

Feb 14 : Basis and dimension.

So we have seen that having a spanning set for a subspace U is useful:

- if $U = \text{span}\{\vec{u}_1, \dots, \vec{u}_m\}$ then U is automatically a subspace
- it's easy to produce lots of vectors in U : just take any linear combination of elements of the spanning set $\{\vec{u}_1, \dots, \vec{u}_n\}$.

Last time, we saw:

A set $\{\vec{u}_1, \dots, \vec{u}_n\}$ is LD if and only if at least one vector is in the span of the rest.

Which implies:

If your spanning set is LD, you can remove redundant vectors, one at a time, without changing its span, until your result is a LI spanning set.

Fantastic! But how do we get a spanning set in the first place?

Theorem: Enlarging LI sets

suppose $\{\vec{u}_1, \dots, \vec{u}_k\}$ is an LI set in a vector space V . Let $\vec{v} \in V$. Then

$\{\vec{u}_1, \dots, \vec{u}_k, \vec{v}\}$ is LI if and only if

$$\vec{v} \notin \text{span}\{\vec{u}_1, \dots, \vec{u}_k\}.$$

A Bigger LI
set with a
BIGGER SPAN

This is just stating our theorem about reducing spanning sets another way, actually.

Eg) The set $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \right\} \subseteq M_{2,2}$ is LI (easy to see just 2 vectors)

Since any linear combination of these 2 matrices has zeros in row 2, $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \notin \text{span}\{\vec{u}_1, \vec{u}_2\}$

\therefore by theorem $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \right\}$ is LI. \square

Eg) The set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is LI, since solving $a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ means necessarily that $a=b=c=0$.

Can we get a bigger LI set in \mathbb{R}^3 ?

No: since $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, EVERY vector in \mathbb{R}^3 is already in $\text{span}\{\vec{i}, \vec{j}, \vec{k}\}$, any bigger

Set $\{\vec{i}, \vec{j}, \vec{k}, \vec{v}\}$ must be LD.
 We have maxed out: $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^3$
 and that is as big as we can get.

(eg) Suppose $\{\vec{u}, \vec{v}\}$ is an LI set in \mathbb{R}^2 . This part is an algebraic argument for a fact we have seen geometrically, FYI.
 Let's prove that $\text{span}\{\vec{u}, \vec{v}\} = \mathbb{R}^2$.

Say $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$.

If $u_1 = v_1 = 0$ then \vec{u} and \vec{v} would be parallel.
 So let's organize it so that $u_1 \neq 0$.
 We claim that $\text{span}\{\vec{u}, \vec{v}\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2$.

To prove this we have to solve

$$a \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + b \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \& \quad a' \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + b' \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} u_1 & v_1 & | & 1 \\ u_2 & v_2 & | & 0 \end{bmatrix}}_{\neq 0} \sim \underbrace{\begin{bmatrix} u_1 & v_1 & | & 1 \\ 0 & v_2 - \frac{u_2 v_1}{u_1} & | & -\frac{u_2}{u_1} \end{bmatrix}}_{\neq 0}$$

If this were 0, we'd have $v_2 = \frac{u_2 v_1}{u_1}$

$$\Leftrightarrow v_2 = \frac{v_1}{u_1} u_2 \quad \text{and} \quad v_1 = \frac{v_1}{u_1} u_1$$

so $(u_1, u_2) \parallel (v_1, v_2)$
 are parallel

\therefore This is NOT zero, so it will be a leading one (when we finish row reducing)

and so the system will ALWAYS be CONSISTENT.
It doesn't even matter what's on the right side -
there are no zero rows in the coefficient matrix,
so no way can the system be inconsistent

∴ yes, you can solve for $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as linear
combinations of \vec{u} and \vec{v}

$$\therefore \underbrace{\text{span}\{\vec{u}, \vec{v}\}}_{\text{and since } \vec{u}, \vec{v} \in \mathbb{R}^2} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2.$$

and since $\vec{u}, \vec{v} \in \mathbb{R}^2$, this says it is all \mathbb{R}^2 .

