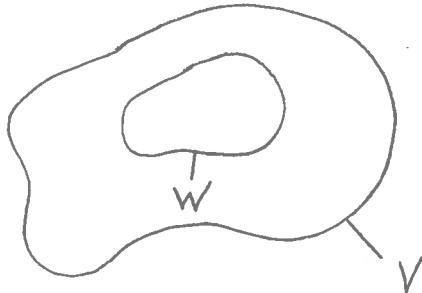


# Vertretung von Monica

Today: §5 and §6

Next Tuesday: §6 and §7



Let  $V$  be a vector space and let  $W$  be a subset of  $V$ . We equip  $W$  with the operations from  $V$ ; and if this turns  $W$  into a vector space, we call it a subspace of  $V$ .

Subspace Test A subset  $W$  of  $V$  is a subspace if and only if (i)  $\vec{0} \in W$ , (ii)  $W$  is closed under addition, (iii)  $W$  is closed under scalar multiplication.

$$W_1 = \{(3x, -x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$$

$$W_2 = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\} \subseteq \mathbb{R}^2$$

$$W_3 = \{(x, y) \in \mathbb{R}^2 \mid x+y=0\} \subseteq \mathbb{R}^2$$

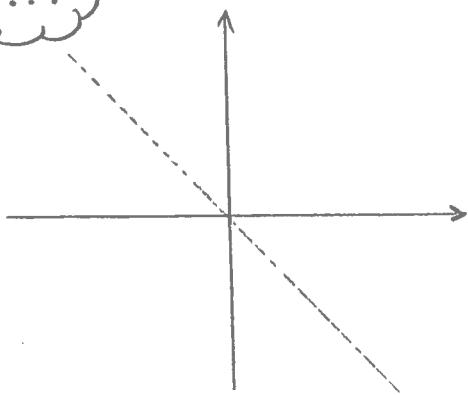
$$W_4 = \{(x, 1) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$$

$$W_5 = \{\vec{u} \in \mathbb{R}^2 \mid \|\vec{u}\| \leq 1\} \subseteq \mathbb{R}^2$$

"surrounding  
space"

We saw:  $W_1$  is a subspace,  $W_2$  and  $W_5$  are not.

$W_3 \text{ ??? }$



(i)  $(0,0) \in W_3$  because  $0+0=0$ . ✓

(ii) Let  $\vec{u}, \vec{v} \in W_3$ . Claim:  $\vec{u} + \vec{v} \in W_3$ .

Since  $\vec{u}, \vec{v} \in W_3$ ,  $\vec{u} = (x, y)$  with  $x+y=0$  and  $\vec{v} = (x', y')$  with  $x'+y'=0$ . Now, observe that:

$$\vec{u} + \vec{v} = (x+x', y+y')$$

$$\text{and } (x+x') + (y+y') = \underbrace{x+y}_0 + \underbrace{x'+y'}_0 = 0.$$

So,  $\vec{u} + \vec{v} \in W_3$ . ✓

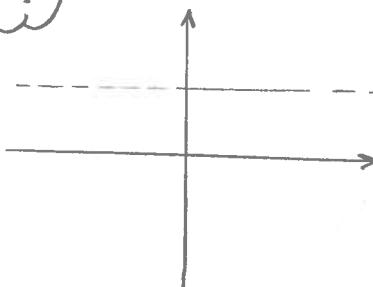
(iii) Let  $k \in \mathbb{R}$ ,  $\vec{u} \in W_3$ . Claim:  $k\vec{u} \in W_3$ . Since  $\vec{u} \in W_3$ ,  $\vec{u} = (x, y)$  with  $x+y=0$ . Now, observe that:

$$k\vec{u} = k(x, y) = (kx, ky) \text{ and } kx + ky = k\underbrace{(x+y)}_0 = 0.$$

So,  $k\vec{u} \in W_3$ .  $\checkmark$

Yes,  $W_3$  is a subspace!

$W_4$  ???



(i)  $(0,0) \notin W_4$  ↗

No,  $W_4$  is not a subspace!

Recall It suffices to show that one property fails.

We could alternatively have said:

" $(0,1), (1,1) \in W_4$  but  $(0,1) + (1,1) = (1,2) \notin W_4$ "

So,  $W_4$  is not closed under addition. ↗ "

Counterexample  
with numbers

But, checking if the zero vector is  
in a subset is typically very easy.

Do that first !!!

OPTIONAL - DONE!

Three more subspaces...

① Surrounding Space:  $M_{22} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mid \text{all } a_{ij} \in \mathbb{R} \right\}$

with addition and scalar multiplication  
introduced last time

Subset:  $\mathcal{Y} = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \subseteq M_{22}$

Yes,  $\mathcal{Y}$  is a subspace!

Complicated proof: p.56 of textbook!  
Simple proof: Later in this class!

(3)

$\mathcal{S}$  is called the space of symmetric  $2 \times 2$  matrices.

