gives us

$$W = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

by our theorem (span, part 2).

That is, we identified a redundant vector that we could throw out without changing our span!

Caution: It was also true that  $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$  but we can't throw away BOTH of these vectors: they depend on each other. Throw away vectors ONE AT A TIME.

Theorem (reducing spanning sets)

Suppose  $W = \text{span} \{v_1, \dots, v_m\}$ . If  $v_m \in \text{span} \{v_1, \dots, v_{m-1}\}$  then  $W = \text{span} \{v_1, \dots, v_{m-1}\}$ .

Proof: So by definition of span, we know  $\text{span} \{v_1, \dots, v_{m-1}\} \subseteq W$ . Conversely, each  $v_1, \dots, v_{m-1}$  is in  $\text{span} \{v_1, \dots, v_m\}$  (clear) and the hypothesis gives us  $v_m \in \text{span} \{v_1, \dots, v_{m-1}\}$ . Therefore by our thm (span part 2),  
 $W = \text{span} \{v_1, \dots, v_m\} \subseteq \text{span} \{v_1, \dots, v_{m-1}\}$ .

Since the sets are contained in one another they must be equal.  $\square$

Point: If your spanning set is LD, then you can keep removing vectors like this, one at a time, until what's left is LI — and you will keep the same span.

The flip side is also true:

Theorem (Enlarging LI sets)

Suppose  $\{v_1, \dots, v_m\}$  is LI, in a subspace  $W$ .

Then  $\{v, v_1, \dots, v_m\}$  is LI

if and only if

$v \notin \text{span}\{v_1, \dots, v_m\}$ .

example: We can see that  $\{1+x^2, 1+x\}$  is LI since these polynomials are not multiples of each other.

Also  $x^3 \notin \text{span}\{1+x^2, 1+x\}$  since the spanning set does not contain any polynomials with  $x^3$  as a summand.

$\therefore$  by the theorem  $\{x^3, 1+x^2, 1+x\}$  is also LI.

Point: So if we have a spanning set for a vector space  $U$  we can remove redundant vectors one at a time until we have an LI spanning set.

On the other hand, we could build an LI spanning set by choosing vectors in "new directions" until we get

A spanning set... assuming finitely many vectors would be enough.

Q: will we get the same number of vectors in a LI spanning set every time? Does it depend on which method we use, or the choices we make?

A detailed example:

$$\text{Let } W = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} \right\}.$$

Reduce this spanning set to a LI spanning set.

$$\text{First note that } \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\} \subseteq W$$

$$\therefore W = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} \right\} \quad \text{by the thm (reducing spanning sets)}$$

Are these LI? If not, which vector can we remove?

$$\text{We solve } a \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\Leftrightarrow \begin{cases} a - b + c = 0 \\ 2a + b + 5c = 0 \\ -a + b - c = 0 \end{cases}$$

notice the shortcut!

$\Leftrightarrow$  augmented matrix

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 1 & 5 & 0 \\ -1 & 1 & -1 & 0 \end{array} \right]$$

$$\begin{array}{l} \sim \\ -2R_1 + R_2 \\ R_1 + R_3 \end{array} \left[ \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

There is no leading 1 in the third column

∴ ∞ many solutions

∴ LD.

We want to choose a vector to remove, so we want an actual dependence equation. ∴ continue solving.

$$\begin{array}{l} \sim \\ \frac{1}{3}R_2 \end{array} \left[ \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_2+R_1} \left[ \begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This says:  $a + 2c = 0$  (c is a free variable)  
 $b + c = 0.$

∴ if we set  $c = 1$  (for example) we get

$$\begin{array}{l} a = -2 \\ b = -1 \\ c = 1 \end{array}$$

(any c would work:  
EXCEPT  $c = 0$ !)

as a solution, which gives

$$-2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

check!

For sure the third vector has a nonzero coefficient, because we chose  $c = 1$  (its coefficient). So:

$$\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \therefore \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

So by our theorem:  
 $W = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$

and this spanning set is LI since it consists of just 2 vectors, and they are not scalar multiples of each other.

$$\text{So } W = \left\{ s \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \mid s, t \in \mathbb{R} \right\}$$

which is the parametric form of a plane in  $\mathbb{R}^3$ .  
□