

Feb 28

Quick recap of basis, linear independence, span, dimension and coordinates
Then: matrix multiplication!

Recall: If V is a vector space and $\{v_1, \dots, v_n\}$ is a set of vectors in V then:

- span $\{v_1, \dots, v_n\} = \{a_1v_1 + \dots + a_nv_n \mid a_i \in \mathbb{R}\}$ the set of all linear combinations of $\{v_1, \dots, v_n\}$
 - a subspace of V .
- eg $\text{span} \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right] = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in M_{2 \times 2} \mid a, b \in \mathbb{R} \right\}$

- $\{v_1, \dots, v_n\}$ is linearly independent (LI) if the ONLY solution to $a_1v_1 + \dots + a_nv_n = \vec{0}$ is $a_1 = 0, a_2 = 0, \dots, a_n = 0$.

eg $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is not linearly independent because $\underbrace{2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}_{\text{a nontrivial dependence relation}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

- A basis for U is a LI spanning set.

\Leftrightarrow a biggest LI set in U

\Leftrightarrow a smallest spanning set for U

- $\dim(U) = \#\text{elements in a basis for } U$.

eg $W = \{(x, y, z, t) \in \mathbb{R}^4 \mid x+2y=0\}$

$$= \{(-2y, y, z, t) \mid y, z, t \in \mathbb{R}\}$$

$$= \text{span}\{(-2, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$$

If $a(-2, 1, 0, 0) + b(0, 0, 1, 0) + c(0, 0, 0, 1) = 0$ then
 $(-2a, a, b, c) = 0 \Rightarrow c=0, a=0, b=0 \therefore \text{LI}$

$\therefore \{(-2, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ is a basis for W .

$$\therefore \dim W = 3. \quad \square$$

Every vector space has a basis. We are interested
 in finite-dimensional vector spaces \therefore finite bases.

Thm Given a spanning set, you can reduce it to a basis.

eg $U = \left\{ \begin{bmatrix} a+b & b+c \\ a+2b+c & b+c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} = \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\}$

but $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \therefore U = \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\}$

\uparrow
 2 elements
 not scalar multiples
 $\therefore \text{LI} \therefore \text{basis}$

$$\therefore \dim U = 2$$

Thm: Given an LI set in U , you can extend it to a basis.

(Just find a vector in U which is not in the span of what you already have).

Thm: If U is a subspace of V then $0 \leq \dim U \leq \dim V$ & $\dim U = \dim V \Leftrightarrow U = V$.

Thm: If $\dim U = m$ and you have a set of m LI vectors in U , then it's a basis.

If $\dim U = m$ and you have a spanning set with m vectors, it's a basis.

#elements in any LI set in $U \leq \dim U \leq$ #elements in any spanning set of U .

Application: Coordinate vectors

eg $U = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ basis $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

$$= \left\{ a \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$
$$= \left\{ \begin{bmatrix} a & b \\ a+b & b \end{bmatrix} \in M_{2,2} \mid a, b \in \mathbb{R} \right\}$$

So the coordinate vector of coordinate vector of $\begin{bmatrix} a & b \\ a+b & b \end{bmatrix}$ is $\underbrace{(a, b)}_{\text{shorthand; convenient}}$. $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ is $(1, 2)$ & $= 1 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

eg) With respect to the basis $\{(-2, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$
 of W , the coordinates of
 $(-2, 1, 4, 5)$ are $\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$

$$1(-2, 1, 0, 0) + 4(0, 0, 1, 0) + 5(0, 0, 0, 1)$$

When you have a basis then to describe
 a vector, or to add vectors, you only need
 coordinates.

$$\text{eg } \vec{u} \in A = \begin{bmatrix} 5 & 7 \\ 12 & 7 \end{bmatrix}, \vec{v} \in B = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix} \quad A+B = \begin{bmatrix} 8 & 6 \\ 14 & 6 \end{bmatrix}$$

$\vec{u} = (5, 7)$ $\vec{v} = (3, -1)$ $\vec{u} + \vec{v} = (8, 6)$

fewer calculations.

$$3A = \begin{bmatrix} 15 & 21 \\ 36 & 21 \end{bmatrix} \quad 3\vec{u} = (15, 21)$$

eg) You create waveforms

$$\vec{u} = 0.5 \sin(0.5x) + 0.75 \cos(0.25x) + 0.5 \sin(3x)$$

$$\vec{v} = -0.5 \sin(0.5x) + \cos(4x) - 0.25 \sin(2x)$$

Then you can store all their linear combinations

as the coordinate vectors (a, b) and work with those instead of storing

$$\frac{1}{2}a\sin(\frac{1}{2}x) + \frac{3}{4}a\cos(\frac{1}{3}x) + \frac{1}{7}a\sin(\frac{7}{2}x) = \frac{1}{2}b\sin(\frac{1}{2}x) \\ + b\cos(4x) - \frac{1}{4}b\sin(2x).$$

□

So we have spent a lot of time getting to know vectors, and vector spaces. Next

steps:

- matrix multiplication

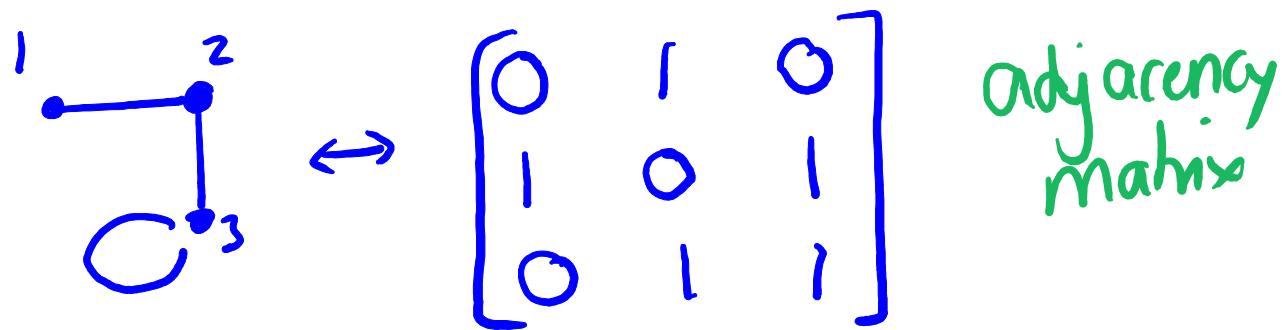
(and how matrices transform vectors)

Recall: A matrix is a table of numbers.
It is of size $m \times n$ if it has m rows
and n columns.

eg $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is a 3×2 matrix.
"three by two"

What can a matrix mean? LOTS of things:

- the coefficients of a linear system
- a table of values
- a network ("graph" in computer science)



- a dynamical system for a probabilistic state machine (aka Markov chain)
(each entry is the probability of passing from state i to state j)
- a rotation or reflection, ^{or scaling} of geometric objects
("linear transformation")

Different ways to "see" a matrix:

- as a table of numbers (data)
- as a collection of columns (vectors)
- as a collection of rows (also vectors)
- as "generalized numbers": things you can add & multiply!

How to multiply matrices:

1) Why would you multiply matrices?

e.g. total cost = cost/component \times components

#	Components		
	A	B	C
α	5	1	3
β	1	2	1
↑ products	↑ 2 units of β go into matrix β		

	cost	volume
A	1	3
B	10	1
C	2	10

↑
\$2 per unit of C.

so what is cost & volume of products α & β ?

$$P = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 \\ 10 & 1 \\ 2 & 10 \end{bmatrix}$$

$$d: \begin{bmatrix} 5 & 1 & 3 \\ A & B & C \end{bmatrix} \begin{bmatrix} 1 \\ 10 \\ 2 \end{bmatrix}_{\text{cost/unit}} = 5 \times 1 + 1 \times 10 + 3 \times 2 = 21 \text{ \$}$$

Unit needed

L

$$PC = \alpha \begin{bmatrix} 5 \times 1 + 1 \times 10 + 3 \times 2 & 5 \times 3 + 1 \times 1 + 3 \times 10 \\ \beta \begin{bmatrix} 1 \times 1 + 2 \times 10 + 1 \times 2 & 1 \times 3 + 2 \times 1 + 1 \times 10 \end{bmatrix} \end{bmatrix}$$

L
The entry in row i & column j of PC comes from the dot product of row i of P & column j of C .

$$\therefore PC = \begin{bmatrix} 21 & 46 \\ 23 & 15 \end{bmatrix}$$

(so α, β cost about the same but α is 3 times larger.)

Matrix multiplication:

If A is an $m \times n$ matrix and
 B is an $n \times p$ matrix
then AB is an $m \times p$ matrix.
The (i,j) entry of AB is the dot product of
row i of A with column j of B .

So "AB" only makes sense if
 # columns of A = # rows of B.

eg)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \end{bmatrix}$$

like coefficient matrix of a linear system

$$= x \begin{bmatrix} 1 \\ 4 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \end{bmatrix} + z \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

a linear combination of column vectors

eg) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ not allowed

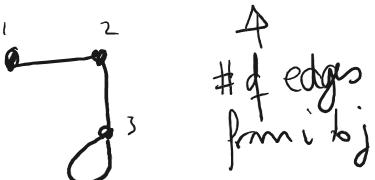
$$\begin{matrix} 2 \times 3 \\ \underbrace{\hspace{1cm}}_{\neq} \end{matrix} \quad \begin{matrix} 2 \times 1 \\ \end{matrix}$$

eg) $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} x+3y+5z & 2x+4y+6z \end{bmatrix}$

$$\begin{matrix} 1 \times 3 \\ \end{matrix} \quad \begin{matrix} 3 \times 2 \\ \end{matrix} \quad = x \begin{bmatrix} 1 & 2 \end{bmatrix} + y \begin{bmatrix} 3 & 4 \end{bmatrix} + z \begin{bmatrix} 5 & 6 \end{bmatrix}$$

a linear combination of row vectors.

eg) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$



of edges from i to j

paths of length 2 from vertex i to vertex j.

Several strange things about matrix multiplication

- Sometimes AB is allowed but BA is not.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 22 \\ 39 \end{bmatrix}$$

$\underset{A}{\underset{\neq}{\underset{2 \times 1}{}}} \quad \underset{B}{\underset{\neq}{\underset{2 \times 2}{}}}$

B A not allowed

- Sometimes AB & BA are both allowed but $AB \neq BA$

eg) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 7 & 1 \end{bmatrix}$

$\underset{AB}{}$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 4 & 6 \end{bmatrix}$$

BA

eg) $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ as 3×1 matrices

$\vec{u}\vec{v}$ not allowed

$\vec{u}^T = [1 \ 2 \ 3]$ transpose = swap rows & columns

$$U^T v = [1 \ 2 \ 3] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = [1 \ -2 + 3] = [2] = 2$$

1×1 matrix = number

dot product $\vec{u} \bullet \vec{v}$

$$U^T v = V^T u \quad (\text{check!})$$

$$V U^T = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} [1 \ 2 \ 3] = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix}$$

$3 \times 1 \quad 1 \times 3$

nothing at all
to do with
the dot product.

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