MAT1341 C : Instructor Monica Nevins
Monday, March 6, 2017 : Test #3

Duration: 75 minutes

Family name: Solutions

First name: ______________________________

Student number : ______________________________

DGD Section number : ______________

Please read the following instructions carefully.

- You have 75 minutes to complete this exam.
- This is a closed book exam. No notes, calculators, cell phones or related devices of any kind are permitted. All such devices, including cell phones, must be stored in your bag under your desk for the duration of the exam.
- Read each question carefully — you will save yourself time and grief later on.
- Questions 1 and 2 are multiple choice, worth 1 point each. **Record your answers to the multiple choice questions in the boxes provided.**
- Questions 3–5 are long answer, with point values as indicated. **The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.**
- Question 6 is a bonus question, worth 3 points. **The bonus question is more difficult; do not attempt it until you are satisfied that you have completed the rest of the test to the best of your ability.**
- Where it is possible to check your work, do so.
- Good luck!

Faculté des sciences
Mathématiques et statistique

Faculty of Science
Mathematics and Statistics

613–562–5864
613–562–5776
www.uOttawa.ca
585 King Edward
Ottawa
ON K1N 6N5

Marker's use only:

Question	Marks
1 & 2 (/2)	
3 (/6)	
4 (/6)	
5 (/6)	
6 (/3) (bonus)	
Total (/20)	

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1. (1 point) Which of the following are correct statements about sets of vectors in a vector space V?

(1) The set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent if and only if setting $a_1 = a_2 = a_3 = 0$ gives $a_1\vec{u} + a_2\vec{v} + a_3\vec{w} = \vec{0}$. False – of course setting $a_i = 0$ gives $\vec{0}$, always!

(2) The set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent if and only if there is a solution to the equation $a_1\vec{u} + a_2\vec{v} + a_3\vec{w} = \vec{0}$ different from $a_1 = a_2 = a_3 = 0$. TRUE

(3) If $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent then $\dim(V) \geq 3$. TRUE: size LI set in V $\leq \dim V$

(4) If $U = \mathsf{span}\{\vec{u}, \vec{v}, \vec{w}\}$ and $\dim(U) = 3$ then $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent. FALSE: theorem says in this case, LI

A. All are true.
B. Only (1) and (3) are true.
C. Only (2) and (3) are true.
D. Only (2), (3) and (4) are true.
E. Only (2) and (4) are true.
F. Only (3) and (4) are true.

Your answer: C

2. (1 point) Suppose X is a subspace of $\mathbb{R}^6$ such that $X \neq \{\vec{0}\}$ and $X \neq \mathbb{R}^6$. Which of the following are always true?

(1) X has a spanning set consisting of 6 vectors. TRUE: $\dim X \leq 5$ but you can add more vectors to a basis & still span

(2) X contains a linearly independent set of 6 vectors. FALSE: that would mean $\dim X \geq 6$

(3) X contains fewer than 6 vectors. FALSE (every nonzero vector space has ∞ many vectors)

(4) There is a basis of X which spans $\mathbb{R}^6$. FALSE (or else $X = \mathbb{R}^6$)

(5) $1 \leq \dim(X) \leq 5$. TRUE (thm from class)

A. Only (1) and (4) are true.
B. Only (1), (3) and (5) are true
C. Only (3) and (5) are true.
D. Only (2) and (4) are true.
E. Only (2) and (5) are true.
F. Only (1) and (5) are true.

Your answer: F

3. (6 points) Consider the subspace $W = \{(x,y,z,t) \in \mathbb{R}^4 \mid x+y+z+t = 0\}$ of $\mathbb{R}^4$. Consider the following four vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}.$$

(a) Each of these vectors are elements of W. Explain why $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

(b) Find at least one non-trivial linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ that results in the zero vector, by setting up and solving an appropriate linear system of equations using row reduction. Verify that your answer is correct.

(c) Express one of the vectors in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ as a linear combination of the rest.

(d) Assuming the remaining vectors are linearly independent, give a basis of W and explain why your answer is correct.

(a) $W \subsetneq \mathbb{R}^4$ is a subspace of $\mathbb{R}^4$ and not equal to $\mathbb{R}^4$ so $\dim W \leq 3$. A theorem in class says that the size of any LI set in $W \leq \dim W$. There are 4 vectors in this set but $\dim W < 4$, so they cannot be LI; this set is LD.