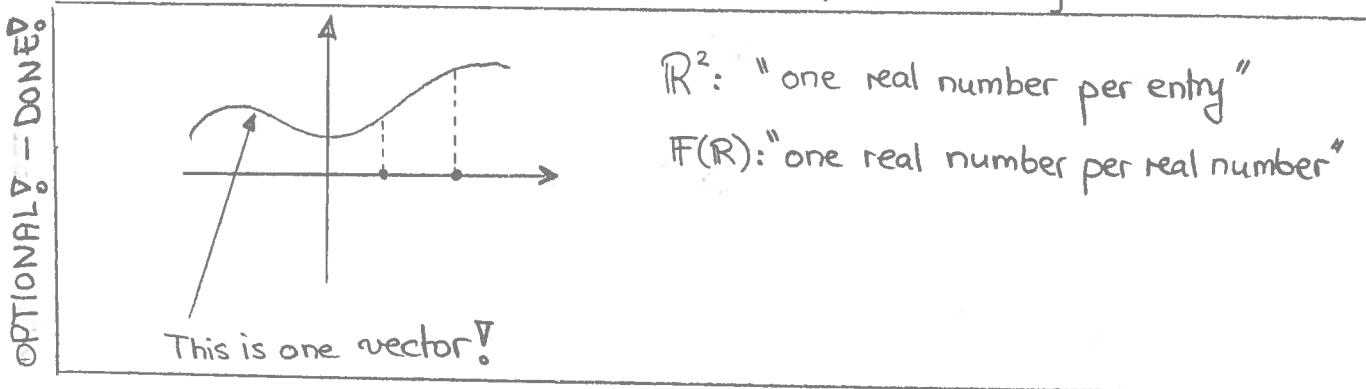
There is a fancy way to describe it: The transpose of an  $m \times n$  matrix  $A$  is the  $n \times m$  matrix  $A^T$  whose rows are the columns of  $A$ . E.g.:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Therefore,  $\mathcal{S} = \{ A \in M_{2 \times 2} \mid A^T = A \}$ .

(2)

Surrounding space:  $F(\mathbb{R}) = \{ \text{all functions from } \mathbb{R} \text{ to } \mathbb{R} \}$



Subset:  $P_2 = \{ p(x) = a + bx + cx^2 \mid a, b, c \in \mathbb{R} \} \subseteq F(\mathbb{R})$

Yes,  $P_2$  is a subspace! Simple proof: Later in this class!

SKIPPED

Maybe recall  $P$  here. However, our theorem about spans does not show that  $P$  is a subspace. So, we have to go the hard way and run the subspace test (p.56 of textbook).

(3)

Surrounding space:  $\mathbb{R}^3$  with standard operations

Subset:  $W = \{ (x, y, z) \in \mathbb{R}^3 \mid x - 2y + z = 0 \} \subseteq \mathbb{R}^3$

That's similar to  $W_3$ . But, we will now go a slightly different way to see that  $W$  is a subspace.

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid x - 2y + z = 0\} \quad (*)$$

DONE!  
BUT STUDENTS  
SEEM TO PREFER  
THE AD-HOC WAY!



Homogeneous linear system:

$$\left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \end{array} \right] \text{ RREF}$$

$\uparrow \quad \uparrow \quad \uparrow$   
s      t

$$x = 2s - t, \quad y = s, \quad z = t$$

$$= \{(2s - t, s, t) \mid s, t \in \mathbb{R}\}$$

$$= \{s \cdot (2, 1, 0) + t \cdot (-1, 0, 1) \mid s, t \in \mathbb{R}\}$$

$$= \text{span} \{(2, 1, 0), (-1, 0, 1)\} \quad (**)$$

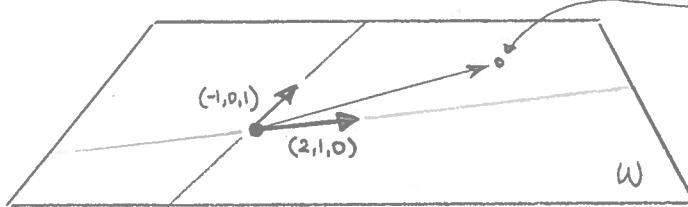
SKIPPED? MONICA, MENTION IT PLEASE!

### Advantages

(\*) Easy to check if some vector is in  $W$ :

$$(47, 23, -1) \in W \text{ because } 47 - 2 \cdot 23 - 1 = 0$$

(\*\*) Finite list of vectors describing  $W$ :



"linear combination"  
of  $(2, 1, 0)$  and  
 $(-1, 0, 1)$ !"

Definition Let  $\vec{v}_1, \dots, \vec{v}_m$  be vectors of a vector space  $V$ . The set of all linear combinations of  $\vec{v}_1, \dots, \vec{v}_m$  is called the span of  $\vec{v}_1, \dots, \vec{v}_m$ . In other words:

$$\text{span} \{\vec{v}_1, \dots, \vec{v}_m\} = \{a_1 \vec{v}_1 + \dots + a_m \vec{v}_m \mid a_1, \dots, a_m \in \mathbb{R}\}$$

## BIG THEOREM (part 1)

$W = \text{span} \{ \vec{v}_1, \dots, \vec{v}_m \}$  is always a subspace of  $V$ .