Consequence: If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ then $\{\vec{u}, \vec{v}, \vec{w}\}$ is LD.

So: in \mathbb{R}^2 , any LI set has at MOST 2 vectors.
and any spanning set has at LEAST 2 vectors,
so every LI spanning set has exactly 2 vectors.

Recap of what we have so far, in general.

- if $\{\bar{u}_1, \dots, \bar{u}_m\}$ spans U , then any bigger set must be LD.
- if $\{\bar{v}_1, \dots, \bar{v}_k\}$ is LI in U , then no smaller set could span U .

We have an even better, much stronger, result.
(We prove this later, with matrices, say)

Theorem (LI sets are never bigger than spanning sets)

If a vector space V can be spanned by n vectors, then any LI set has at most n vectors.

equivalently:
If an LI set in V has n vectors, then every spanning set of V has at least n vectors.

We can write this as:

$$\text{the size of any LI set in } V \leq \text{the size of any spanning set of } V$$

eg) Since $\text{span}\{1, x, x^2\} = P_2$, the set $\{1+x+x^2, 1-x-x^2, 2+2x, 4-x^2\}$ is LD because it has 4 vectors and a spanning set has 3.

eg) Since the plane $W = \{(x, y, z) \mid x+y+z=0\}$ is the span of 2 vectors $\{(1, 0, -1), (1, -1, 0)\}$ any set of 3 vectors in W is LD.

* Note we say "in W". Of course you can find 3 LI vectors in \mathbb{R}^3 , but they cannot all lie in W.

* The theorem applies to all vector spaces, including subspaces.

So what's the critical balance?

Definition



A set $\{\vec{v}_1, \dots, \vec{v}_n\}$ of vectors in a vector space V is called a basis of V if

① $\{\vec{v}_1, \dots, \vec{v}_n\}$ is LI AND

② $\text{span}\{\vec{v}_1, \dots, \vec{v}_n\} = V$.

So a basis is

- a LI spanning set
- a maximal LI set in V
- a minimal spanning set of V .

examples:

- 1) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis of \mathbb{R}^2
- 2) $\{1, x, x^2\}$ is a basis of P_2
- 3) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis of $M_{2,2}$
- 4) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $W = \{(x,y,z) \mid x+y+z=0\}$

BUT

- 75) $\{(1,0)\}$ is not a basis for \mathbb{R}^2 because it does not span all of \mathbb{R}^2 .
(It is a basis for a line in \mathbb{R}^2)

- 76) $\{(1,0), (0,1), (1,1)\}$ is not a basis for \mathbb{R}^2 because it is NOT LI.

Putting this together with our big inequality ("LI sets smaller than spanning sets"), yields:

Theorem: All bases of V have the same size.

That is: if $\{\tilde{v}_1, \dots, \tilde{v}_m\}$ is one basis of V and $\{\tilde{u}_1, \dots, \tilde{u}_k\}$ is another basis of V then $m=k$. (But the bases can be different!)

Cool. So we have:

* Definition * If V has a finite basis $\{\tilde{v}_1, \dots, \tilde{v}_n\}$ then the dimension of V is n , and we write $\dim V = n$.

In this case, we can say V is finite-dimensional.
(If it doesn't have a finite basis, then it is infinite-dimensional.)

Examples:

- 1) $\dim \mathbb{R}^2 = 2$ since $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis with 2 elements.
- 2) $\dim P_2 = 3$ since $\{1, x, x^2\}$ is a basis with 3 elements
- 3) $\dim M_{22} = 4$ since $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis with 4 elements.
- 4) We can check that $\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^n$ "the n standard basis vectors"
- is a basis (check: LI and spans \mathbb{R}^n)
 $\therefore \dim \mathbb{R}^n = n$
- 5) The space of ALL polynomials of any degree is infinite dimensional. But $P_n = \text{span} \{ \underbrace{1, x, x^2, \dots, x^n}_{n+1 \text{ vectors}} \}$ is $n+1$ -dimensional.
- 6) A plane through the origin in \mathbb{R}^3 is 2-dimensional since it has a LI spanning set (i.e. a BASIS) consisting of 2 vectors.