(b) We solve

$$a\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + c\begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longleftrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ -1 & -1 & 0 & -3 & 0 \end{array}\right] \begin{array}{l} \\ -R1+R2 \\ R1+R3 \\ R1+R4 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & -1 & 2 & -2 & 0 \end{array}\right] \begin{array}{l} \\ R2 \leftrightarrow R3 \\ \& \; R2+R4 \\ -\tfrac{1}{3}R3 \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{array}\right] \begin{array}{l} -2R3+R1 \\ -R3+R2 \\ \\ -3R3+R4 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$a + d = 0$
$b + 2d = 0$
$c = 0$
d is free.

(Continue your answer on the next page if necessary.)

(Continue your answer to Q3 on this page if needed.)

Choose $d=1$: then $a=-1$, $b=-2$, $c=0$, $d=1$ is a solution:

$$-1\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}-2\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}+0\begin{bmatrix}2\\-1\\-1\\0\end{bmatrix}+1\begin{bmatrix}1\\1\\1\\-3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

is a nontrivial dependence relation.

(c) In (b) we showed

$$-\vec{v}_1-2\vec{v}_2+\vec{v}_4=\vec{0}$$

$$\therefore \quad \vec{v}_4=\vec{v}_1+2\vec{v}_2$$

(d) If $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ are LI in W, then $\dim W \geq 3$ by theorem from class. Since $W \subseteq \mathbb{R}^4$ and $W \neq \mathbb{R}^4$ (since, for example, $(1,0,0,0) \notin W$) we have $\dim W = 3$.

$\therefore$ $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ is an LI set ~~in~~ in W with $\dim W$ vectors so by a theorem from class ~~it~~ is a basis for W.

4. (6 points) Consider the following elements of P_2:

$$f(x) = x - 4, \quad g(x) = x^2 - 3x \quad \text{and} \quad h(x) = x^2 - 12.$$

(a) Let $U = \mathsf{span}\{f, g\}$. Show that $h \in U$.

(b) Show that $\{f, g\}$ is linearly independent. (You may use the fact that $\{1, x, x^2\}$ is linearly independent.)

In parts (c) and (d) you may use your results from (a) and (b).

(c) Let $W = \mathsf{span}\{f, g, h\}$. Show that $\{f, g\}$ is a basis for W.

(d) Find the coordinate vector of h with respect to the basis $\{f, g\}$.

(a) We need to find $a, b \in \mathbb{R}$ so that $h = af + bg$.

i.e. $x^2 - 12 = a(x-4) + b(x^2 - 3x)$

$= b x^2 + (a - 3b)x - 4a$

$\therefore$ $b = 1$ and $a = 3$; check:

$x^2 - 12 = 3(x-4) + (x^2 - 3x)$ ✓

$\in \text{span}\{x-4, x^2-3x\}$

(b) Suppose $a, b \in \mathbb{R}$ such that $af + bg = 0$.

Then $a(x-4) + b(x^2 - 3x) = 0$

$\Rightarrow bx^2 + (a-3b)x - 4a = 0$

$\Rightarrow b = 0$ & $-4a = 0$ by LI of $\{1, x, x^2\}$

$\Rightarrow a = b = 0$ is the only solution

$\Rightarrow \{f, g\}$ is LI.

(c) Since by (a), $h \in \text{span}\{f,g\}$, by a theorem from class, $W = \text{span}\{f, g, h\} = \text{span}\{f, g\}$.

By (b), $\{f, g\}$ is LI.

$\therefore$ $\{f,g\}$ is an LI spanning set of W

(Continue your answer on the next page if necessary.)

$\therefore$ $\{f,g\}$ is a basis of W.

(Continue your answer to Q4 on this page if needed.)

(d) From (a), we know that $h(x) = 3f(x) + g(x)$

$\therefore$ the coordinate vector of h with respect to the basis $B = \{f, g\}$ of W is

$$[h]_B = (3, 1).$$

5. (6 = 4 × 1.5 points) State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement. Then justify your answer:

- If you say the statement may be false, you must give an explicit counterexample (with numbers or functions!).
- If you say the statement is always true, then you CANNOT use an example to justify your response — you must give a clear general explanation using the theory from the course. If you use a theorem from class, you **must state the theorem**.

(a) Suppose $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^5$. If $2\vec{u} + 3\vec{v} + 0\vec{w} = \vec{0}$ then $\{\vec{u}, \vec{w}\}$ is linearly independent.