Proof Let's run the subspace test.

(i)  $\vec{0} = 0 \cdot \vec{v}_1 + \dots + 0 \cdot \vec{v}_m \in W \quad \checkmark$

(ii) Let  $\vec{u}, \vec{v} \in W$ . Claim:  $\vec{u} + \vec{v} \in W$ . Since  $\vec{u}, \vec{v} \in W$ ,

$$\vec{u} = a_1 \vec{v}_1 + \dots + a_m \vec{v}_m \quad \text{and} \quad \vec{v} = b_1 \vec{v}_1 + \dots + b_m \vec{v}_m. \quad \text{Therefore,}$$

$$\vec{u} + \vec{v} = (a_1 + b_1) \vec{v}_1 + \dots + (a_m + b_m) \vec{v}_m \in W. \quad \checkmark$$

(iii) Let  $k \in \mathbb{R}, \vec{u} \in W$ . Claim:  $k\vec{u} \in W$ . Since  $\vec{u} \in W$ ,

$$\vec{u} = a_1 \vec{v}_1 + \dots + a_m \vec{v}_m. \quad \text{Therefore, } k\vec{u} = (ka_1) \vec{v}_1 + \dots + (ka_m) \vec{v}_m \in W. \quad \checkmark$$

Cool, keep in mind:

"Spans are subspaces!"

Back to our examples

③  $W = \{ (x, y, z) \in \mathbb{R}^3 \mid x - 2y + z = 0 \} = \dots$

$$\dots = \text{span} \{ (2, 1, 0), (-1, 0, 1) \} \Rightarrow W \text{ subspace.}$$

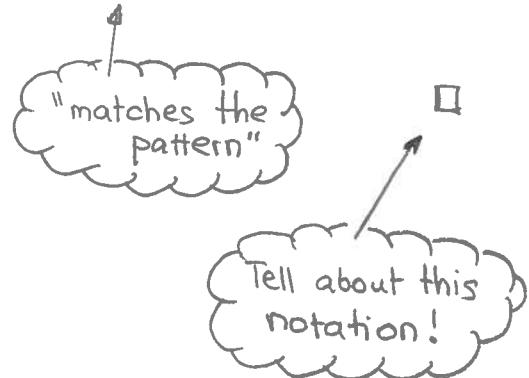
①  $\mathcal{S} = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \subset \mathbb{M}_{2 \times 2}$

$$= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \Rightarrow \mathcal{S} \text{ subspace.}$$

②  $\mathbb{P}_2 = \{ a + bx + cx^2 \mid a, b, c \in \mathbb{R} \}$

$$= \text{span} \{ 1, x, x^2 \} \Rightarrow \mathbb{P}_2 \text{ subspace.}$$



Exercise Identify  $W_1$  and  $W_3$  as spans, and conclude that they are subspaces. \*

Questions  $W = \text{span}\{(1,0), (0,1)\}$  END OF CLASS !!!

- How many elements does  $W$  have?
- Can you identify  $W$ ?

$$W = \text{span}\{(5,7)\}$$

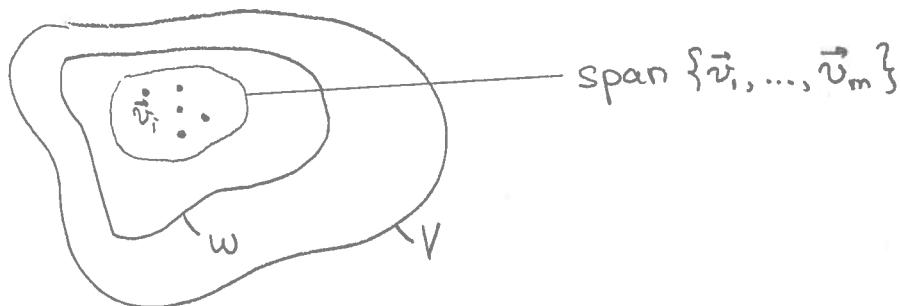
$$W = \text{span}\{(10,14)\}$$

} same questions!!!

### Outlook

#### BIG THEOREM (part 2)

If  $W$  is any subspace which contains  $\vec{v}_1, \dots, \vec{v}_m$ , then  $W$  contains  $\text{span}\{\vec{v}_1, \dots, \vec{v}_m\}$ . That means,  $\text{span}\{\vec{v}_1, \dots, \vec{v}_m\}$  is the smallest subspace containing  $\vec{v}_1, \dots, \vec{v}_m$ .



This will allow us to characterize all subspaces of  $\mathbb{R}^2$  and  $\mathbb{R}^3$ .

\* Solutions  $W_1 = \{(3x, -x) \mid x \in \mathbb{R}\} = \{x \cdot (3, -1) \mid x \in \mathbb{R}\} = \text{span}\{(3, -1)\}$

$$W_3 = \{(x, y) \mid x+y=0\} = \{(-s, s) \mid s \in \mathbb{R}\}$$

$$= \{s \cdot (-1, 1) \mid s \in \mathbb{R}\} = \text{span}\{(-1, 1)\}$$