True or false: FALSE

Justification:

For example, if $\vec{u} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{w} = \vec{0}$

then $2\vec{u} + 3\vec{v} + 0\vec{w} = \vec{0}$ but $\{\vec{u}, \vec{w}\}$ is LD

because it contains the zero vector.

Moral: knowing one dependence relation is not the same as knowing ALL dependence relations.

(b) Let $U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \;\middle|\; a + 2b + 3c = 0 \quad \text{and} \quad d = 0 \right\}$. Then $\dim(U) = 2$.

True or false: TRUE

Justification:

$$U = \left\{ \begin{bmatrix} -2b-3c & b \\ c & 0 \end{bmatrix} \;\middle|\; b, c \in \mathbb{R} \right\} \quad \text{since } \begin{array}{l} a = -2b-3c \\ d = 0 \end{array}$$

$$= \left\{ b \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} \;\middle|\; b, c \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

These 2 vectors are not multiples of each other $\therefore$ LI. $\therefore$ U has an LI spanning set with 2 vectors $\Rightarrow \dim U = 2$.

(c) Suppose $\vec{u}, \vec{v} \in \mathbb{R}^3$. Then it is always possible to find a vector $\vec{w} \in \mathbb{R}^3$ such that the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is a basis for $\mathbb{R}^3$.

True or false: FALSE

Justification:

suppose $\vec{u} = \vec{0}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then for any choice of $\vec{w} \in \mathbb{R}^3$, the set $\{\vec{u}, \vec{v}, \vec{w}\}$ will be LD because it contains the $\vec{0}$.

$\therefore$ it can never be a basis.

(d) If $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent then $\{\vec{u}, \vec{w}\}$ must also be linearly independent.

True or false: TRUE

Justification:

Either: we showed a theorem in class that any subset of an LI set is also LI.

or: suppose $a\vec{u} + b\vec{w} = \vec{0}$.

then $a\vec{u} + 0\vec{v} + b\vec{w} = \vec{0}$.

Since $\{\vec{u}, \vec{v}, \vec{w}\}$ is LI, all these coefficients have to be 0.

$\therefore$ $a = 0$, $b = 0$ is the only solution

$\therefore$ $\{\vec{u}, \vec{w}\}$ is LI.

6. (Bonus, max 3 points) Let V be a vector space. Suppose $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a basis for a subspace U of V, and suppose $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is basis for a subspace W of V. Suppose further that $\vec{v}$ is a nonzero vector lying in both U and W. Carefully prove that if

$$X = \mathsf{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{v}\}$$

then $\dim(X) \leq 5$.

First note that $\vec{v} \in U$ means $\vec{v} \in \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ so by a theorem from class, we can reduce our spanning set for X by removing $\vec{v}$:

$$X = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{w}_1, \vec{w}_2, \vec{w}_3\}.$$

Next, since $\vec{v} \in U$ and $\vec{v} \in W$, we can write:

$$\vec{v} = a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3$$
$$\vec{v} = d\vec{w}_1 + e\vec{w}_2 + f\vec{w}_3.$$

$$\therefore \quad a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 = d\vec{w}_1 + e\vec{w}_2 + f\vec{w}_3.$$

Since $\vec{v} \neq \vec{0}$, at least one of the coefficients a, b or c is $\neq 0$. Let's renumber the vectors so that it's the first, so $a \neq 0$.

Then we have:

$$a\vec{u}_1 = -b\vec{u}_2 - c\vec{u}_3 + d\vec{w}_1 + e\vec{w}_2 + f\vec{w}_3$$
$$\Rightarrow \vec{u}_1 = -\frac{b}{a}\vec{u}_2 - \frac{c}{a}\vec{u}_3 + \frac{d}{a}\vec{w}_1 + \frac{e}{a}\vec{w}_2 + \frac{f}{a}\vec{w}_3$$
$$\Rightarrow \vec{u}_1 \in \text{span}\{\vec{u}_2, \vec{u}_3, \vec{w}_1, \vec{w}_2, \vec{w}_3\}.$$

So again, by the theorem of reducing spanning sets,

$$X = \text{span}\{\vec{u}_2, \vec{u}_3, \vec{w}_1, \vec{w}_2, \vec{w}_3\}.$$

Since X has a spanning set with 5 vectors, by thm from class, $\dim X \leq 5$.